2d Cohomological Hall algebras and Kac–Moody Lie algebras

Plan : I - Constructible derived category

II - 2-Calabi–Yau categories and their moduli stacks

III - Cohomological Hall algebra structure and the BPS associative algebra

IV - A glimpse into 3d-cohomological algebras [Quivers with potential]

V - Generalised Kac-Moody Lie algebras

VI - The BPS algebra by generators and relations & PBW theorem

VII - The strictly seminilpotent CoHA

VIII - Proof

Motivation / Overview

Goal: Study $H_*^{BM}(\mathfrak{M}_{\mathcal{A}})$ for $\mathcal{A}$ 2CY Abelian category

representation theory, Nakajima quiver varieties

Study of counting polynomials arising in the study of the classical Hall algebra $H_{Q,\mathbb{F}_q}$.

$\mathcal{A} = \operatorname{Rep} \Pi_Q$

$\mathcal{A} = \operatorname{Higgs}^{sst}(C)$ and $\mathcal{A} = \operatorname{Rep} \pi_1(S)$

$H_*^{BM}(\mathfrak{M}_{\mathcal{A}})$

$\mathcal{A} =$ sheaves on K3 or Abelian surfaces

Nonabelian Hodge theory

enumerative geometry

$\mathcal{A}$ as above will be referred to as **2CY Abelian categories**

This lecture series: technical background and structural results.

Main results I would like to explain

$$\underset{\substack{\text{subalgebra} \\ \text{defined using a} \\ \text{"less" perverse filtration}}}{\mathrm{BPS}_{\mathcal{A},\mathrm{Alg}}} \subset \underset{\text{2d - CoHA structure}}{H_*^{BM}(\mathfrak{M}_{\mathcal{A}})}$$

<u>Theorem A</u>: $\mathrm{BPS}_{\mathcal{A},\mathrm{Alg}} \underset{\text{algebras}}{\cong} U(\underbrace{\mathfrak{n}^+_{\mathcal{A}}})$

a generalized Kac-Moody Lie algebra in the sense of Borcherds

generators: $IH(\mathcal{M}_{\mathcal{A}}, a)$ $\quad a \in \Sigma_{\mathcal{A}} \subset \pi_0(\mathcal{M}_{\mathcal{A}})$

$IH(\mathcal{M}_{\mathcal{A}}, \ell a)$ $\quad a \in \Sigma_{\mathcal{A}},\ (a,a)=0,\ \ell \geq 2$

relations: "Serre relations"

<u>Theorem B</u>: $H_*^{BM}(\mathfrak{M}_{\mathcal{A}}) \underset{\text{v. spaces}}{\cong} \mathrm{Sym}\left(\mathfrak{n}^+_{\mathcal{A}} \otimes H^*_{\mathbb{C}^*}(pt)\right)$

In fact, I would like to explain sheafified versions of Theorems A and B.

$JH : \mathfrak{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{A}}$ "good moduli space"

$BPS_{\mathcal{A},Alg}$ can be upgraded to $\mathcal{BPS}_{\mathcal{A},Alg} \in \mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$ algebra object

$H_*^{BM}(\mathfrak{M}_{\mathcal{A}})$ can be upgraded to $JH_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{A}}} \in \mathcal{D}_c^+(\mathcal{M}_{\mathcal{A}})$ algebra object

To formulate and prove the upgrades of Theorems A and B to categories of sheaves, we need to define GKM algebras in $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$.

Today: Constructible derived categories

Geometry of moduli stacks of objects in 2CY categories.

I - Constructible derived category

We are in the complex setting.

① Constructible sheaves

X : $\mathbb{C}$-algebraic variety

$Sh(X, \mathbb{Q})$: Abelian category of all sheaves of $\mathbb{Q}$-vector spaces on X

$\mathcal{F} \in Sh(X,\mathbb{Q})$

$$\left\{ \begin{array}{c} \text{category of opens in} \\ X, \text{ w/ inclusions} \\ \text{as morphisms} \end{array} \right\} \xrightarrow[\mathcal{F}]{\text{contravariant functor}} \mathbb{Q}-\mathrm{Vect}$$

$$\begin{array}{ccc} U \subset X & \longmapsto & \mathcal{F}(U) \\ \text{analytic open} & & \end{array}$$

$\mathcal{D}(X,\mathbb{Q}) := \mathcal{D}(Sh(X,\mathbb{Q}))$ derived category of an Abelian category (Verdier).

