Cohomological Hall algebras and nonabelian Hodge isomorphism for stacks

joint work with Ben Davison and Sebastian Schlegel Mejia.

C smooth projective curve $/\mathbb{C}$ genus g

$$\to \underset{\text{Betti}}{\mathcal{M}^{B}_{g,r,d}} \overset{\sim}{\longleftrightarrow} \mathcal{M}^{Dol}_{r,d}(C) \overset{\sim}{\longleftrightarrow} \mathcal{M}^{dR}_{r,d}(C)$$

$(\underset{\uparrow\, rk}{r}, \underset{\uparrow\, \text{degree}}{d}) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$.

NAHT All 3 spaces are homeomorphic.
in particular, they have isomorphic cohomology / BM homology

better behaved for singular or non compact spaces.

Questions: ① Can we compare the moduli stacks?
② Can we at least compare their BM homology?

Today: Yes for ②.

① Betti. $C \ni c$

$$\pi_1(C,c) = \langle x_i, y_i, 1 \leq i \leq g \mid \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1} = 1 \rangle.$$

character variety:

$$\mathcal{M}^{B}_{g,r,0} = \langle M_i, N_i \in GL_r^{2g} \mid \prod M_i N_i M_i^{-1} N_i^{-1} = I \rangle \,/\!\!/\, GL_r.$$

algebraic variety.

character stack $\mathfrak{M}^{B}_{g,r,0} = [R_{g,r,0}/GL_r]$.

studying the GL_r-equivariant geometry of $R^B_{g,r,0}$.

② Dolbeault C.

$(\mathcal{F}, \theta)$

v.b r, d Higgs field

$\theta : \mathcal{F} \to \mathcal{F} \otimes K_C$ $\mathcal{O}_C$-linear.

can. bundle

stability : $\mathcal{G} \subset \mathcal{F}$, inequality.

$g \geq 2$

$\mathcal{M}^{Dol}_{r,d}(C)$ alg. variety parametrising polystable Higgs bundles r, d.

* irreducible
* smooth when (r,d) coprime
* $2(g-1)r^2 + 2$
* $\mathcal{M}^{Dol}_{r,d}(C) = \underbrace{R^{Dol}_{r,d}(C)}// GL_r$

parametrises framed (sst) Higgs bundles.

* $\mathfrak{M}^{Dol}_{r,d}(C) = \left[\underbrace{R^{Dol}_{r,d}(C)}/ GL_r\right]$.

③ de Rham · $d = 0$

$(\mathcal{F}, \nabla)$

v.b. connection

$\nabla : \mathcal{F} \to \mathcal{F} \otimes K_C$

Leibniz rule.

$\mathcal{M}^{dR}_{r,0}(C) = R^{dR}_{r,0}(C) // GL_r$

$\mathfrak{M}^{dR}_{r,0}(C) := \left[R^{dR}_{r,0}(C) / GL_r\right]$.

NAHT $\quad d = 0.$

Hitchin
Corlette
Donaldson
Simpson
...

$$\begin{tikzcd}
 & R^B_{g,r,0} \arrow[d] \arrow[ddl, bend right, dash] \arrow[ddr, bend left, dash, "\text{homeomorphism.}"] & \\
 & \mathcal{M}^B_{g,r,0} & \\
\mathcal{M}^{Dol}_{r,0}(\mathbb{C}) \arrow[ur, dash, "\sim"] \arrow[rr, dash, "\sim", "\text{real analytic}"'] & & \mathcal{M}^{dR}_{r,0}(\mathbb{C}) \arrow[ul, "\text{Riemann-Hilbert}\ \text{complex analytic}"'] \\
R^{Dol}_{r,0}(\mathbb{C}) \arrow[u] \arrow[rr, dash] & & R^{dR}_{r,0} \arrow[u]
\end{tikzcd}$$

are set-theoretic GL_r-equivariant bijections (Simpson).

not homeo. (counterexample by Simpson).

$$H_*^{BM}(\mathfrak{M}^B_{g,r,0}) \cong H_*^{BM}(\mathfrak{M}^{dR}_{r,0}(\mathbb{C})).$$

Question: How to deal with the remaining arrows?

$$\begin{tikzcd}
\boxed{\mathfrak{M}^{Dol}_{r,0}(\mathbb{C})} \arrow[d, "JH^{Dol}"'] & \boxed{\mathfrak{M}^B_{g,r,0}} \arrow[d, "JH^B"] \\
\mathcal{M}^{Dol}_{r,0}(\mathbb{C}) \arrow[r, "\sim", "\varphi"'] \arrow[dr, "\pi"'] & \mathcal{M}^B_{g,r,0} \arrow[d, "\pi"] \\
 & pt
\end{tikzcd}$$

ver is some coh. shift.

$\mathcal{A}_r^{Dol} := JH_*^{Dol} \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}^{Dol}_{r,0}(C)} \in \mathcal{D}_c^+(\mathcal{M}^{Dol}_{r,0}(C))$
SI

$\mathcal{A}_r^B := JH_*^B \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{g,n,d}} \in \mathcal{D}_c^+(\mathcal{M}^B_{n,0}(C))$

we want to compare $\mathcal{A}_r^{Dol}$ and $\mathcal{A}_r^B$.

$$\mathcal{A}^{Dol} = \bigoplus_{r\geq 1} \mathcal{A}_r^{Dol} \qquad \mathcal{M}^{Dol}(C) := \bigsqcup_{r\geq 1} \mathcal{M}^{Dol}_{r,0}(C)$$

$$\mathcal{A}^B = \bigoplus_{r\geq 1} \mathcal{A}_r^B. \qquad \mathcal{M}_g^B := \bigsqcup_{r\geq 1} \mathcal{M}^B_{g,n,0}(C).$$

Thm (Davison - H - Schlegel Mejia)

1. we have a cohomological Hall algebra structure on $\mathcal{A}^\Delta$
$\boxed{\Delta \in \{Dol, B\}}$.

