

# Cohomological integrality for 0-dimensional sheaves on surfaces

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$C$  a smooth quasi-projective curve.

$\text{Sym}^n(C)$  is smooth:  $C = \mathbb{A}^1$  affine line;  $\text{Sym}^n C \cong \text{Spec } \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$   
 $\cong \text{Spec } \mathbb{C}[e_1, \dots, e_n]$

$e_i$  = elementary symmetric functions, algebraically independent.

$\text{Coh}_n(C)$  = stack of length  $n$  coherent sheaves on  $C$ .

$C = \mathbb{A}^1 = \text{Spec } (\mathbb{C}[x])$  coherent sheaf =  $\mathbb{C}[x]$ -module  
finite length = finite dimensional /  $\mathbb{C}$   
 $\leadsto V$  f.d.  $\mathbb{C}$ -vector space with  $f \in \text{End}(V)$ .

$\Rightarrow \text{Coh}_n(C) \simeq \left[ \text{Mat}_{n \times n}(\mathbb{C}) / \text{GL}_n(\mathbb{C}) \right]$  smooth stack.

Prop:  $\dim \text{Coh}_n(C) = 0$  [easy to see for  $C = \mathbb{A}^1$ ]

• length 1:  $\text{Coh}_1(C) \cong C \times \text{BC}^*$

•  $\text{Coh}_n(C)$  is a smooth stack

Hilbert-Chow morphism

• It has a good moduli space  $\text{Coh}_n(C) \longrightarrow \text{Sym}^n(C)$   
 $\mathcal{F} \longmapsto \text{supp}(\mathcal{F})$ .

For  $C = \mathbb{A}^1$ ,  $\left[ \text{Mat}_{n \times n}(\mathbb{C}) / \text{GL}_n(\mathbb{C}) \right] \longrightarrow \text{Sym}^n(\mathbb{A}^1)$   
 $M \longmapsto$  eigenvalues of  $M$   
or  
characteristic polynomial of  $M$

Cohomological integrality:

$$H^*(\text{Coh}_{f.e}(C), \mathbb{Q}) \cong \text{Sym} \left( H^*(C, \mathbb{Q}) \otimes H^*(\text{pt}/\mathbb{C}^*) \right)$$

vector spaces

[Meinhardt-Reineke]

Poincaré polynomials

$$\sum_{\substack{n \in \mathbb{N} \\ i \in \mathbb{Z}}} \dim H^i(\text{Coh}_n(C)) t^i q^n = \text{PE} \left( q \frac{1 + 2g t + t^2}{1 - t^2} \right)$$

$P_C(t)$  if  $C$  sm. proj. genus  $g$

$$= \text{PE} \left( q \frac{P_C(t)}{1 - t^2} \right)$$

$$C = \mathbb{A}^1 : \text{PE} \left( \frac{q}{1 - t^2} \right) = \text{PE} \left( q \sum_{l \geq 0} t^{2l} \right)$$

$$= \prod_{l \geq 0} \frac{1}{1 - q t^{2l}}$$

$$= \sum_{m, n} a_{m, n} q^n t^{2m}$$

# partitions of  $m$  having less than  $n$  nonzero parts or equal to

$$\sum_{n \in \mathbb{N}} \dim H^*(\text{pt}/GL_n) q^n$$

SII

$$\Phi[e_1, \dots, e_n]$$

degree  $l$                        $2n$

$$\dim \Phi[e_1, \dots, e_n] = \sum_{l \geq 0} \left\{ \begin{matrix} \text{partitions of } l \\ (d_1, \dots, d_r) \\ d_i \leq n \end{matrix} \right\} t^{2l}$$

●  $S$  smooth quasi-projective surface  $A^2, P^2, K3/Abelian$

$$\text{Sym}^n(S) = S^n / \mathbb{C}_n \quad \text{singular } (n \geq 2)$$

$$= \{ \{x_1, \dots, x_n\} \subset S \}^{\text{unordered}}$$

Smooth locus:  $\text{Sym}^n(S) \setminus \Delta$  big diagonal

$\text{Sym}^n S$  symplectic singularity.