Reminder of its construction

$$\mathcal{C}(\underbrace{Sh(X,\mathbb{Q})}_{\text{Abelian category}}) = \text{category of complexes of sheaves}$$

$$\to C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \to \dots$$

w/ $d^i \circ d^{i-1} = 0$

morphisms $C^\bullet \xrightarrow{f^\bullet} D^\bullet$ are $f^\bullet = (f^i : C^i \to D^i)_{i \in \mathbb{Z}}$

making all squares commute:

$$d^i f^i = f^{i+1} d^i.$$

cohomology functors $H^i : \mathcal{C}(Sh(X,\mathbb{Q})) \longrightarrow Sh(X,\mathbb{Q})$

$$H^i(C^\bullet) = \ker d^i / \operatorname{im} d^{i-1} .$$

quasi-isomorphisms : $f^\bullet : C^\bullet \to D^\bullet$ s.t. $H^i(f^\bullet) : H^i(C^\bullet) \to H^i(D^\bullet)$ is an isomorphism $(\forall i \in \mathbb{Z})$

qis = quasi-isomorphisms of $\mathcal{E}(Sh(X,\mathbb{Q}))$

$\mathcal{D}(Sh(X,\mathbb{Q})) := \mathcal{E}(Sh(X,\mathbb{Q}))[\text{qis}^{-1}]$ localization of categories (Verdier)

⚠ Not Abelian anymore

Verdier worked out what structure we have on $\mathcal{D}(Sh(X,\mathbb{Q}))$.

We obtain a triangulated category

That is : $\mathcal{D}$ additive category

$[1] : \mathcal{D} \to \mathcal{D}$ automorphism (translation functor)

+ class of distinguished triangles $X \to Y \to Z \to X[1]$.

Satisfying some axioms.

TR1

TR2

TR3

TR4

For the derived category of an Abelian category $\mathcal{A} = Sh(X,\mathbb{Q})$, the class of distinguished triangles is generated by

$$X^\bullet \xrightarrow[f]{} Y^\bullet \to Z^\bullet \to X^\bullet[1]$$

$$Z^n = Y^n \oplus X^{n+1}$$

$$d^n = \begin{pmatrix} d_Y^n & f^n \\ 0 & -d_X^{n+1} \end{pmatrix} : Y^n \oplus X^{n+1} \longrightarrow Y^{n+1} \oplus X^{n+2}$$

Cohomology functors descend to $\mathcal{D}(Sh(X, \mathbb{Q}))$:

$$H^i : \mathcal{D}(Sh(X, \mathbb{Q})) \longrightarrow Sh(X, \mathbb{Q}).$$

$$C^\bullet \longmapsto \ker d^i / \operatorname{im} d^{i-1}$$

$\mathcal{F} \in Sh(X, \mathbb{Q})$ is called constant if there is a $\mathbb{Q}$-vector space A s.t.

$$\mathcal{F}(U) = A \qquad \forall\, U \subset X$$

and restriction maps are given by id_A.

locally constant if any $x \in X$ has an analytic open neighbourhood $U \subset X$ s.t. $\mathcal{F}|_U$ is constant.

locally constant sheaves with finite dimensional fibers are also called local systems.

$\mathcal{F} \in Sh(X, \mathbb{Q})$ is called constructible if there is a finite stratification $X = \bigsqcup_{i \in I} X_i$ such that

$\mathcal{F}|_{X_i}$ is locally constant

$\forall x \in X$, $\mathcal{F}_x$ is finite-dimensional

$Sh_c(X, \mathbb{Q}) \subset Sh(X, \mathbb{Q})$: full subcategory of constructible sheaves.

It is Abelian

Constructible derived category:

$\mathcal{D}_c(X, \mathbb{Q})$ = full subcategory of $\mathcal{D}(X, \mathbb{Q})$ of complexes which can be represented by complexes of sheaves $\mathcal{F}^\bullet$ with $H^i(\mathcal{F}^\bullet) \in Sh_c(X, \mathbb{Q})$ $\forall i \in \mathbb{Z}$.

It is still triangulated.

(2) 6-functor formalism

For $f: X \longrightarrow Y$ a morphism between $\mathbb{C}$-algebraic varieties, we have adjoint pairs of functors

(f^*, f_*)

$(f^!, f_!)$

$(\otimes, \mathcal{H}om)$

<u>Remark</u>: When a functor is left/right exact, it is derived on the right/left.

Deriving an exact functor does not do anything to it.
(right and left)

Verdier duality $\mathbb{D}: \mathcal{D}_c(X, \mathbb{Q})^{op} \longrightarrow \mathcal{D}_c(X, \mathbb{Q})$

$$\mathbb{D} f^* \simeq f^! \mathbb{D}$$

$$\mathbb{D} f_* \simeq f_! \mathbb{D}$$

If $f: X \longrightarrow pt$,

$$f_* \mathbb{Q}_X = H^*_{sing}(X, \mathbb{Q})$$

③ Perverse sheaves

There is a general formalism of t-structures to extract Abelian categories from triangulated ones.

