Lecture 3: brief summary

$$\underset{\text{Abelian}}{\mathcal{E}} \subset \underset{\text{dg}}{\mathcal{D}}$$

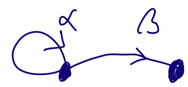
$$\underset{\text{1-Artin}}{\mathfrak{M}_{\mathcal{E}}} \underset{\text{open}}{\subset} \mathfrak{M}_{\mathcal{D}}$$

$$\mathfrak{M}_{\mathcal{E}} \xrightarrow{JH} \mathcal{M}_{\mathcal{E}}$$

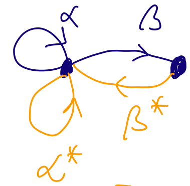
Today.

$\mathcal{E} = \operatorname{rep} \Pi_Q$ preprojective algebra

$Q = (Q_0, Q_1)$ quiver



$\overline{Q}$ double



$$\rho = [\alpha, \alpha^*] + [\beta, \beta^*] \in \mathbb{C}\overline{Q}$$

$$\Pi_Q = \mathbb{C}\overline{Q} / \langle\langle \rho \rangle\rangle \quad \text{preprojective algebra.}$$

$d \in \mathbb{N}^{Q_0}$

$$X_{\overline{Q},d} = T^*\left(\bigoplus_{\alpha \in Q_1} \operatorname{Hom}_{\mathbb{C}}\left(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}}\right)\right)$$

$$GL_d := \prod_{i \in Q_0} GL_{d_i}$$

$$\mu_d : X_{\overline{Q},d} \longrightarrow \mathfrak{gl}_d$$

$$(x_\alpha, x_{\alpha^*})_{\alpha \in Q_1} \mapsto \sum_{\alpha \in Q_1} [x_\alpha, x_{\alpha^*}] .$$

$$\mathfrak{M}_{\Pi_Q,d} := \left[\mu_d^{-1}(0) / GL_d \right]$$

$$\Big\downarrow JH$$

$$\mathcal{M}_{\Pi_Q,d} := \mu_d^{-1}(0) /\!\!/ GL_d .$$

$$JH_{\mathcal{E}} : \mathfrak{M}_{\mathcal{E}} \longrightarrow \mathcal{M}_{\mathcal{E}}$$

$$\mathcal{A}_{\mathcal{E}} := JH_* \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathcal{E}}} \in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{E}})$$ sheafified CoHA

$$^{p}\mathcal{H}^0(\mathcal{A}_{\mathcal{E}}) =: \mathcal{BPS}_{\mathcal{E},\mathrm{Alg}} \in \mathrm{Perv}(\mathcal{M}_{\mathcal{E}})$$ sheafified BPS algebra

BPS algebra by generators and relations

Theorem A (Davison–H–Schlegel Mejia, 2023)

$$\mathcal{BPS}_{\Pi_Q,\mathrm{Alg}} \cong U(\mathfrak{n}^+_{\Pi_Q}) \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$$

where $\mathfrak{n}^+_{\Pi_Q} \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$ is the positive part of a GKM generated by

$$\begin{cases} \mathcal{IC}(\mathcal{M}_{\Pi_Q,d}) & d \in \Sigma_{\Pi_Q} \\ \mathcal{IC}(\mathcal{M}_{\Pi_Q,d}) & d \in \Sigma_{\Pi_Q},\ \ell \geq 2, \\ & (d,d)_Q = 0. \end{cases}$$

$$(-,-)_Q : \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_0} \longrightarrow \mathbb{Z}$$ Euler form of the dg-preprojective algebra $\mathcal{G}_2(Q)$, determines the relations.

$(-,-)_Q$ is symmetrized Euler form of Q.

PBW theorem

Theorem B (Davison - H- Schlegel Mejia)
The PBW map
$$\mathrm{Sym}_{\boxdot}\left(\mathcal{BPS}^+_{\Pi_Q} \otimes H^*_{\mathbb{C}^*}(pt)\right) \longrightarrow \mathcal{A}_{\Pi_Q}$$
is an isomorphism in $\mathcal{D}^+_c(\mathcal{M}_{\Pi_Q})$.

