A concrete approach to higher and derived stacks

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Back to first principles

Building blocks:
- commutative rings (classical AG)
- dg/simplicial/$E_\infty$-rings (derived AG)
- or non-commutative, analytic, $C^\infty$, . . .

Affine schemes give “wrong” colimits:

$$
\begin{align*}
\mathbb{G}_m & \xrightarrow{z} \mathbb{A}^1 \\
\mathbb{A}^1 & \xrightarrow{z^{-1}} \mathbb{A}^1
\end{align*}
$$

$$
\mathbb{A}^0 \rightarrow \text{Spec } \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})
$$

Need ambient category to glue/quotient.
- Schemes $\subset$ ringed spaces
- Algebraic stacks $\subset$ functors on Aff
- Quasi-coherent sheaves $\subset\mathcal{O}_X$-modules (enough injectives)

Nice categories, with many nasty objects.

- Who cares about arbitrary sheaves on the big affine site?
- Or about arbitrary $\mathcal{O}_X$-modules?

Are there smaller ambient categories?
Čech nerves

For affine presentation $U \to \mathcal{X}$ of Artin stack with affine diagonal, the Čech nerve

$$\check{\mathcal{X}}_n := \underbrace{U \times \mathcal{X} U \times \mathcal{X} \ldots \times \mathcal{X} U}_{n+1}$$

recovers $\mathcal{X}$, $H^*(\mathcal{X}, \mathcal{O}_\mathcal{X})$ from affine diagram.

- Quasi-compact, semi-separated scheme $X$, affine cover $\{U_i\}_i$, take $U = \bigsqcup_i U_i$, then

$$\check{X}_n = \bigsqcup_{i_0, \ldots, i_n} U_{i_0} \cap \ldots \cap U_{i_n}.$$
Simplicial objects

- $|\Delta^n| := \{x \in \mathbb{R}^{n+1}_+ : \sum_{i=0}^n x_i = 1\}$.

- For topological space $X$, $\text{Sing}(X)_n := \text{Hom}(|\Delta^n|, X)$, so

$$\text{Sing}(X)_0 \xleftarrow{\partial_1} \text{Sing}(X)_1 \xleftarrow{\sigma_0} \text{Sing}(X)_0 \xleftarrow{\partial_0} \text{Sing}(X)_1 \xleftarrow{\sigma_1} \text{Sing}(X)_2 \ldots,$$

relations like $\partial_i \sigma_i = \text{id}$.

- Any diagram of this form is called simplicial.
Higher algebraic stacks

Technical assumption: from now on, everything is assumed strongly quasi-compact (quasi-compact, quasi-separated . . . )

- Every algebraic $n$-stack can be resolved by a simplicial affine scheme ($s$Aff).

\[
X_0 \leftarrow X_1 \leftarrow X_2 \cdots X_3 \cdots 
\]

- Equivalently cosimplicial ring

\[
A^0 \leftarrow A^1 \leftarrow A^2 \cdots A^3 \cdots 
\]

- Which simplicial affines arise this way?
- What about morphisms?
Derived $n$-stacks

- Algebraic derived $n$-stack resolved by a simplicial derived affine ($sd\text{Aff}$).
- $d\text{Aff}/\mathbb{Q} \simeq (\text{CDG}^{\leq 0} A_{\mathbb{Q}})^{\text{opp}}$.
- Simplicial rings over any base, $\rightsquigarrow$ simplicial cosimplicial affine:

\[
\begin{array}{cccccc}
X_0^0 & \hookrightarrow & X_1^0 & \hookrightarrow & X_2^0 & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
X_0^1 & \hookleftarrow & X_1^1 & \hookleftarrow & X_2^1 & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\vdots & & \vdots & & \vdots & \\
\end{array}
\]
Aside: Can derived moduli spaces avoid simplices?

Could they have set-valued moduli functors?

Well, what could $\mathbb{A}^1$ give (as a functor on $\text{CDG}^{\leq 0} A$, say)?

- $A \mapsto A^0$ not homotopy-invariant.
- $A \mapsto H^0 A$ not left-exact.
- Need $\pi_i \mathbb{A}^1 (A) = H^{-i} A$. 
Simplices and horns

- $m$-simplex $\Delta^m$ is simplicial set with
  $$\text{Hom}_{\text{sSet}}(\Delta^m, X) = X_m.$$  

- Boundary $\partial \Delta^m = \bigcup_{i=0}^{m} \partial^i \Delta^{m-1} \subset \Delta^m$.

- $k$th horn $\Lambda^{m,k} = \bigcup_{i=0, \ i \neq k}^{m} \partial^i \Delta^{m-1}$.

