Cohomological integrality for 0-dimensional sheaves on surfaces

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- C a smooth quasi-projective curve.

$\mathrm{Sym}^n(C)$ is smooth: $C = \mathbb{A}^1$ affine line; $\mathrm{Sym}^n C \cong \mathrm{Spec}\, \mathbb{C}[x_1, \dots, x_n]^{S_n}$

$\cong \mathrm{Spec}\, \mathbb{C}[e_1, \dots, e_n]$

e_i = elementary symmetric functions, algebraically independent.

$\mathrm{Coh}_n(C)$ = stack of length n coherent sheaves on C.

$C = \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$ coherent sheaf = $\mathbb{C}[x]$-module

finite length = finite dimensional / $\mathbb{C}$

$\rightsquigarrow$ V f.d. $\mathbb{C}$-vector space with $f \in \mathrm{End}(V)$.

$$\Rightarrow \mathrm{Coh}_n(C) \simeq \left[\mathrm{Mat}_{n\times n}(\mathbb{C}) / \mathrm{Gl}_n(\mathbb{C}) \right]$$ smooth stack.

Prop: · $\dim \mathrm{Coh}_n(C) = 0$ [easy to see for $C = \mathbb{A}^1$]

- length 1: $\mathrm{Coh}_1(C) \cong C \times B\mathbb{C}^*$
- $\mathrm{Coh}_n(C)$ is a smooth stack
- It has a good moduli space

Hilbert-Chow morphism

$$\begin{tikzcd} \mathrm{Coh}_n(C) \arrow[r] & \mathrm{Sym}^n(C) \\ \mathcal{F} \arrow[r, mapsto] & \mathrm{supp}(\mathcal{F}). \end{tikzcd}$$

For $C = \mathbb{A}^1$,

$$\begin{tikzcd} \left[\mathrm{Mat}_{n\times n}(\mathbb{C}) / \mathrm{Gl}_n(\mathbb{C}) \right] \arrow[r] & \mathrm{Sym}^n(\mathbb{A}^1) \\ M \arrow[r, mapsto] & \text{eigenvalues of } M \end{tikzcd}$$

or

characteristic polynomial of M

Cohomological integrality:

$$H^*\left(\mathrm{Coh}_{f.l}(C), \mathbb{Q}^{\overset{0}{\cancel{\mathrm{vir}}}}\right) \overset{\text{vector spaces}}{\simeq} \mathrm{Sym}\left(H^*(C,\mathbb{Q}) \otimes H^*(\mathrm{pt}/\mathbb{C}^*)\right)$$

[Meinhardt-Reineke]

Poincaré polynomials

$$\sum_{\substack{n\in\mathbb{N}\\ i\in\mathbb{Z}}} \dim H^i(\mathrm{Coh}_n(C))\, t^i q^n = \mathrm{PE}\left(q\frac{\overbrace{1+2gt+t^2}^{P_C(t)}}{1-t^2}\right)$$

if C sm. proj. genus g

$$= \mathrm{PE}\left(\frac{q P_C(t)}{1-t^2}\right)$$

$$C = \mathbb{A}^1 : \quad \mathrm{PE}\left(\frac{q}{1-t^2}\right) = \mathrm{PE}\left(q\sum_{\ell\geq 0} t^{2\ell}\right)$$

$$= \prod_{\ell\geq 0} \frac{1}{1-qt^{2\ell}}$$

$$= \sum_{m,n} a_{m,n} q^n t^{2m}$$

$a_{m,n}$ = # partitions of m having less than or equal to n nonzero parts

$$\sum_{n\in\mathbb{N}} \dim H^*(\mathrm{pt}/GL_n)\, q^n$$

$$H^*(\mathrm{pt}/GL_n) \cong \mathbb{C}[e_1, \dots, e_n]$$

degree: e_1 — 2, e_n — $2n$

$$\dim \mathbb{C}[e_1, \dots, e_n] = \sum_{\ell\geq 0} \left\{\begin{array}{c}\text{partitions of } \ell\\ (\lambda_1, \dots)\\ \lambda_i \leq n\end{array}\right\} t^{2\ell}$$

S smooth quasi-projective surface $\mathbb{A}^2, \mathbb{P}^2,$ K3 / Abelian

$\mathrm{Sym}^n(S) = S^n/\mathfrak{S}_n$ singular $(n \geq 2)$

$= \{\{x_1, \dots, x_n\} \subset S\}$ unordered.

Smooth locus: $\mathrm{Sym}^n(S) \setminus \Delta$ big diagonal

$\mathrm{Sym}^n S$ symplectic singularity.

