Weight decompositions on étale fundamental groups

J.P.Pridham*

November 22, 2006

Abstract

Let X be a smooth or proper variety defined over a finite field. The geometric étale fundamental group $\pi_1(X,\bar{x})$ of X is a normal subgroup of the Weil group, so conjugation gives it a Weil action. We consider the pro-$\mathbb{Q}_l$-algebraic completion of $\pi_1(X,\bar{x})$ as a non-abelian Weil representation. Lafforgue's Theorem and Deligne's Weil II theorems imply that this affine group scheme is mixed, in the sense that its structure sheaf is a mixed Weil representation. When X is smooth, weight restrictions apply, affecting the possibilities for the structure of this group. This gives new examples of groups which cannot arise as étale fundamental groups of smooth varieties.

Contents

Introduction

Let X be a connected variety defined over a finite field $k=\mathbb{F}_q$, equipped with a point $x \in X(\mathbb{F}_q)$, and let l be a prime not dividing q. The embedding $i:\pi_1(X,\bar{x}) \hookrightarrow W(X_k,x)$ of

*The author is supported by Trinity College, Cambridge.

the fundamental group into the Weil group gives a conjugation action of the Weil group on the fundamental group. The Weil conjecture for $\mathrm{H}^1(X, \mathbb{Q}_l)$ can be re-expressed by considering $\mathrm{H}^1(X, \mathbb{Q}_l)^\vee$ as the l-adic Weil representation V, universal among continuous group homomorphisms

$$\pi_1(X, x) \to V,$$

and then stating that this Weil representation is mixed.

We consider a non-abelian version of this, by defining the pro-$\mathbb{Q}_l$-algebraic group ${}^W\varpi_1(X, \bar{x})$ to be the universal object classifying continuous $W(X_k, x)$-equivariant homomorphisms

$$\pi_1(X, \bar{x}) \to G(\mathbb{Q}_l),$$

where G ranges over all algebraic groups G over $\mathbb{Q}_l$ equipped with continuous $W(X_k, x)$-actions. Representations of ${}^W\varpi_1(X, \bar{x})$ are precisely $\pi_1(X, \bar{x})$-subrepresentations of $W(X_k, x)$-representations.

We say that an algebraic Weil action on a pro-algebraic group G is mixed if the structure sheaf $O(G)$ is a sum of mixed Weil representations.

The Levi decomposition for pro-algebraic groups allows us to write

$${}^W\varpi_1(X, \bar{x}) \cong R_u({}^W\varpi_1(X, \bar{x})) \rtimes {}^W\varpi_1^{\mathrm{red}}(X, \bar{x}),$$

where $R_u({}^W\varpi_1(X, \bar{x}))$ is the pro-unipotent radical of ${}^W\varpi_1(X, \bar{x})$ and ${}^W\varpi_1^{\mathrm{red}}(X, \bar{x})$ is the pro-reductive completion of ${}^W\varpi_1(X, \bar{x})$. This decomposition is unique up to conjugation by $R_u({}^W\varpi_1(X, \bar{x}))$.

In Section 1, we use Lafforgue's Theorem to show that for any variety, the Weil action on ${}^W\varpi_1^{\mathrm{red}}(X, \bar{x})$ is pure of weight zero. Deligne's Weil II theorems then show that, if X is smooth or proper, the Weil action on ${}^W\varpi_1(X, \bar{x})$ is mixed. This can be thought of as a direct analogue of the non-abelian Hodge theorems of [Sim92]. One consequence is that for any morphism $f : X \to Y$ of varieties over $\mathbb{F}_q$, with X smooth, and $\mathbb{V}$ any semisimple constructible $\mathbb{Q}_l$-local system underlying a Weil sheaf on Y, the pullback $f^{-1}\mathbb{V}$ is semisimple.

The rest of the paper is dedicated to studying the Weil action on $R_u({}^W\varpi_1(X, \bar{x}))$ when X is smooth or proper, and thus establishing restrictions on the structure of the fundamental group. In order to study the pro-unipotent extension ${}^W\varpi_1(X, \bar{x}) \to {}^W\varpi_1^{\mathrm{red}}(X, \bar{x})$, we use deformation-theoretic machinery. The group $R_u({}^W\varpi_1(X, \bar{x}))$ is the universal deformation

$$\rho : \pi_1(X, \bar{x}) \to U \rtimes {}^W\varpi_1^{\mathrm{red}}(X, \bar{x})$$

of the canonical representation

$$\rho_0 : \pi_1(X, \bar{x}) \to {}^W\varpi_1^{\mathrm{red}}(X, \bar{x}),$$

for U pro-unipotent. In [Pri05b], a theory of deformations over nilpotent Lie algebras with G-actions was developed, and this enables us to analyse our scenario. The essential philosophy is that all the concepts for deformations over Artinian rings, developed by Schlessinger in [Sch68] and later authors, can be translated to this context.

In Section 2 we introduce the notion of Simplicial Deformation Complexes (SDCs), which will fulfil the rôle played by twisted DGAs in [Pri05b]. In Section 3, the SDC associated to the l-adic cohomology on a scheme X is defined. It is shown that this can be used to recover pro-unipotent radical $R_u(\varpi_1(X,\bar{x}))$ of the pro-$\mathbb{Q}_l$-algebraic fundamental group.

In Section 4, we use Deligne's Weil II theorems to study $R_u({}^W\varpi_1(X,\bar{x}))$. If X is smooth and proper, then the weight decomposition on $R_u({}^W\varpi_1(X,\bar{x}))$ splits the lower central series filtration, and it is quadratically presented, in the sense that its Lie algebra can be defined by equations of bracket length two. If X is merely smooth, then $R_u({}^W\varpi_1(X,\bar{x}))$ is defined by equations of bracket length at most four. Since rigid representations of the fundamental group extend to Weil representations, these properties are used to give new examples of groups which cannot occur as fundamental groups of smooth varieties.

This generalises the results of [Pri04a] on deforming reductive representations of the fundamental group, and of [Pri04b] on the pro-$\mathbb{Q}_l$-unipotent fundamental group $\pi_1(X,\bar{x})\otimes\mathbb{Q}_l$. In both of these examples, we are taking a reductive representation

$$\rho_0 : \pi_1(X,\bar{x}) \to G,$$

and considering deformations

$$\rho : \pi_1(X,\bar{x}) \to U \rtimes G$$

of ρ_0, for U unipotent. Effectively, [Pri04a] considers only $U = \exp(\mathrm{Lie}(G)\otimes\mathfrak{m}_A)$, for $\mathfrak{m}_A$ a maximal ideal of an Artinian local $\mathbb{Q}_l$-algebra, while [Pri04b] considers only the case when $G = 1$. Since taking $U = R_u(\varpi_1(X,\bar{x}))$ pro-represents this functor when $G = \varpi_1^{\mathrm{red}}(X,\bar{x})$, both examples can be understood in terms of the structure of $R_u(\varpi_1(X,\bar{x}))$.

The structure result in the smooth and proper case is much the same as those established in [Hai98] and [Pri05b] for fundamental groups of compact Kähler manifolds. Likewise, [Pri04a] and [Pri04b] were the analogues in finite characteristic of Goldman and Millson's results on Kähler representations ([GM88]) and Morgan's restrictions on fundamental groups of smooth varieties ([Mor78]), respectively.

1 The pro-algebraic fundamental group as a Weil representation

1.1 Algebraic actions

All pro-algebraic groups in this paper will be defined over fields of characteristic zero (usually $\mathbb{Q}_l$). All representations of pro-algebraic groups will be finite-dimensional.

Definition 1.1. Given a pro-algebraic group G, let $O(G)$ denote global sections of the structure sheaf of G. This is a sum of $G\times G$-representations, the actions corresponding to right and left translation. Let $E(G)$ be the dual of $O(G)$ — this is a pro-$G\times G$-representation. In fact, since any coalgebra is the sum of its finite-dimensional subcoalgebras, $E(G)$ is an inverse limit of finite-dimensional (non-commutative) algebras.

$E(G)$-modules then correspond to pro-G-representations, and for a morphism $G\to H$ and a pro-G-representation V, we define

$$\mathrm{Ind}_G^H V := V\hat{\otimes}_{E(G)}E(H).$$

Definition 1.2. Given a discrete group Γ acting on a pro-algebraic group G, we define ${}^{\Gamma}G$ to be the maximal quotient of G on which Γ acts algebraically. This is the inverse limit $\varprojlim_{\alpha} G_{\alpha}$ over those surjective maps

$$G \to G_{\alpha},$$

with G_{α} algebraic, for which the Γ-action descends to G_{α}.

Lemma 1.3. *The representations of ${}^{\Gamma}G$ are precisely those G-representations which arise as G-subrepresentations of (finite-dimensional) $G \ltimes \Gamma$-representations.*

Proof. Given ${}^{\Gamma}G \xrightarrow{\theta} \mathrm{GL}(V)$, there must exist an algebraic quotient group G_{α} of G to which Γ descends, with θ factoring as ${}^{\Gamma}G \to G_{\alpha} \to \mathrm{GL}(V)$. Now, since G_{α} is an algebraic group, $\mathrm{Aut}(G_{\alpha})$ is also, and there is a homomorphism $G \ltimes \Gamma \to H_{\alpha} := G_{\alpha} \ltimes \mathrm{Aut}(G_{\alpha})$. Since $G_{\alpha} \hookrightarrow H_{\alpha}$, the G_{α}-representation V is a subrepresentation of the pro-H_{α}-representation $\mathrm{Ind}_{G_{\alpha}}^{H_{\alpha}} V$, so for some quotient representation $\mathrm{Ind}_{G_{\alpha}}^{H_{\alpha}} V \to W$, the composition $V \to W$ must be injective. Thus V is a subrepresentation of the $G \ltimes \Gamma$-representation W.