- Partial matching objects
  $$M_{\Lambda^{m,k}} X := \text{Hom}_{\text{sSet}}(\Lambda^{m,k}, X) = \left\{ x \in \prod_{0 \leq i \leq m}^{i \neq k} X_{m-1} : \partial_i x_j = \partial_j x_{i+1}, \ \forall i \geq j \right\}.$$
| $|\Delta^n|$ | $|\partial \Delta^n|$ | $|\Lambda^{n,0}|$ | $|\Lambda^{n,1}|$ | $|\Lambda^{n,2}|$ |
|-------|-------------|-------------|-------------|-------------|
| $n=0$ | 0           | N/A         | N/A         | N/A         |
|       | $\varnothing$ |             |             |             |
| $n=1$ | 0-1         |             |             |             |
|       | 2           |             |             |             |
| $n=2$ | 0-1         | 0-1         | N/A         | N/A         |
Duskin–Glenn $n$-hypergroupoids

- Horn-fillers
  \[ X_m \rightarrow M_{\Lambda^m,k} X \]
  are surjective for all $m, k$, and isomorphisms for $m > n$.

- Relative $n$-hgpds $X/Y$:
  \[ X_m \rightarrow M_{\Lambda^m,k} X \times (M_{\Lambda^m,k} Y) Y_m \]
  are surjective for all $m, k$, and isomorphisms for $m > n$. 
1-hgpds are nerves of groupoids.

Relative 0-hgpds are Cartesian:

\[ X_m \cong X_0 \times_{Y_0} Y_m. \]

\[ n \)-hgpds determined by \( X_{\leq n+1} \), but have to check conditions at \( X_{n+2} \).

\[ n = 1 \) case: objects \( X_0 \), morphisms \( X_1 \), composition \( X_{\leq 2} \), associativity \( X_{\leq 3} \).

Relative \( n = 0 \) case: \( f_0 : X_0 \to Y_0 \) gives fibres, \( f_1 \) gluing data, \( f_2 \) cocycle condition.
$n$-stacks the Grothendieck way

- Can define $n$-hypergroupoids in any category $\mathcal{A}$ with finite limits and a class $\mathcal{C}$ of covering maps.
- Affine schemes and smooth/étale surjections $\leadsto$ Artin/DM $n$-hgpds.

**Theorem [P]** $n$-geometric Artin$_{DM}$ stacks $\leftrightarrow$ hypersheafifications $X^\#$ of Artin$_{DM}$ $n$-hgpds $X$.

[HAG2 $n$-geometric stacks ($\mathcal{X} \to \mathcal{X}^{S^{n-1}}$ affine)]

$\subset$ Lurie $n$-stacks ($\mathcal{X} \simeq \mathcal{X}^{S^{n+1}}$) $\subset$ $(n + 2)$-geom stacks]
Derived $n$-geometric stacks

- Subtleties: replace isos with quasi-isos,
  - require Reedy fibrant: matching maps $X_m \to M_{\partial \Delta^m} X$ fibrations (i.e. quasi-free)
  - alternatively, use homotopy limits.

- Derived $\text{Artin}^\text{DM}_n$ $n$-hgpd in $sd\text{Aff}$ is Reedy fibrant with smooth/étale horn-fillers.

- (HAG2): $A_\bullet \to B_\bullet$ smooth/étale if $H_0 A \to H_0 B$ is so, and $H_* B \cong H_* A \otimes_{H_0 A} H_0 B$.

**Theorem [P]** Derived $n$-geometric $\text{Artin}^\text{DM}_n$ stacks $\leftrightarrow X^\#$ for derived $\text{Artin}^\text{DM}_n$ $n$-hgpd $X$. 
Example: Reedy fibrant replacement of $\Delta^1$

- Need $X_* \in sd\text{Aff}$ with (i) $\Delta^1 \xrightarrow{\sim} X_m$, and (ii) $X_m \to M_{\partial \Delta^m} X$ quasi-free ($m = 1$ is $X_1 \to X_0 \times X_0$).
- Let $NC_* (\Delta^m, k)$ be normalised chains (gen’d by non-deg simplices).
- Set $X_m = \text{Spec} k[NC_* (\Delta^m, k)]$.
- (i) $NC_* (\Delta^m, k) \xrightarrow{\sim} k$, and (ii) $NC_* (\partial \Delta^m, k) \hookrightarrow NC_* (\Delta^m, k)$.
Sketch proof of theorem

→ Given $n$-hgpds $\mathcal{X}_\bullet$, define $\text{Dec}_+ \mathcal{X}$ by
\[(\text{Dec}_+ \mathcal{X})_m \coloneqq \mathcal{X}_{m+1}.\]
Then $\partial_{\bullet+1} : (\text{Dec}_+ \mathcal{X})_\bullet \to \mathcal{X}_\bullet$ is relative
$(n - 1)$-hgpds, and $(\text{Dec}_+ \mathcal{X})_\bullet^\# \simeq \mathcal{X}_0$. Thus $\mathcal{X}_0 \to \mathcal{X}^\#$ is $n$-atlas, by induction.