$S = \mathbb{A}^2 = \mathrm{Spec}\, \mathbb{C}[x, y]$.

finite length coherent sheaf on S $\Leftrightarrow$ $V \in \mathrm{mod}\, \mathbb{C}[x,y]$

$\dim_{\mathbb{C}} V < \infty$

$\Leftrightarrow$ V f. dim $\mathbb{C}$. vspace with two commuting endomorphisms (actions of x and y)

length $n \in \mathbb{N}$ $\rightsquigarrow$ $\dim_{\mathbb{C}} V = n$.

$$\mathrm{Coh}_n(\mathbb{A}^2) \simeq \left[\{(M, N) \in \mathrm{Mat}_{n \times n}^{\times 2} \mid MN = NM\} / GL_n \right]$$

simultaneous conjugation.

$$\simeq \left[C(\mathfrak{gl}_n) / GL_n \right]$$

$C(\mathfrak{gl}_n) =$ " commuting variety ".

very complicated algebraic variety.

simultaneous diagonalisation of matrices $\Rightarrow$ there is an open substack $(\mathbb{C}^2)^n \setminus \Delta / \mathfrak{S}_n \times \mathrm{pt}/(\mathbb{C}^*)^n = \{(x_1, y_1), \dots, (x_n, y_n)$ pairwise distinct$\}/\mathfrak{S}_n$.

of $\mathrm{Coh}_n(\mathbb{A}^2)$

For general S, $\mathrm{Coh}_n(S)$ is a global analogue of the commuting variety.

<u>Prop</u>: * $\dim \mathrm{Coh}_n(S) = n$

$\mathrm{vir}\dim \mathrm{Coh}_n(S) = 0$

↖ to account for the fact that not smooth

* length 1: $\mathrm{Coh}_1(S) \cong S \times B\mathbb{C}^*$

* For $n \geq 2$, $\mathrm{Coh}_n(S)$ is a singular stack

* It has a good moduli space

$$\begin{tikzcd}
\mathrm{Coh}_n(S) \arrow[r] & \mathrm{Sym}^n S \\
\mathcal{F} \arrow[r, mapsto] & \mathrm{supp}\, S
\end{tikzcd}$$

For $S = \mathbb{A}^2$, $\left[C(\mathfrak{gl}_n) / GL_n \right] \longrightarrow S^n\mathbb{A}^2$

$$\left(\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, \begin{pmatrix} b_1 & & * \\ & \ddots & \\ 0 & & b_n \end{pmatrix} \right) \longmapsto \left\{ (a_i, b_i)_{1 \leq i \leq n} \right\}.$$

Actually, this part of the story carries over to higher dimensional varieties but the stack and its moduli space have increasingly severe singularities.

Cohomological integrality (Kapranov–Vasserot, Davison, Davison-H-Schlegel Mejia)

$$H^{BM}_{-*}\left(\mathrm{Coh}_{f.l.}(S), \mathbb{Q}^{\mathrm{vir}}\right) \simeq \mathrm{Sym}\left(\bigoplus_{n\geq 0} H^{*+2}(S,\mathbb{Q}) \otimes H^{*}(\mathrm{pt}/\mathbb{C}^{*})\right)$$

some well-chosen shifts.

$$\mathbb{Q}^{\mathrm{vir}}_{\mathrm{Coh}_n(S)} = \mathbb{Q}_{\mathrm{Coh}_n(S)}\left[\mathrm{virdim}\,\mathrm{Coh}_n(S)\right].$$
$$= \mathbb{Q}_{\mathrm{Coh}_n(S)}$$

Check for length 1 :

$$H^{BM}_{-*}\left(S\times B\mathbb{C}^{*}, \mathbb{Q}^{\mathrm{vir}}\right) = H^{*}\Big(\overbrace{S\times B\mathbb{C}^{*}}^{\text{smooth of dim 1}}, \underset{\substack{\| \\ \mathbb{Q}[2]}}{\mathbb{D}\mathbb{Q}}\Big)$$

$$\underset{\text{Künneth.}}{=} H^{*+2}(S,\mathbb{Q})\otimes H^{*}(\mathrm{pt}/\mathbb{C}^{*})$$

Consequence: * $H^{BM}(\mathrm{Coh}_{f.l}(S))$ has pure mixed Hodge structure if and only if $H^{*}(S,\mathbb{Q})$ has pure MHS.