$$S = A^2 = \text{Spec } \mathbb{C}[x, y]$$

finite length coherent sheaf on  $S \Leftrightarrow$

$$V \in \text{mod } \mathbb{C}[x, y]$$

$$\dim_{\mathbb{C}} V < \infty$$

}

$\Leftrightarrow V$  f. fin.  $\mathbb{C}$ -vspace  
with two commuting  
endomorphisms (actions of  
 $x$  and  $y$ )

$$\text{length } n \in \mathbb{N} \quad \Leftrightarrow \quad \dim_{\mathbb{C}} V = n.$$

$$\text{Coh}_n(A^2) \cong \left[ \left\{ \sum (M, N) \in \text{Mat}_{n \times n}^{\mathbb{C}} \mid MN = NM \right\} / \left[ \begin{array}{c} GL_n \\ \text{simultaneous} \\ \text{conjugation} \end{array} \right] \right]$$

$$\cong \left[ \mathbb{C}(\text{log } n) / GL_n \right]$$

$\mathbb{C}(\text{log } n) =$  "commuting variety".  
very complicated algebraic variety.

simultaneous diagonalisation of matrices  $\Rightarrow$  there is an open  
substack  $(\mathbb{C}^2)^n \setminus \Delta / \mathbb{C}_n \times \text{pt} / \mathbb{C}_n = \{ (x_1, y_1), \dots, (x_n, y_n) \}$  pairwise distinct  $\mathbb{C}_n$ .

of  $\text{Coh}_n(\mathbb{A}^2)$

For general  $S$ ,  $\text{Coh}_n(S)$  is a global analogue of the commuting variety.

Prop: \*  $\dim \text{Coh}_n(S) = n$

vir  $\dim \text{Coh}_n(S) = 0$

↑ to account for the fact that not smooth

\* length 1:  $\text{Coh}_1(S) \cong S \times \mathbb{B}\mathbb{P}^*$

\* For  $n \geq 2$ ,  $\text{Coh}_n(S)$  is a singular stack

\* It has a good moduli space

$$\begin{array}{ccc} \text{Coh}_n(S) & \longrightarrow & \text{Sym}^n S \\ \neq & \longleftarrow & \text{supp } S \end{array}$$

For  $S = \mathbb{A}^2$ ,  $\left[ \text{C}(\text{opln}) / \text{GL}_n \right] \rightarrow S^n \mathbb{A}^2$

$$\left( \begin{pmatrix} a_1 & * \\ & a_n \\ 0 & a_n \end{pmatrix}, \begin{pmatrix} b_1 & * \\ & b_n \\ 0 & b_n \end{pmatrix} \right) \mapsto \{ (a_i, b_i)_{1 \leq i \leq n} \}.$$

Actually, this part of the story carries over to higher dimensional varieties but the stack and its moduli space have increasingly severe singularities.

Cohomological integrality (Kapranov-Vasserot, Davison,  
Davison-H-Schlegel Meja)

$$H_{-*}^{BM}(\text{Coh}_{f.e.}(S), \mathcal{Q}^{\text{vir}}) \cong \text{Sym} \left( \bigoplus_{h \geq 0} H^{*+2}(S, \mathcal{Q}) \otimes H^*(\text{pt}/\mathbb{C}^*) \right)$$

some well-chosen shifts.

$$\begin{aligned} \mathcal{Q}_{\text{Coh}_n(S)}^{\text{vir}} &= \mathcal{Q}_{\text{Coh}_n(S)} [\text{vir dim Coh}_n(S)] \\ &= \mathcal{Q}_{\text{Coh}_n(S)} \end{aligned}$$

Check for length 1:

$$\begin{aligned} H_{-*}^{BM}(S \times B\mathbb{C}^*, \mathcal{Q}^{\text{vir}}) &= H^*(\overbrace{S \times B\mathbb{C}^*}^{\text{smooth of dim 1}}, \underbrace{\mathbb{D}\mathcal{Q}}_{\mathbb{Q}[2]}) \\ &= H^{*+2}(S, \mathcal{Q}) \otimes H^*(\text{pt}/\mathbb{C}^*) \end{aligned}$$

Künneth.