Conversely, Let $V \leq W$ be G-representations, with W a $G \ltimes \Gamma$-representation. If we let G_{α} be the image of $G \to \mathrm{GL}(W)$, then the adjoint action of Γ on $\mathrm{GL}(W)$ restricts to an action on G_{α}. Since the action of G on W preserves V, there is an algebraic map $G_{\alpha} \to \mathrm{GL}(V)$, as required. □

Definition 1.4. Given a pro-algebraic group G, we will denote its reductive quotient by G^{red}; this is the universal object among quotients $G \to H$, with H reductive algebraic. Representations of G^{red} correspond to semisimple representations of G. We write $\mathrm{R_u}(G)$ for the kernel of $G \to G^{\mathrm{red}}$ — this is called the pro-unipotent radical of G.

Lemma 1.5. ${}^{\Gamma}(G^{\mathrm{red}}) = ({}^{\Gamma}G)^{\mathrm{red}}$. *We will hence denote this group by* ${}^{\Gamma}G^{\mathrm{red}}$.

Proof. Note that in both cases, representations correspond to those semisimple G-representations which arise as G-subrepresentations of (finite-dimensional) $G \ltimes \Gamma$-representations. □

The Levi decomposition, proved in [HM69], states that for every pro-algebraic group G, the surjection $G \to G^{\mathrm{red}}$ has a section, unique up to conjugation by $\mathrm{R_u}(G)$, inducing an isomorphism $G \cong \mathrm{R_u}(G) \rtimes G^{\mathrm{red}}$.

Lemma 1.6. *Given a pro-algebraic group G, an automorphism F of G, and an element $g \in G$, the action of F on G is algebraic if and only if the action of $\mathrm{ad}_g \circ F$ is algebraic.*

Proof. First note that we have an isomorphism from $G \rtimes \langle \mathrm{ad}_g \circ F \rangle$ to $G \rtimes \langle F \rangle$ fixing G, given by sending $\mathrm{ad}_g \circ F$ to $g \cdot F$. Hence, by Lemma 1.3, ${}^{F}G = {}^{\mathrm{ad}_g \circ F}G$. □

Corollary 1.7. *The action of F on G is algebraic if and only if the corresponding actions on G^{red} and $\mathrm{R_u}(G)$ are.*

Proof. Without loss of generality, by the previous lemma, we may assume that F must preserve the Levi decomposition (following conjugation by a suitable element of $\mathrm{R_u}(G)$). Write $F = F^{\mathrm{red}}F^u$, for $F^u : \mathrm{R_u}(G) \to \mathrm{R_u}(G)$, and $F^{\mathrm{red}} : G^{\mathrm{red}} \to G^{red}$. By Lemma 1.3 and Tannaka duality, ${}^F G$ is the image of $G \to (G \rtimes \langle F \rangle)^{\mathrm{alg}}$, the latter group being the pro-algebraic completion of $G \rtimes \langle F \rangle$.

Then note that we have an embedding

$$(G \rtimes \langle F \rangle)^{\mathrm{alg}} \hookrightarrow (\mathrm{R_u}(G) \rtimes \langle F^u \rangle)^{\mathrm{alg}} \rtimes (G^{\mathrm{red}} \rtimes \langle F^{\mathrm{red}} \rangle)^{\mathrm{alg}},$$

so the map from G to the group on the left is an embedding if and only if the maps from $G^{\mathrm{red}}, \mathrm{R_u}(G)$ to the groups on the right are embeddings. $\square$

Lemma 1.8. *Let F act on $G \ltimes U$, for G reductive and U pro-unipotent, with F preserving and acting algebraically on G. If we also assume that $\mathrm{Hom}_G(V, U/[U,U])$ is finite-dimensional for all G-representations V, then F acts algebraically on $G \ltimes U$.*

Proof. By the previous lemma, it suffices to show that F acts algebraically on U. Let S be the set of isomorphism classes of irreducible representations of G. Since F acts algebraically on G, the F-orbits in S are all finite. Let $\mathfrak{u} := \mathrm{Lie}(U)$, and take the canonical decomposition $\mathfrak{u} = \prod_{s \in S} \mathfrak{u}_s$ of $\mathfrak{u}$ as a G-representation. Let $T = S/F$ be the set of F-orbits in S, giving a weaker decomposition $\mathfrak{u} = \prod_{t \in T} \mathfrak{u}_t$, where $\mathfrak{u}_t = \prod_{s \in t} \mathfrak{u}_s$.

F is then an automorphism of $\mathfrak{u}$ respecting this decomposition; let H be the group of all such automorphisms. We then have an embedding

$$U \rtimes \langle F \rangle \hookrightarrow U \rtimes H,$$

so it suffices to show that the group H is pro-algebraic, since this embedding must then factor through $(U \rtimes \langle F \rangle)^{\mathrm{alg}}$.

Choose a G-equivariant section to the map $\mathfrak{u} \to \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$, and let its image be V. The group H is a closed subspace of the space of all linear maps $\mathrm{Hom}_T(V, \mathfrak{u})$ preserving the T-decomposition. The hypothesis implies that V_s is finite-dimensional for all $s \in S$, so V_t must be finite-dimensional for all $t \in T$, the F-orbits being finite. Thus H is an affine group scheme, i.e. a pro-algebraic group, as required. $\square$

Lemma 1.9. *If G is a pro-algebraic group, and we regard $O(G)$ as a sum of G-representations via the left action, then for any G-representation V, $V^\vee \cong \mathrm{Hom}_G(V, O(G))$, with the G-action on $V^\vee$ coming from the right action on $O(G)$.*

Proof. This follows immediately from [DMOS82] II Proposition 2.2, which states that G-representations correspond to $O(G)$-comodules. Under this correspondence, $\alpha \in V^\vee$ is associated to the morphism which sends $v \in V$ to the function $g \mapsto \alpha(g \cdot v)$. $\square$

Lemma 1.10. *If an endomorphism F acts on a pro-algebraic group G and compatibly on a G-representation V (i.e. $F(g \cdot v) = (Fg) \cdot (Fv)$), then the dual action of F on $V^\vee$ corresponds to the action on $\mathrm{Hom}_G(V, O(G))$ which sends θ to the composition*

$$V \xrightarrow{F} V \xrightarrow{\theta} O(G) \xrightarrow{F^*} O(G)$$

1.2 Weil actions

Let $k = \mathbb{F}_q$, take a connected variety X_k/k, and let $X = X_k \otimes_k \bar{k}$. Fix a closed point x of X, and denote the associated geometric point $x \otimes_{k(x)} \bar{k} \to X$ by $\bar{x}$. Without loss of generality (increasing q if necessary), we assume that $k(x) \subset \mathbb{F}_q$. Let l be a prime not dividing q, and consider the pro-$\mathbb{Q}_l$-algebraic completion $\varpi_1(X, \bar{x})$ of the étale fundamental group $\pi_1(X, \bar{x})$ of X. This is the universal object classifying continuous homomorphisms

$$\pi_1(X, \bar{x}) \to G(\mathbb{Q}_l),$$

where G ranges over all algebraic groups G over $\mathbb{Q}_l$.

Recall that the Frobenius element gives a canonical generator of $\pi_1(\operatorname{Spec} k) \cong \hat{\mathbb{Z}}$, and that the Weil group $W(X_k, x)$ is defined by

$$W(X_k, x) = \pi_1(X_k, \bar{x}) \times_{\hat{\mathbb{Z}}} \mathbb{Z},$$

which has $\pi_1(X, \bar{x})$ as a normal subgroup. Observe that the conjugation action of $W(X_k, x)$ on $\pi_1(X, \bar{x})$ then extends by universality to an action of $W(X_k, x)$ on $\varpi_1(X, \bar{x})$. Let $F_x \in W(X_k, x)$ be the Frobenius element associated to x.