A choice of such t-structure on $\mathcal{D}_c(X,\mathbb{Q})$ produces the category of perverse sheaves

(Beilinson-Bernstein-Deligne-Gabber, 1983)

$\mathcal{F} \in \mathcal{D}_c(X,\mathbb{Q})$ is called perverse if it satisfies the

- support condition

$\forall k \in \mathbb{Z}, \ \dim\{x \in X \mid H^k(i_x^* \mathcal{F}) \neq 0\} \leq -k$ $\qquad {}^p\mathcal{D}_c^{\leq 0}(X,\mathbb{Q})$

- cosupport condition = support condition for $\mathbb{D}\mathcal{F}$

$\forall k \in \mathbb{Z}, \ \dim\{x \in X \mid H^k(i_x^! \mathcal{F}) \neq 0\} \leq -k$ $\qquad {}^p\mathcal{D}_c^{\geq 0}(X,\mathbb{Q})$

$\mathrm{Perv}(X) = {}^p\mathcal{D}^{\leq 0}(X,\mathbb{Q}) \cap {}^p\mathcal{D}^{\geq 0}(X,\mathbb{Q})$ is an Abelian category.

It is Noetherian and Artinian : all its objects are of finite length.

Examples of perverse sheaves

- $\mathbb{Q}_X[\dim X]$ for smooth, equidimensional X
- $(i_x)_* \mathbb{Q}_{pt}$ for $i_x : pt \to X$ inclusion of $x \in X$
- $\mathcal{L}[\dim X]$ for $\mathcal{L}$ local system on smooth, equidim. X

Truncation functors

Define ${}^p\mathcal{D}_c^{\leq i}(X,\mathbb{Q}) = {}^p\mathcal{D}_c^{\leq 0}(X,\mathbb{Q})[-i]$

${}^p\mathcal{D}_c^{\geq i}(X,\mathbb{Q}) = {}^p\mathcal{D}_c^{\geq 0}(X,\mathbb{Q})[-i]$.

The perverse t-structure gives functors

${}^p\tau_{\leq i} : \mathcal{D}_c(X,\mathbb{Q}) \longrightarrow {}^p\mathcal{D}_c^{\leq i}(X,\mathbb{Q})$

right adjoint to the natural inclusion ${}^p\mathcal{D}_c^{\leq i}(X,\mathbb{Q}) \longrightarrow \mathcal{D}_c(X,\mathbb{Q})$

and ${}^p\tau_{\geq i} : \mathcal{D}_c(X,\mathbb{Q}) \longrightarrow {}^p\mathcal{D}_c^{\geq i}(X,\mathbb{Q})$

left adjoint to the natural inclusion ${}^p\mathcal{D}_c^{\geq i}(X,\mathbb{Q}) \longrightarrow \mathcal{D}_c(X,\mathbb{Q})$.

We obtain the perverse cohomology functors

$${}^p\mathcal{H}^i := {}^p\tau_{\leq 0}\,{}^p\tau_{\geq 0}[i] : \mathcal{D}_c(X,\mathbb{Q}) \longrightarrow \mathrm{Perv}(X).$$

Intermediate extension

$j : U \longrightarrow X$ open immersion.

Functor $j_{!*} : \mathrm{Perv}(U) \longrightarrow \mathrm{Perv}(X)$ constructed as follows.

$\mathcal{F} \in \mathrm{Perv}(U)$

$j_! \mathcal{F} \longrightarrow j_* \mathcal{F}$ morphism in $\mathcal{D}_c(X, \mathbb{Q})$

${}^p\mathcal{H}^0 (j_! \mathcal{F}) \xrightarrow{\varphi} {}^p\mathcal{H}^0 (j_* \mathcal{F})$ morphism in $\mathrm{Perv}(X)$

$j_{!*} \mathcal{F} := \mathrm{im}\, \varphi \in \mathrm{Perv}(X)$

Classification of simple objects:

X $\mathbb{C}$-algebraic variety.

$Y \overset{i}{\subset} X$ irreducible, closed

$U \overset{j}{\subset} Y$ smooth open

$\mathcal{L}$ irreducible local system on U

$\mathcal{IC}(\mathcal{L}) := j_{!*} \mathcal{L}[\dim Y] \in \mathrm{Perv}(Y)$

$i_* \mathcal{IC}(\mathcal{L}) \in \mathrm{Perv}(X)$ is a simple perverse sheaf..

All simple perverse sheaves on X are obtained this way.