Today: lecture 4 : VII – The strictly seminilpotent CoHA
VIII – Applications
- → Nonabelian theory for stacks
- → Positivity of cuspidal polynomials of quivers
- → Decomposition of the cohomology of Nakajima quiver varieties.

VII - The strictly seminilpotent CoHA

$Q = (Q_0, Q_1)$ quiver $\qquad \Pi_Q$ preprojective algebra

$$JH : \mathfrak{M}_{\Pi_Q} \longrightarrow \mathcal{M}_{\Pi_Q} \qquad \text{good moduli space.}$$

① 2d CoHA for a saturated submonoid

$$\mathcal{N} \xhookrightarrow{i} \mathcal{M}_{\Pi_Q} \quad \text{s.t.} \qquad \begin{tikzcd} \mathcal{N} \times \mathcal{N} \arrow[r, "\oplus"] \arrow[d, "i \times i"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{N} \arrow[d, "i"] \\ \mathcal{M}_{\Pi_Q} \times \mathcal{M}_{\Pi_Q} \arrow[r, "\oplus"'] & \mathcal{M}_{\Pi_Q} \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{N} \arrow[r, hook] \arrow[d, "JH"'] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{M}_{\Pi_Q} \arrow[d, "JH"] \\ \mathcal{N} \arrow[r, hook, "i"'] & \mathcal{M}_{\Pi_Q} \end{tikzcd}$$

$$\mathcal{A} = i^! \mathcal{A}_{\Pi_Q} \underset{\text{base-change}}{\cong} JH\, \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{N}} .$$

has an induced algebra structure, $i^! m$.

Interesting saturated submonoids

$$\begin{aligned} \mathbb{N}^{Q_0} &\hookrightarrow \mathcal{M}_{\Pi_Q} \\ d &\longmapsto 0_d \end{aligned} \qquad \mathfrak{N} =: \mathfrak{M}^{nil}_{\Pi_Q} \qquad \text{“fully nilpotent 2d CoHA”}$$

$$\mathcal{M}^{SSN}_{\Pi_Q} \hookrightarrow \mathcal{M}_{\Pi_Q} \qquad \text{submonoid of semisimple}$$

representations of Π_Q whose only arrows acting nontrivially are loops $\alpha \in Q_1$.

$$i^! \mathcal{A}_{\Pi_Q} =: \mathcal{A}_{\Pi_Q}^{SSN} \quad \text{"strictly seminilpotent 2d CoHA".}$$

$i^!$ is a right adjoint, so is left t-exact.

Proposition: $\mathcal{A}_{\Pi_Q}$ is in cohomological degrees ≥ 0

$$\Rightarrow \mathcal{A}_{\Pi_Q}^{SSN} \in \mathcal{D}_c^{\geq 0}(\mathcal{M}_{\Pi_Q}^{SSN}).$$

2) The strictly seminilpotent top-CoHA

$\mathcal{M}\times\mathcal{M}\xrightarrow{\oplus}\mathcal{M}$ monoid in $\mathbb{C}$-schemes, $\pi_0(\mathcal{M})\cong\mathbb{N}^{Q_0}$

$\mathcal{D}_c^{\geq 0}(\mathcal{M})\xrightarrow{H^0}\mathrm{Vect}_{\mathbb{Q}}$ is monoidal (Künneth formula).

$\Rightarrow$ $H^0(\mathcal{A}_{\Pi_Q}^{SSN})\in\mathrm{Vect}_{\mathbb{Q}}$ is an algebra object.

It can be described as a GKM algebra.

Theorem (H) $Q=(Q_0,Q_1)$

$Q_0 = Q_0^{real}\sqcup Q_0^{im}$

vertices without loops

vertices with ≥ 1 loops

$\Phi^+ = Q_0^{real}\sqcup(Q_0^{im}\times\mathbb{Z}_{\geq 1})\subset\mathbb{N}^{Q_0}$.