Lemma 1.11. *If $W := W(X_k, x)$ and $F := F_x$, then ${}^W\varpi_1(X, \bar{x}) = {}^F\varpi_1(X, \bar{x})$, with representations of this group being those continuous $\pi_1(X, \bar{x})$-representations which arise as $\pi_1(X, \bar{x})$-subrepresentations of Weil representations.*

Proof. By Lemma 1.3, representations of ${}^W\varpi_1(X, \bar{x})$ are continuous $\pi_1(X, \bar{x})$-subrepresentations of $\pi_1(X, \bar{x}) \ltimes W(X_k, x)$-representations. These are precisely $\pi_1(X, \bar{x})$-subrepresentations of $W(X_k, x)$-representations. Since $W(X_k, x) = \pi_1(X, \bar{x}) \ltimes \langle F_x \rangle$, these are the same as representations of ${}^F\varpi_1(X, \bar{x})$. By Tannaka duality ([DMOS82]), this determines the quotient groups ${}^W\varpi_1(X, \bar{x}), {}^F\varpi_1(X, \bar{x})$ of $\varpi_1(X, \bar{x})$, which must then be equal. □

Lemma 1.12. *${}^W\varpi_1(X, \bar{x})$ is the image of the homomorphism $\varpi_1(X, \bar{x}) \xrightarrow{i} \mathcal{W}(X_k, x)$, where $\mathcal{W}(X_k, x)$ is the pro-algebraic completion of the Weil group $W(X_k, x)$.*

Proof. Representations of $\operatorname{Im}(i)$ are those $\varpi_1(X, \bar{x})$ representations V for which $V \to \operatorname{Ind}_{\varpi_1(X, \bar{x})}^{\mathcal{W}(X_k, x)}$ is injective. By Lemmas 1.3 and 1.11, these are the same as ${}^W\varpi_1(X, \bar{x})$-representations. □

Definition 1.13. Given a pro-$\mathbb{Q}_l$-algebraic group G, equipped with an algebraic action of the Weil group $W(X_k, x)$, we will say that this Weil action on G is mixed (resp. pure of weight w) if $O(G)$ is a sum of finite-dimensional Weil representations which are mixed (resp. pure of weight $-w$). Note that if $O(G)$ is pure, then it is pure of weight 0, since the unit map $\mathbb{Q}_l \to O(G)$ must be Weil equivariant, so we always have a subspace of weight 0.

Theorem 1.14. *The natural Weil action on ${}^W\varpi_1^{\mathrm{red}}(X, \bar{x})$ is pure (of weight 0).*

Proof. Since ${}^W\varpi_1^{\mathrm{red}}(X, \bar{x})$ is reductive, its category of representations is generated under addition by the irreducible representations. Tannaka duality ([DMOS82]) states that $O({}^W\varpi_1^{\mathrm{red}}(X, \bar{x}))$ must then be dual to the pro-vector space of endomorphisms of the fibre functor from the category of representations to the category of vector spaces. Similarly, $O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x})) \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l$ classifies $\bar{\mathbb{Q}}_l$-representations, and is dual to the fibre functor from representations over $\bar{\mathbb{Q}}_l$. By Schur's Lemma, scalar multiplications are the only endomorphisms of irreducible representations over $\bar{\mathbb{Q}}_l$.

If we write $\mathrm{End}(V)$ for the space of endomorphisms of the vector space underlying V, there is then an isomorphism of ${}^W\varpi_1^{\mathrm{red}}(X,\bar{x}) \times {}^W\varpi_1^{\mathrm{red}}(X,\bar{x})$-representations

$$O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x})) \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l \cong \bigoplus_{V \in T} \mathrm{End}(V),$$

where T is the set of all isomorphism classes of irreducible representations of ${}^W\varpi_1^{\mathrm{red}}(X,\bar{x})$ over $\bar{\mathbb{Q}}_l$. By Lemmas 1.3 and 1.11, it follows that an irreducible representation of $\pi_1(X,\bar{x})$ which is a subrepresentation of some $W(X_k,x)$-representation. This is the same as underlying a $W(X_{\mathbb{F}_q^n},x)$-representations for some n, since $W(X_{\mathbb{F}_q^n} = \pi_1(X,\bar{x}) \ltimes \langle F_x^n \rangle$.

From Lafforgue's Theorem ([Del80] Conjecture 1.2.10, proved in [Laf02] Theorem VII.6 and Corollary VII.8), every irreducible Weil representation over $\bar{\mathbb{Q}}_l$ is of the form

$$V \cong P \otimes \bar{\mathbb{Q}}_l^{(b)},$$

for some pure representation P of weight zero. Now,

$$\mathrm{End}(V) \cong V^\vee \otimes V \cong P^\vee \otimes P,$$

which is a pure $W(X_{\mathbb{F}_q^n},x)$-representation of weight 0. Therefore

$$\sum_i \mathrm{End}((F_x^\sharp)^i V) = \sum_{i=0}^{n-1} \mathrm{End}((F_x^\sharp)^i V) \le O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x})) \otimes \bar{\mathbb{Q}}_l)$$

is a pure Weil subrepresentation of weight 0. Hence $O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x})) \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l$ and $O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ are also pure of weight 0, as required. □

Lemma 1.15. *If X is a smooth or proper variety, then ${}^W\varpi_1(X,\bar{x})$ is the universal group G fitting in to the diagram*

$$\varpi_1(X,\bar{x}) \to G \to {}^W\varpi_1^{\mathrm{red}}(X,\bar{x}),$$

with $\ker(G \to {}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ *pro-unipotent.*

Proof. Since G and ${}^W\varpi_1(X,\bar{x})$ are both quotients of $\varpi_1(X,\bar{x})$, with $G \to {}^W\varpi_1(X,\bar{x})$, it suffices to show that the composition $G \to w(X_k,x)$ is an embedding, or equivalently that the Frobenius action on G is algebraic. By Lemma 1.8, it then suffices to show that $\mathrm{Hom}_{{}^W\varpi_1^{\mathrm{red}}(X,\bar{x})}(V, U/[U,U])$ is finite-dimensional for all ${}^W\varpi_1^{\mathrm{red}}(X,\bar{x})$-representations V, where U is the pro-unipotent radical of G.

Now, observe that $\mathrm{Hom}_{{}^W\varpi_1^{\mathrm{red}}(X,\bar{x})}(U/[U,U], V)$ is the space of outer G-derivations to V, which in turn is the space of outer $\varpi_1(X,\bar{x})$-representations to V. By Theorem 3.4, this is just

$$\mathrm{H}^1(\varpi_1(X,\bar{x}), V) = \mathrm{H}^1(\pi_1(X,\bar{x}), V) = \mathrm{H}^1(X, \mathbb{V}),$$

which is finite-dimensional. □

Proposition 1.16. *The Weil action on G is mixed if and only if the the induced actions on G^{red} and on the continuous dual vector space $(\mathrm{R_u}(G)/[\mathrm{R_u}(G), \mathrm{R_u}(G)])^\vee$ are mixed.*

Proof. We first choose a Levi decomposition $G = G^{\mathrm{red}} \ltimes \mathrm{R_u}(G)$. The Weil action will not usually preserve this decomposition. However, for each $y \in X$, we may choose an element $u_y \in \mathrm{R_u}(G)$ such that $F'_y := \mathrm{ad}_{u_y} \circ F_y$ does preserve this Levi decomposition. The key point is that u_y acts unipotently on $O(G)$.

Now, for any Weil representation V, the weight a subrepresentation $\mathcal{W}_a(V)$ of V is defined as the intersection of the weight $n(y)a$ F_y-subrepresentations $\mathcal{W}_{n(y)a}(V, F_y)$ of V, for all $y \in X$ and $|k(y)| = q^{n(y)}$. Since ad_{u_y} acts unipotently on $O(G)$, we deduce that

$$\mathcal{W}_{n(y)a}(V, F_y) = \mathcal{W}_{n(y)a}(V, F'_y),$$

for all $y \in X$.

If we write $\mathfrak{u}$ for the (pro-nilpotent) Lie algebra of $\mathrm{R_u}(G)$, and let $\mathfrak{u}^\vee$ denote its continuous dual, then the isomorphism $\mathrm{R_u}(G) \cong \exp(\mathfrak{u})$ and the Levi decomposition give us an isomorphism

$$O(G) \cong O(G^{\mathrm{red}})[\mathfrak{u}^\vee] = \bigoplus_n O(G^{\mathrm{red}}) \otimes \mathrm{Symm}^n(\mathfrak{u}^\vee),$$

which is F'_y equivariant for all $y \in X$.

To say that a Weil representation is mixed is the same as saying that

$$V = \bigoplus_{a \in \mathbb{Z}} (\bigcap_{y \in X} \mathcal{W}_{n(y)a}(V, F_y)),$$

and we have seen that for $V = O(G)$ it is equivalent to replace F_y by F'_y. Since $O(G^{\mathrm{red}})$ is mixed, and this property is respected by sums and tensor operations, it suffices to show that $\mathfrak{u}^\vee$ is mixed for the F'_y. This is the same as being mixed for the natural action of the F_y on $\mathfrak{u}^\vee$, so it suffices to show that the latter is a mixed Weil representation.

Consider the lower central series filtration $\Gamma_n\mathfrak{u}$ of $\mathfrak{u}$ given by

$$\Gamma_1\mathfrak{u} := \mathfrak{u}, \quad \Gamma_{n+1}\mathfrak{u} = [\mathfrak{u}, \Gamma_n\mathfrak{u}],$$

so that $\mathfrak{u} = \varprojlim \mathfrak{u}/\Gamma_n\mathfrak{u}$. If $\mathfrak{u}_n^\vee := (\mathfrak{u}/\Gamma_{n+1}\mathfrak{u})^\vee$, then $\mathfrak{u}^\vee = \sum \mathfrak{u}_n^\vee$, and it only remains to show that the latter are mixed. Now there is a canonical map

$$\mathfrak{u}_n^\vee/\mathfrak{u}_{n-1}^\vee \hookrightarrow \mathrm{CoLie}_n(\mathfrak{u}_1^\vee),$$

where CoLie_n is the degree n homogeneous part of the free co-Lie algebra functor. Since this is a tensor operation, the right-hand side is mixed ($\mathfrak{u}_1^\vee$ being mixed by hypothesis).