Fundamental theorem in the theory: the BBDG decomposition theorem

Let $\mathcal{F} \in \mathrm{Perv}(X)$ be a simple perverse sheaf and $f: X \longrightarrow Y$ a projective morphism between complex algebraic varieties.

Then $f_* \mathcal{F} \in D_c^b(Y, \mathbb{Q})$ is a semisimple complex,

that is

$$f_* \mathcal{F} \simeq \bigoplus_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(\mathcal{F})[-i] \quad \text{and}$$

$${}^p\mathcal{H}^i(\mathcal{F}) \in \mathrm{Perv}(Y) \text{ is a semisimple perverse sheaf.}$$

Mixed Hodge modules

In these lectures, I will keep thing more elementary by working with constructible sheaves.

It is possible to enhance thing by working with mixed Hodge modules: $MHM(X)$ for X an algebraic variety $/\mathbb{C}$.

$$\mathrm{rat} : MHM(X) \longrightarrow \mathrm{Perv}(X) \qquad \text{faithful exact functor}$$

$$\mathcal{D}^+(MHM(X)) \longrightarrow \mathcal{D}^+(\mathrm{Perv}(X)) \underset{\substack{\text{Beilinson}\\\text{equivalence}}}{\simeq} \mathcal{D}^+_c(X).$$

MHM are crucial for purity arguments (via the weight structure) to obtain semisimplicity of the objects considered.

④ Monoidal structures

Monoids

$\mathcal{M}$ = monoid in the category of complex schemes finite type, separated connected components.

$\oplus : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ monoid map.

$\eta : pt \to \mathcal{M}$ unit.

e.g. ⓐ $\mathcal{M} = \mathbb{N}^{Q_0}$ seen as $\bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathrm{Spec}(\mathbb{C})$.

$\oplus$ usual map.

$\eta : pt \to 0 \in \mathbb{N}^{Q_0}$.

ⓑ $\mathcal{M} = \bigsqcup_{n \in \mathbb{N}} \mathbb{C}^n / S_n$

$\oplus_{m,n} : \mathbb{C}^m / S_m \times \mathbb{C}^n / S_n \to \mathbb{C}^{m+n} / S_{m+n}$.

$\eta : pt \xrightarrow{\sim} \mathbb{C}^0 / S_0 \simeq pt$

Commutative monoid $\quad sw : \mathcal{M}^{\times 2} \to \mathcal{M}^{\times 2}$.

$(x, y) \longmapsto (y, x)$

$\oplus \circ sw = \oplus$.

Monoidal structures

$\mathcal{M}$ monoid in $\mathbb{C}$-schemes. | For simplicity, assume $\pi_0(\mathcal{M}) \overset{\text{as monoids.}}{=} \mathbb{N}^{Q_0}$
| Assume $\mathcal{M}_0 = pt$ $\quad 0 \in \mathbb{N}^{Q_0}$

$\mathcal{F}, \mathcal{G} \in \mathcal{D}^+_c(\mathcal{M}, \mathbb{Q})$

Define $\mathcal{F} \boxdot \mathcal{G} := \oplus_* (\mathcal{F} \boxtimes \mathcal{G})$

Fact: this gives a monoidal structure on $\mathcal{D}^+_c(X, \mathbb{Q})$

unit: $\eta_* \mathbb{Q}_{pt}$.

If $\oplus$ is commutative, we get a symmetric monoidal structure.

All monoidal structures appearing will be symmetric.

~~Associative~~ Algebra objects:

$(\mathcal{A}, m, \eta)$ with $\mathcal{A} \in \mathcal{D}^+_c(\mathcal{M}, \mathbb{Q})$

$m : \mathcal{A} \boxdot \mathcal{A} \longrightarrow \mathcal{A}$ multiplication map

$\eta : \mathbb{1} \to \mathcal{A}$ unit

satisfying the standard associativity and unitality constraints

$$\begin{tikzcd}
\mathcal{A} \boxdot \mathcal{A} \boxdot \mathcal{A} \arrow[r, "m \boxdot \mathrm{id}_{\mathcal{A}}"] \arrow[d, "\mathrm{id}_{\mathcal{A}} \boxdot m"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{A} \boxdot \mathcal{A} \arrow[d, "m"] \\
\mathcal{A} \boxdot \mathcal{A} \arrow[r] & \mathcal{A}
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{A} \simeq \mathcal{A} \boxdot \mathbb{1} \arrow[r, "\mathrm{id}_{\mathcal{A}} \boxdot \eta"] \arrow[dr, "\mathrm{id}_{\mathcal{A}}"'] & \mathcal{A} \boxdot \mathcal{A} \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathcal{A}
\end{tikzcd}
$$

