TD 5

$\mathfrak{g}$ abelian $\quad$ $\mathfrak{g}$ solvable $\quad$ · $\mathfrak{g}$ not semisimple. In particular $\mathfrak{g} = \mathbb{C}$ is not a simple Lie algebra by def.

ex 5.1: 1- $\mathfrak{g}$ semisimple. $\quad$ $\mathfrak{a} \subset \mathfrak{g}$ abelian ideal.

$$\mathfrak{a} \subset \mathfrak{r} = \text{radical} = 0$$

so $\mathfrak{a} = 0$.

Conversely assume no nontrivial abelian ideal.

$$\mathfrak{r} \subset \mathfrak{g}$$

If $\mathfrak{g}$ isn't semisimple, $\quad 0 \neq \mathfrak{r} \supset D\mathfrak{r} \supset D^2\mathfrak{r} \supset \cdots \supset D^s\mathfrak{r} = 0$

$$D^{s-1}\mathfrak{r} \neq 0.$$

decreasing sequence of ideals of $\mathfrak{g}$

$$0 \neq D^{s-1}\mathfrak{r} \text{ abelian ideal of } \mathfrak{g}.$$

$\mathfrak{a} \subset \mathfrak{g}$ ideal

$[\mathfrak{a}, \mathfrak{a}]$ ideal.

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Dor $\quad a, a' \in \mathfrak{a}, z \in \mathfrak{g}$

$[z, [a, a']]$

$= -[a, [a', z]]$

$- [a', [z, a]]$

$\in \mathfrak{a}$.

2- $\quad \Delta \subset \mathfrak{h}^*$ root system. Δ generates $\mathfrak{h}^*$.

Let $h \in \mathfrak{h}$, $\quad \alpha(h) = 0 \quad \forall \alpha \in \Delta$.

* $[h, \mathfrak{h}] = 0$ $\quad$ since $\mathfrak{h}$ is commutative

* $x \in \mathfrak{g}_\alpha, \alpha \in \Delta$, $\quad [h, x] = \alpha(h) \cdot x = 0$.

$\Rightarrow h \in Z(\mathfrak{g}). = 0 \therefore h = 0$. $\quad$ so $\quad \text{Vect}(\Delta) = \mathfrak{h}^*$.

ex 5.2: $\quad$ 2. $\quad [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} h_\alpha$ $\quad$, $h_\alpha \in \mathfrak{h}$

$$\alpha(h_\alpha) \neq 0.$$

$f \curvearrowright V \overset{v.s.}{=} \bigoplus V_\lambda$

$\cup$

$W = \bigoplus W_\lambda$

$W_\lambda = W \cap V_\lambda$.

If $0 \neq \mathfrak{a} \subset \mathfrak{g}$ abelian ideal.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

$\mathfrak{h} \overset{ad}{\curvearrowright} \mathfrak{a}$ action.

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{a} \cap \mathfrak{g}_\alpha)$$

If $\mathfrak{a} \cap \mathfrak{g}_\alpha \neq 0$ for some $\alpha \in \Delta$, then $\mathfrak{g}_\alpha \subset \mathfrak{a}$

$\mathbb{C} h_\alpha = [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \subset \mathfrak{a}$. So $h_\alpha \in \mathfrak{a}$. But $\underset{x \in \mathfrak{g}_\alpha \setminus \{0\}}{[h_\alpha, x]} = \overset{\overset{0}{\neq}}{\alpha(h)} \cdot x$

h_α, x do not commute, but $h_\alpha, x \in \mathfrak{a}$: contradiction.

So $\mathfrak{g}$ is semisimple.

ex 5.3 : $\Delta \subset \mathfrak{h}^*$ indec if not of the form $\Delta = \Delta_1 \sqcup \Delta_2$,
$\Delta_i \neq \phi$, $\forall \alpha, \beta \in \Delta$, $\alpha + \beta \notin \Delta \cup \{0\}$.