2. $BPS^\Delta_{Alg} := {}^p\mathcal{H}^0(\mathcal{A}^\Delta) \in \mathrm{Perv}(\mathcal{M}^\Delta) \leftarrow$ MHM.
is an algebra object.

3. $BPS^\Delta_{Alg} = U(BPS^\Delta_{Lie})$
$\uparrow$
enveloping alg

NAHT
IE complexes are topological invariants.

$$BPS^\Delta_{Lie} := \mathrm{Free}_{Lie}\left(\bigoplus_r \mathcal{IE}(\mathcal{M}^\Delta_{r,0})\right).$$

4. PBW - iso:

$$\mathrm{Sym}\left(BPS^\Delta_{Lie} \otimes \underbrace{H^*(B\mathbb{C}^*)}_{\mathbb{C}[u]}\right) \xrightarrow{\sim} \mathcal{A}^\Delta$$

monoidal structure $\mathcal{F}, \mathcal{G} \in \boxed{\mathcal{D}_c^+(\mathcal{M}^\Delta)}$

$$\mathcal{F} \boxdot \mathcal{G} := \oplus_*(\mathcal{F} \boxtimes \mathcal{G}) \qquad \oplus : \mathcal{M}^\Delta \times \mathcal{M}^\Delta \to \mathcal{M}^\Delta$$

Corollary: $\boxed{\mathcal{A}^{B} \cong \mathcal{A}^{Dol}} \in D_c^{+}(\mathcal{M}^{\Delta})$ $\Delta \in \{B, Dol\}$.

$$\begin{array}{c} \pi_* \mathcal{A}^{B} \cong \pi_* \mathcal{A}^{Dol} \\ \| \\ \boxed{H_*^{BM}(\mathfrak{M}^{B}_{g,\bullet}) \cong H_*^{BM}(\mathfrak{M}_{\bullet}^{Dol}(C))}. \end{array}$$

Essential ingredient: • local description of the maps $JH^{\Delta} : \mathfrak{M}^{\Delta} \to \mathcal{M}^{\Delta}$.

• comes from the fact that, in a precise sense,

the categories • $\mathrm{Higgs}^{\mu\text{-sst}}(C)$ are 2CY Abelian categories.

• $\mathrm{Rep}\, \pi_1(C, c)$

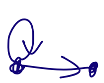
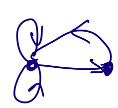
$x \in \mathcal{M}^{\Delta}$ $\mathcal{F} = \bigoplus_{i=1}^{s} \mathcal{F}_i^{m_i}$ $\mathcal{F}_i$ are pairwise nonisomorphic, simple. $m_i > 0$.

$\underline{\mathcal{F}} = \{\mathcal{F}_1, \dots, \mathcal{F}_s\}$

$\overline{Q}_{\underline{\mathcal{F}}}$ = Ext - quiver of $\underline{\mathcal{F}}$.

vertices $\underline{\mathcal{F}}$; $\#\{\mathcal{F}_i \longrightarrow \mathcal{F}_j\} = \dim \mathrm{Ext}^1(\mathcal{F}_i, \mathcal{F}_j)$.

[$\overline{Q}_{\underline{\mathcal{F}}}$ is the double of some quiver $Q_{\underline{\mathcal{F}}}$

$Q =$  $\rightsquigarrow$ $\overline{Q} =$ 

$\Pi_{Q_{\underline{\mathcal{F}}}}$ = preprojective algebra of $Q_{\underline{\mathcal{F}}}$.

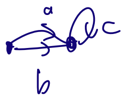
2CY categories
Ben Davison.

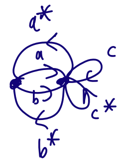
$$\begin{tikzcd} \mathfrak{M}^{\Delta} \arrow[d, "JH^{\Delta}"'] & U/GL_{m_i} \arrow[l] \arrow[r] \arrow[d] & \mathfrak{M}_{\Pi_{Q_{\underline{\mathcal{F}}}}, \underline{m}} \arrow[d, "JH"] \\ \mathcal{M}^{\Delta} & U/\!/GL_{m_i} \arrow[l] \arrow[r] & \mathcal{M}_{\Pi_{Q_{\underline{\mathcal{F}}}}, \underline{m}} \end{tikzcd}$$

s.t. horizontal maps are étale.

$Q \rightsquigarrow \Pi_Q = \mathbb{C}\overline{Q}/\langle \rho \rangle$ $\qquad \rho = [a,a^*] + [b,b^*] + [c,c^*] \in \mathbb{C}\overline{Q}$

↑
oriented graph



$\overline{Q} =$ 

$\mathcal{A}^{Dol}$ contains $H^{BM}_*(\mathcal{M}(r,d))$

B

$\mathcal{M}^{Dol}$