We next observe that if

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence of ind-Weil representations with any two mixed, then the third is; this completes the proof. ∎

Theorem 1.17. *If X is smooth or proper, then the natural Weil action on ${}^W\varpi_1(X,\bar{x})$ is mixed of non-positive weight.*

Proof. By Theorem 1.14 and Proposition 1.16, it suffices to show that the Weil action on $(\mathrm{R_u}({}^W\varpi_1(X,\bar{x}))/[\mathrm{R_u}({}^W\varpi_1(X,\bar{x})),\mathrm{R_u}({}^W\varpi_1(X,\bar{x}))])^\vee$ is mixed of non-negative weight. By Lemma 1.15 and Lemma 1.9, we may alternatively describe this as

$$\mathrm{H}^1(X,\mathbb{O}({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))),$$

where $\mathbb{O}({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ is the sheaf on X corresponding to the vector space $O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ equipped with its left $\varpi_1^{\mathrm{red}}(X,\bar{x})$-action. The $\pi_1(X,\bar{x})$-action on $(\mathrm{R_u}({}^W\varpi_1(X,\bar{x}))/[\mathrm{R_u}({}^W\varpi_1(X,\bar{x})),\mathrm{R_u}({}^W\varpi_1(X,\bar{x}))])^\vee$ then comes from the right action on $O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$, and by Lemma 1.10 the Frobenius action comes from the natural Frobenius action on $O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$.

Now, as in Theorem 1.14, we may write

$$O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))\otimes_{\mathbb{Q}_l}\bar{\mathbb{Q}}_l\cong\bigoplus_{V\in T}\mathrm{End}(V),$$

where T is the set of all isomorphism classes of irreducible representations of ${}^W\varpi_1^{\mathrm{red}}(X,\bar{x})$. This is a sum of Weil representations, and each V extends to a representation of $W(X_{\mathbb{F}_q^n},x)$ for some n, automatically compatible with the Frobenius action on $O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ (which then corresponds to the adjoint action). Since a Weil representation is pure of weight w if and only if the restricted $W(X_{\mathbb{F}_q^n},x)$-representation is so, it suffices to show that the $W(X_{\mathbb{F}_q^n},x)$-representation

$$\mathrm{H}^1(X,\mathbb{V}^\vee)\otimes V$$

is mixed for each irreducible $\pi_1(X,\bar{x})$-representation with $(F^n)^*V\cong V$.

The group $W(X_{\mathbb{F}_q^n},x)$ acts on $\mathrm{H}^1(X,\mathbb{V}^\vee)$ by composing the canonical map $W(X_{\mathbb{F}_q^n},x)\to\mathbb{Z}$ with the Frobenius action arising from the Weil structure of V. By Lafforgue's Theorem, we may assume that V is pure of weight zero (by Schur's Lemma, note that different choices of Frobenius action on V all give the same adjoint action on $\mathrm{End}(V)$). From Deligne's Weil II theorems ([Del80] Corollaries 3.3.4 – 3.3.6), it then follows that $\mathrm{H}^1(X,\mathbb{V}^\vee)$ is mixed of non-negative weight, so $\mathrm{H}^1(X,\mathbb{V}^\vee)\otimes V$ must also be mixed of non-negative weight, V being pure of weight 0. □

Corollary 1.18. *If X is smooth, then the quotient map ${}^W\varpi_1(X,\bar{x})\to{}^W\varpi_1^{\mathrm{red}}(X,\bar{x})$ has a unique Weil-equivariant section.*

Proof. In this case, the weights of $\mathrm{H}^1(X,\mathbb{V}^\vee)\otimes V$ are strictly positive (1 or 2), so $O({}^W\varpi_1(X,\bar{x}))/O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ is of strictly positive weights, giving us a decomposition

$$O({}^W\varpi_1(X,\bar{x}))=\mathcal{W}_0O({}^W\varpi_1(X,\bar{x}))\oplus\mathcal{W}_+O({}^W\varpi_1(X,\bar{x})).$$

Projection onto $\mathcal{W}_0O({}^W\varpi_1(X,\bar{x}))=O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ yields the section. □

Corollary 1.19. *If $f:X\to Y$ is a morphism of connected varieties over $\bar{\mathbb{F}}_p$, with X smooth, and $\mathbb{V}$ a semisimple constructible $\bar{\mathbb{Q}}_l$-local system underlying a Weil sheaf on Y, then $f^{-1}\mathbb{V}$ is semisimple.*

Proof. If $\mathbb{V}$ is of rank n, then it corresponds to a homomorphism ${}^{W}\varpi(Y,\bar{y})^{\mathrm{red}} \to \mathrm{GL}(n,\mathbb{Q}_l)$, or equivalently

$$O(\mathrm{GL}_n) \to O({}^{W}\varpi(Y,\bar{y})^{\mathrm{red}}) \le \mathcal{W}_0 O({}^{W}\varpi(Y,\bar{y})),$$

so $f^{-1}\mathbb{V}$ must correspond to

$$O(\mathrm{GL}_n) \to \mathcal{W}_0 O({}^{W}\varpi_1(X,\bar{x})) = O({}^{W}\varpi_1^{\mathrm{red}}(X,\bar{x})),$$

as f commutes with Frobenius. Therefore $f^{-1}\mathbb{V}$ is semisimple. □

2 SDCs over $\mathcal{N}(G)$

Fix a field K of characteristic zero, and recall the following definitions from [Pri05b]. Take a pro-algebraic reductive group G over K, and define $\mathcal{N}(G)$ to be the category whose objects are pairs $(\mathfrak{u},\rho)$, where $\mathfrak{u}$ is a finite-dimensional nilpotent Lie algebras over K, and $\rho : G \to \mathrm{Aut}(\mathfrak{u})$ is a representation to the group of Lie algebra automorphisms of $\mathfrak{u}$. A morphism θ from $(\mathfrak{u},\rho)$ to $(\mathfrak{u}',\rho')$ is a morphism $\theta : \mathfrak{u} \to \mathfrak{u}'$ of Lie algebras such that $\theta \circ \rho = \rho'$. Observe that $\mathcal{N}(G)$ is an Artinian category, and write $\widehat{\mathcal{N}}(G)$ for the category $\mathrm{pro}(\mathcal{N}(G))$.

The formal definitions, properties and constructions of SDCs over $\mathcal{N}(G)$ hold in exactly the same manner as those defined in [Pri05a] over $\mathcal{C}_\Lambda$. A summary follows.

Definition 2.1. A simplicial deformation complex $S^\bullet$ consists of smooth homogeneous functors $S^n : \mathcal{N}(G) \to \mathrm{Set}$ for each $n \ge 0$, together with maps

$$\begin{array}{ll} \partial^i : S^n \to S^{n+1} & 1 \le i \le n \\ \sigma^i : S^n \to S^{n-1} & 0 \le i < n, \end{array}$$

an associative product $* : S^m \times S^n \to S^{m+n}$, with identity $1 : \bullet \to S^0$, where $\bullet$ is the constant functor $\bullet(\mathfrak{g}) = \bullet$ on $\mathcal{N}(G)$, such that:

1. $\partial^j\partial^i = \partial^i\partial^{j-1} \quad i < j$.
2. $\sigma^j\sigma^i = \sigma^i\sigma^{j+1} \quad i \le j$.
3. $\sigma^j\partial^i = \begin{cases} \partial^i\sigma^{j-1} & i < j \\ \mathrm{id} & i = j,\ i = j+1 \\ \partial^{i-1}\sigma^j & i > j+1 \end{cases}$.
4. $\partial^i(s) * t = \partial^i(s * t)$.
5. $s * \partial^i(t) = \partial^{i+m}(s * t)$, for $s \in S^m$.
6. $\sigma^i(s) * t = \sigma^i(s * t)$.
7. $s * \sigma^i(t) = \sigma^{i+m}(s * t)$, for $s \in S^m$.

Remark 2.2. If we set ω_0 to be the unique element of $S^1(0)$, then, since 0 is the initial object in $\mathcal{N}(G)$, we may set $\partial^0(s) = \omega_0 * s$, and $\partial^{n+1}(s) = s * \omega_0$. $S^\bullet$ then becomes a cosimplicial complex.

Definition 2.3. Let $t_S^\bullet$ be the tangent space of $S^\bullet$, i.e. $t_S^n(V) = S^n(V\epsilon)$, for $V \in \mathrm{Rep}(G)$.

Definition 2.4. Define the Maurer-Cartan functor MC_S by

$$\mathrm{MC}_S(\mathfrak{g}) = \{\omega \in S^1(\mathfrak{g}) \colon \omega * \omega = \partial^1(\omega)\}.$$

Lemma 2.5. *S^0 is a group under multiplication.*

Now, if $\omega \in \mathrm{MC}_S(\mathfrak{g})$ and $g \in S^0(\mathfrak{g})$, then $g * \omega * g^{-1} \in \mathrm{MC}_S(\mathfrak{g})$. We may therefore make the following definition:

Definition 2.6.

$$\mathrm{Def}_S = \mathrm{MC}_S/S^0,$$

the quotient being with respect to the adjoint action. The deformation groupoid

$$\mathfrak{Def}_S$$

has objects MC_S, and morphisms given by S^0.

Lemma 2.7. *The action $S^0 \times S^n \to S^n$ is faithful (i.e. $s * h = t$ for some t only if $s = 1$).*

This implies that, for all $\omega \in \mathrm{MC}_S(\mathfrak{g})$, $\sigma^0(\omega) = 1$.

