A concrete approach to higher and derived stacks

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Back to first principles

Building blocks:

- commutative rings (classical AG)
- dg/simplicial/E_∞-rings (derived AG)
- or non-commutative, analytic, $\mathcal{C}^\infty$, ...

Affine schemes give "wrong" colimits:

$$\begin{array}{ccccc} & & \mathbb{A}^1 & & \\ & \overset{z}{\nearrow} & & \searrow & \\ \mathbb{G}_m & & & & \mathbb{A}^0 = \operatorname{Spec}\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \\ & \underset{z^{-1}}{\searrow} & & \nearrow & \\ & & \mathbb{A}^1 & & \end{array}$$

Need ambient category to glue/quotient.

- Schemes $\subset$ ringed spaces
- Algebraic stacks $\subset$ functors on Aff
- Quasi-coherent sheaves $\subset$ $\mathscr{O}_X$-modules (enough injectives)

Nice categories, with many nasty objects.

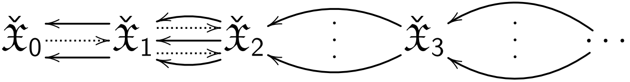
- Who cares about arbitrary sheaves on the big affine site?
- Or about arbitrary $\mathscr{O}_X$-modules?

Are there smaller ambient categories?

Čech nerves

For affine presentation $U \to \mathfrak{X}$ of Artin stack with affine diagonal, the Čech nerve

$$\check{\mathfrak{X}}_n := \underbrace{U \times_{\mathfrak{X}} U \times_{\mathfrak{X}} \ldots \times_{\mathfrak{X}} U}_{n+1}$$



recovers $\mathfrak{X}$, $\mathrm{H}^*(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ from affine diagram.

- Quasi-compact, semi-separated scheme X, affine cover $\{U_i\}_i$, take $U = \coprod_i U_i$, then

$$\check{X}_n = \coprod_{i_0,\ldots,i_n} U_{i_0} \cap \ldots \cap U_{i_n}.$$

Simplicial objects

- $|\Delta^n| := \{x \in \mathbb{R}_+^{n+1} : \sum_{i=0}^n x_i = 1\}$.
- For topological space X, $\mathrm{Sing}(X)_n := \mathrm{Hom}(|\Delta^n|, X)$, so

$$\mathrm{Sing}(X)_0 \underset{\partial_0}{\overset{\partial_1}{\underset{\sigma_0}{\leftleftarrows\!\!\!\!\rightarrow}}} \mathrm{Sing}(X)_1 \begin{smallmatrix}\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow\end{smallmatrix} \mathrm{Sing}(X)_2 \ldots,$$

  relations like $\partial_i \sigma_i = \mathrm{id}$.
- Any diagram of this form is called simplicial.

Higher algebraic stacks

Technical assumption: from now on, everything is assumed strongly quasi-compact (quasi-compact, quasi-separated . . .)

- Every algebraic n-stack can be resolved by a simplicial affine scheme (sAff).

$$X_0 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_2 \;\substack{\longleftarrow \\ \vdots \\ \longleftarrow}\; X_3 \;\substack{\longleftarrow \\ \vdots \\ \longleftarrow}\; \cdots,$$

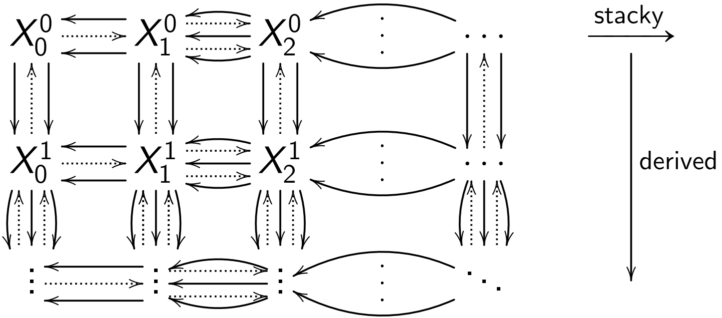
- Equivalently cosimplicial ring

$$A^0 \;\substack{\longrightarrow \\ \dashleftarrow \\ \longrightarrow}\; A^1 \;\substack{\longrightarrow \\ \dashleftarrow \\ \longrightarrow \\ \dashleftarrow \\ \longrightarrow}\; A^2 \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; A^3 \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; \cdots,$$

- Which simplicial affines arise this way?
- What about morphisms?