Lie algebra objects

$(\mathcal{L}, \ b = [-,-]) \qquad \mathcal{L} \in \mathcal{D}^+_c(\mathcal{M}, \mathcal{R})$

$$b : \mathcal{L} \boxdot \mathcal{L} \longrightarrow \mathcal{L}$$

satisfying * antisymmetry :

$$b \circ sw \simeq -b$$

* Leibniz identity

$$
\begin{tikzcd}
\mathcal{L} \boxdot \mathcal{L} \boxdot \mathcal{L} \arrow[r, "\mathrm{id}_{\mathcal{L}} \boxdot {[-,-]}"] \arrow[rr, bend right, "{[-,[-,-]]} =: b^{(3)}"'] & \mathcal{L} \boxdot \mathcal{L} \arrow[r, "{[-,-]}"] & \mathcal{L}
\end{tikzcd}
$$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

$$b^{(3)} + b^{(3)} \circ (1\,2\,3) + b^{(3)} \circ (2\,1\,3) = 0$$

Monoidal functors

If $F : D_c^+(\mathcal{M}, \mathbb{Q}) \longrightarrow (\mathcal{T}, \otimes)$ is a monoidal functor, $(\mathcal{A}, m, \eta)$ an algebra / Lie algebra object in $D_c^+(\mathcal{M}, \mathbb{Q})$, $(F(\mathcal{A}), F(m), F(\eta))$ is an algebra / Lie alg. object in $\mathcal{T}$.

e.g. ① Derived global sections

$$H^* : D_c^+(\mathcal{M}, \mathbb{Q}) \longrightarrow \underbrace{\mathbb{Z}\text{-graded}}_{\text{super}} \text{ vector spaces}$$

is monoidal $\qquad\qquad\qquad\qquad \cong D_c^+(pt)$

Indeed, $H^*(\mathcal{M}, \mathcal{F} \boxdot \mathcal{G}) = H^*(\mathcal{M}, \oplus_*(\mathcal{F} \boxtimes \mathcal{G}))$

$$= p_* \oplus_* (\mathcal{F} \boxtimes \mathcal{G})$$

$$= p_* \mathcal{F} \otimes p_* \mathcal{G} \quad \text{in } D_c^+(pt)$$

② Pullback

If $\mathcal{N} \xrightarrow{f} \mathcal{M}$ is a saturated submonoid in the category of $\mathbb{C}$-schemes with finite type, separated connected components,

$$f^! : D_c^+(\mathcal{M}, \mathbb{Q}) \longrightarrow D_c^+(\mathcal{N}, \mathbb{Q}) \text{ is monoidal.}$$

Indeed:

$$\begin{tikzcd} \mathcal{N}\times\mathcal{N} \arrow[r, "\oplus"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{N} \arrow[d, "f"] \\ \mathcal{M}\times\mathcal{M} \arrow[r, "\oplus"'] & \mathcal{M} \end{tikzcd}$$

$$f^! \underbrace{\oplus_*(\mathcal{F}\boxtimes\mathcal{G})}_{\mathcal{F}\,\square\,\mathcal{G}} \underset{\text{base-change}}{\cong} \oplus_* f^!(\mathcal{F}\boxtimes\mathcal{G})$$

$$\cong (f^!\mathcal{F})\,\square\,(f^!\mathcal{G})$$

compatibility $\boxtimes$ with $f^!$.

③ Pushforward (generalises ①)

$$f : \mathcal{M} \longrightarrow \mathcal{N}$$

monoidal functors

$$f_* : D^+_c(\mathcal{M}) \longrightarrow D^+_c(\mathcal{N}) \quad \text{in general}$$

$$f_! : D^+_c(\mathcal{M}) \longrightarrow D^+_c(\mathcal{N}) \quad \text{if } \oplus_{\mathcal{N}} \text{ and } \oplus_{\mathcal{M}} \text{ are proper.}$$

Proof:

$$\begin{tikzcd} \mathcal{M}\times\mathcal{M} \arrow[r, "\oplus_{\mathcal{M}}"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{M} \arrow[d, "f"] \\ \mathcal{N}\times\mathcal{N} \arrow[r, "\oplus_{\mathcal{N}}"'] & \mathcal{N} \end{tikzcd}$$ commutes.

$\boxtimes$

Now: Some constructions.

Upshot: everything works as for classical algebras in Vect.

