NOTES CHARACTERISING HIGHER AND DERIVED STACKS CONCRETELY

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Abstract. This is an informal summary of the main concepts in [Pri], based on notes of various seminars. It gives constructions of higher and derived stacks without recourse to the extensive theory developed by Toën, Vezzosi and Lurie. Explicitly, higher stacks are described in terms of simplicial diagrams of affine schemes, which are analogous to atlases for manifolds.

Introduction

The need for simplicial objects. Any scheme X gives rise to a functor from rings to sets, sending A to $X(A) = \mathrm{Hom}(\mathrm{Spec}\, A, X)$. Likewise, any algebraic stack gives a functor from rings to groupoids. When X is a moduli space or stack, the points of X have a geometric meaning. For instance

$$\mathcal{H}ilb_Y(A) = \{\text{closed subschemes of } Y \times \mathrm{Spec}\, A, \text{ flat over } A\}.$$

In derived algebraic geometry, the basic building blocks are simplicial rings, or equivalently in characteristic 0, dg rings. Thus derived moduli spaces and stacks have to give rise to functors on such derived rings $d\mathrm{Alg}$.To understand what such a functor must be, we start with a derived ring A and look at possible functors associated to $\mathrm{Spec}\, A$.

(1) The obvious candidate is the functor $\mathrm{Hom}_{d\mathrm{Alg}}(A, -)\colon d\mathrm{Alg} \to \mathrm{Set}$. This is clearly no good, as it does not map quasi-isomorphisms to isomorphisms, even when A is a polynomial ring.
(2) This suggests the functor $\mathrm{Hom}_{\mathrm{Ho}(d\mathrm{Alg})}(P, -)\colon d\mathrm{Alg} \to \mathrm{Set}$, where $\mathrm{Ho}(d\mathrm{Alg})$ is the homotopy category (given by formally inverting quasi-isomorphisms). This works well for infinitesimal derived deformation theory (as in [Man]), but is not left-exact. Thus it cannot sheafify, so will not give a good global theory.
(3) The solution is to look at the derived Hom-functor $\mathbf{R}\underline{\mathrm{Hom}}$, which takes values in simplicial sets (to be defined in the sequel). This maps quasi-isomorphisms to weak equivalences, and has good exactness properties. Thus even if we start with a moduli problem without automorphisms, the derived problem leads us to consider simplicial sets.

Where do simplicial objects come from? Simplicial resolutions of schemes will be familiar to anyone who has computed Čech cohomology. Given a quasi-compact semi-separated scheme Y, we may take a finite affine cover $U = \coprod_i U_i$ of Y, and define the simplicial scheme $\check{Y}$ to be the Čech nerve $\check{Y} := \mathrm{cosk}_0(U/Y)$. Explicitly,

$$\check{Y}_n = \overbrace{U \times_Y U \times_Y \ldots \times_Y U}^{n+1} = \coprod_{i_0,\ldots,i_n} U_{i_0} \cap \ldots \cap U_{i_n},$$

so $\check{Y}_n$ is an affine scheme, and $\check{Y}$ is the unnormalised Čech resolution of Y.

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Given a quasi-coherent sheaf $\mathscr{F}$ on Y, we can then form a cosimplicial complex $\check{C}^n(Y,\mathscr{F}) := \Gamma(\check{Y}_n,\mathscr{F})$, and of course Zariski cohomology is given by

$$\mathrm{H}^i(Y,\mathscr{F}) \cong \mathrm{H}^i\check{C}^\bullet(Y,\mathscr{F}).$$

Likewise, if $\mathfrak{Y}$ is a semi-separated Artin stack, we can choose a presentation $U \to \mathfrak{Y}$ with U affine, and set $\check{Y} := \mathrm{cosk}_0(U/\mathfrak{Y})$, so

$$\check{Y}_n = \overbrace{U \times_{\mathfrak{Y}} U \times_Y \ldots \times_Y U}^{n+1}.$$

Resolutions of this sort were used by Olsson in [Ols] to study quasi-coherent sheaves on Artin stacks.

Questions.

(1) Which simplicial affine schemes correspond to schemes, Artin stacks or Deligne–Mumford stacks in this way?

(2) What about higher stacks?

[For an example of a higher stack, moduli of perfect complexes $\mathscr{F}$ on X will give an n-stack provided we restrict to complexes with $\mathrm{Ext}^i_X(\mathscr{F},\mathscr{F}) = 0$ for all $i \leq -n$. Similarly, a 2-stack governs moduli of stacky curves.]

(3) What about derived schemes and stacks?