Derived n-stacks

- Algebraic derived n-stack resolved by a simplicial derived affine ($sd\mathrm{Aff}$).
- $d\mathrm{Aff}/\mathbb{Q} \simeq (CDG^{\leqslant 0}A_{\mathbb{Q}})^{\mathrm{opp}}$.
- Simplicial rings over any base, $\rightsquigarrow$ simplicial cosimplicial affine:



Aside: Can derived moduli *spaces* avoid simplices?

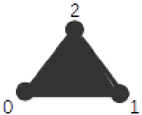
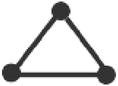
Could they have set-valued moduli functors?

Well, what could $\mathbb{A}^1$ give (as a functor on $CDG^{\leqslant 0}A$, say)?

- $A \mapsto A^0$ not homotopy-invariant.
- $A \mapsto \mathrm{H}^0 A$ not left-exact.
- Need $\pi_i \underline{\mathbb{A}}^1(A) = \mathrm{H}^{-i} A$.

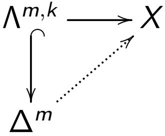
Simplices and horns

- m-simplex Δ^m is simplicial set with
$$\mathrm{Hom}_{s\mathrm{Set}}(\Delta^m, X) = X_m.$$
- Boundary $\partial\Delta^m = \bigcup_{i=0}^m \partial^i \Delta^{m-1} \subset \Delta^m$.
- kth horn $\Lambda^{m,k} = \bigcup_{\substack{i=0,\\ i\neq k}}^m \partial^i \Delta^{m-1}$.
- Partial matching objects
$$M_{\Lambda^{m,k}} X := \mathrm{Hom}_{s\mathrm{Set}}(\Lambda^{m,k}, X) =$$
$$\{x \in \prod_{\substack{0\leqslant i\leqslant m\\ i\neq k}} X_{m-1} \ : \ \partial_i x_j = \partial_j x_{i+1}, \forall i \geqslant j\}.$$

	n=0	n=1	n=2
$\lvert\Delta^n\rvert$			
$\lvert\partial\Delta^n\rvert$	Ø		
$\lvert\Lambda^{n,0}\rvert$	N/A		
$\lvert\Lambda^{n,1}\rvert$	N/A		
$\lvert\Lambda^{n,2}\rvert$	N/A	N/A	

Duskin–Glenn n-hypergroupoids

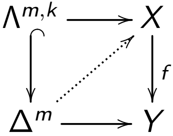
- Horn-fillers



$$X_m \to M_{\Lambda^{m,k}} X$$

are surjective for all m, k, and isomorphisms for $m > n$.

- Relative n-hgpds X/Y:



$$X_m \to M_{\Lambda^{m,k}} X \times_{(M_{\Lambda^{m,k}} Y)} Y_m$$

are surjective for all m, k, and isomorphisms for $m > n$.

- 1-hgpds are nerves of groupoids.
- Relative 0-hgpds are Cartesian:

$$X_m \cong X_0 \times_{Y_0} Y_m.$$

- n-hgpd determined by $X_{\leqslant n+1}$, but have to check conditions at X_{n+2}.
- $n = 1$ case: objects X_0, morphisms X_1, composition $X_{\leqslant 2}$, associativity $X_{\leqslant 3}$.
- Relative $n = 0$ case: $f_0 \colon X_0 \to Y_0$ gives fibres, f_1 gluing data, f_2 cocycle condition.

n-stacks the Grothendieck way

- Can define n-hypergroupoids in any category $\mathcal{A}$ with finite limits and a class $\mathbf{C}$ of covering maps.
- Affine schemes and smooth/étale surjections $\rightsquigarrow$ Artin/DM n-hgpds.

