BPS Lie algebra of 2CY categories and positivity of cuspidal polynomials

w/ Ben Davison and Sebastian Schlegel Mejia in Edinburgh

$A =$ finitely presented algebra / field $k = \{ \mathbb{C}, \mathbb{F}_q$

$$= \frac{k\langle x_1, \dots, x_n \rangle}{\langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle}$$

module variety / representation space

$d \in \mathbb{N}$, $\quad X_{A,d} = \{ M_1, \dots, M_n \in \mathrm{Mat}_{d\times d}(k) \mid f_1(M_1, \dots, M_n) = \dots = f_m(M_1, \dots, M_n) = 0 \}$

$\circlearrowleft$

$GL_d(k)$ by simultaneous conjugation.

Question : what (algebraic structures) can we extract from the geometry of $X_{A,d} \circlearrowleft GL_d$?

<u>Various answers</u>, governed by the homological dimension of A and the geometric objects considered.

$\rightsquigarrow$ it could be interesting to study the geometric aspects of the action of functors defined algebraically, e.g. reflection functors, translation functors.

$\mathfrak{M}_{A,d} = X_{A,d} / GL_d \quad$ " quotient stack", parametrise all representations of A

$\downarrow JH$

$\mathcal{M}_{A,d} = X_{A,d} /\!/ GL_d \quad$ space of semisimple representations of A = closed GL_d-orbits in $X_{A,d}$.

<u>Example</u>: $A = \mathbb{C}[x] = \mathbb{C}\circlearrowleft$ h. dim $= 1$

$$\begin{tikzcd} M \arrow[d, mapsto] & \mathfrak{M}_{A,d} \arrow[d] & = \mathrm{Mat}_{d\times d}/\mathrm{GL}_d \\ \text{coefficients of } \chi_M & \mathcal{M}_{A,d} & = \mathrm{Mat}_{d\times d}/\!/\mathrm{GL}_d = \mathbb{C}^n/\mathfrak{S}_n \simeq \mathbb{C}^n \end{tikzcd}$$

$A = \mathbb{C}[x,y] = \Pi_{\circlearrowleft}$ h. dim 2

$$\begin{tikzcd} (M,N) \arrow[d, mapsto] \in & \mathfrak{M}_{A,d} \arrow[d] & = \{(M,N) \in \mathrm{Mat}_{d\times d} \mid [M,N]=0\}/\mathrm{GL}_d \\ \text{coefficients of } (\chi_M, \chi_N). & \mathcal{M}_{A,d} & = (\mathbb{C}^2)^n/\mathfrak{S}_n \quad \text{symplectic singularity} \end{tikzcd}$$

In general: more complicated to have an explicit description

<u>biased (and incomplete list of) references:</u>

$A = \mathbb{C}Q$, Q quiver: Reineke, Hille, King, more generally: A smooth

$A = \Pi_Q$, Q quiver Crawley-Boevey, Nakajima, more generally: A 2CY
Geiss, Leclerc, Schröer, ...

as in David's talk for $n=2$.

In this talk, we start with a quiver.

$Q = (Q_0, Q_1)$ quiver

$\mathbb{F}_q$; $H_{Q,\mathbb{F}_q}$ Hall algebra bialgebra

$H^{cusp}_{Q,\mathbb{F}_q} = \bigoplus_{d \in \mathbb{N}^{Q_0}} H^{cusp}_{Q,\mathbb{F}_q}[d]$ primitive elements

Π_Q preprojective algebra

$\mathcal{M}_{\Pi_Q} = \bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathcal{M}_{\Pi_Q, d}$ moduli space of semisimple reps of Π_Q

$\mathcal{A}_{\Pi_Q} := \bigoplus_{d \in \mathbb{N}^{Q_0}} H^{BM}_*(\mathfrak{M}_{\Pi_Q}, \mathbb{Q})$ cohomological Hall algebra

$\cup$

$BPS_{\Pi_Q, Alg} \supset BPS_{\Pi_Q, Lie}$ constructed using filtrations

$C_{Q,d}(q) := \dim_{\mathbb{C}} H^{cusp}_{Q,\mathbb{F}_q}[d] = IP(\mathcal{M}_{\Pi_Q,d}) := \sum_{\ell \in \mathbb{Z}} \dim IH^i(\mathcal{M}_{\Pi_Q,d}) t^\ell$ intersection Poincaré polynomial

$\begin{cases} t = q^{-1/2} \\ d \in \Sigma_{\Pi_Q} \end{cases}$

$\in \mathbb{Q}[q]$

Bozec-Schiffmann

$\in \mathbb{N}[q]$

$BPS_{\Pi_Q, Lie}$ is a GKM algebra generated by $IH(\mathcal{M}_{\Pi_Q, d})$ $d \in \Sigma_{\Pi_Q}$ with $(\mathbb{N}^{Q_0} \times \mathbb{Z})$-character

$\mathrm{ch}\, BPS_{\Pi_Q, Lie} = \sum_{d \in \mathbb{N}^{Q_0}} \underbrace{A_{Q,d}(t^{-2})}_{\in \mathbb{N}[t, t^{-1}]} z^d$

<u>Upshot</u>:

$\mathfrak{n}^+$ pos. part of GKM.