1- Δ indec $\Rightarrow$ $\mathfrak{g}$ simple

[$\mathfrak{g}$ not simple $\Rightarrow$ Δ decomposable.

$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_1, \mathfrak{g}_2 \neq 0$ and semisimple.
ideals of $\mathfrak{g}$.

$$\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \bigoplus_{\alpha \in \Delta_1} \mathfrak{g}_{1\alpha} \qquad \mathfrak{g}_2 = \mathfrak{h}_2 \oplus \bigoplus_{\alpha \in \Delta_2} \mathfrak{g}_{2\alpha}$$

$$\mathfrak{g} = (\mathfrak{h}_1 \oplus \mathfrak{h}_2) \oplus \bigoplus_{\alpha \in \Delta_1} \mathfrak{g}_{1\alpha} \oplus \bigoplus_{\alpha \in \Delta_2} \mathfrak{g}_{2\alpha}.$$

$$\Delta = \underset{\mathfrak{h}_1^*}{\underset{\cap}{\Delta_1}} \sqcup \underset{\mathfrak{h}_2^*}{\underset{\cap}{\Delta_2}} \subset \mathfrak{h}^* \qquad \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$$

$$\mathfrak{h}^* \simeq \mathfrak{h}_1^* \oplus \mathfrak{h}_2^*.$$

$\alpha \in \Delta_1$, $\beta \in \Delta_2$. Show that $\alpha + \beta \notin \Delta \cup \{0\}$.

$\alpha + \beta \neq 0$ since $\mathfrak{h}_1^* \cap \mathfrak{h}_2^* = 0$ so α, β are linearly independ

If $\alpha + \beta = \gamma \in \Delta = \Delta_1 \sqcup \Delta_2$

so . for ex. $\gamma \in \Delta_1$ and $\overset{\mathfrak{h}_1^*}{\overset{\cup}{\alpha - \gamma}} = \overset{\mathfrak{h}_2^*}{\overset{\cup}{-\beta}}$

but $\mathfrak{h}_1^* \cap \mathfrak{h}_2^* = (0)$.

2- Δ is indecomposable $\Leftrightarrow$ $\forall \alpha, \beta \in \Delta$, can find $\gamma_1, \dots, \gamma_s \in \Delta$

$$\underset{\alpha}{\underset{\parallel}{\gamma_1}} \gamma_2 \cdots\cdots \gamma_{s-1} \underset{\beta}{\underset{\parallel}{\gamma_s}} \qquad (*)$$

$$\gamma_i + \gamma_{i+1} \in \Delta \cup \{0\}$$

$\boxed{\Leftarrow}$

If $\Delta = \Delta_1 \sqcup \Delta_2$ decomposable. Take $\alpha \in \Delta_1$, $\beta \in \Delta_2$.

If $\exists\, \gamma_1 \dots \gamma_s$ as in the r.h.s of $(*)$,

$\exists i$

$$\underset{\Delta_1}{\underset{\cap}{\alpha}} = \gamma_1 \gamma_2 \cdot \underset{\Delta_1}{\underset{\cap}{\underset{\wr}{\gamma_i}}} \underset{\Delta_2}{\underset{\cap}{\gamma_{i+1}}} \gamma_s = \underset{\Delta_2}{\underset{\cap}{\beta}} \qquad \gamma_i + \gamma_{i+1} \in \Delta \cup \{0\}.$$

By def of a decomposition of Δ, cannot have this.

$\rightarrow$ no path between α and β.

have to prove

$\boxed{\Rightarrow}$ $\exists\, \alpha, \beta \in \Delta$, no path between α and $\Rightarrow$ Δ decomposable

Assume $\nearrow$.

Define $\Delta_1 = \{\alpha' \in \Delta \mid \exists \text{ path between } \alpha \text{ and } \alpha'\} \ni \alpha$

$\Delta_2 = \{\alpha' \in \Delta \mid \nexists \text{ path between } \alpha \text{ and } \alpha'\} \ni \beta$

$\Delta = \underset{\alpha}{\underset{\rotatebox{90}{$\in$}}{\Delta_1}} \sqcup \underset{\beta}{\underset{\rotatebox{90}{$\in$}}{\Delta_2}}$ Show: if $\gamma \in \Delta_1$, $\delta \in \Delta_2$,

$\gamma + \delta \notin \Delta \cup \{0\}$.