Theorem [P] n-geometric $\substack{\text{Artin}\\\text{DM}}$ stacks $\leftrightarrow$ hypersheafifications $X^\sharp$ of $\substack{\text{Artin}\\\text{DM}}$ n-hgpds X.

[HAG2 n-geometric stacks ($\mathfrak{X} \to \mathfrak{X}^{S^{n-1}}$ affine)

$\subset$ Lurie n-stacks ($\mathfrak{X} \simeq \mathfrak{X}^{S^{n+1}}$) $\subset$ $(n+2)$-geom stacks]

Derived n-geometric stacks

- Subtleties: replace isos with quasi-isos,
  - require Reedy fibrant: matching maps $X_m \to M_{\partial\Delta^m} X$ fibrations (i.e. quasi-free)
  - alternatively, use homotopy limits.
- Derived $\substack{\text{Artin}\\\text{DM}}$ n-hgpd in $sd\mathrm{Aff}$ is Reedy fibrant with $\substack{\text{smooth}\\\text{étale}}$ horn-fillers.
- (HAG2): $A_\bullet \to B_\bullet$ smooth/étale if $\mathrm{H}_0 A \to \mathrm{H}_0 B$ is so, and $\mathrm{H}_* B \cong \mathrm{H}_* A \otimes_{\mathrm{H}_0 A} \mathrm{H}_0 B$.

Theorem [P] Derived n-geometric $\substack{\text{Artin}\\\text{DM}}$ stacks $\leftrightarrow X^\sharp$ for derived $\substack{\text{Artin}\\\text{DM}}$ n-hgpds X.

Example: Reedy fibrant replacement of $\mathbb{A}^1$

- Need $X_\bullet \in sd\mathrm{Aff}$ with (i) $\mathbb{A}^1 \xrightarrow{\sim} X_m$, and (ii) $X_m \to M_{\partial\Delta^m}X$ quasi-free ($m = 1$ is $X_1 \to X_0 \times X_0$).
- Let $NC_\bullet(\Delta^m, k)$ be normalised chains (gen'd by non-deg simplices).
- Set $X_m = \operatorname{Spec} k[NC_\bullet(\Delta^m, k)]$.
- (i) $NC_\bullet(\Delta^m, k) \simeq k$, and (ii) $NC_\bullet(\partial\Delta^m, k) \hookrightarrow NC_\bullet(\Delta^m, k)$.

Sketch proof of theorem

- $\rightarrow$ Given n-hgpd $X_\bullet$, define $\mathrm{Dec}_+ X$ by $(\mathrm{Dec}_+ X)_m := X_{m+1}$. Then $\partial_{\bullet+1} \colon (\mathrm{Dec}_+ X)_\bullet \to X_\bullet$ is relative $(n-1)$-hgpd, and $(\mathrm{Dec}_+ X)^\sharp_\bullet \simeq X_0$. Thus $X_0 \to X^\sharp$ is n-atlas, by induction.
- $\leftarrow$ Complicated induction, $2^n - 1$ steps ($n = 1$ is just Čech nerve). If $\mathfrak{X}_{\leqslant n}$ affine, can resolve $\mathfrak{X}_{n+1} \to M_{\partial\Delta^{n+1}}\mathfrak{X}$, upsetting $\mathfrak{X}_{\leqslant n}$ (so $f(n+1) = f(n) + 1 + f(n)$).

Aside: dg schemes

- Semi-separated dg scheme $X = (X^0, \mathscr{O}_X) \rightsquigarrow$ derived Zariski 1-hgpd, by Reedy fibrant replacement of

$$\check{X}_i := \mathrm{Spec}\, \Gamma(\check{X}^0{}_i, \mathscr{O}_X),$$

for Čech nerve $\check{X}^0$ of X^0.

- $\check{X}_\bullet$ quasi-isomorphic to completion along $\pi^0 X = \mathbf{Spec}_{(X^0)} \mathrm{H}^0 \mathscr{O}_X$ if Noetherian.