- $U(\mathfrak{n}^+)$ is a bialgebra with primitive elements $\mathfrak{n}$
- $U_q(\mathfrak{n}^+)$ ——— $\mathfrak{n}^+ / [\mathfrak{n}^+, \mathfrak{n}^+]$
$\cong$
$[\mathfrak{n}^+, \mathfrak{n}^+]^\perp$ for some scalar product on $\mathfrak{n}^+$.

Cuspidal polynomials

$Q=(Q_0,Q_1)$

$$H_{Q,\mathbb{F}_q} = \mathrm{Fun}_{\mathbb{C}}\left(\mathrm{Rep}(Q,\mathbb{F}_q)/\sim, \mathbb{C}\right)$$

Ringel Hall algebra
[geometric construction of quantum groups]

product, coproduct :

"$p_* r^*$" "$r_* p^*$"

$$\begin{tikzcd}
 & \mathcal{E}xact(Q,\mathbb{F}_q) \arrow[dl, "r"'] \arrow[dr, "p"] & \\
\mathrm{Rep}(Q,\mathbb{F}_q)\times\mathrm{Rep}(Q,\mathbb{F}_q) & & \mathrm{Rep}(Q,\mathbb{F}_q)
\end{tikzcd}$$

$$0\to M\to E\to N\to 0 \quad \mapsto (M,N), \quad \mapsto E$$

Δ = coproduct

$$H^{cusp}_{Q,\mathbb{F}_q} = \{ f\in H_{Q,\mathbb{F}_q} \mid \Delta f = f\otimes 1 + 1\otimes f\}$$

cuspidal functions

Thm (Bozec-Schiffmann 2017)

$$C_{Q,d}(q) := \dim_{\mathbb{C}} H^{cusp}_{Q,\mathbb{F}_q}[d] \in \mathbb{Q}(q)$$

$$\in \mathbb{Z}[q] \quad \text{if } \langle d,d\rangle_Q \neq 0$$

Euler form of Q

They conjectured $C_{Q,d}(q)\in\mathbb{N}[q]$ if $\langle d,d\rangle_Q<0$ (known if $=1$)

Theorem (Davison-H-Schlegel Mejia): The conjecture holds.

This is a refinement of the positivity property for Kac polynomials.

M-graded characters

$\mathbb{N}^{Q_0} \ni$ $d \in \mathbb{N}^{Q_0}$.

abs. indec

$$A_{Q,d}(q) = \# \{ \text{d-dim rep of } Q / \mathbb{F}_q \} / \text{iso} \qquad \in \mathbb{N}[q]$$

Hausel-Letellier-Rodriguez Villegas 2013

$$\mathrm{ch}\, H_{Q,\mathbb{F}_q} = \underbrace{\mathrm{Exp}_{z,q}}_{} \left(\sum_{d \in \mathbb{N}^{Q_0}} A_{Q,d}(q) z^d \right)$$

plethystic exponential : sends ch V on ch(Sym V).

Describing the algebra structure of $H_{Q,\mathbb{F}_q}$: Sevenhant - Van den Bergh 2001 (not for today)

Positive part of generalised Kac–Moody algebras

M, $(-,-) : M \times M \to \mathbb{Z}$ monoid with bilinear form

$M = \mathbb{N}^{Q_0}$, $(-,-)$ = symmetrised Euler form of Q

= Euler form of Π_Q - preprojective algebra of Q

$R^+ = \{ m \in M \mid (m,m) \leq 2 \}$ positive roots

$$\Sigma = \left\{ m \in R^+ \;\middle|\; \forall\, m = \sum_{j=1}^{\ell} m_j \text{ nontrivial},\ 2-(m,m) > \sum_{j=1}^{\ell} \left(2-(m_j,m_j)\right) \right\}$$ primitive simple positive roots

$$\phi^+ = \Sigma \cup \left\{ \ell m : \begin{array}{l} m \in \Sigma,\ (m,m)=0 \\ \ell \geq 1 \end{array} \right\}$$ simple positive roots.