Definition 2.8. For $V \in \mathrm{Rep}(G)$, define the cohomology groups of S to be

$$\mathrm{H}^i(S, V) := \mathrm{H}^i(t_S^\bullet(V)),$$

the cohomology groups of the cosimplicial complex $t_S^\bullet(V)$.

Lemma 2.9. *The tangent space of MC_S is $V \mapsto \mathrm{Z}^1(t_S^\bullet(V))$, with the action of S^0 giving $\nu(s) = \partial^1(s) - \partial^0(s)$.*

Lemma 2.10. *$V \mapsto \mathrm{H}^2(S, V)$ is a complete obstruction space for MC_S.*

[Pri05b] Theorems 2.28 to 2.32 now imply:

Theorem 2.11. *Def_S is a deformation functor, with tangent space $\mathrm{H}^1(S)$ and complete obstruction space $\mathrm{H}^2(S)$. Moreover, if $\mathrm{H}^0(S) = 0$, then Def_S is homogeneous.*

Proof. [Pri05b] Theorem 2.29 and Corollary 2.32. □

Theorem 2.12. *If $\phi : S \to T$ is a morphism of SDCs, and*

$$\mathrm{H}^i(\phi) : \mathrm{H}^i(S) \to \mathrm{H}^i(T)$$

are the induced maps on cohomology, then:

1. *If $\mathrm{H}^1(\phi, V)$ is bijective, and $\mathrm{H}^2(\phi, V)$ injective for all $V \in \mathrm{Rep}(G)$, then $\mathrm{Def}_S \to \mathrm{Def}_T$ is étale.*

2. *If also* $\mathrm{H}^0(\phi, G)$ *is surjective for all* $V \in \mathrm{Rep}(G)$*, then* $\mathrm{Def}_S \to \mathrm{Def}_T$ *is an isomorphism.*

3. *Provided condition 1 holds,* $\mathfrak{Def}_S \to \mathfrak{Def}_T$ *is an equivalence of functors of groupoids if and only if* $\mathrm{H}^0(\phi, V)$ *is an isomorphism for all* $V \in \mathrm{Rep}(G)$*.*

Call a morphism $\phi : S \to T$ a quasi-isomorphism if the $\mathrm{H}^i(\phi) : \mathrm{H}^i(S) \to \mathrm{H}^i(T)$ are all isomorphisms.

Definition 2.13. Given a morphism $\phi : S \to T$ of SDCs, define the groupoid

$$\mathfrak{Def}_\phi$$

to be the fibre of the morphism

$$\mathfrak{Def}_S \to \mathfrak{Def}_T$$

over the unique point $x_0 \in \mathrm{MC}_T(0)$.

Explicitly, $\mathfrak{Def}_\phi(\mathfrak{g})$ has objects

$$\{(\omega, h) \in \mathrm{MC}_S(\mathfrak{g}) \times T^0(\mathfrak{g}) \quad : \quad h\phi(\omega)h^{-1} = x_0\},$$

and morphisms

$$S^0(\mathfrak{g}), \quad \text{where} \quad g(\omega, h) = (g\omega g^{-1}, h\phi(g)^{-1}).$$

Theorem 2.14. *Let* $S^n_\phi(A)$ *be the fibre of* $S^n(A) \xrightarrow{\phi} T^n(A)$ *over* x_0^n*. Then* $S^\bullet_x$ *is an SDC, and the canonical map*

$$\mathfrak{Def}_{S_\phi} \to \mathfrak{Def}_\phi$$

is an equivalence of groupoids.

2.1 Constructing SDCs

Throughout this section, we will consider functors $\mathcal{D} : \mathcal{N}(G) \to \mathrm{Cat}$. We will *not* require that these functors satisfy (H0) (the condition that $F(0) = \bullet$).

Definition 2.15. Given a functor $\mathcal{D} : \mathcal{N}(G) \to \mathrm{Cat}$, and an object $D \in \mathrm{Ob}\,\mathcal{D}(0)$, define $\mathfrak{Def}_{\mathcal{D},D} : \mathcal{N}(G) \to \mathrm{Grpd}$ by setting $\mathfrak{Def}_{\mathcal{D},D}(\mathfrak{g})$ to be the fibre of $\mathcal{D}(\mathfrak{g}) \to \mathcal{D}(0)$ over (D, id).

Definition 2.16. We say a functor $\mathcal{E} : \mathcal{N}(G) \to \mathrm{Cat}$ has uniformly trivial deformation theory if the functor $\mathrm{Mor}\,\mathcal{E}$ is smooth and homogeneous, and the functor $\mathrm{Cmpts}\,\mathcal{E}$ is constant, i.e for $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$ in $\mathcal{N}(G)$, $\mathcal{E}(\mathfrak{g}) \to \mathcal{E}(\mathfrak{h})$ is full and essentially surjective.

Assume that we are given functors $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E} : \mathcal{N}(G) \to \mathrm{Cat}$, and a diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] \arrow[d, shift right, "V"'] \arrow[d, phantom, "\dashv"] & \mathcal{A} \arrow[l, shift left, "F"] \arrow[d, shift right, "V"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{B} \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] \arrow[u, shift right, "G"'] & \mathcal{E}, \arrow[l, shift left, "F"] \arrow[u, shift right, "G"']
\end{tikzcd}$$

where $\mathcal{E}$ has uniformly trivial deformation theory, the horizontal adjunctions are monadic and the vertical adjunctions comonadic. Let

$$\top_{\mathrm{h}} = UF \qquad\qquad \bot_{\mathrm{h}} = FU$$
$$\bot_{\mathrm{v}} = VG \qquad\qquad \top_{\mathrm{v}} = GV,$$

with

$$\eta : 1 \to \top_{\mathrm{h}}, \quad \gamma : \bot_{\mathrm{v}} \to 1, \quad \varepsilon : \bot_{\mathrm{h}} \to 1 \text{ and } \alpha : 1 \to \top_{\mathrm{v}}.$$

Assume moreover that the following identities hold:

$$\begin{aligned} GU &= UG \\ FV &= VF \\ UV &= VU \end{aligned}$$

$$V\varepsilon = \varepsilon V \qquad\qquad U\alpha = \alpha U$$
$$V\eta = \eta V \qquad\qquad U\gamma = \gamma U.$$

Theorem 2.17. *For $D \in \mathrm{Ob}\,\mathcal{D}(0)$, let $E = UVD \in \mathrm{Ob}\,\mathcal{E}(0)$, and write $E(\mathfrak{g})$ for the image of E under the map $\mathrm{Ob}\,\mathcal{E}(0) \to \mathrm{Ob}\,\mathcal{E}(\mathfrak{g})$. Set $S^n(\mathfrak{g})$ to be the fibre*

$$S^n(\mathfrak{g}) = \mathrm{Hom}_{\mathcal{E}(\mathfrak{g})}(\top_{\mathrm{h}}^n E(\mathfrak{g}), \bot_{\mathrm{v}}^n E(\mathfrak{g}))_{UV(\alpha_D^n \circ \varepsilon_D^n)}$$

of

$$\mathrm{Hom}_{\mathcal{E}(\mathfrak{g})}(\top_{\mathrm{h}}^n E(\mathfrak{g}), \bot_{\mathrm{v}}^n E(\mathfrak{g})) \to \mathrm{Hom}_{\mathcal{E}(0)}(\top_{\mathrm{h}}^n E, \bot_{\mathrm{v}}^n E)$$

over $UV(\alpha_D^n \circ \varepsilon_D^n)$.

We give S^n the structure of an SDC as in [Pri05a] Theorem 2.5.

Then

$$\mathfrak{Def}_{\mathcal{D},D} \simeq \mathfrak{Def}_S.$$

Proof. As for [Pri05a] Theorem 2.5. □

3 Principal homogeneous spaces

Let $k = \mathbb{F}_q$, take a connected variety X_k/k, and let $X = X_k \otimes_k \bar{k}$. Fix a closed point x of X, and denote the associated geometric point $x \otimes_{k(x)} \bar{k} \to X$ by $\bar{x}$. Without loss of generality (increasing q if necessary), we assume that $k(x) \subset \mathbb{F}_q$. Given a semisimple continuous $\pi_1(X, x)$-representation V over $\mathbb{Q}_l$, let $\mathbb{V}$ denote the corresponding semisimple local system on X.

Definition 3.1. Given a pro-l group K, define a constructible principal (resp. faithful) K-sheaf to be a principal (resp. faithful) K-sheaf $\mathbb{D}$, such that

$$\mathbb{D} = \varprojlim_{K \to F \text{ finite}} F \times^K \mathbb{D}.$$

Given an l-adic pro-Lie group G, a constructible principal (resp. faithful) G-sheaf is a G-sheaf $\mathbb{B}$ for which there exists a constructible principal (resp. faithful) K-sheaf $\mathbb{D}$, for some $K \leq G$ compact, with $\mathbb{B} = G \times^K \mathbb{D}$ (observe that compact and totally disconnected is equivalent to pro-finite).

Lemma 3.2. *Given an l-adic pro-Lie group G, the category of continuous representations*

$$\rho : \pi_1(X, \bar{x}) \to G$$

is equivalent to the category of constructible principal G-sheaves on X.