Assume $\gamma + \delta \in \Delta \cup \{0\}$.

$\alpha \xrightarrow{\gamma_i} \underbrace{\gamma\ \delta}_{\gamma+\delta \in \Delta\cup\{0\}}$ $\rightarrow$ path between α and δ: contradiction.

so $\gamma + \delta \notin \Delta \cup \{0\}$. and $\Delta = \Delta_1 \sqcup \Delta_2$ is a dec. of Δ

[$\mathfrak{g}$ ss $\rightarrow$ $\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_s$

ind. root syst., canonical decomposition.

$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$, $\mathfrak{g}_i$ simple.]

ex 5.4. V $\mathbb{C}$-vspace. $B \in (V \otimes V)^*$

1. $\mathfrak{o}_{V,B} = \{a \in \mathfrak{gl}(V) \mid B(au, v) + B(u, av) = 0 \quad \forall u, v \in V\}$

$\mathfrak{gl}(V) \curvearrowright (V \otimes V)^* \ni b$ $[a \in \mathfrak{gl}(V).\ a \cdot b(u,v) = -b(a\cdot u, v) - b(u, a \cdot v)]$

$\mathfrak{o}_{V,B} = \mathrm{Stab}_{\mathfrak{gl}(V)}(B) = \{a \in \mathfrak{gl}(V) \mid a \cdot B = B\}$.

sublie algebra.

2- Choose a basis of V $\{v_i\}_{1 \le i \le n}$, M the matrix of B in this basis

$M = (B(v_i, v_j))_{1 \le i,j \le n}$

$v \in V$ $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ x_i are coordinates of v in $\{v_i\}$,

$u \in V$

$B(u,v) = {}^t u B v \in \mathbb{C}$.

transpose of M

$\left(\overset{\curvearrowleft\ \curvearrowright}{\diagdown}\right) = a^T$

$$\mathfrak{o}_{n,M} = \{ a \in \mathfrak{gl}_n(\mathbb{C}) \mid a^T M + Ma = 0 \}$$

M is symm n.d.

3- Over alg. closed field : $\exists\, A \in GL_n(\mathbb{C})$ s.t.

$$A^T M A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

"any quad form over a.c. field is equivalent to $x_1^2 + \dots + x_n^2$"
of dim n, nondeg.

4. $$M = \begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix} \in \mathfrak{gl}_n(\mathbb{C}).$$

$\mathfrak{o}_{n,M} = \mathfrak{so}_n(\mathbb{C})$
orthogonal Lie algebra

$a \in \mathfrak{gl}_n$. $\quad a^T M + Ma = 0$

$\iff a + a' = 0$

a' = transpose of a w.r.t. the antidiagonal.

$$\begin{pmatrix} a & \diagup \\ & -a \end{pmatrix}$$

$$a = \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$$

$$\begin{pmatrix} x & z \\ y & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z & x \\ t & y \end{pmatrix}$$

$$Ma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} z & t \\ x & y \end{pmatrix}$$

$$\begin{pmatrix} z+z & t+x \\ t+x & y+y \end{pmatrix} = 0$$

5- $\mathfrak{so}_2 = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \in \mathfrak{gl}_2 \right\} \simeq \mathbb{C}$ not $\overset{\text{semi}}{\text{simple}}$.

6- $n \geq 3$. $\mathfrak{so}_n$ is semisimple.

$n = 2N+1$

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & a_N & & & & \\ & & & 0 & & & \\ & & & & -a_N & & \\ & & & & & \ddots & \\ 0 & & & & & & -a_1 \end{pmatrix}, a_1, \dots, a_N \in \mathbb{C} \right\}$$

$\subset \mathfrak{so}_n$

$n = 2N$.

