Counting representations of algebras

joint with Fabian Korthauer (Düsseldorf)

Algebras 1

k field $\qquad k = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{F}_q$

$\mathbb{F}_q$: finite field

A k-algebra : k vector space

\+ multiplication law

$m : A \times A \longrightarrow A$

$\left\{ \begin{array}{l} \forall a, b, c \in A \\ * \text{ associative} \\ (ab)c = a(bc) \\ * \text{ unit} \\ 1_A \cdot a = a \end{array} \right.$

Examples: k

k^n $(n \geq 1)$ for the componentwise multiplication

$\mathrm{Mat}_n(k)$ matrix multiplication

Algebras 2

Systematic ways to produce algebras

G group $\rightsquigarrow$ $k[G]$ group algebra

$$k[G] = \bigoplus_{g \in G} k \cdot e_g$$ basis indexed by G

$g, h \in G$ $\qquad e_g e_h = e_{gh}$ $\qquad k$-linearly extended

G finite

$G = \mathbb{Z}$,

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle$$
$$\|$$

$$G = PSL(2,\mathbb{Z}) := SL(2,\mathbb{Z}) \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$ modular group.

ping-pong lemma

$$\cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} := \{a_1 b_1 a_2 b_2 \dots a_n b_n, \ a_i \in \mathbb{Z}/2\mathbb{Z}, b_i \in \mathbb{Z}/3\mathbb{Z}\}$$

Algebras 3

Free algebras: $F_I := k\langle x_i,\ i \in I\rangle$ $\quad$ I set

polynomials in variables x_i, $i \in I$
that do not commute

A k-algebra, $\quad A \cong \dfrac{k\langle x_i,\ i \in I\rangle}{\langle\langle f_j(x),\ j \in J\rangle\rangle_{2\text{-sided}}}$.

I, J finite: $\quad$ finitely presented algebra

$F_n := k\langle x_1, \dots, x_n\rangle$ $\quad$ free algebra on n generators

$$k[PSL(2,\mathbb{Z})] \cong \frac{F_2}{\langle\langle x_1^2 = 1 = x_2^3\rangle\rangle}$$

Representations of algebras

vector spaces are well-understood.

For any $n \in \mathbb{N}$, k^n vector space of dimension n.

$\mathrm{Mat}_n(k)$ algebra of linear transformations of k^n.

A some k-algebra

How to get a grasp on A?

$\to$ realize A as an algebra of linear transformations of vector spaces

That is: study maps $A \longrightarrow \mathrm{Mat}_n(k)$ compatible with the products.

Identify two such maps if they arise from different choices of bases of k^n.

Counting representations of algebras: example

$G = \mathbb{Z}/2\mathbb{Z}$, $A = k[G]$, k field, $2 \in k \setminus \{0\}$.

$A \longrightarrow \mathrm{Mat}_n(k)$ is a choice of $M \in \mathrm{Mat}_n(k)$ s.t. $M^2 = I_n$.

linear algebra: M is diagonalizable

$$M \sim \begin{pmatrix} \underbrace{1 \cdots 1}_{\ell} & & 0 \\ & \underbrace{-1 \cdots -1}_{m} & \\ 0 & & \end{pmatrix}$$ in a diagonalization basis of k^n.

$m + \ell = n$

We have $n+1$ choices of such M up to identifications.

More generally, if G is a finite group,

$\#\{k[G] \longrightarrow \mathrm{Mat}_n(k)\} / \text{identification}$ is a nonnegative integer, does not depend on k. $(*)$

What if G is infinite?

The set of algebra maps $k[G] \xrightarrow{\varphi} \mathrm{Mat}_n(k)$ is usually infinite, even after identifications

e.g. if $G = \mathbb{Z}$, such a φ amounts to choosing

$\varphi(1) \in GL_n(k)$: ∞ many choices if k is infinite

We take k a finite field : $k = \mathbb{F}_q$ for $q = p^r$ p prime number.

if $q = p$ $k \cong \mathbb{Z}/p\mathbb{Z}$

Then, $\# \mathrm{Mat}_n(k) = q^{(n^2)}$ is finite : the set of possible φ's is finite :

Counting the maps $k[G] \to \mathrm{Mat}_n(k)$ makes sense again.