* In terms of Poincaré polynomials:

$$\sum_{\substack{i\in\mathbb{Z}\\ n\in\mathbb{N}}} \dim H^{BM}_{-i}\big(\underbrace{\mathrm{Coh}_n(S)}_{\text{very singular stack}},\mathbb{Q}\big)\, t^{i} q^{n} = \mathrm{PE}\left(\frac{q\,t^{-2}\,\overbrace{P_S(t)}^{\text{Poincaré pol of } S}}{(1-q)\,(1-t^{2})}\right)$$

S smooth!

Question: How to prove such a cohomological integrality iso?

→ Using cohomological Hall algebras.

Cohomological Hall algebras

Construct an algebra structure on the Borel-Moore homology of some stacks classifying objects in categories.

$\mathcal{A}$ Abelian category
$$\mathcal{A} = \begin{cases} \operatorname{Rep} Q & Q \text{ quiver} \\ \operatorname{Coh}(C) & C \text{ sm. proj. curve} \\ \operatorname{Rep} \Pi_Q & \Pi_Q \text{ preproj. algebra of } Q \\ \operatorname{Coh}(S) & S \text{ quasi-proj. curve} \end{cases}$$

e.g.

$Q = (\underset{\text{vertices}}{Q_0}, \underset{\text{arrows}}{Q_1})$ quiver $\mathcal{A} = \operatorname{Rep} Q.$

representation V of Q V_i $i \in Q_0$ vectorspace

$V_i \xrightarrow{x_\alpha} V_j$ linear map $\alpha \in Q_1$.

$$\dim V := (\dim V_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$$

representation space : $\displaystyle\bigoplus_{i \xrightarrow{\alpha} j \in Q_1} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) =: X_{Q,d}$

$$\circlearrowleft \prod_{i \in Q_0} GL_{d_i} =: GL_d$$

stack of representations: $\mathfrak{M}_{Q,d} = [X_{Q,d}/GL_d]$.

In general:
- $\mathfrak{M}_{\mathcal{A}}$ stack of objects in $\mathcal{A}$
- $\mathrm{Exact}_{\mathcal{A}}$ stack of exact sequences of objects of $\mathcal{A}$

$$0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$$

$$\begin{cases} \ker f = 0 \\ \operatorname{im} g = M \\ \ker g = \operatorname{im} f \end{cases}.$$

$= $ stack of subobjects $\{N \subset E\}$.

e.g. $Q = \bullet$ $\qquad \mathfrak{M}_Q = \bigsqcup_{d\in\mathbb{N}} pt/GL_d$

$$\mathrm{Exact}_Q = \bigsqcup_{d,e\in\mathbb{N}} pt/P_{d,e} \qquad P_{d,e} = \left\{ \begin{array}{c} e\{ \\ \end{array} \begin{pmatrix} * & * \\ 0 & \underbrace{*}_{d} \end{pmatrix} \right\}$$

$\subset GL_{d+e}$
parabolic

correspondence

$$\begin{tikzcd} & \mathrm{Exact}_{\mathcal{A}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} & & \mathfrak{M}_{\mathcal{A}} \end{tikzcd}$$

$$\begin{tikzcd} & 0 \to N \to E \to M \to 0 \arrow[dl] \arrow[dr] & \\ (M,N) & & E \end{tikzcd}$$

p is proper. $\qquad$ e.g. if $\mathcal{A} = \mathrm{Vect}$, e.g.

$$\begin{tikzcd} & pt/P_{d,e} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ pt/GL_d \times GL_e & & pt/GL_{d+e} \end{tikzcd}$$

In general: p is proper: fiber of $p \cong GL_{d+e}/P_{d,e}$ flag variety.

q is smooth if $\mathcal{A}$ has homological dimension 1

eg above: fiber is $P_{d,e}/GL_d \times GL_e \simeq \left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right\}$.

q is only "quasi-smooth" if $\mathcal{A}$ has homological dimension 2

In any case, we can define a pullback map

$$H^{BM}(\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}}) \xrightarrow{q^*} H^{BM}(\mathrm{Exact}_{\mathcal{A}})$$

and a pushforward map $H^{BM}(\mathrm{Exact}_{\mathcal{A}}) \to H^{BM}(\mathcal{M}_{\mathcal{A}})$.

which once combined give the CoHA multiplication

$m = p_* q^*$ on $H^{BM}(\mathcal{M}_{\mathcal{A}})$.

This is the CoHA of $\mathcal{A}$.