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & & & & 0 \\ & \ddots & & & & \\ & & a_N & & & \\ & & & -a_N & & \\ & & & & \ddots & \\ 0 & & & & & -a_1 \end{pmatrix} \right\}$$

$\mathfrak{h}$ abelian subalgebra, $\subset \underset{\substack{\shortparallel_{ss} \\ \mathfrak{so}_n}}{\mathfrak{g}^{ss}}$, maximal.

$a_1, \dots, a_N$ s.t. $a_1, \dots, a_N, 0, -a_N, \dots, -a_1$ are distinct, take

$x \in \mathfrak{gl}_n$, $[x, \mathrm{diag}(a_1, \dots, a_N, 0, -a_N, \dots, -a_1)] = 0 \Rightarrow x$ diagonal

$\rightarrow \mathfrak{h}$ maximal: $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{so}_n$.

basis of $\mathfrak{h}^*$: $\varepsilon_1, \dots, \varepsilon_N \in \mathfrak{h}^*$ basis of $\mathfrak{h}^*$. $\varepsilon_i \begin{pmatrix} a_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & a_N & & & & \\ & & & 0 & & & \\ & & & & -a_N & & \\ & & & & & \ddots & \\ 0 & & & & & & -a_1 \end{pmatrix} = a_i$,

$n = 2N+1$.

$$\boxed{\varepsilon_i = -\varepsilon_{n+1-i}. \qquad \varepsilon_{\frac{n+1}{2}} = 0}$$

Find root space decomposition of $\mathfrak{so}_n$: eigenvectors for $\mathfrak{h} \circlearrowright \mathfrak{so}_n$.

$E_{ij} = i\begin{pmatrix} 0 & \overset{j}{\vdots} & 0 \\ \cdots & 1 & \\ 0 & & 0 \end{pmatrix}$ answer: $E_{ij} - E_{n+1-j, n+1-i}$ root: $\boxed{\varepsilon_i - \varepsilon_j}$

$1 \leq i, j \leq n$ $i \neq j$.

$$\mathfrak{so}_n = \mathfrak{h} \oplus \sum_{i \neq j} \mathbb{C} \cdot (E_{ij} - E_{n+1-j, n+1-i}) \longrightarrow \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} \cdots 1 & \\ & -1 \end{pmatrix} \end{array}$$

if $i \rightsquigarrow n+1-j$
$j \rightsquigarrow n+1-i$

roots of $\mathfrak{so}_n$: $n = 2N+1$ $\Delta_{\mathfrak{so}_n} = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i, -\varepsilon_i, \varepsilon_i + \varepsilon_j, -\varepsilon_i - \varepsilon_j \}$ $\boxed{1 \leq i, j \leq N, i \neq j}$

$\varepsilon_i - \varepsilon_j$ $1 \leq i, j \leq n$, $i \neq j$.

$\varepsilon_i - \varepsilon_j$ if $1 \leq i, j \leq N$

if $j = n+1-j'$

$\varepsilon_i + \varepsilon_{j'}$ $1 \leq j \leq N$

$n = 2N$ $\Delta_{\mathfrak{so}_n} = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j, -\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq N, i \neq j \}$

$\mathrm{Span}\, \Delta_{\mathfrak{so}_n} \ni \varepsilon_i$ $1 \leq i \leq N$.

$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \overset{?}{=} \mathbb{C} h_\alpha$ $\mathfrak{g}_\alpha = \mathbb{C} \cdot (E_{ij} - E_{n+1-j, n+1-i})$

$\mathfrak{g}_{-\alpha} = \mathbb{C} \cdot (E_{ji} - E_{n+1-i, n+1-j})$

$\alpha = \varepsilon_i - \varepsilon_j$

$1 \leq i, j \leq N$
$i \neq j$

$$h_\alpha = [E_{i\overset{\downarrow}{j}} - E_{n+1-j,n+1-i} \,,\, E_{ji} - E_{n+1-i,\overset{\downarrow}{n+1-j}}]$$

$$= E_{ii} + E_{n+1-j,n+1-j}$$
$$- E_{jj} - E_{n+1-i,n+1-i}$$

$\alpha = \varepsilon_i - \varepsilon_j$

$\alpha(h_\alpha) = 2 \neq 0.$

$\Rightarrow$ so_n is semi simple. If $n = 2$: something does not work.