Cartan matrix: $\left((m,n)\right)_{m,n \in \phi^+}$

Assumptions: $\begin{cases} (m,m) = 2 & \text{if } (m,m) > 0 \\ (m,n) \leq 0 & \text{if } m \neq n \end{cases}$

If V is a $\Phi^+ \times \mathbb{Z}$-graded vector space

$$\mathrm{ch} V = \sum_{m \in \Phi^+} \underbrace{\dim V[m]}_{\in\, \mathbb{N}[t,t^{-1}]} z^m$$

$\mathbb{N}[t,t^{-1}]$ encodes the $\mathbb{Z}$-grading.

we construct a Lie algebra $\mathfrak{n}_V$.

It is generated by V with the relations

$$\begin{cases} [v,w]=0 & \text{if } (\deg_m v, \deg_n(w)) = 0 \\ \mathrm{ad}(v)^{1-(m,n)}(w) = 0 & \text{if } (\deg_n(v), \deg_n(v)) = 2. \end{cases}$$

$\rightsquigarrow$ positive part of a generalised KM Lie algebra.

$$\mathrm{ch}\, \mathfrak{n}_V = \sum_{m \in \Lambda} \dim \mathfrak{n}_V[m]\, z^m \quad \text{varies with } V.$$

Question : what are its possible values?

<u>Theorem</u> (Bozec-Schiffmann)

If $\exists V$ such that $\mathrm{ch}\, \mathfrak{n}_V = \sum_{d \in \mathbb{N}^{Q_0}} A_{Q,d}(t) z^d$, then

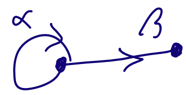
$\mathbb{N}^{Q_0} \times \mathbb{Z}$-graded

$C_{Q,d}(q) = \dim_{\mathbb{C}} V[d]$. if $\langle d,d \rangle \neq 0$.

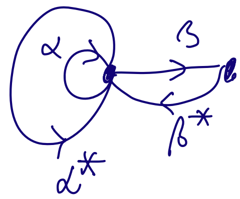
To solve the positivity conjecture, it suffices to find such a V.

Cohomological Hall algebras

$Q = (Q_0, Q_1)$ quiver



$\overline{Q}$ doubled quiver



$\rho = [\alpha, \alpha^*] + [\beta, \beta^*] \in \mathbb{C}\overline{Q}$ preprojective relation

$\Pi_Q = \mathbb{C}\overline{Q} / \rho$ preprojective algebra

$$X_{\overline{Q}, d} = T^* X_{Q,d} \xrightarrow{\mu_d} \mathfrak{gl}_d \qquad \text{moment map.}$$

$$(x_\alpha, x_{\alpha^*})_{\alpha \in Q_1} \longmapsto \sum_{\alpha \in Q_1} [x_\alpha, x_{\alpha^*}]$$

$\mu_d^{-1}(0)$ is the rep space of d-dimensional reps of Π_Q.

$\mathfrak{M}_{\Pi_Q, d} = \mu_d^{-1}(0) / GL_d$ stack of reps of Π_Q.

$$\begin{tikzcd} \mathfrak{M}_{\Pi_Q, d} \arrow[d, "JH"'] \\ \mathcal{M}_{\Pi_Q, d} \end{tikzcd}$$

$\mathcal{M}_{\Pi_Q, d}$ moduli space of reps of Π_Q

$$\mathcal{A}_{\Pi_Q} = \bigoplus_{d \in \mathbb{N}^{Q_0}} H^{BM}(\mathfrak{M}_{\Pi_Q, d}, \mathbb{Q}^{vir}) \quad + \quad \text{algebra structure}$$

$$\begin{tikzcd} & \mathfrak{Exact}_{\Pi_Q} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathfrak{M}_{\Pi_Q} \times \mathfrak{M}_{\Pi_Q} & & \mathfrak{M}_{\Pi_Q} \end{tikzcd}$$

"$q_* p^*$" gives $\mathcal{A}_{\Pi_Q}$ an associative multiplication.

$$\mathcal{A}_{\Pi_Q} = H^*(\underbrace{JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\Pi_Q}}}_{\in \mathcal{D}^+_c(\mathcal{M}_{\Pi_Q})})$$

Constructing p^* is a difficult task.

Question: Describing this algebra: open question.