1. Hypergroupoids

1.1. Simplicial sets.

Definition 1.1. Define $|\Delta^n|$ to be the geometric n-simplex $\{x \in \mathbb{R}^{n+1}_+ : \sum_{i=0}^n x_i = 1\}$. Write $\partial^i \colon |\Delta^{n-1}| \to |\Delta^n|$ for the inclusion of the ith face, and $\sigma^i \colon |\Delta^{n+1}| \to |\Delta^n|$ for the map given by collapsing the edge $(i, i+1)$.

Definition 1.2. Given a topological space X, define $\mathrm{Sing}(X)_n$ to be the set of continuous maps from $|\Delta^n|$ to X. These fit into a diagram

$$\begin{tikzcd} \mathrm{Sing}(X)_0 \arrow[r, "\sigma_0", dotted] & \mathrm{Sing}(X)_1 \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] \arrow[r, shift left, dotted] \arrow[r, shift right, dotted] & \mathrm{Sing}(X)_2 \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] & \vdots & \mathrm{Sing}(X)_3 \arrow[ll, bend right] \arrow[ll, bend left] & \ldots & \ldots, \end{tikzcd}$$

where the arrows satisfy various relations such as $\partial_i\sigma_i = \mathrm{id}$. (Note that contravariance has turned superscripts into subscripts).

Any diagram of this form is called a simplicial set. We will denote the category of simplicial sets by $\mathbb{S}$. We can define simplicial diagrams in any category similarly, while cosimplicial diagrams are given by reversing all the arrows.

If $A_\bullet$ is a simplicial abelian group, then note that setting $d := \sum_{i=0}^n (-1)^i\partial_i \colon A_n \to A_{n-1}$ gives maps satisfying $d^2 = 0$, so $A_\bullet$ becomes a chain complex.

Definition 1.3. The combinatorial n-simplex $\Delta^n \in \mathbb{S}$ is characterised by the property that $\mathrm{Hom}_{\mathbb{S}}(\Delta^n, K) \cong K_n$ for all simplicial sets K. Its boundary $\partial\Delta^n \subset \Delta^n$ is given by $\bigcup_{i=0}^n \partial^i(\Delta^{n-1})$, and the kth horn $\Lambda^{n,k}$ is given by $\bigcup_{\substack{0\leq i\leq n\\ i\neq k}} \partial^i(\Delta^{n-1})$.

Definition 1.4. There is a geometric realisation functor $|-| \colon \mathbb{S} \to \mathrm{Top}$, left adjoint to Sing. This is characterised by the properties that it preserves colimits and that $|\Delta^n| = |\Delta^n|$.

Draw a picture of $|\Lambda^{2,k}|$ or $|\Lambda^{3,k}|$ and you will see the reasoning for both the term horn and the notation Λ.

Definition 1.5. A map $f \colon K \to L$ in $\mathbb{S}$ is said to be a weak equivalence if $|f| \colon |K| \to |L|$ is a weak equivalence (i.e. induces isomorphisms on all homotopy groups).

Note that the canonical maps $|\mathrm{Sing}(X)| \to X$ and hence $K \to \mathrm{Sing}(|K|)$ are always weak equivalences, so $\mathbb{S}$ and Top have the same homotopy theory.

1.2. **Hypergroupoids.**

Definition 1.6. A (Duskin–Glenn) n-hypergroupoid (often also called a weak n-groupoid) is an object $X \in \mathbb{S}$ for which the partial matching maps

$$X_m \to \mathrm{Hom}_{\mathbb{S}}(\Lambda^{m,k}, X)$$

are surjective for all $m \geq 1$, and isomorphisms for all $m > n$.

The first condition is equivalent to saying that X is a Kan complex, or fibrant.

Examples 1.7.
(1) A 0-hypergroupoid is just a set $X = X_0$.
(2) A 1-hypergroupoid X is the nerve $B\Gamma$ of a groupoid Γ, given by

$$(B\Gamma)_n = \coprod_{x_0,\ldots,x_n} \Gamma(x_0, x_1) \times \Gamma(x_1, x_2) \times \ldots \times \Gamma(x_{n-1}, x_n).$$

Thus Γ can be recovered by taking objects X_0, morphisms X_1, source and target $\partial_0, \partial_1 : X_1 \to X_0$, identity $\sigma_0 : X_0 \to X_1$ and multiplication

$$X_1 \times_{\partial_0, X_0, \partial_1} X_1 \xrightarrow{(\partial_2,\partial_0)^{-1}} X_2 \xrightarrow{\partial_1} X_1.$$

Equivalently, Γ is the fundamental groupoid $\pi_f X$ of X.
(3) Under the Dold-Kan correspondence between non-negatively graded chain complexes and abelian groups, n-hypergroupoids in abelian groups correspond to chain complexes concentrated in degrees $[0, n]$.

