Hall algebras and vertex algebras in enumerative geometry

April 11, 2023

Abstract

This is the list of topics for the workshop on Skye on "Hall and vertex algebras in enumerative geometry", April 17th – April 22nd 2023.

1 Organization

Participants can choose topics among the list of 16 topics below so that the topics can be distributed. In addition to the 16 topics, there will be 2 research talks and 3 night sessions, the latter of a more spontaneous nature. The night sessions will consist in question sessions on a paper or topic one or more participants have a deep knowledge of.

To make the process as smooth as possible and allow plenty of time for the preparation of the presentations, the topics will be attributed at regular intervals. We will let you know if some subjects have been unsucessful and still need a speaker. Participants are not expected to prepare more that one topic.

It could be nice to have, say a week in advance (April 10th), some notes for the talks. Therefore, we ask participants to send to the organizers a 4-page long description of their talk. These notes can then be shared with participants in advance to get an idea of the contents of the talks.

There will be one free afternoon for a hike. This day will be decided last minute to have the best possible weather.

The list of references can be completed and any suggestion is welcome.

2 Topics

2.1 Vertex algebras, definition and motivations (Magdalena)

1. Vertex algebras
2. Virasoro Lie algebra
3. Vertex operator algebras
4. Lie algebra associated to a vertex algebra

Possible references are [FB04; Joy21]. One can also refer to existing reading seminars or lecture series on the subject, for example

1. Reading seminar in Edinburgh in 2021, https://sites.google.com/view/readinggrouponvertexalgebras/home

2. Joyce's lecture series and notes, https://people.maths.ox.ac.uk/~joyce/VertexAlgebras2021/index.html

3. Book in progress of Arakawa and Moreau [AM21] https://www.imo.universite-paris-saclay.fr/~anne.moreau/CEMPI-arc_space-vertex_algebras.pdf, in particular the Chapter 3 for examples.

Also, it is good to refer to papers using vertex algebras in the context of enumerative geometry, as [BLM22].

1. Space of physical states?

2. What is the Borcherds Lie algebra of a lattice vertex algebra?

2.2 Vertex algebras, examples (Robert)

1. Commutative vertex algebras (=associative algebras with derivation)

2. Vertex algebras associated to affine Kac–Moody algebras

3. Lattice vertex algebras

4. Kac–Moody Lie algebras from lattice vertex Lie algebras

The references are the same as in §2.1.

2.3 Vertex coalgebras, vertex Lie algebras, vertex enveloping algebras (Felix Thimm)

1. Vertex coalgebras

2. Vertex Lie algebras

3. Vertex enveloping algebras

Possible references are [Lat21; Pri99]

2.4 Quantization of vertex algebras (Felix Kung)

Goal: explaining the quantization of vertex algebras as in [EK00].

2.5 Arc spaces and Poisson vertex algebras (Nikola)

1. definition of arc spaces

2. arc spaces and vertex algebra structure

Reference: Chapter 1 and 4 in the book [AM21]https://www.imo.universite-paris-saclay.fr/~anne.moreau/CEMPI-arc_space-vertex_algebras.pdf

2.6 Double Poisson vertex algebras (Thibault)

Goal: explaining the contents of the paper [DKV15], and in the unlikely case there is time left, the multiplicative version. [FV21].

2.7 W-algebras (Alyosha)

1. W-algebras
2. $W_k(\mathfrak{gl}_r)$ and its current algebra [SV13, §8.1-8.4] and references therein.

A reference may be [Ara17]

2.8 Vertex algebra structure on the (co)homology of moduli stacks (Marco)

Goal: explaining how to obtain a vertex algebra structure as in [Joy21, §4] and and to construct a Lie algebra on the (co)homology of the rigidified moduli stack.

2.9 Wall-crossing with Joyce's Lie algebra (Sebastian)

Explain the construction of the class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}$ and the idea behind wall-crossing following [Joy21].

2.10 Shuffle realizations of quantum groups (Shivang)

1. Quotient realization of quantum groups (as for example in [Lus10, Chapters 1, 2]).
2. Shuffle algebra realization of quantum groups [Ros98]
3. More recent shuffle algebra realizations of quantum groups following [Neg21] (it would be nice to see the definition of "slope subalgebras"), [NSS21; Neg21].

2.11 CoHA of a quiver and shuffle algebras (Patrick)

1. CoHA of a quiver (without potential)
2. Explicit formula for the product

Possible references are [Lat21; KS10]

2.12 CoHA of a preprojective algebra and shuffle algebras (Noah)

1. Preprojective algebra of a quiver
2. Cohomological Hall algebra of a preprojective algebra [YZ18; SV20; DHM22].
3. Embedding in a shuffle algebra [SV20].