(equivalently, the maps $G \to GL_n(k)$)

Examples

$G = \mathbb{Z}$. $\quad \#\{\mathbb{F}_q[\mathbb{Z}] \to \mathrm{Mat}_n(\mathbb{F}_q)\}/\sim \; = \# \; GL_n(\mathbb{F}_q) / GL_n(\mathbb{F}_q)$

conjugation action.
set-theoretic quotient

$n = 1$: $\quad q-1$

G finite group: $\#\{\mathbb{F}_q[G] \to \mathrm{Mat}_n(\mathbb{F}_q)\}/\sim$ is an integer, independent of q (∗), depending on the number of conjugacy classes in G. / equal to it if we restrict to <u>simple</u> G-representations

G arbitrary: hard to say anything!

Counting representations of virtually free groups

Theorem (H-Korthauer) Let $G = PSL(2,\mathbb{Z})$. $2 \nmid q$

Then $\#\{\mathbb{F}_q[G] \to \mathrm{Mat}_n(\mathbb{F}_q)\}/_\sim \in \mathbb{N}[q]$

$\langle x_1, x_2 \mid x_1^2 = 1 = x_1^3 \rangle$ $\quad \langle x_1, x_2 \rangle \subset PSL(2,\mathbb{Z})$ has finite index.

$\shortparallel$

$PSL(2,\mathbb{Z})$ is virtually free : it has a free subgroup of finite index.

Virtually free groups : * free groups $\mathbb{Z}$, $\mathbb{Z}^{*r} := \overbrace{\mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z}}^{r}$

* amalgamated products of finite groups :

$$G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} \dots \underset{H_{r-1}}{*} G_r \quad , \quad G_i, H_i \text{ finite}$$

* "HNN-extensions" of finite groups.

Higman, Neumann, Neumann
Bernhard Hanna.

Actually, we prove:

Theorem (H-Korthauer) Let G be a virtually free group. Then, for any $n \geq 0$

$$P_{G,n}(q) := \#\left\{\mathbb{F}_q[G] \longrightarrow \mathrm{Mat}_n(\mathbb{F}_q)\right\}^{\circ} / \sim \quad \in \mathbb{Z}[q].$$

If G is an amalgamated product of Abelian groups, then

$$P_{G,n}(q) \in \mathbb{N}[q].$$

Actually, we deduce this theorem from a theorem regarding smooth algebras A over $\mathbb{F}_q$.

Example

$G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ $\qquad$ $q = p^r$, $p \neq 2$ prime

A representation of G of dim n is a pair of independent group morphisms

$$f, g : \mathbb{Z}/2\mathbb{Z} \longrightarrow GL_n(\mathbb{F}_q)$$

Such a map is equivalent to a decomposition

$$\mathbb{F}_q^n \cong \underbrace{A}_{\text{eigenvalue } 1} \oplus \underbrace{B}_{\text{eigenvalue } -1}$$

So we are counting <u>pairs</u> of such decompositions up to the natural action of $GL_n(\mathbb{F}_q)$.

$\to$ the theorem tells us this is a polynomial in $\mathbb{N}[q]$.

elementary solution?

Noncommutative geometry

A a k algebra (finitely presented)

$$A \cong \frac{k\langle x_1, \dots, x_r \rangle}{\langle\langle f_1, \dots, f_s \rangle\rangle}$$

A produces geometric spaces $\mathrm{Rep}_n(A)$, $n \geq 0$

"representation space"

$$\mathrm{Rep}_n(A) := \{ M_1, \dots, M_n \in \mathrm{Mat}_n(k) \mid f_1(M_1, \dots, M_r) = \dots = f_s(M_1, \dots, M_r) = 0 \}$$

$\circlearrowleft$

$GL_n(k)$, simultaneous conjugation.

$$\text{orbits} \xleftrightarrow{1:1} \{ \text{rep of } A \text{ of dim } n \} / \sim .$$

Smoothness: A smooth $\Rightarrow$ $\mathrm{Rep}_n(A)$ is smooth $(\forall n \in \mathbb{N})$

Counting orbits

Our counting problem reduces to compute

$$\# \, \mathrm{Rep}_n(A) / GL_n(k) \qquad k = \mathbb{F}_q$$

set-theoretic quotient.

This is not well-behaved and not geometric.

The quotient stack / groupoid $\mathfrak{M}_{A,n} = \left[\mathrm{Rep}_n(A) / GL_n(k) \right]$ is a geometric objects.

Volume

$$\mathrm{vol}(\mathfrak{M}_{A,n}(k)) = \sum_{\underbrace{x \in \mathrm{Rep}_n(A)/GL_n(k)}_{\text{set-theoretic}}} \frac{1}{\mathrm{Stab}_{GL_n(k)}(x)} \neq \sum_{\underbrace{x \in \mathrm{Rep}_n(A)/GL_n(k)}_{\text{set-theoretic}}} 1$$

We get the wrong number!