Partial answers ① it look like $U(\mathfrak{n}_{\Pi_Q}[u])$ for some Lie algebra $\mathfrak{n}$ (Davison)

② $U(\mathfrak{n}) \hookrightarrow \mathcal{A}_{\Pi_Q}$ is recovered as

$H^*({}^{p}\mathcal{H}^0(\mathcal{A}_{\Pi_Q})) \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$ (heart of a t-structure on $\mathcal{D}^+_c(\mathcal{M}_{\Pi_Q})$)

(Davison-H-Schlegel Mejia)

Thm ① $\mathfrak{n}$ is isomorphic to the generalised Kac-Moody Lie algebra associated to

$$\left(\mathbb{N}^{Q_0}, (-,-), \bigoplus_{d \in \Phi^+} IC(\mathcal{M}_{\Pi_Q, d})\right).$$

symm Euler form

(Davison + ε)

② $$\mathrm{ch}\, \mathfrak{n} = \sum_{d \in \mathbb{N}^{Q_0}} A_{Q,d}(q^{-2}) z^d.$$

Corollary $\left[\begin{array}{l} C_{Q,d}(q) \in \mathbb{N}[q] \quad \text{for } \langle d,d\rangle_Q < 0 \\ = \displaystyle\sum_{j \in \mathbb{Z}} \mathbb{H}^j(\mathcal{M}_{\Pi_Q,d})\, q^{-j/2}. \end{array}\right.$

Proof of the theorem (in a random order)

- Work with $\mathcal{A}_{\Pi_Q} \in D^+_c(\mathcal{M}_{\Pi_Q})$ constructible complex

  and $\mathcal{BPS}_{\Pi_Q} \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$

  algebra objects for the monoidal structure given by

  $\mathcal{F} \boxdot \mathcal{G} := \oplus_*(\mathcal{F} \boxtimes \mathcal{G})$

  $\oplus : \mathcal{M}_{\Pi_Q} \times \mathcal{M}_{\Pi_Q} \to \mathcal{M}_{\Pi_Q}$ direct sum of semisimple modules.

- neighbourhood theorem:

  $M = \bigoplus_{i=1}^{r} M_i^{\oplus m_i} \in \mathrm{Rep}(\Pi_Q)$ semisimple ; $\dim M = d$.

  $x \in \mathcal{M}_{\Pi_Q,d}$ corresponding to M

$\underline{M} = \{M_1, \dots, M_r\}$

Ext - quiver of $\underline{M}$: $\overline{Q}_{\underline{M}} = (\underline{M}, \text{arrows})$

$\# \mathrm{Ext}^1(M_i, M_j)$ arrows $M_i \to M_j$

Choose $Q_{\underline{M}}$ such that its double is $\overline{Q}_{\underline{M}}$.

$$\begin{tikzcd}
\mathfrak{M}_{\Pi_Q,d} \arrow[d] & U/G \arrow[l, hook'] \arrow[r, hook] \arrow[d] \arrow[dl, phantom, "\lrcorner"] \arrow[dr, phantom, "\llcorner"] & \mathfrak{M}_{\Pi_{Q_{\underline{M}}},m} \arrow[d] \\
\mathcal{M}_{\Pi_Q,d} & U/\!/G \arrow[l, hook', "i_{\underline{M}}"', "\text{étale neighbourhood of } x"] \arrow[r, hook, "i_0", "\text{étale neighbourhood of } 0_m"'] & \mathcal{M}_{\Pi_{Q_{\underline{M}}},m}
\end{tikzcd}$$

- $i_{\underline{n}}^* \mathcal{BPS}_{\Pi_Q, d} \cong i_0^* \mathcal{BPS}_{\Pi_{Q_n}, m}$.
- Compatibility of CoHA multiplications
- Define GKM Borcherds algebras in the symmetric monoidal category $(\mathrm{Perv}(\mathcal{M}_{\Pi_Q}), \boxdot)$
- There is a GKM in this category, Bor_{Π_Q}, associated to $\left(\mathbb{N}^{Q_0}, (-,-)_{\Pi_Q}, \bigoplus_{d \in \Phi^+} \mathcal{SE}(\mathcal{M}_{Q,d})\right)$, with a canonical morphism $\mathrm{Bor}_{\Pi_Q} \xrightarrow{\Psi_Q} \mathcal{BPS}_{\Pi_Q}$
- It is an isomorphism: If not we take $\mathcal{F} \in \ker \Psi \oplus \operatorname{coker} \Psi$. $x \in \operatorname{supp} \mathcal{F}$ such that $i_x^! \mathcal{F} \neq 0$.
  we pull back to $U/\!/G$: Ψ_{Q_n} is not an isomorphism.

$\rightarrow$ induction on Q by iterating this procedure

$\rightarrow$ We eventually use the description of the top-CoHA of the strictly seminilpotent stack.

<u>Theorem (H)</u>: $H_0^{\mathrm{BM}}\left(\mathfrak{M}_{\Pi_Q}^{\mathrm{SSN}}, \mathbb{Q}^{\mathrm{vir}}\right) \simeq \mathcal{U}(\underbrace{\mathfrak{n}_Q^+})$

some GKM defined by Bozec in his thesis