Free algebra

For $\mathcal{G} \in \mathrm{Perv}(\mathcal{M}_{>0})$, $\mathcal{M}_{>0} = \bigsqcup_{d \in \mathbb{N}^{Q_0}\setminus\{0\}} \mathcal{M}_d$ we define

$$\mathrm{Free}_{\boxdot}(\mathcal{G}) := \bigoplus_{n \geq 0} \mathcal{G}^{\boxdot n}.$$

It has the product $m: \mathrm{Free}_{\boxdot}(\mathcal{G}) \boxdot \mathrm{Free}_{\boxdot}(\mathcal{G}) \longrightarrow \mathrm{Free}_{\boxdot}(\mathcal{G})$ induced by

$$\mathcal{G}^{\boxdot m} \boxdot \mathcal{G}^{\boxdot n} \cong \mathcal{G}^{\boxdot(m+n)} \quad \forall\, m, n \in \mathbb{N}.$$

Free Lie algebra $\mathrm{Free}_{\boxdot\text{-Lie}}(\mathcal{G}) :=$ subobject of $\mathrm{Free}_{\boxdot}(\mathcal{G})$ generated by $\mathcal{G}, [\mathcal{G},\mathcal{G}], [\mathcal{G},[\mathcal{G},\mathcal{G}]], \ldots$

Ideal $\mathcal{F} \in \mathrm{Perv}(\mathcal{M})$ algebra object

$\mathcal{J} \subset \mathcal{F}$ subobject.

$\mathcal{J}$ is a 2-sided ideal if

$$\begin{tikzcd} \mathcal{F}\boxdot\mathcal{J}\boxdot\mathcal{F} \arrow[r, "\mathrm{id}_A\boxdot f\boxdot \mathrm{id}_A"] \arrow[drr, dashed, "\text{factors}"'] & \mathcal{F}\boxdot\mathcal{F}\boxdot\mathcal{F} \arrow[r, "m"] & \mathcal{F} \\ & & \mathcal{J} \arrow[u, "f"'] \end{tikzcd}$$

etc. for Lie ideal....

Envelopping algebra :

$\mathcal{L} \in (\mathrm{Perv}(\mathcal{M}), \boxdot)$ Lie algebra object.

$$U(\mathcal{L}) = \mathrm{Free}_{\boxdot}(\mathcal{L}) / \mathcal{I}$$

where $\mathcal{I} \subset \mathrm{Free}_{\boxdot}(\mathcal{L})$ is the 2-sided ideal generated by the image of

$$\mathcal{L} \boxdot \mathcal{L} \xrightarrow[{[-,-] \oplus (m \circ sw - m)}]{} \mathcal{L} \oplus (\mathcal{L} \boxdot \mathcal{L}) \subset \mathrm{Free}_{\boxdot}(\mathcal{L})$$

Symmetric algebras

$$\mathrm{Sym}_{\boxdot}(\mathcal{F}) := \bigoplus_{n \geq 0} \mathrm{Sym}^n_{\boxdot}(\mathcal{F})$$

PBW theorem :

$$\begin{tikzcd} \mathrm{Sym}_{\boxdot}(\mathcal{L}) \arrow[r, "\text{\textquotedblleft iterated multiplication\textquotedblright}"] \arrow[dr, "\text{mono.}"'] & U(\mathcal{L}) \\ & \mathrm{Free}_{\boxdot}(\mathcal{L}) \arrow[u, "\text{alg. map}"'] \end{tikzcd}$$

is an isomorphism of perverse sheaves.

<u>Proof</u>: $U(\mathcal{L})$ is filtered by the images of the maps

$$\bigoplus_{n \leq m} \mathcal{L}^{\boxdot n} \to U(\mathcal{L}).$$

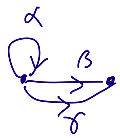
The associated graded is exactly $\mathrm{Sym}_{\boxdot}(\mathcal{L})$.

II- 2-Calabi–Yau categories and their moduli stacks

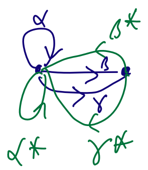
Note: I am not a derived algebraic geometer: I have a pedestrian approach.

① Examples

ⓐ Preprojective algebras. $Q = (Q_0, Q_1)$ quiver (Q_0: vertices, Q_1: arrows)

e.g. 

$\overline{Q} = (Q_0, Q_1 \sqcup Q_1^{op})$ double quiver



$$\rho = \sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha) \in \mathbb{C}\overline{Q} \text{ path algebra of } \overline{Q}$$

$$\Pi_Q := \mathbb{C}\overline{Q} / \langle\langle \rho \rangle\rangle$$

Thm (Crawley-Boevey) Π_Q is a 2CY algebra, if Q not Dynkin ADE

Rk: If Q is Dynkin ADE, work with Ginzburg dg-algebra instead.