Proof. As in [Pri04a] Section 2.2. $\square$

Fix a Zariski-dense representation $\rho_0 : \pi_1(X, \bar{x}) \to G$, for G a reductive pro-algebraic group over $\mathbb{Q}_l$. For $\mathfrak{g} \in \mathcal{N}(G)$, we therefore have a functorial equivalence of groupoids between $\mathfrak{R}_{\rho_0}(\exp(\mathfrak{g}))$, the groupoid of continuous representations

$$\rho : \pi_1(X, \bar{x}) \to \exp(\mathfrak{g}) \rtimes G$$

deforming the canonical representation

$$\rho_0 : \pi_1(X, \bar{x}) \to G,$$

and $\mathfrak{B}_{\mathbb{B}_0}(\exp(\mathfrak{g}))$, the groupoid of constructible principal $\exp(\mathfrak{g}) \rtimes G$-sheaves $\mathbb{B}$ with $\mathbb{B} \times^{\exp(\mathfrak{g}) \rtimes G} G \cong \mathbb{B}_0$, the constructible principal G-sheaf corresponding to ρ_0.

Recall the following definitions of the Godement resolution:

Since X is a variety, let

$$X' = \coprod_{x \in |X|} \bar{x}.$$

We have maps $u_x : \bar{x} \to X$, giving a map $u : X' \to X$.

Definition 3.3. For a constructible locally free $\mathbb{Z}_l$-sheaf $\mathscr{F}$, define

$$\mathrm{C}^n(X, \mathscr{F}) := \varprojlim_m \mathrm{C}^n(X, \mathscr{F}/l^m \mathscr{F}),$$

where

$$\mathrm{C}^n(X, \mathscr{F}) := \Gamma(X', (u^* u_*)^n u^* \mathscr{F}),$$

for sheaves $\mathscr{F}$ on X, and

$$\mathscr{C}^n(\mathscr{F}) := (u_* u^*)^{n+1} \mathscr{F},$$

so that $\mathrm{C}^n(X, \mathscr{F}) = \Gamma(X, \mathscr{C}^n(\mathscr{F}))$.

For a constructible locally free $\mathbb{Q}_l$-sheaf $\mathscr{F} \otimes \mathbb{Q}_l$ of finite rank, define

$$\begin{aligned} \mathscr{C}^n(\mathscr{F} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) &:= \mathscr{C}^n(\mathscr{F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \\ \mathrm{C}^n(X, \mathscr{F} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) &:= \mathrm{C}^n(X, \mathscr{F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \end{aligned}$$

Note that this construction is independent of the choice of $\mathscr{F}$, since $\mathscr{F}$ is of finite rank.

As in [Pri04a] Lemma 2.6, the Mittag-Leffler condition implies that

$$\mathrm{H}^i(\mathrm{C}^n(X, \mathscr{F} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)) = \mathrm{H}^i(X, \mathscr{F} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l),$$

provided that $\mathrm{H}^i(X, \mathscr{F}), \mathrm{H}^{i-1}(X, \mathscr{F})$ are of finite rank, e.g. if X is a smooth or proper variety.

Given $\mathfrak{g} \in \mathcal{N}(G)$, the representation $\rho_0 : \pi_1(X, \bar{x}) \to G$ gives $\mathfrak{g}$ the structure of a semi-simple $\pi_1(X, \bar{x})$-representation. Let $\mathscr{g}$ be the corresponding sheaf of Lie algebras on X.

Theorem 3.4. *The functorial groupoid $\mathfrak{g} \mapsto \mathfrak{R}_{\rho_0}(\exp(\mathfrak{g}))$, is governed by the SDC*

$$S^n(\mathfrak{g}) = \exp(\mathrm{C}^n(X, \mathscr{g})),$$

with product given by the Alexander-Whitney cup product

$$g * h = (\partial^{m+n} \ldots \partial^{m+2} \partial^{m+1} g) \cdot (\partial^0)^m h,$$

for $g \in \mathrm{C}^m$, $h \in \mathrm{C}^n$. This SDC has tangent space

$$V \mapsto \mathrm{H}^1(X, \mathbb{V}),$$

and obstruction space

$$V \mapsto \mathrm{H}^2(X, \mathbb{V})$$

if X is smooth or proper.

Proof. We have the functorial comonadic adjunction:

$$\begin{tikzcd}
\text{Faithful Constructible } (\exp(\mathfrak{g}) \rtimes G)\text{-Sheaves}(X) \arrow[d, shift right, "u^*"'] \\
\text{Faithful Constructible } (\exp(\mathfrak{g}) \rtimes G)\text{-Sheaves}(X') \arrow[u, shift right, "u_*"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

using the canonical injection $(\exp(\mathfrak{g}) \rtimes G) \to u^*(\exp(\mathfrak{g}) \rtimes G)$ to make $u_*\mathbb{B}$ a faithful $(\exp(\mathfrak{g}) \rtimes G)$-sheaf, and where u^* is defined by:

$$u^*\mathbb{D} = (\exp(\mathfrak{g}) \rtimes G) \times^K (\varprojlim_F u^*(F \times^K \mathbb{D})),$$

with K and $\mathbb{D}$ as in Definition 3.1.

Observe that the second category has uniformly trivial deformation theory, so that, by Theorem 2.17, deformations are described by the SDC

$$S^n(\mathfrak{g}) = \mathrm{Hom}(u^*\mathbb{B}_0(\mathfrak{g}), (u^*u_*)^n u^*\mathbb{B}_0((\mathfrak{g})))^{\exp(\mathfrak{g}) \rtimes G}_{u^*\alpha^n},$$

where $\mathbb{B}_0(\mathfrak{g})$ is the constructible principal $\exp(\mathfrak{g}) \rtimes G$-sheaf corresponding to the representation $\rho_0 : \pi_1(X, \bar{x}) \to \exp(\mathfrak{g}) \rtimes G$, and $\alpha^n : \mathbb{B} \to (u^*u_*)^n\mathbb{B}$ is the canonical map associated to the adjunction.

There is then an isomorphism of SDCs

$$\begin{array}{rcl}
\Gamma(X', (u^*u_*)^n u^* \exp(\mathscr{g})) & \to & S^n(\mathfrak{g}), \\
g & \mapsto & (b \mapsto g \cdot b),
\end{array}$$

where the SDC structure of the term on the left is given by the Alexander-Whitney cup product.

The SDC on the left can be rewritten as

$$S^n(\mathfrak{g}) = \exp(\mathrm{C}^n(X, \mathscr{g})),$$

as required. □

As in [Pri05b], if we set $G = \varpi_1^{\mathrm{red}}(X, \bar{x})$, then the Lie algebra of $R_u(\varpi_1(X, \bar{x}))$ is a hull for the functor Def_S, since it is a hull for $\mathfrak{R}$. If we set $G = {}^W\varpi_1^{\mathrm{red}}(X, \bar{x})$, and X is smooth or proper, then $R_u({}^W\varpi_1(X, \bar{x}))$ is similarly a hull for Def_S, by Lemma 1.15.

4 Frobenius actions

4.1 Geometric Frobenius

Given (X,x) as in the previous section, with X smooth or proper, define an SDC on $\mathcal{N}({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ by
$$S^n(\mathfrak{g}) = \exp(\mathrm{C}^n(X, \mathscr{g})),$$
whose hull must then be $R_u({}^W\varpi_1(X,\bar{x}))$.

There is a Frobenius action $F : X \to X$, and a compatible Frobenius action on X', the actions and isomorphisms combining to give a Frobenius action of SDCs over $\mathcal{N}({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$, given by:

$$\mathrm{C}^n(X, \mathscr{g}) \xrightarrow{F^\sharp} \mathrm{C}^n(X, F^*\mathscr{g}).$$

We will now investigate the Frobenius action this induces on the groupoid $\mathfrak{Def}_S(\mathfrak{g})$. As in Section 3, this groupoid is equivalent to the groupoid whose objects are principal homogeneous $\exp(\mathfrak{g}) \rtimes {}^W\varpi_1^{\mathrm{red}}(X,\bar{x})$-spaces $\mathbb{B}$, such that

$$\mathbb{B} \times^{(\exp(\mathfrak{g}) \rtimes {}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))} {}^W\varpi_1^{\mathrm{red}}(X,\bar{x}) \cong \mathbb{B}_0,$$

and whose morphisms preserve these isomorphisms, where $\mathbb{B}_0$ corresponds to the canonical representation $\pi_1(X,\bar{x}) \to {}^W\varpi_1^{\mathrm{red}}(X,\bar{x})$. Under $F^\sharp$, the torsor $\mathbb{B}$ is sent to the torsor $F^*\mathbb{B}$ in $\mathfrak{Def}_S(F^*\mathfrak{g})$.

Remark 4.1. The Weil action on ${}^W\varpi_1(X,\bar{x})$ gives a Frobenius action (via $F_x \in W(X_k, x)$) on $R_u({}^W\varpi_1(X,\bar{x}))$. This then gives a Frobenius action

$$\mathrm{Hom}(R_u({}^W\varpi_1(X,\bar{x})), \exp(\mathfrak{g})) \xrightarrow{F^\sharp} \mathrm{Hom}(R_u({}^W\varpi_1(X,\bar{x})), \exp(F^*\mathfrak{g})).$$

As in [Pri04b] Remark 2.23, the hull morphism

$$\mathrm{Hom}(R_u({}^W\varpi_1(X,\bar{x})), \exp(\mathfrak{g})) \to \mathrm{Def}_S(\mathfrak{g})$$

is Frobenius equivariant under these actions.