$\mathcal{Y}_d=\mathbb{Q}\quad\forall d\in\Phi^+$

$\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ GKM associated to Q by Bozec

$$H^0(\mathcal{A}_{\Pi_Q}^{SSN})\cong U(\mathfrak{n}^+)$$

Ingredients to prove this:

* Lusztig category $\mathcal{P}$: perverse sheaves on $\mathfrak{M}_Q$ stack of representations of Q
* Bozec crystal structure for $H^0(\mathcal{A}_{\Pi_Q}^{SSN})$.
* Characteristic cycle map $CC: K_0(\mathcal{P})\to H^0(\mathcal{A}_{\Pi_Q}^{SSN})$

VIII - Applications

① Decomposition of the cohomology of Nakajima quiver varieties

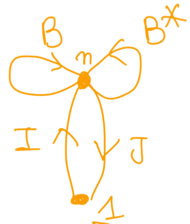
Nakajima, 1990s

geometric construction of representations of Kac-Moody algebras in the cohomology of quiver varieties

* uses a small part of the cohomology [middle degree]
* quivers without loops

Heisenberg Lie algebra action on the cohomology of Hilbert schemes of points on $\mathbb{C}^2$, $\mathrm{Hilb}^n(\mathbb{C}^2)$, $n \geq 0$

* $\mathrm{Hilb}^n(\mathbb{C}^2)$ can be seen as Nakajima quiver variety of the quiver



ADHM quiver

$[B, B^*] + IJ = 0$

+ stability :

= framed Jordan quiver

Today : * Put both situations in the same context of BPS Lie algebra actions on coh. of quiver varieties
* Give a description of this cohomology as direct sum

of modules over the BPS Lie algebra.

<u>Generalized Kac-Moody algebras</u> — quick reminder.

$M = \mathbb{N}^{Q_0}$ monoid

$(-,-) : M \times M \longrightarrow \mathbb{Z}$ bilinear form

$\Phi^+ \subset \mathbb{N}^{Q_0} \setminus \{0\}$ "simple positive roots"

$\mathcal{G} = \bigoplus_{d \in \Phi^+} \mathcal{G}_d$ "vector space of positive Chevalley generators"

$\mathcal{G}$ is $\Phi^+ \times \mathbb{Z}$-graded with fin. dim graded pieces

$\mathfrak{g}$: Lie algebra generated by $\mathcal{G} \oplus \underset{\underset{\mathbb{Q}^{Q_0}}{\parallel}}{\mathfrak{h}} \oplus \mathcal{G}^\vee$, with a lot of relations.

triangular decomposition : $\mathfrak{g} \underset{\text{v-spaces}}{\cong} \overbrace{\mathfrak{n}^- \oplus \mathfrak{h}}^{\mathfrak{b}^-} \oplus \mathfrak{n}^+$

enveloping algebra : $U(\mathfrak{g}) \underset{\text{v.spaces.}}{\simeq} \underbrace{U(\mathfrak{n}^-) \otimes U(\mathfrak{h})}_{= U(\mathfrak{b}^-)} \otimes U(\mathfrak{n}^+)$

Lowest representations

$f : \mathfrak{h} \to \mathbb{Q}$ linear form

induces $U(\mathfrak{h}) \simeq \mathrm{Sym}(\mathfrak{h}) \longrightarrow \mathbb{Q}$ algebra homomorphism

$U(\mathfrak{b}^-) \twoheadrightarrow U(\mathfrak{h}) \longrightarrow \mathbb{Q}$ 1-dimensional rep. of $U(\mathfrak{b}^-)$

$M_f := U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} \mathbb{Q}$ induced representation

$\qquad\quad \ni$
$\quad 1 \otimes 1$ lowest weight vector

$L_f :=$ quotient by the maximal proper $\mathfrak{g}$-submodule.

$\rightarrow$ simple representation of $\mathfrak{g}$ of lowest weight f, and lowest weight vector $\overline{1 \otimes 1}$.

Nakajima quiver varieties

* Very influential family of Hyperkähler varieties.
* Construction from a quiver.
* Variation of GIT gives often/sometimes (partial) crepant resolutions of symplectic singularities
* Recovers many significant situations in representation theory:

(1) Springer resolution for GL_n:

$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \cdots \arrow[r, dash] & n-1 \arrow[d, dash] \\ & & & & \boxed{n} \end{tikzcd}$$

$$\begin{tikzcd} T^*(G/B) \arrow[d] \\ \mathfrak{g} \supset \mathcal{N} \text{ nilp. cone} \end{tikzcd}$$

(2) Minimal resolutions of surface singularities

$$\begin{tikzcd} \widetilde{\mathbb{C}^2/\Gamma} \arrow[d] \\ \mathbb{C}^2/\Gamma \end{tikzcd}$$

Γ finite group

Q = McKay quiver of Γ

(3) Slodowy slices.