Consequence:  $H_{-*}^{BM}(\text{Coh}_{f.e.}(S))$  has pure mixed Hodge structure  
if and only if  $H^*(S, \mathcal{Q})$  has pure MHS.

\* In terms of Poincaré polynomials:

$$\sum_{\substack{i \in \mathbb{Z} \\ n \in \mathbb{N}}} \dim H_{-i}^{BM}(\text{Coh}_n(S), \mathcal{Q}) t^i q^n = \text{PE} \left( \frac{q t^{-2} P_S(t)}{(1-q)(1-t^2)} \right)$$

very singular stack Poincaré pol of S  
S smooth!

Question: How to prove such a cohomological integrality iso?

→ Using cohomological Hall algebras.

## ● Cohomological Hall algebras

Construct an algebra structure on the Borel-Moore homology of some stacks classifying objects in categories.

$\mathcal{A}$  Abelian category

{	Rep $\mathcal{Q}$	$\mathcal{Q}$ quiver
	Coh(C)	C sm. proj. curve
	Rep $\Pi \alpha$	$\Pi \alpha$ preproj. algebra of $\mathcal{Q}$
	Coh(S)	S quasi-proj. curve

e.g.

$\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$  quiver  
 vertices      arrows

representation  $V$  of  $\mathcal{Q}$

$\mathcal{A} = \text{Rep } \mathcal{Q}$ .

$V_i$   $i \in \mathcal{Q}_0$  vector space

$V_i \xrightarrow{\alpha_j} V_j$  linear map  $\alpha \in \mathcal{Q}_1$ .

$\dim V := (\dim V_i)_{i \in \mathcal{Q}_0} \in \mathbb{N}^{\mathcal{Q}_0}$

representation space:  $\bigoplus_{i \rightarrow_j \alpha \in \mathcal{Q}_1} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) =: X_{\mathcal{Q}, d}$

$\curvearrowright \prod_{i \in \mathcal{Q}_0} GL_{d_i} =: G_{\mathcal{Q}, d}$

stack of representations:  $\pi_{Q,d} = [X_{Q,d} / GL_d]$ .

In general: \*  $\pi_{\mathcal{A}}$  stack of objects in  $\mathcal{A}$

\*  $\text{Exact}_{\mathcal{A}}$  stack of exact sequences of objects of  $\mathcal{A}$

$\mathcal{A}$

$$0 \rightarrow N \xrightarrow{f} E \xrightarrow{g} M \rightarrow 0$$

$$\begin{cases} \ker f = 0 \\ \text{img } g = M \\ \ker g = \text{img } f \end{cases}$$

= stack of subobjects  $\{N \subset E\}$ .

eg.  $\mathcal{Q} = \bullet$

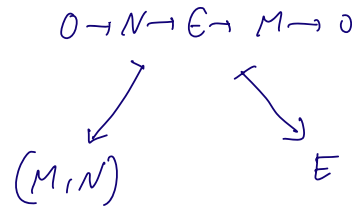
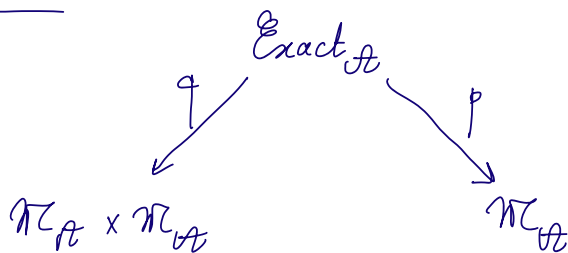
$$\pi_{\mathcal{Q}} = \coprod_{d \in \mathbb{N}} \text{pt} / GL_d$$