4.2 Structure of the fundamental group

Definition 4.2. As in Theorem 1.17, $\mathbb{O}({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ is the sheaf of algebras on X corresponding to the vector space $O({}^W\varpi_1^{\mathrm{red}}(X,\bar{x}))$ equipped with its left $\varpi_1^{\mathrm{red}}(X,\bar{x})$-action. From now on, we will simply denote this sheaf by $\mathbb{O}$. This is a pure Weil sheaf of weight 0. The Frobenius actions on the cohomology groups $\mathrm{H}^i(X,\mathbb{O})$ combine with the right $\varpi_1^{\mathrm{red}}(X,\bar{x})$-actions to make them mixed Weil representations.

Theorem 4.3. *There is an isomorphism*

$$\mathrm{Lie}(R_u({}^W\varpi_1(X,\bar{x}))) \cong L(\mathrm{H}^1(X,\mathbb{O})^\vee)/(f(\mathrm{H}^2(X,\mathbb{O})^\vee)),$$

where $L(V)$ is the free pro-nilpotent Lie algebra on generators V, and

$$f : \mathrm{H}^2(X,\mathbb{O})^\vee \to \Gamma_2 L(\mathrm{H}^1(X,\mathbb{O})^\vee)$$

preserves the (Frobenius) weight decompositions of [Del80] 3.3.7. The resulting weight decomposition on $R_u(\varpi_1(X,\bar{x}))$ is the same as the natural Weil weight decomposition of Theorem 1.17.

Moreover, the quotient map

$$f : \mathrm{H}^2(X,\mathbb{O})^\vee \to \Gamma_2/\Gamma_3 \cong \bigwedge^2(\mathrm{H}^1(X,\mathbb{O})^\vee)$$

is dual to half the cup product

$$\mathrm{H}^1(X,\mathbb{O}) \times \mathrm{H}^1(X,\mathbb{O}) \xrightarrow{\frac{1}{2}\cup} \mathrm{H}^2(X,\mathbb{O}).$$

Proof. This is essentially the same as [Pri04b] Theorem 3.2, making use of Lemma 1.9. To see that the Frobenius decomposition corresponds to the Weil decomposition, note that the action of $F_x \in W(X_k, x)$ determines the Weil decomposition. $\square$

Corollary 4.4. *If X is smooth and proper, then*

$$\mathrm{Lie}(R_u({}^W\varpi_1(X,\bar{x})))$$

is quadratically presented.

In fact, there is an isomorphism of Weil representations

$$\mathrm{Lie}(R_u({}^W\varpi_1(X,\bar{x}))) \cong L(\mathrm{H}^1(X,\mathbb{O})^\vee)/(\check{\cup}\,\mathrm{H}^2(X,\mathbb{O})^\vee)),$$

where $\check{\cup}$ is dual to the cup product.

Proof. This follows since, under these hypotheses, [Del80] Corollaries 3.3.4–3.3.6 imply that $\mathrm{H}^1(X,\mathbb{O})$ is pure of weight 1, and $\mathrm{H}^2(X,\mathbb{O})$ is pure of weight 2. $\square$

Corollary 4.5. *If X is smooth, then*

$$\mathrm{Lie}(R_u({}^W\varpi_1(X,\bar{x})))$$

is a quotient of the free pro-nilpotent Lie algebra $L(\mathrm{H}^1(X,\mathbb{O})^\vee)$ by an ideal which is finitely generated by elements of bracket length $2, 3, 4$.

Proof. This follows since, under these hypotheses, [Del80] Corollaries 3.3.4–3.3.6 imply that $\mathrm{H}^1(X,\mathbb{O})$ is of weights 1 and 2, while $\mathrm{H}^2(X,\mathbb{O})$ is of weights $2, 3$ and 4. $\square$

Corollary 4.6. *Let G be an arbitrary reductive $\mathbb{Q}_l$- algebraic group, acting on a unipotent $\mathbb{Q}_l$-algebraic group U defined by homogeneous equations, i.e. $\mathfrak{u} \cong \mathrm{gr}\,\mathfrak{u}$ as Lie algebras with G-actions.*

1. *If X is smooth and proper, and*

$$\rho_2 : W(X_k, x) \to (U/[U,[U,U]]) \rtimes G$$

is a Zariski-dense representation, then

$$\rho_1 : \pi_1(X,x) \to (U/[U,U]) \rtimes G$$

lifts to a representation

$$\rho : \pi_1(X,x) \to U \rtimes G.$$

2. If X is merely smooth, and

$$\rho_4 : W(X_k, x) \to (U/\Gamma_5 U) \rtimes G$$

is a Zariski-dense representation, then

$$\rho_1 : \pi_1(X, x) \to (U/[U,U]) \rtimes G$$

lifts to a representation

$$\rho : \pi_1(X, x) \to U \rtimes G.$$

Proof. As for [Pri05b] Corollary 6.4. ☐

Remarks 4.7. Note that Corollaries 4.4 and 4.5 imply the results of both [Pri04a] and of [Pri04b]. The pro-unipotent completion $\pi_1(X, \bar{x}) \otimes \mathbb{Q}_l$ studied in [Pri04b] is just the maximal quotient $\theta_\sharp R_u(\varpi_1(X, \bar{x}))$ of $R_u(\varpi_1(X, \bar{x}))$, for $\theta : \varpi_1^{\mathrm{red}}(X, \bar{x}) \to 1$, on which $\pi_1(X, \bar{x})$ acts trivially.

The problem considered in [Pri04a] is to fix a reductive representation $\rho_0 : W(X_k, x) \to G(\mathbb{Q}_l)$, and consider lifts $\rho : \pi_1(X, \bar{x}) \to G(A)$, for Artinian rings A. The hull of this functor is the functor

$$A \mapsto \mathrm{Hom}_{\pi_1(X,\bar{x})}(R_u(\varpi_1(X, \bar{x})), \exp(\mathfrak{g} \otimes \mathfrak{m}_A)),$$

where $\mathfrak{g}$ is the Lie algebra of G, regarded as the adjoint representation. It follows that this hull then has generators $\mathrm{Hom}_{\pi_1(X,\bar{x})}(\mathfrak{g}, H_1)$, and relations

$$\mathrm{Hom}_{\pi_1(X,x)}(\mathfrak{g}, H_2) \to \mathrm{Symm}^2 \mathrm{Hom}_{\pi_1(X,x)}(\mathfrak{g}, H_1)$$

given by composing the coproduct and the Lie bracket, where

$$H_i = \mathrm{H}^i(X, \mathbb{O}(\varpi_1^{\mathrm{red}}(X, \bar{x})))^\vee.$$

Definition 4.8. A representation $\rho : \Gamma \to G$ of a pro-finitely generated group Γ is said to be rigid if the orbit $G(\rho) \subset \mathrm{Hom}(\Gamma, G)$ under the conjugation action is open in the l-adic topology. Observe that this is equivalent to the condition that $\mathrm{H}^1(\Gamma, \mathrm{Lie}(G)) = 0$, since this is the dimension of the quotient space at $[\rho]$.

A representation is properly rigid if the representation to the Zariski closure of its image is rigid.

The following lemma is inspired by the observation in [Sim92] that rigidity ensures that a local system on a complex projective variety is a variation of Hodge structure.

Lemma 4.9. *Every properly rigid representation $\rho : \pi_1(X, \bar{x}) \to G$ extends to a representation of $W(X_{k'}, x)$, for some finite extension $k \subset k'$.*

Proof. Replace G by the Zariski closure of the image of ρ. If we give the set $\mathbb{N}$ the multiplicative ordering, then it becomes a poset, and $F_x^n \rho_{n \in \mathbb{N}}$ is a net in $\mathrm{Hom}(\pi_1(X, \bar{x}), G)$. Since $F_x^n \to 1$, this net tends to ρ. Since $G(\rho)$ is an open neighbourhood of ρ, there exists an n for which $F_x^n \rho \in G(\rho)$; let $F_x^n = \mathrm{ad}_g(\rho)$. We may now define a representation

$$\pi_1(X, \bar{x}) \rtimes \langle F_x^n \rangle \xrightarrow{(\rho, g)} G,$$

noting that the former group is $W(X_{k'}, x)$, for $k \subset k'$ a degree n extension. ☐

Remark 4.10. Observe that the lemma remains true under the weaker hypothesis that $\mathrm{Im}\,(\mathrm{H}^1(\pi_1(X,\bar{x}), \mathrm{Lie}(\mathrm{Im}\,\rho)) \to \mathrm{H}^1(\pi_1(X,\bar{x}), \mathrm{Lie}(G))) = 0$.