Refined / Relative cohomological Hall algebra

If we have a map $\varpi : \mathfrak{M}_{\mathcal{A}} \longrightarrow \mathcal{M}$ and $\mathcal{M}$ is an algebraic variety with maybe infinitely many connected components and a monoid structure $\oplus : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ s.t.

$$\begin{tikzcd}
& \mathfrak{Exact}_{\mathcal{A}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} \arrow[d, "\varpi \times \varpi"] & \circlearrowright & \mathfrak{M}_{\mathcal{A}} \arrow[d, "\varpi"] \\
\mathcal{M} \times \mathcal{M} \arrow[rr, "\oplus"] & & \mathcal{M}
\end{tikzcd} ,$$

Then a similar procedure gives a multiplication on

$$\varpi_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\mathcal{A}}} \in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}})$$

The monoidal structure on $\mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}})$ is given by

$$\mathcal{F} \boxdot \mathcal{G} = \oplus_* (\mathcal{F} \boxtimes \mathcal{G}).$$

e.g of ϖ: $\mathcal{M} = pt$, $\mathcal{M} = \pi_0(\mathfrak{M}_{\mathcal{A}})$, ...

Sometimes, there is a universal ϖ with $\mathcal{M}$ an algebraic space, it is the good moduli space: $JH : \mathfrak{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{A}}$.

We study $H^{BM}(\mathfrak{M}_{\mathcal{A}})$ through the richer object $JH_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{A}}} =: \mathcal{A}$
$\in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}})$
constructible complex

The BPS associative algebra

We can use the relative CoHA $\mathcal{A} = \mathrm{JH}_* \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathcal{A}}}$ to define a smaller, more manageable algebra : the BPS associative algebra.

$\mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}})$ has the perverse t-structure and associated coh. functors.

${}^{\mathfrak{p}}\mathcal{H}^i$ $i \in \mathbb{Z}$.

Prop : If $\mathcal{A}$ is a 2CY category (i.e. $\mathrm{Ext}^{2-i}(M,N) \cong \mathrm{Ext}^i(N,M)^*$ functorially in M, N),

${}^{p}\mathcal{H}^i(\mathcal{A}) = 0$ if $i < 0$.

$\Rightarrow$ ${}^{\mathfrak{p}}\mathcal{H}^0(\mathcal{A})$ is an algebra object in $(\mathrm{Perv}(\mathcal{M}_{\mathcal{A}}), \boxdot)$.

$$\overset{\mathrm{BPS}}{\shortparallel}$$

$\mathrm{BPS} := H^*(X, {}^{p}\mathcal{H}^0(\mathcal{A}))$ associative algebra.

Theorem (DHS) $\exists$ Generalised Kac-Moody datum on $\underbrace{\pi_0(\mathcal{M}_{\mathcal{A}})}_{\text{monoid}}$ with bilinear form $(-,-)_{\mathcal{A}}$ (Euler form on $\mathcal{A}$) s.t.

$\mathrm{BPS} \cong U(\mathfrak{n}^+)$ $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ corresponding GKM.

More precisely, BPS is generated by $\mathrm{IH}^*(\mathcal{M}_{\mathcal{A},a})$ for $a \in R^+ \subset \pi_0(\mathcal{M}_{\mathcal{A}})$ subject to Serre type relations $\forall\, x \in \mathrm{IH}^*(\mathcal{M}_{\mathcal{A},a}), y \in \mathrm{IH}^*(\mathcal{M}_{\mathcal{A},b})$

$$\begin{cases} [x,y] = 0 & \text{if } (a,b) = 0 \\ \mathrm{ad}(x)^{1-(a,b)}(y) = 0 & \text{if } (a,a) = 2. \end{cases}$$

Notation: $\mathrm{BPS}_{\mathrm{Lie}} := \mathfrak{n}^+$

Cohomological integrality for 2CY categories

<u>Thm</u> (DHS) $\mathcal{A}$ 2CY

$$H^{BM}_*(\mathfrak{M}_{\mathcal{A}}, \mathbb{Q}^{vir}) \cong \mathrm{Sym}\left(\mathrm{BPS}_{Lie} \otimes H^*(B\mathbb{C}^*)\right)$$

<u>Construction of the morphism</u> $\longleftarrow$:

$$\mathrm{BPS}_{Lie} \subset \mathrm{BPS} \subset H^{BM}_*(\mathfrak{M}_{\mathcal{A}}, \mathbb{Q}^{vir})$$

$$\circlearrowright$$

$H^*(B\mathbb{C}^*)$ (first Chern class of the det line bundle on $\mathfrak{M}_{\mathcal{A}}$)

$$\to \quad \mathrm{BPS}_{Lie} \otimes H^*(B\mathbb{C}^*) \longrightarrow H^{BM}_*(\mathfrak{M}_{\mathcal{A}}, \mathbb{Q}^{vir}).$$

Using alg structure,

$$\mathrm{Sym}\left(\mathrm{BPS}_{Lie} \otimes H^*(B\mathbb{C}^*)\right) \longrightarrow H^{BM}_*(\mathfrak{M}_{\mathcal{A}}, \mathbb{Q}^{vir}).$$

<u>Goal</u>: this morphism is an iso.