$$\text{Exact}_{\mathcal{Q}} = \coprod_{d, e \in \mathbb{N}} \text{pt} / P_{d,e}$$

$$P_{d,e} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

$\subset GL_{d+e}$   
parabolic

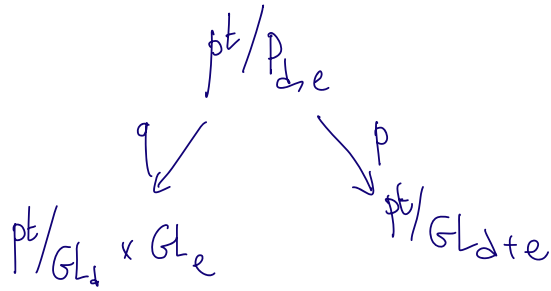
correspondence



$p$  is proper.

e.g. if  $\mathcal{A} = \text{Vect}$ ,

e.g.



In general:  $p$  is proper: fiber of  $p \cong GL_{d+e}/P_{d,e}$  flag variety.

$q$  is smooth if  $\mathcal{O}_T$  has homological eq above, fiber is  $P_{d,e}/GL_d \times GL_e \cong \left\{ \begin{smallmatrix} 0 & * \\ \infty & * \end{smallmatrix} \right\}^p$  dimension 1

$q$  is only "quasi-smooth" if  $\mathcal{O}_T$  has homological dimension 2  
In any case, we can define a pullback map

$$H^{BM}(\mathcal{N}_{\mathcal{R}} \times \mathcal{N}_{\mathcal{O}_T}) \xrightarrow{q^*} H^{BM}(\text{Exact}_{\mathcal{O}_T})$$

and a pushforward map  $H^{BM}(\text{Exact}_{\mathcal{O}_T}) \rightarrow H^{BM}(\mathcal{N}_{\mathcal{O}_T})$ .

which once combined give the CoHA multiplication

$$m = p_* q^* \text{ on } H^{BM}(\mathcal{N}_{\mathcal{R}}).$$

This is the CoHA of  $\mathcal{O}_T$ .



## Refined / Relative cohomological Hall algebra

If we have a map  $\varpi: \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{M}$  and  $\mathcal{M}$  is an algebraic variety with maybe infinitely many connected components and a monoid structure  $\oplus: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  s.t.

$$\begin{array}{ccc}
 & \text{Exact}_{\oplus} & \\
 q \swarrow & & \searrow p \\
 \mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} & \hookrightarrow & \mathcal{M}_{\mathcal{A}} \\
 \downarrow \varpi \times \varpi & \circlearrowleft & \downarrow \varpi \\
 \mathcal{M} \times \mathcal{M} & \xrightarrow{\oplus} & \mathcal{M}
 \end{array}$$

then a similar procedure gives a multiplication on

$$\varpi_* \mathbb{D}\mathcal{R}_{\mathcal{M}_{\mathcal{A}}}^{\text{vir}} \in \mathcal{D}_c^+(\mathcal{M}_{\mathcal{A}})$$

The monoidal structure on  $\mathcal{D}_c^+(\mathcal{M}_{\mathcal{A}})$  is given by

$$\mathcal{F} \boxtimes \mathcal{G} = \varpi_* (\mathcal{F} \boxtimes \mathcal{G}).$$

e.g. of  $\varpi: \mathcal{M} = \text{pt}, \mathcal{M}_{\mathcal{A}} = \pi_0(\mathcal{M}_{\mathcal{A}}), \dots$

Sometimes, there is a universal  $\varpi$  with  $\mathcal{M}$  an algebraic space, it is the good moduli space:  $\mathcal{JH}: \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{M}_{\text{good}}$ .