Stack of objects: $d \in \mathbb{N}^{Q_0}$

$$X_{Q,d} = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}}) \circlearrowleft$$

$$X_{\overline{Q},d} \cong T^* X_{Q,d} \circlearrowleft GL_d = \prod_{i \in Q_0} GL_{d_i}$$

Hamiltonian

$$\mu_d : T^*X_{Q,d} \longrightarrow \mathfrak{gl}_d \quad \text{moment map}$$
$$(x_\alpha, x_{\alpha^*})_{\alpha\in Q_1} \mapsto \sum_{\alpha\in Q_1} [x_\alpha, x_{\alpha^*}]$$

$$\mathfrak{M}_{\Pi_{Q,d}} := \left[\mu_d^{-1}(0)/GL_d\right] \quad \text{quotient stack}$$

$$\begin{tikzcd} \mathfrak{M}_{\Pi_{Q,d}} \arrow[d, "JH"'] \\ \mathcal{M}_{\Pi_{Q,d}} \end{tikzcd}$$

$$\mathcal{M}_{\Pi_{Q,d}} := \mu_d^{-1}(0)/\!/GL_d \quad \text{affine GIT quotient.}$$

(a') Multiplicative versions of preprojective algebras

(b) <u>Sheaves on symplectic surfaces</u>

S K3 or Abelian surface
or $S = T^*C$ for C smooth projective curve.

H polarization

$Coh^{H\text{-}ss}_{p(t)}(S)$ semistable sheaves on S w/ normalized Hilbert polynomial $p(t)$.

$$\begin{tikzcd} \mathfrak{M}^{H\text{-}ss}_{p(t)}(S) \arrow[d, "JH"] \\ \mathcal{M}^{H\text{-}ss}_{p(t)}(S) \end{tikzcd}$$

Classical constructions using Quot-schemes.

(c) S Riemann surface, of genus g

$$\pi_1(S,x) \simeq \left\{ x_i, y_i : 1 \leq i \leq g \;\middle|\; \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1} = 1 \right\}$$

ordered product

Thm (Davison) Let $g \geq 1$. $\mathrm{Rep}\,\pi_1(S,x)$ is 2CY.

Construction of moduli stacks and spaces is a particular case of multiplicative preprojective algebra.

Use the multiplicative moment map

$$\mu_n : GL_n^{2g} \longrightarrow GL_n \qquad n \geq 1$$

$$(M_i, N_i)_{1 \leq i \leq g} \longmapsto \prod_{i=1}^{g} M_i N_i M_i^{-1} N_i^{-1}$$

$$\mathfrak{M}_{g,n} = \left[\mu_n^{-1}(\mathrm{Id}_n) / GL_n \right]$$

$$\downarrow \mathrm{JH}$$

$$\mathcal{M}_{g,n} = \mu_n^{-1}(\mathrm{Id}_n) /\!/ GL_n .$$

② 2-Calabi-Yau categories

We put all categories as above under the umbrella of what we call

2-Calabi-Yau Abelian categories.

$\mathcal{E}$ = "ambient" pretriangulated dg-category

$\mathfrak{M}_{\mathcal{E}}$ = derived moduli stack of objects in $\mathcal{E}$

$\mathcal{A} \subset H^0(\mathcal{E})$ Abelian category s.t.

$$\underset{\substack{\text{1-Artin} \\ \text{substack}}}{\mathfrak{M}_{\mathcal{A}}} \overset{\text{open}}{\subset} \mathfrak{M}_{\mathcal{E}} .$$

Good moduli space

We assume that $\mathfrak{M}_{\mathcal{A}}$ has a good moduli space in the sense of Alper-Rydh-Hall:

$$JH_{\mathcal{A}} : \mathfrak{M}_{\mathcal{A}} \longrightarrow \underbrace{M_{\mathcal{A}}}_{\text{usually algebraic space}}$$

Assume: finite type, separated $\mathbb{C}$-scheme.

In particular, $JH_{\mathcal{A}}$ is universal among maps to an algebraic space.

⊕-morphism

$\oplus : \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} \longrightarrow \mathfrak{M}_{\mathcal{A}}$ directsum, induces (by universality of $JH_{\mathcal{A}}$) $\oplus : M_{\mathcal{A}} \times M_{\mathcal{A}} \longrightarrow M_{\mathcal{A}}$. finite map

2 Calabi-Yau structure

$\forall x_1, \dots, x_r \in \mathcal{M}_{\mathcal{A}}$, corresponding to simple objects $\mathcal{F}_1, \dots, \mathcal{F}_r \in \mathcal{A}$, the full dg-subcategory of $\mathcal{C}$ generated by $\mathcal{F}_1, \dots, \mathcal{F}_r$ has a right 2-Calabi-Yau structure [Brav-Dyckerhoff].