<u>Proof</u>: ① Work at the relative level, ie $\mathcal{BPS}$ instead of BPS
$\mathcal{A}$ instead of $H^{BM}_*(\mathfrak{M}_{\mathcal{A}}, \mathbb{Q}^{vir})$.

② Use the "local neighbourhood theorem" of Davison for 2CY categories

③ Use the description of the top-CoHA of the SSN cone for preprojective algebras of quivers.

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② **Thm** (Davison) $\mathcal{A}$ 2CY category

$JH : \mathfrak{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{A}}$ good m.s

$x \in \mathcal{M}_{A}$, $x \rightsquigarrow \mathcal{F} = \bigoplus_{j=1}^{r} \mathcal{F}_j^{\oplus m_j}$.

$\underline{\mathcal{F}} = \{\mathcal{F}_1, \dots, \mathcal{F}_r\}$ pairwise distinct objects of $\mathcal{A}$.

$\overline{Q}_{\underline{\mathcal{F}}}$: Ext. quiver of $\underline{\mathcal{F}}$

$= \left((\overline{Q}_{\underline{\mathcal{F}}})_0, (\overline{Q}_{\underline{\mathcal{F}}})_1 \right)$

$(\overline{Q}_{\underline{\mathcal{F}}})_0 = \underline{\mathcal{F}}$,

$\forall i, j$, we have $\mathrm{ext}^1(\mathcal{F}_i, \mathcal{F}_j)$ arrows $\mathcal{F}_i \to \mathcal{F}_j$.

$\exists\, Q_{\underline{\mathcal{F}}}$ s.t. $\overline{Q}_{\underline{\mathcal{F}}}$ is the double of $Q_{\underline{\mathcal{F}}}$.

$$\begin{tikzcd}
(\mathfrak{M}_{\mathcal{A}}, x) \arrow[d, "JH"'] & ([U/GL_m], y) \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathfrak{M}_{\Pi_{Q_{\underline{\mathcal{F}}}}}, 0_m) \arrow[d] \\
(\mathcal{M}_{\mathcal{A}}, x) & (U/\!/GL_m, y) \arrow[l] \arrow[r] & (\mathcal{M}_{\Pi_{Q_{\underline{\mathcal{F}}}}}, 0_m)
\end{tikzcd}$$

w/ étale horizontal maps.

③ $Q = (Q_0, Q_1)$ quiver

Π_Q preprojective algebra

Lagrangian substack

$$\begin{tikzcd}
\mathfrak{M}^{SSN}_{\Pi_Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{M}_{\Pi_Q} \arrow[d] \\
\mathcal{M}^{SSN}_{\Pi_Q} \arrow[r, "\imath"'] & \mathcal{M}_{\Pi_Q}
\end{tikzcd}$$

$\{$ s.simple reps of Π_Q s.t. only <u>loops</u> in Q_1 act possibly by $\neq 0$.

$\imath^! \mathcal{A} \in \mathcal{D}^+_c(\mathcal{M}^{SSN}_{\Pi_Q})$.

$H^0(\imath^! \mathcal{A}) \subset H^*(\imath^! \mathcal{A})$ is a subalgebra

It has a linear basis given by fund. classes of irr. components of $\mathfrak{M}_{\Pi_Q}$.

$$I = (Q_0^{re} \times \{1\}) \sqcup (Q_1^{im} \times \mathbb{Z}_{\geq 1})$$

vertices w/o loops — vertices w/ at least one loop.

<u>Thm</u> (H) $H^0(\imath^! \mathcal{A}) \cong U(\mathfrak{n}_Q^+)$ where $\mathfrak{n}_Q^+$ is the Lie algebra generated by e_i, $i \in I$ w/ relations

$$\begin{cases} (e_i, e_j) = 0 & \text{if } (i,j) = 0 \\ \mathrm{ad}(e_i)^{1-(i,j)}(e_j) = 0 & \text{if } i \in Q_0^{re} \times \{1\}. \end{cases}$$

induced by the symmetrised Euler form of Q