We study  $H^{\text{BM}}(\mathcal{M}_{\mathcal{A}})$  through the richer object  $\mathcal{JH}_* \mathbb{D}\mathcal{R}_{\mathcal{M}_{\mathcal{A}}}^{\text{vir}} := \mathcal{I} \in \mathcal{D}_c^+(\mathcal{M}_{\text{good}})$  constructible complex

## The BPS associative algebra

We can use the relative CoHA  $\mathcal{A} = \mathcal{J}H_* \mathcal{D}Q_{\mathcal{M}_A}^{vir}$  to define a smaller, more manageable algebra: the BPS associative algebra.

$\mathcal{D}_c^+(\mathcal{M}_A)$  has the perverse t-structure and associated coh. functors.

$$\mathcal{P}H^i \quad i \in \mathbb{Z}.$$

Prop: If  $\mathcal{A}$  is a DCY category (i.e.  $\text{Ext}^{2-i}(M, N) \cong \text{Ext}^i(N, M)^*$  functorially in  $M, N$ ),

$$\mathcal{P}H^i(\mathcal{A}) = 0 \quad \text{if } i < 0.$$

$\Rightarrow \mathcal{P}H^0(\mathcal{A})$  is an algebra object in  $(\text{Per}(\mathcal{M}_A), \boxplus)$ .

!!  
BPS

BPS :=  $H^*(X, \mathcal{P}H^0(\mathcal{A}))$  associative algebra.

Theorem (DHS)  $\exists$  Generalised Kac-Moody datum on  $\pi_0(\mathcal{M}_A)$  with bilinear form  $(-|-)_{\mathcal{A}}$  (Euler form on  $\mathcal{A}$ ) s.t.  $\pi_0(\mathcal{M}_A)$  is a monoid.

BPS  $\cong U(\pi^+)$   $\sigma_{\mathcal{Y}} = \pi^- \oplus \mathfrak{h} \oplus \pi^+$  corresponding GK.  $\pi^+$

More precisely, BPS is generated by  $H^*(\mathcal{M}_{A,a})$  for  $a \in \mathbb{R}^+ \subset \pi_0(\mathcal{M}_A)$  subject to Serre type relations  $\forall x \in H^*(\mathcal{M}_{A,a}), y \in H^*(\mathcal{M}_{A,b})$

$$\left\{ \begin{array}{l} [x, y] = 0 \quad \text{if } (a, b) = 0 \\ \text{ad}(x)^{1-(a,b)}(y) = 0 \quad \text{if } (a, a) = 2. \end{array} \right.$$

Notation:  $\text{BPS}_{\text{Lie}} := \pi^+$

## Cohomological integrality for LCY categories

Chm (DHS) A LCY

$$\left[ \begin{array}{l} H_*^{\text{BM}}(\mathcal{M}_A, \mathbb{Q}^{\text{vir}}) \cong \text{Sym}(BPS_{\text{Lie}} \otimes H^*(BC^*)) \end{array} \right.$$

Construction of the morphism  $\longleftarrow$  :

$$BPS_{\text{Lie}} \subset BPS \subset H_*^{\text{BM}}(\mathcal{M}_A, \mathbb{Q}^{\text{vir}})$$

$\uparrow$   
 $H^*(BC^*)$  (first Chern class of the det  
line bundle on  $\mathcal{M}_A$ )

$$\rightarrow BPS_{\text{Lie}} \otimes H^*(BC^*) \rightarrow H_*^{\text{BM}}(\mathcal{M}_A, \mathbb{Q}^{\text{vir}})$$

Using alg structure,

$$\text{Sym}(BPS_{\text{Lie}} \otimes H^*(BC^*)) \rightarrow H_*^{\text{BM}}(\mathcal{M}_A, \mathbb{Q}^{\text{vir}}).$$

Goal: this morphism is an iso.

Proof: ① Work at the relative level, i.e.  $BPS$  instead of  $BPS_{\text{Lie}}$   
 $\mathcal{M}$  instead of  $H_*^{\text{BM}}(\mathcal{M}_A, \mathbb{Q}^{\text{vir}})$ .