$i=1$
$j=2$

$n=2.$ $E_{11} + E_{22} - E_{22} - E_{11} = 0,$

$7.$ $n = 3$: root system : $\Delta = \{\varepsilon_1, -\varepsilon_1\}.$ indecomposable.
$\varepsilon_1 - \varepsilon_1$ path.

$\Rightarrow$ Same root system as sl_2.

$sl_2 \simeq so_3.$

$$so_3 = \begin{pmatrix} a & c & 0 \\ d & 0 & -c \\ 0 & -d & -a \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$\rightarrow$ set the rels of sl_2

$$\begin{cases} [h,e] = 2e \\ [h,f] = -2f \\ [e,f] = h. \end{cases}$$

$\cdot$ $n \geqslant 5.$ $\qquad \alpha \quad -\alpha$

$n = 2N+1.$ $\quad \Delta so_n$

$n > 5$

$$\begin{tikzcd} \varepsilon_i - \varepsilon_j \arrow[rr, bend left, <->] & \varepsilon_j & \varepsilon_i \end{tikzcd}$$

$\varepsilon_i - \varepsilon_j \qquad \varepsilon$

$$\begin{tikzcd} \varepsilon_i+\varepsilon_j \arrow[rr, bend left] & -\varepsilon_k-\varepsilon_i & \varepsilon_j+\varepsilon_k \end{tikzcd}$$

$\varepsilon_j-\varepsilon_k \qquad \varepsilon_j-\varepsilon_i$

i, j, k distinct.

$\varepsilon_i+\varepsilon_j, \varepsilon_k-\varepsilon_i, \varepsilon_j-\varepsilon_k, \varepsilon_i-\varepsilon_j$

$\varepsilon_j+\varepsilon_k \qquad \varepsilon_j-\varepsilon_i \qquad \varepsilon_i-\varepsilon_k \qquad \varepsilon_i.$

$n \geq 5.$

cannot do this for $n=5$.

$\varepsilon_i+\varepsilon_j, -\varepsilon_j,$

$\varepsilon_i \qquad \varepsilon_i-\varepsilon_j.$

$n=5$: $=2.2\cdot 1$ $\quad \mathfrak{h}$ has dim 2. $\varepsilon_i, \varepsilon_j$

$\varepsilon_i+\varepsilon_j, -\varepsilon_i, \varepsilon_i-\varepsilon_j$

$\varepsilon_j \qquad -\varepsilon_j$

eg so_4. $\quad \Delta = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_1\} \overset{\Delta_{sl_2} \subseteq}{} \sqcup \{\varepsilon_1+\varepsilon_2, -\varepsilon_1-\varepsilon_2\} \overset{sl_2 \subseteq}{}.$

decomposition,

$\to \quad so_4 \simeq sl_2 \oplus sl_2$

explicit iso:

$x = E_{11}-E_{44}$

$y = E_{22}-E_{33}$

$e = E_{12}-E_{34} \qquad g = E_{31}-E_{42}$

$f = E_{13}-E_{24} \qquad h = E_{21}-E_{43}.$

commute $\langle x-y, e, h\rangle \simeq sl_2 \quad \Rightarrow$ explicit iso.

$\langle x+y, f, g\rangle \simeq sl_2.$

S.S.