← Complicated induction, $2^n - 1$ steps
($n = 1$ is just Čech nerve). If $\mathcal{X}_{\leq n}$ affine, can resolve $\mathcal{X}_{n+1} \to M_{\partial \Delta^{n+1}} \mathcal{X}$, upsetting $\mathcal{X}_{\leq n}$ (so $f(n + 1) = f(n) + 1 + f(n)$).
Aside: dg schemes

- Semi-separated dg scheme

\[ X = (X^0, \mathcal{O}_X) \leadsto \text{derived Zariski 1-hgpd, by Reedy fibrant replacement of} \]

\[ \hat{X}_i := \text{Spec } \Gamma(\hat{X}^0_i, \mathcal{O}_X), \]

for Čech nerve \( \hat{X}^0 \) of \( X^0 \).

- \( \hat{X}_\bullet \) quasi-isomorphic to completion along

\[ \pi^0 X = \text{Spec } (X^0) \mathcal{H}^0 \mathcal{O}_X \] if Noetherian.
Aside: derived schemes

- Derived scheme is derived Artin/DM $n$-stack $\mathfrak{X}$ with underived truncation $\pi^0\mathfrak{X} \simeq Y$, a scheme.
- No ambient scheme (unlike dg schemes).
- When $Y$ semi-separated, $\mathfrak{X}$ given by CDGAs $\mathcal{A}^{\leq 0}$ on $\tilde{Y}$ with $H^0\mathcal{A} = O_Y$ and $H^*\mathcal{A}$ Cartesian/quasi-coherent. Zariski 1-hgpd is fibrant replacement of

\[ \tilde{X}_i := \text{Spec } \Gamma(\tilde{Y}_i, \mathcal{A}^\bullet). \]
Trivial $n$-hypergroupoids

To calculate morphisms or sheaves (functor of points), we need to refine atlases. $X$ is a trivial $n$-hgpd over $Y$ if the matching maps

$$X_m \to M_{\partial \Delta^m X} \times (M_{\partial \Delta^m Y}) Y_m$$

are surjective for all $m$ and isos for $m \geq n$. [Thus determined by $X_{<n} \to Y_{<n}$.]

Smooth Étale surjections $\rightsquigarrow$ trivial $\text{Artin}_{\text{DM}} n$-hgpds.
Main theorem

Theorem (P)

The $\infty$-category of $(n, P)$-geometric stacks is the localisation of the category of $(n, P)$-hypergroupoids with respect to trivial $(n, P)$-hypergroupoids.

- $P$ any property like (derived) Artin/DM.
Morphisms

More explicitly, for $Y$ a (derived) Artin $n$-hgpds, mapping space is

$$\text{map}(X^\#, Y^\#)_m = \lim_{\alpha} \text{Hom}(X_\alpha \times \Delta^m, Y)$$

for $\{X_\alpha \to X\}_\alpha$ any weakly initial system ($\approx$ universal cover) of trivial (derived) DM $n$-hgpds.

N.B. $Y^\#(A) = \text{map}(\text{Spec } A, Y)$. 
Sheaves

- Complexes $\mathcal{F}_\bullet$ of $\mathcal{O}_X$-mods on $X^\#$ with qu-coh homology are quasi-Cartesian complexes of qu-coh sheaves on $X$.
- Given by complexes $\mathcal{F}_\bullet(X_m)$ of $\mathcal{O}(X_m)$-mods and compatible quasi-isos

$$\partial^i : \partial_i^* \mathcal{F}_\bullet(X_m) \to \mathcal{F}_\bullet(X_{m+1}).$$

- Same definition works for any sheaves satisfying descent w.r.t. smooth/étale morphisms.
Stacks in other settings

- Zhu’s Lie $n$-groupoids are $n$-hgpds in manifolds, w.r.t. smooth submersions.
- $\mathcal{C}$-manifolds work just as well.
- $C^\infty$-rings for singular Lie $n$-gpds.
- Simplicial/dg analytic rings for derived analytic geometry.
- Poisson DGAs for Poisson str. on DM stacks (pro-Poisson needed for Artin).
- Pro-Artinian rings for formal stacks.
- $\text{ind}(d\text{Aff})$ for de Rham stack etc.
Deformations

How to deform a (derived) Artin $n$-geometric stack $\mathcal{X}$?

1. Write $\mathcal{X} = X^\#$, $X$ an Artin $n$-hgpd.
2. Calculate deformations of $X$.
3. Answer is invariant under trivial Artin $n$-hgpd $\tilde{X} \to X$.
4. Get cotangent complex (quasi-Cartesian)

$$\mathbb{L}^X = \mathrm{Tot}(\Omega^1_X \to i^*\Omega^1_{X^{\Delta^1}} \to i^*\Omega^1_{X^{\Delta^2}} \to \ldots).$$