Roughly, this means that we have bi-functorial isomorphisms
$$\mathrm{Hom}_{H^0(\mathcal{C})}(\mathcal{F}, \mathcal{G}[i]) \cong \mathrm{Hom}_{H^0(\mathcal{C})}(\mathcal{G}, \mathcal{F}[2-i])^*$$
$$\forall \mathcal{F}, \mathcal{G} \in \mathcal{D}$$

RHom complex: If $X = \mathrm{Spec}(A)$ (k-algebra), X-points of $\mathfrak{M}_{\mathcal{C}}$ are pseudo-perfect $\mathcal{C} \otimes_{\mathbb{C}} A$-module N.

For N, N' such points, $\mathrm{RHom}_{\mathcal{C} \otimes_{\mathbb{C}} A}(N, N')$ is a dg-A module.

$\rightsquigarrow$ defines the RHom complex on $\mathfrak{M}_{\mathcal{C}}^{\times 2}$ and, by restriction, on $\mathfrak{M}_{\mathcal{A}}^{\times 2}$.

$$\mathcal{E} := \mathrm{RHom}[1]$$

Stack of short exact sequences

$$\mathfrak{Exact}_{\mathcal{A}} \simeq \mathrm{Tot}(\mathcal{E}).$$

$$\begin{tikzcd}
\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} \arrow[d, "JH_{\mathcal{A}} \times JH_{\mathcal{A}}"'] & \mathfrak{Exact}_{\mathcal{A}} \arrow[l, "q"', "\text{quasi-smooth}"] \arrow[r, "p", "\text{proper}"'] \arrow[d, phantom, "\circlearrowright"] & \mathfrak{M}_{\mathcal{A}} \arrow[d, "JH_{\mathcal{A}}"] \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \arrow[rr, "\oplus"'] & & \mathcal{M}_{\mathcal{A}}
\end{tikzcd}$$

③ The local neighbourhood theorem

Ext quivers

$\mathcal{A}$ finite length Abelian category

$M = \bigoplus_{i \in I} S_i^{\oplus m_i}$ semisimple object in $\mathcal{A}$

S_i pairwise non-isomorphic simples in $\mathcal{A}$

$m_i \in \mathbb{Z}_{\geq 0}$.

$Q = (Q_0, Q_1)$ Ext-quiver of M / of $\{S_i, i \in I\}$

$Q_0 = I$

and $\# \{ i \to j \} := \dim \mathrm{Ext}^1(S_i, S_j)$

Dimension vector for Q: $(m_i, i \in I) \in \mathbb{N}^{Q_0}$.

Local neighbourhood theorem

Upshot: locally, the map $\mathrm{JH} : \mathfrak{M}_{\mathcal{A}} \to \overset{x \atop \uparrow}{\mathcal{M}_{\mathcal{A}}}$ looks like the map $\mathrm{JH} : \mathfrak{M}_{\Pi_Q} \to \mathcal{M}_{\Pi_Q}$ for Q quiver.

More precisely

<u>Thm:</u> [Davison] $\mathcal{A}$ 2CY as above

$$M = \bigoplus_{1\leq i\leq r} M_i^{\oplus m_i} \quad \text{semisimple object}$$

$$\underline{M} = \{M_i : 1\leq i\leq r\}$$

$Q_{\underline{M}}$ quiver such that $\overline{Q}_{\underline{M}}$ is the Ext-quiver of $\underline{M}$.

There exists a finite type, $\mathbb{C}$-algebraic variety U with a $GL_{\underline{m}}$-action s.t.

we have a diagram of pointed spaces/stacks with étale horizontal maps and Cartesian squares

$$\begin{tikzcd}
(\mathfrak{M}_{\Pi_{Q_{\underline{M}}}}, 0_m) \arrow[d, "JH_{\Pi_{Q_{\underline{n}}}}"'] & {[U/GL_m]} \arrow[l] \arrow[r] \arrow[d] & (\mathfrak{M}_{\mathcal{A}}, x) \arrow[d, "JH_{\mathcal{A}}"] \\
(\mathcal{M}_{\Pi_{Q_{\underline{n}}}}, 0_m) & U/\!/GL_m \arrow[l] \arrow[r] & (\mathcal{M}_{\mathcal{A}}, x)
\end{tikzcd}$$

In addition, we have compatibility with the RHom-complexes:

$$RHom_{\Pi_{Q_{\underline{n}}}}\Big|_{(JH_{\Pi_{Q_{\underline{n}}}} \times JH_{\Pi_{Q_{\underline{n}}}})^{-1}(0_{\underline{m}} \times 0_{\underline{n}})} \simeq RHom_{\Pi_{Q_{\underline{n}}}}\Big|_{(JH_{\mathcal{A}} \times JH_{\mathcal{A}})^{-1}(\{x,y\})}$$