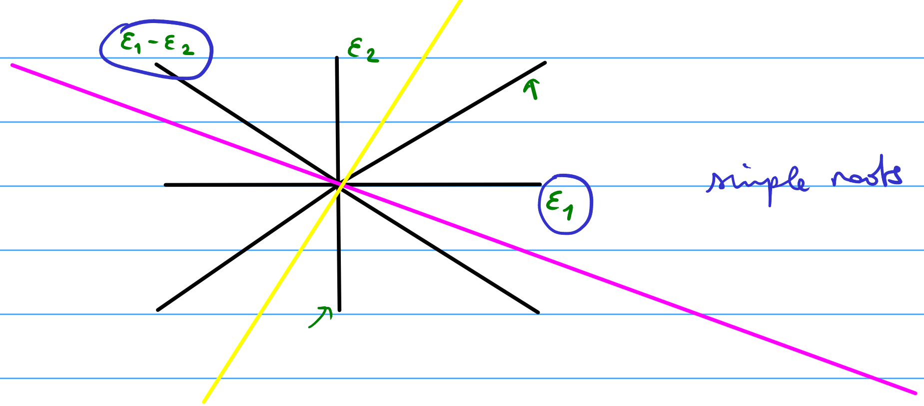
n

$$M = \begin{pmatrix} &&&&&1\\ &&&&\iddots&\\ &&&1&&\\ &&-1&&&\\ &\iddots&&&&\\ -1&&&&& \end{pmatrix} \quad \cdots$$

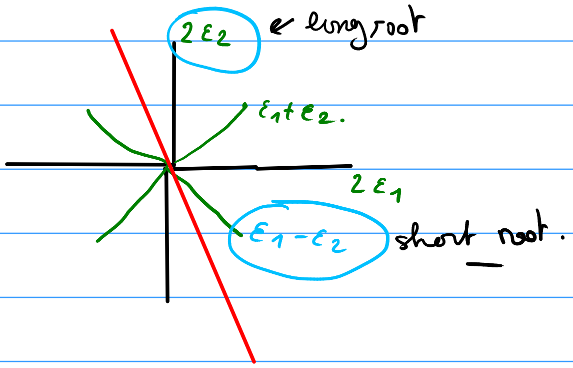
n

$\mathrm{rk}\, so_5 = \dim \mathfrak{h} = 2$

B_2 : so_5



C_2 : sp_4 . roots : $\{ 2\varepsilon_1 , 2\varepsilon_2 , \varepsilon_1 + \varepsilon_2 , -\varepsilon_1 + \varepsilon_2 , \dots \}$



5.4 $a , b \in o_{V,B}$ $[a,b] \overset{?}{\in} o_{V,B}$.

$u,v \in V$

$$B([a,b]u , v) + B(u, [a,b]v)$$

$$\shortparallel$$

$$B(abu,v) - B(bau, v) + B(u, abv) - B(u,bav)$$

$\shortparallel$ 0

$= 0$

$\Rightarrow [a,b] \in o_{V,B}$.

$V \circlearrowleft gl(V)$. $gl(V) \curvearrowright V \otimes V \ni u \otimes v$

$a \in gl(V)$

$a \cdot (u \otimes v) = au \otimes v + u \otimes av$.

and $gl(V) \curvearrowright V^*$: $a \cdot f = f(-a \cdot \ )$

$a \in gl(V)$ $f \in V^*$

$gl(V) \curvearrowright (V \otimes V)^*$: $a \cdot B = - B(a \cdot - , -) - B(-, a \cdot -)$

$$(a\cdot B)(u,v) = -B(au,v) - B(u,av).$$

$$\mathfrak{g}_{V,B} = \mathrm{Stab}_{\mathfrak{gl}(V)}(B) = \{a \in \mathfrak{gl}(V) \mid a\cdot B = 0\}$$

$$= \text{sub-Lie algebra.}$$

$\mathfrak{g} \curvearrowright \underset{\substack{\rotatebox{90}{$\in$} \\ w}}{W}$. $\quad \mathrm{Stab}_{\mathfrak{g}}(w)$ is a sublie alg of $\mathfrak{g}$

$$\|$$

$$\{a\in\mathfrak{g} \mid a\cdot w = 0\}.$$