2.13 CoHA of a quiver with potential (Tanguy)

1. Vanishing cycle sheaf, critical cohomology, Thom-Sebastiani theorem
2. Critical cohomological Hall algebra

The references are [KS10; DM20].

2.14 The BPS Lie algebra (Sarunas)

1. Perverse filtration on the critical cohomological Hall algebra
2. The relative BPS Lie algebra
3. Yangian-type PBW theorem for the critical CoHA

The references are [DM20; Dav20].

2.15 CoHAs and vertex (co)algebras (Woonam)

Explain the contents of the paper [Lat21], in particular the interactions between vertex algebras, vertex coalgebras, and cohomological Hall algebra structure on the (co)homology of the stack of representations of a quiver.

2.16 AGT conjectures, quantum AGT conjectures (Alyosha)

The idea is to explain what are the AGT conjectures and how to prove them following Schiffmann–Vasserot [SV13].

1. Affine W-algebra of $\mathfrak{gl}_r$ at some level [SV13]
2. Action of the W-algebra on the cohomology [SV13]
3. Quantum version [Neg18]

This talk is by nature more an overview of the AGT conjectures as it is a deep subject.

3 Research talks

3.1 Arkadij Bojko: Wall-crossing for Calabi-Yau fourfolds and applications

Abstract: As will be discussed in multiple other talks during the workshop, Joyce's vertex algebras are a powerful new ingredient added to the existing theory of wall-crossing for sheaves on surfaces. My work focuses on proving wall-crossing in two dimensions higher, in which case the enumerative invariants were defined for Calabi-Yau fourfolds. In this setting, there are many open conjectures with a scarce amount of tools to address them, so it is desirable that the end result can have many concrete applications. For this purpose, I introduce yet another new structure into the picture - formal families of vertex algebras. Apart from being a natural extension of the theory, they allow to wall-cross with insertions instead of just the plain virtual fundamental classes.

To make the whole machinery work with (polynomial) Bridgeland stability conditions and sheaf-counting classes for fourfolds, I require a different approach to the proofs used in the surface case. In the talk, I will discuss the main difficulties that I encountered, and I will present examples using the complete package.

Slides of the author on the subject: https://drive.google.com/file/d/1-AKEgRgOEZI6iEqCjdu7HSA0NE29RmiP/view

3.2 Chenjing Bu: Counting sheaves on curves

Abstract: Counting vector bundles on algebraic curves has been a classical problem. These counting invariants were first computed by Witten using physical methods, and his formulae were later proved by Jeffrey and Kirwan. In this talk, I will introduce a new viewpoint towards this problem, using Joyce's enumerative invariants obtained via his vertex algebra. This also extends the invariants to the case where the rank and degree are not coprime. Computing these invariants using Joyce's wall-crossing formulae reveals an interesting new structure. Namely, the invariants can be expressed as a divergent infinite sum, which can be assigned a finite value via a regularization process.

4 Open questions sessions

The participants can ask some questions about the subjects below. The speakers can choose the questions for which they would like to give an answer. This should also benefit to the speakers and so suggestions or ideas are welcome. Some very rough examples cooked late in the evening could be:

1. You use the hypothesis H in a crucial way to prove A, can you detail a bit more how it is used?
2. I have in mind the situation of a quiver with relations. Is it possible to adapt your formalism to this situation?
3. All the categories you consider are of small homological dimension. Is there something to say or expected in higher dimensions?
4. You consider various cohomology theories. What about Morava K-theories?

I have no idea whether these questions make sense or not. The organizers will make sure that this does not turn into an examination!

4.1 Virasoro constraints, Arkadij, Woonam

Participants are encouraged to get an idea of what Virasoro constraints are and gather questions they could have.

Some references are: [BLM22], with a video: https://www.youtube.com/watch?v=o80E6psv3_M we encourage people to watch, [Mor+20], [Mor22], [Van23], [Giv01], [Tel12], one other seminar talk: https://www.youtube.com/watch?v=qqL6wzNboh4.

We recommend to concentrate on the first three references for the question session.

4.2 Counting objects in categories, Chenjing

Chenjing has two very interesting papers on enumerative geometry of categories of geometric nature: [Bu22; Bu23]. We encourage people to have a look at them and think about questions they could have.

4.3 Cohomological Hall algebras, Ben

CoHAs became a big subject. We have several talks by participants on them but it is worth having a question session. A partial list of references is [KS10; SV20; SV17; DM20; Dav20]. You may as well search in the arXiv "CoHA".

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