Aside: derived schemes

- Derived scheme is derived Artin/DM n-stack $\mathfrak{X}$ with underived truncation $\pi^0\mathfrak{X} \simeq Y$, a scheme.
- No ambient scheme (unlike dg schemes).
- When Y semi-separated, $\mathfrak{X}$ given by CDGAs $\mathscr{A}^{\leqslant 0}$ on $\check{Y}$ with $\mathrm{H}^0\mathscr{A} = \mathscr{O}_Y$ and $\mathrm{H}^*\mathscr{A}$ Cartesian/quasi-coherent. Zariski 1-hgpd is fibrant replacement of

$$\check{X}_i := \mathrm{Spec}\,\Gamma(\check{Y}_i, \mathscr{A}^\bullet).$$

Trivial n-hypergroupoids

To calculate morphisms or sheaves (functor of points), we need to refine atlases.

X is a trivial n-hgpd over Y if the matching maps

$$X_m \to M_{\partial\Delta^m} X \times_{(M_{\partial\Delta^m} Y)} Y_m \qquad \begin{array}{ccc} \partial\Delta^m & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^m & \longrightarrow & Y \end{array}$$

are surjective for all m and isos for $m \geqslant n$.

[Thus determined by $X_{<n} \to Y_{<n}$.]

$\substack{\text{Smooth} \\ \text{Étale}}$ surjections $\rightsquigarrow$ trivial $\substack{\text{Artin} \\ \text{DM}}$ n-hgpds.

Main theorem

Theorem (P)

The ∞-category of $(n, \mathbf{P})$-geometric stacks is the localisation of the category of $(n, \mathbf{P})$-hypergroupoids with respect to trivial $(n, \mathbf{P})$-hypergroupoids.

- $\mathbf{P}$ any property like (derived) Artin/DM.

Morphisms

More explicitly, for Y a (derived) Artin n-hgpd, mapping space is

$$\mathrm{map}(X^\sharp, Y^\sharp)_m = \varinjlim_{\alpha} \mathrm{Hom}(X_\alpha \times \Delta^m, Y)$$

for $\{X_\alpha \to X\}_\alpha$ any weakly initial system ($\approx$universal cover) of trivial (derived) DM n-hgpds.

N.B. $Y^\sharp(A) = \mathrm{map}(\mathrm{Spec}\ A, Y)$.

Sheaves

- Complexes $\mathscr{F}_\bullet$ of $\mathscr{O}_X$-mods on $X^\sharp$ with qu-coh homology are *quasi-Cartesian* complexes of qu-coh sheaves on X.
- Given by complexes $\mathscr{F}_\bullet(X_m)$ of $O(X_m)$-mods and compatible quasi-isos

$$\partial^i \colon\ \partial_i^* \mathscr{F}_\bullet(X_m) \to \mathscr{F}_\bullet(X_{m+1}).$$

- Same definition works for any sheaves satisfying descent w.r.t. smooth/étale morphisms.

Stacks in other settings

- Zhu's Lie *n*-groupoids are *n*-hgpds in manifolds, w.r.t. smooth submersions.
- $\mathbb{C}$-manifolds work just as well.
- $\mathcal{C}^\infty$-rings for singular Lie *n*-gpds.
- Simplicial/dg analytic rings for derived analytic geometry.
- Poisson DGAs for Poisson str. on DM stacks (pro-Poisson needed for Artin).
- Pro-Artinian rings for formal stacks.
- ind(dAff) for de Rham stack etc.

Deformations

How to deform a (derived) Artin n-geometric stack $\mathfrak{X}$?

1. Write $\mathfrak{X} = X^\sharp$, X an Artin n-hgpd.
2. Calculate deformations of X.
3. Answer is invariant under trivial Artin n-hgpds $\tilde{X} \to X$.
4. Get cotangent complex (quasi-Cartesian)

$$\mathbb{L}^X = \mathrm{Tot}(\Omega^1_X \to i^*\Omega^1_{X^{\Delta^1}} \to i^*\Omega^1_{X^{\Delta^2}} \to \ldots).$$