Proposition 4.11. *If X is smooth and proper, and $\Gamma := \pi_1(X,\bar{x}) = \Delta \rtimes \Lambda$, let H be the Zariski closure of the image of Λ in $\mathrm{Aut}(\Delta \otimes \mathbb{Q}_l)$. If H is reductive, $\mathrm{H}^1(\Lambda, \mathrm{Lie}(H)) = 0$, and $\mathrm{Hom}_\Lambda(\Delta/[\Delta,\Delta], \mathrm{Lie}(H)) = 0$, then $\Delta \otimes \mathbb{Q}_l$ is quadratically presented.*

Proof. First observe that the representation $\rho : \Gamma \to \Lambda \to H$ is rigid. This follows because the condition $\mathrm{H}^1(\Lambda, \mathrm{Lie}(H)) = 0$ ensures that $\Lambda \to H$ is rigid, so any representation $\Gamma \to H \ltimes \epsilon\mathrm{Lie}(H)$ must be conjugate to one which restricts to ρ on Λ. The image of Δ must also lie in $\mathrm{Lie}(H)$, so the representation is determined by an element of $\mathrm{Hom}_\Lambda(\Delta/[\Delta,\Delta], \mathrm{Lie}(H)) = 0$, so it must be ρ. Therefore, by Lemma 4.9, ρ extends to a Weil representation (possibly after changing the base field).

Therefore ρ factors as $\pi_1(X,\bar{x}) \to {}^W\varpi_1^{\mathrm{red}}(X,\bar{x}) \xrightarrow{\theta} H$ Now, by Corollary 4.4, we know that

$$\theta_\sharp Lie(R_u({}^W\varpi_1(X,\bar{x})))$$

must be quadratically presented. The proof now proceeds as in [Pri05b] Proposition 6.7. ☐

Example 4.12. Let $\mathfrak{d}$ be the free $\hat{\mathbb{Z}}$-module

$$\mathfrak{d} := \hat{\mathbb{Z}}x \oplus \hat{\mathbb{Z}}y \oplus \frac{1}{2}\hat{\mathbb{Z}}[x,y],$$

which has the structure of a Lie algebra, with $[x,y]$ in the centre. The Campbell-Baker-Hausdorff formula enables us to regard $\Delta := \exp(\mathfrak{d})$ as the profinite group with underlying set $\mathfrak{d}$ and product

$$a \cdot b = a + b + \frac{1}{2}[a,b],$$

since all higher brackets vanish.

Let $\exp(\mathfrak{h}) := \Delta \otimes \mathbb{Q}_l$; this is isomorphic to the three-dimensional l-adic Heisenberg group.

Observe that $\mathrm{Aut}(\Delta \otimes \mathbb{Q}_l) \cong \mathrm{GL}_2(\mathbb{Q}_l)$, and that $\mathrm{SL}_2(\hat{\mathbb{Z}})$ acts on Δ by the formula:

$$A(v,w) := (Av, (\det A)w) = (Av, w),$$

for $v \in \hat{\mathbb{Z}}x \oplus \hat{\mathbb{Z}}y$ and $w \in \frac{1}{2}\hat{\mathbb{Z}}[x,y]$.

The group

$$\Gamma := \Delta \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}});$$

cannot be the geometric fundamental group of any smooth proper variety defined over a finite field.

Proof. We wish to show that $\Gamma \to \mathrm{Aut}(\Delta \otimes \mathbb{Q}_l)$ is properly rigid. For this, it will suffice to show that $\Lambda \to \mathrm{Aut}(\Delta \otimes \mathbb{Q}_l)$ is properly rigid, and that $\mathrm{Hom}_\Lambda(\Delta/[\Delta,\Delta], \mathfrak{sl}_2(\mathbb{Q}_l)) = 0$.

To prove the first, observe that

$$\mathrm{SL}_2(\hat{\mathbb{Z}}) = \prod_{\nu \text{ prime}} \mathrm{SL}_2(\mathbb{Z}_\nu),$$

and that only pro-l groups contribute to cohomology. We need to show that the only derivations $\mathrm{SL}_2(\mathbb{Z}_l) \to \mathfrak{sl}_2(\mathbb{Q}_l)$ are inner derivations. Now, for N sufficiently large, $\exp : l^N\mathfrak{sl}_2(\mathbb{Z}_l) \to \mathrm{SL}_2(\mathbb{Z}_l)$ converges, and it follows from the simplicity of $\mathfrak{sl}_2(\mathbb{Z}_l)$ that any derivation must agree with an inner derivation when restricted to $\exp(l^N\mathfrak{sl}_2(\mathbb{Z}_l))$. Since this is a subgroup of finite index, and $\mathfrak{sl}_2(\mathbb{Q}_l)$ is torsion-free, the derivation and inner derivation must agree on the whole of $\mathrm{SL}_2(\mathbb{Z}_l)$, as required.

To prove the second, observe that $\mathbb{Q}_l^2$ and $\mathfrak{sl}_2(\mathbb{Q}_l)$ are distinct irreducible $\mathrm{SL}_2(\mathbb{Z}_l)$-representations.

We therefore conclude from the previous proposition that Γ cannot be the fundamental group of any smooth proper variety defined over a finite field, since the action of $\mathrm{SL}_2(\hat{\mathbb{Z}})$ on $\mathfrak{h}$ is semisimple, hence reductive, and $\mathfrak{h}$ is not quadratically presented.

Alternatively, we could use Corollary 4.6 to prove that Γ is not such a group. Let $G = \mathrm{SL}_2(\mathbb{Q}_l)$, $\mathfrak{u} = L(\mathbb{Q}_l^2)$ and $U = \exp(\mathfrak{u})$. Observe that $\mathfrak{h} \cong \mathfrak{u}/[\mathfrak{u}, [\mathfrak{u}, \mathfrak{u}]]$, and let ρ_2 be the standard embedding

$$\rho_2 : \Delta \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}}) \to \exp(\mathfrak{h}) \rtimes \mathrm{SL}_2(\mathbb{Q}_l),$$

which extends to a Weil representation by Lemma 4.9 and the above calculation.

Since all triple commutators vanish in H, this does not lift to a representation

$$\rho : \Delta \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}}) \to U \rtimes \mathrm{SL}_2(\mathbb{Q}_l).$$

□

Note that [Pri04b] cannot be used to exclude this group — the abelianisation of Γ is a torsion group, as SL_2 acts irreducibly on the abelianisation of $\mathfrak{h}$, so $\Gamma \otimes \mathbb{Q}_l = 1$, which is quadratically presented.

Example 4.13. Let $\mathfrak{d}$ be the free $\hat{\mathbb{Z}}$-module on generators

$$x, y, \frac{1}{2}[xy], \frac{1}{12}[x[xy]], \frac{1}{12}[y[xy]], \frac{1}{24}[x[x[xy]]], \frac{1}{24}[x[y[xy]]], \frac{1}{24}[y[y[xy]]];$$

this has a Lie algebra structure, with all quintuple commutators vanishing. We define $\Delta := \exp(\mathfrak{d})$, the group whose underlying set is $\mathfrak{d}$, given a group structure via the truncated Campbell-Baker-Hausdorff formula:

$$a \cdot b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}([a, [a, b]] - [b, [a, b]]) - \frac{1}{24}[a, [b, [a, b]]].$$

Again, let $\Lambda := \mathrm{SL}_2(\hat{\mathbb{Z}})$, acting in the natural way on $\hat{\mathbb{Z}}x \oplus \hat{\mathbb{Z}}y$, with the action extending to the whole of Δ via the laws of Lie algebras. Then

$$\Gamma := \Delta \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}})$$

cannot be the geometric fundamental group of any smooth variety defined over a finite field.

Proof. Let $G = \mathrm{SL}_2(\mathbb{Q}_l)$, and let $\mathfrak{u} = L(\mathbb{Q}_l^2)$ and $U = \exp(\mathfrak{u})$. Observe that

$$\mathfrak{h} := \mathcal{L}(\Delta, \mathbb{Q}_l) \cong \mathfrak{u}/\Gamma_5\mathfrak{u},$$

and let ρ_4 be the standard embedding

$$\rho_4 : \Delta \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}}) \to \exp(\mathfrak{h}) \rtimes \mathrm{SL}_2(\mathbb{Q}_l),$$

which extends to a Weil representation by Lemma 4.9 and calculation in the previous example.

Since all quintuple commutators vanish in H, this does not lift to a representation

$$\rho : \Delta \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}}) \to U \rtimes \mathrm{SL}_2(\mathbb{Q}_l),$$

which gives a contradiction, by Corollary 4.6. □

References

[Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982. Philosophical Studies Series in Philosophy, 20.

[GM88] William M. Goldman and John J. Millson. The deformation theory of representations of fundamental groups of compact Kähler manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (67):43–96, 1988.

[Hai98] Richard M. Hain. The Hodge de Rham theory of relative Malcev completion. *Ann. Sci. École Norm. Sup. (4)*, 31(1):47–92, 1998.

[HM69] G. Hochschild and G. D. Mostow. Pro-affine algebraic groups. *Amer. J. Math.*, 91:1127–1140, 1969.

[Laf02] Laurent Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. *Invent. Math.*, 147(1):1–241, 2002.

[Mor78] John W. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (48):137–204, 1978.

[Pri04a] J. P. Pridham. The deformation theory of representations of the fundamental group of a smooth variety. arXiv math.AG/0401344, 2004.

[Pri04b] J. P. Pridham. The structure of the pro-l-unipotent fundamental group of a smooth variety. arXiv math.AG/0401378, 2004.

[Pri05a] J. P. Pridham. Deformations via simplicial deformation complexes. arXiv math.AG/0311168, 2005.

[Pri05b] J. P. Pridham. The pro-unipotent radical of the pro-algebraic fundamental group of a compact Kähler manifold. arXiv math.CV/0502451, 2005.

[Sch68] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

[Sim92] Carlos T. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, (75):5–95, 1992.