② Use the "local neighbourhood theorem" of Davison for LCY categories

③ Use the description of the top- CohA of the SSN cone for

preprojective algebras of quivers.

② Thm (Davison)  $\mathcal{A}$  2CY category

$$JH: \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{M}_{\mathcal{A}} \quad \text{good m.s}$$

$$x \in \mathcal{M}_{\mathcal{A}}, \quad x \mapsto \mathcal{F} = \bigoplus_{j=1}^n \mathcal{F}_j^{\oplus m_j}$$

$\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_r\}$  pairwise distinct objects of  $\mathcal{A}$ .

$\overline{\mathcal{Q}}_{\mathcal{F}}$  : Ext-quiver of  $\mathcal{F}$

$$= \left( (\overline{\mathcal{Q}}_{\mathcal{F}})_0, (\overline{\mathcal{Q}}_{\mathcal{F}})_1 \right)$$

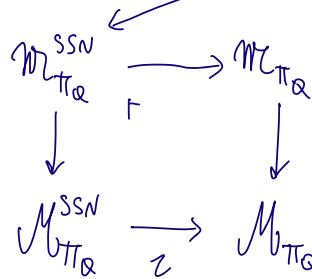
$$(\overline{\mathcal{Q}}_{\mathcal{F}})_0 = \mathcal{F},$$

$\forall i, j$  we have  $\text{ext}^1(\mathcal{F}_i, \mathcal{F}_j)$  arrows  $\mathcal{F}_i \rightarrow \mathcal{F}_j$ .

$\exists \mathcal{Q}_{\mathcal{F}}$  s.t.  $\overline{\mathcal{Q}}_{\mathcal{F}}$  is the double of  $\mathcal{Q}_{\mathcal{F}}$ .

$$\begin{array}{ccccc}
 (\mathcal{M}_{\mathcal{A}|\mathcal{F}}) & \longleftarrow & \left( \frac{U}{G_{\text{hm}}} \right)_{\mathcal{F}} & \longrightarrow & (\mathcal{M}_{\Pi_{\mathcal{Q}_{\mathcal{F}}}})_{\mathcal{O}_m} \\
 \downarrow JH & & \downarrow & & \downarrow \\
 (\mathcal{M}_{\mathcal{A}})_{\mathcal{F}} & \longleftarrow & (U // G_{\text{hm}})_{\mathcal{F}} & \longrightarrow & (\mathcal{M}_{\Pi_{\mathcal{Q}_{\mathcal{F}}}})_{\mathcal{O}_m} \\
 & & \text{w/ étale horizontal maps.} & & 
 \end{array}$$

③  $Q = (Q_0, Q_1)$  quiver  
 the preprojective algebra  $\mathcal{M}_{\Pi_Q}$  Lagrangian substack



s. simple reps of  $\mathcal{M}_{\Pi_Q}$  s.t. only loops in  $Q_1$  act possibly by  $\neq 0$ .

$$i^!A \in D_c^+(\mathcal{M}_{\Pi_Q}^{SSN})$$

$H^0(i^!A) \subset H^*(i^!A)$  is a subalgebra  
 It has a linear basis given by fund. classes of irr. components of  $\mathcal{M}_{\Pi_Q}$ .

$$I = (Q_0^{re} \times \{1\}) \sqcup (Q_1^{im} \times \mathbb{Z}_{\geq 1})$$

$\swarrow$  vertices w/o loops       $\nwarrow$  vertices w/ at least one loop.

Chm (H)  $H^0(i^!A) \cong U(\pi_Q^+)$  where  $\pi_Q^+$  is the Lie algebra generated by  $e_i, i \in I$  w/ relations

$$\begin{cases} (e_i, e_j) = 0 & \text{if } (i, j) = 0 \\ \text{ad}(e_i)^{1-(i, j)}(e_j) = 0 & \text{if } i \in Q_0^{re} \times \{1\} \end{cases}$$

induced by the symmetrised Euler form of  $Q$