(4) Hilbert schemes of n points on $\mathbb{C}^2$ with Hilbert-Chow morphism

$$\begin{tikzcd} n \arrow[loop left, "B"] \arrow[loop right, "B^*"] \arrow[d, shift left, "J"] \\ \boxed{1} \arrow[u, shift left, "I"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hilb}^n(\mathbb{C}^2) \arrow[d, "\text{support.}"] \\ S^n\mathbb{C}^2 \end{tikzcd}$$

Construction: framed representations of preprojective algebras.

$Q = (Q_0, Q_1)$ quiver

$f \in \mathbb{N}^{Q_0}$ framing vector

$Q_f = (Q_0^f, Q_1^f)$ framed quiver:

$$Q_0^f = Q_0 \cup \{\infty\}$$

$$Q_1^f = Q_1 \cup \{\alpha_{i,\ell} : 1 \leq \ell \leq f_i\}_{i \in Q_0}.$$

Π_{Q_f} : preprojective algebra of Q_f

$d \in \mathbb{N}^{Q_0}$

ξ stability parameter (King stability condition) such that M representation of Π_{Q_f} of dim $(d,1)$ is semistable iff it has no nontrivial $(d',1)$-subrepresentations, $d' < d$.

$\mathcal{M}^{\xi\text{-st}}_{\Pi_{Q_f},(d,1)} =: \mathcal{M}(d,f)$ is a $\boxed{\text{smooth}}$ Nakajima quiver variety.

$\xi = 0$: $\mathcal{M}_{\Pi_{Q_f},(d,1)}$ is a (usually) $\boxed{\text{singular}}$ Nakajima quiver variety

$$\mathcal{M}_f := \bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathcal{M}(d,f)$$

$$M_f := \bigoplus_{d \in \mathbb{N}^{Q_0}} H^*\big(\mathcal{M}(d,f), \underbrace{\mathbb{Q}[\dim \mathcal{M}(d,f)]}_{\text{perverse.}}\big)$$

This will be the underlying space of a representation of a Lie algebra:

Double BPS algebra

We saw $\mathrm{BPS}^{3d}_{\Pi_Q, \mathrm{Lie}} \cong \mathfrak{n}^+_{\Pi_Q}$ positive part of some GKM (Chevalley generators + Serre relations)

Define $\mathfrak{g}^{\mathrm{BPS}}_{\Pi_Q} = \mathfrak{n}^+_{\Pi_Q} \oplus \mathfrak{h} \oplus \mathfrak{n}^-_{\Pi_Q}$ to be the full GKM: no geometric construction!

$Q \subset Q_f \Rightarrow \mathfrak{g}^{\mathrm{BPS}}_{\Pi_Q} \subset \mathfrak{g}^{\mathrm{BPS}}_{\Pi_{Q_f}}$ is a Lie subalgebra

Triangular decomposition

$$\mathfrak{g}^{\mathrm{BPS}}_{\Pi_{Q_f}} = \underbrace{\mathfrak{n}^-_{\Pi_{Q_f}}}_{(-\mathbb{N})^{(Q_f)_0}\text{-graded}} \oplus \mathfrak{h}_{Q_f} \oplus \underbrace{\mathfrak{n}^+_{\Pi_{Q_f}}}_{\mathbb{N}^{(Q_f)_0}\text{-graded}}$$

implies that

$$\boxed{\mathfrak{g}^{\mathrm{BPS}}_{\Pi_Q} \text{ acts on } \bigoplus_{d \in \mathbb{N}^{Q_0}} \mathfrak{n}^+_{\Pi_{Q_f}}[d,1]}$$

Moreover,

$$\mathcal{H}^{+}_{\Pi_{Q_f}}[d,1] \overset{def}{=} \mathrm{BPS}^{3d}_{\Pi_{Q_f},(d,1)}$$

$$\underset{\text{Toda}}{\cong} H^*\left(\mathcal{M}(d,f), \mathbb{Q}[\dim \mathcal{M}(d,f)]\right)$$

[BPS cohomology for singular quiver variety does not depend on the stability parameter]

and so, $\displaystyle\bigoplus_{d \in \mathbb{N}^{Q_0}} \mathcal{H}^{+}_{\Pi_{Q_f}}[d,1] = M_f$.