Strategy: Inertia stack

Replace $\mathfrak{M}_{A,n}$ by $I\mathfrak{M}_{A,n}$ "inertia stack"

$\mathfrak{M}_{A,n}$ parametrizes n-dim reps of A

$I\mathfrak{M}_{A,n}$ parametrizes pairs (M, f)

M: n-dim rep of A

$f \in \mathrm{Aut}_A(M)$

Burnside formula: $\mathrm{vol}(I\mathfrak{M}_{A,n}) = \sum_{x \in \mathrm{Rep}_n(A)/GL_n} 1$ right number

right geometric object to consider

Cohomological Donaldson–Thomas theory [CoDT]

$k = \mathbb{C}$

$$\begin{array}{c} \mathrm{Rep}_n(A) \rightsquigarrow \tilde{\mathfrak{M}}_n := \left[\left(T^*\mathrm{Rep}_n(A) \times \mathfrak{gl}_n\right)/GL_n\right] \\ \downarrow \\ \tilde{\mathcal{M}}_n := \left(T^*\mathrm{Rep}_n(A) \times \mathfrak{gl}_n\right)/\!\!/GL_n \end{array}$$

GIT quotient

CoDT produces a perverse sheaf on $\tilde{\mathcal{M}}_n$, $\mathcal{BPS}_n$.

This perverse sheaf satisfies

$$\sum_{i \in \mathbb{Z}} \dim H^i(\tilde{\mathcal{M}}_n, \mathcal{BPS}_n)\, q^{-i/2} = \# \left\{ \begin{array}{l} \text{absolutely indecomposable reps.} \\ \text{of } A \text{ over } \mathbb{F}_q \end{array} \right\} /\sim$$

when this is a polynomial

+ when $[\mathrm{Rep}_n(A)/GL_n]$ is pure

The origins of positivity : purity

Let G be a virtually free group.

$$P_{G,n}(q) := \#\left\{ \begin{matrix} n\text{-dim reps of } G \\ \text{over } \mathbb{F}_q \end{matrix} \right\} / \sim \quad \in \mathbb{Q}[q].$$

$$P_{G,n}(q) \in \mathbb{N}[q] \Longleftarrow \left[\mathrm{Rep}_n(G) / GL_n \right] \text{ has pure cohomology}$$

very indirect route using CoDT

e.g. $G = \mathbb{Z}$, $\quad P_{\mathbb{Z},1}(q) = q-1$

and $\left[\mathrm{Rep}_1(\mathbb{Z}) / \mathbb{C}^* \right] = \left[\mathbb{C}^* / \mathbb{C}^* \right] \cong \mathbb{C}^* \times B\mathbb{C}^*$ does not have pure cohomology.

Purity 2

$G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ should satisfy positivity statement

$$\left[\mathrm{Rep}_n(G) / GL_n \right] \simeq \bigsqcup_{\substack{a+b=n \\ c+d=n}} \left[L_{a,b} \backslash GL_n / L_{c,d} \right]$$

pure cohomology

$$L_{a,b} = \left\{ \begin{array}{c} \\ a\{ \\ b\{ \end{array} \begin{pmatrix} \boxed{*} & 0 \\ 0 & \boxed{*} \end{pmatrix} \right\}$$

$$P_{a,b} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

Purity

$$\begin{array}{l} GL_n / L_{c,d} \\ \downarrow \text{ affine fibration} \\ GL_n / P_{c,d} \end{array}$$

smooth projection $\Rightarrow$ pure (Deligne)

$H^*\left(\left[L_{a,b} \backslash GL_n / L_{c,d} \right] \right) = H^*_{L_{a,b}} \left(GL_n / L_{c,d} \right)$ is pure.

Other algebras or groups?

Let G be a finitely generated group.

Is $\#\{G \longrightarrow GL_n(\mathbb{F}_q)\}/\sim$ a polynomial in q?

Hard to tell in general.

If G has k generators, it is $\leqslant \left(\# GL_n(\mathbb{F}_q)\right)^k$ = polynomial.

$\rightarrow$ We cannot use growth arguments to get a contradiction.

<u>Question</u>: Do other classes of groups exhibit such a polynomial behaviour?

i.e. Find a finitely generated group G and an integer n (necessarily $\geqslant 2$) such that $\#\{G \rightarrow GL_n(\mathbb{F}_q)\}/\sim$ is not polynomial in q in any reasonable sense.