Theorem (Toda + dimensional reduction)

For any quiver Q, any stability parameter $\xi \in \mathbb{Q}^{Q_0}$, any $d \in \mathbb{N}^{Q_0}$

$$\begin{array}{c} \mathcal{M}^{\xi\text{-sst}}_{\Pi_Q,d} \\ \downarrow \pi \quad \text{“affinization morphism”} \\ \mathcal{M}_{\Pi_Q,d} \end{array}$$

$$\pi_* \mathcal{BPS}^{3d,\xi}_{\Pi_Q,d,\text{Lie}} \cong \mathcal{BPS}^{3d}_{\Pi_Q,d,\text{Lie}}$$

Slogan: BPS cohomology is invariant under variation of GIT.

Theorem (Davison-H-Schlegel Mejia)

$$M_f \underset{\mathfrak{g}^{BPS}_{\Pi_Q}\text{-representations}}{\cong} \bigoplus_{(d,1)\in\Sigma_{\Pi_{Q_f}}} IH^*(\mathcal{M}_{\Pi_{Q_f},(d,1)}) \otimes L_{((d,1),-)_{Q_f}}$$

<u>Proof</u>: ① M_f is a semisimple $\mathfrak{g}^{BPS}_{\Pi_Q}$ - module

② Find lowest weight vectors in M_f and identify them with $IH^*(\mathcal{M}_{\Pi_{Q_f},(d,1)})$, $d\in\mathbb{N}^{Q_0}$

$\rightarrow$ automatically of lowest weight $((d,1),-)_{Q_f}$.

③ Show that these l.w vector generate M_f as a $\mathfrak{g}^{BPS}_{\Pi_Q}$ - module.

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① Cartan involution of $\mathfrak{g}^{BPS}_{\Pi_{Q_f}}$ [automorphism of Lie algebra]

Choose a basis B of $\mathfrak{g}$

$\omega : \mathfrak{g}^{BPS}_{\Pi_{Q_f}} \longrightarrow \mathfrak{g}^{BPS}_{\Pi_{Q_f}}$ is defined by sending B to $-B^*$, and acts by the - identity on $\mathfrak{h}$

As part of the GKM package, we have

$$\langle -, - \rangle : \left(\mathfrak{g}^{BPS}_{\Pi_{Q_f}}\right)^{\otimes 2} \longrightarrow \mathbb{Q} \quad \text{invariant, nondegenerate.}$$

and $\langle -, \omega(-) \rangle$ restricts to symmetric positive definite bil form on

$$\mathfrak{n}^+_{\Pi_{Q_f}}, \quad \mathfrak{g}^{BPS}_{\Pi_Q}\text{-invariant}$$

$\Rightarrow$ $\mathfrak{n}^+_{\Pi_{Q_f}}$ is a semisimple $\mathfrak{g}^{BPS}_{\Pi_Q}$-representation

② $IH(\mathcal{M}_{\Pi_{Q_f}, (d,1)}) \underset{\oplus}{\subset} H^*(\mathcal{M}(d,f), \mathbb{Q}[\dim \mathcal{M}(d,f)])$

is a space of lowest weight vectors

[because they are spaces of Chevalley generators]

③ They generate the $\mathfrak{g}^{BPS}_{\Pi_Q}$-module $\mathfrak{n}^+_{\Pi_{Q_f}}$: this comes from the PBW-isomorphism

$$\mathfrak{n}^+_{\Pi_Q} \oplus IH(\mathcal{M}(d,f)) \subset \mathfrak{n}^+_{\Pi_{Q_f}}$$

contains all positive Chevalley generators of $\mathfrak{n}^+_{\Pi_{Q_f}}$.