Properties 1.8.
(1) For an n-dimensional hypergroupoid X, $\pi_m X = 0$ for all $m > n$.
(2) [Pri, Lemma 2.14]: An n-hypergroupoid X is completely determined by its truncation $X_{\leq n+1}$. Explicitly, $X = \mathrm{cosk}_{n+1} X$, where the m-coskeleton $\mathrm{cosk}_m X$ is given by $(\mathrm{cosk}_m X)_i = \mathrm{Hom}((\Delta^i)_{\leq m}, X_{\leq m})$. Moreover, a simplicial set of the form $\mathrm{cosk}_{n+1} X$ is an n-hypergroupoid if and only if it satisfies the conditions of Definition 1.6 up to level $n + 2$.

When $n = 1$, these statements amount to saying that a groupoid is uniquely determined by its objects (level 0), morphisms and identities (level 1) and multiplication (level 2). However, we do not know we have a groupoid until we check associativity (level 3).

There is also a notion of relative n-hypergroupoids $X \to Y$, expressed in terms of (unique) liftings of

$$\begin{tikzcd} \Lambda^{m,k} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^m \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

For example, a relative 0-dimensional hypergroupoid $f \colon X \to Y$ is a Cartesian morphism, in the sense that the maps

$$X_n \xrightarrow{(\partial_i, f)} X_{n-1} \times_{Y_{n-1}, \partial_i} Y_n$$

are all isomorphisms. Given $y \in Y_0$, we can write $F(y) := f_0^{-1}\{y\}$, and observe that f is equivalent to a local system on Y with fibres F. The analogue of Property 1.8.2 above also holds in the relative case. Level 0 gives us the fibres $F(y)$, level 1 gives us the descent data $\theta(z) \colon F(\partial_0 z) \cong F(\partial_1 z)$ for $z \in Y_1$ (thereby determining the local system uniquely), but we do not know we have a groupoid until we check the cocycle condition (level 2): $\theta(\partial_2 w) \circ \theta(\partial_0 w) = \theta(\partial_1 w)$ for all $w \in Y_2$.

2. Higher stacks

We will now show how to develop the theory of higher Artin stacks. For other types of stack, just modify the notion of covering. In particular, for Deligne–Mumford stacks, replace "smooth" with "étale" throughout. For simplicity of exposition, we will assume that everything is quasi-compact, quasi-separated etc. (strongly quasi-compact in the terminology of [TV]) — to allow more general objects, replace affine schemes with arbitrary disjoint unions of affine schemes.

Given a simplicial set K and a simplicial affine scheme X, there is an affine scheme $\mathrm{Hom}_{\mathbb{S}}(K, X)$ with the property that for all rings A, $\mathrm{Hom}_{\mathbb{S}}(K, X)(A) = \mathrm{Hom}_{\mathbb{S}}(K, X(A))$. Explicitly, when $K = \Lambda^{m,k}$ this is given by the equaliser of a diagram

$$\prod_{\substack{0\le i\le m\\ i\neq k}} X_{m-1} \Longrightarrow \prod_{0\le i<j\le m} X_{m-2}.$$

Definition 2.1. Define an Artin n-hypergroupoid to be a simplicial affine scheme $X_\bullet$, such that the partial matching maps

$$X_m \to \mathrm{Hom}_{\mathbb{S}}(\Lambda^{m,k}, X)$$

are smooth surjections for all k, m, and isomorphisms for all $m > n$ and all k.

The idea of using such objects to model higher stacks is apparently originally due to Grothendieck, buried somewhere in [Gro].

Remark 2.2. Note that hypergroupoids can be defined in any category containing pullbacks along covering morphisms. Zhu uses this to define Lie n-groupoids (taking the category of manifolds, with coverings given by submersions). A similar approach could be used to define higher topological stacks, taking surjective local fibrations as the coverings in the category of topological spaces. Similar constructions could be made in non-commutative geometry and in synthetic differential geometry.

Given any Artin n-hypergroupoid X over R, there is an associated functor $X\colon \mathrm{Alg}_R \to \mathbb{S}$, given by $X(A)_n := X_n(A)$. The following is [Pri, Proposition 3.13 and Theorem 4.9]:

Theorem 2.3. *If X is an Artin n-hypergroupoid X over R, then its hypersheafification $X^\sharp\colon \mathrm{Alg}_R \to \mathbb{S}$ is an n-geometric Artin stack in the sense of* [TV, Definition 1.3.3.1]*. Every n-geometric Artin stack arises in this way.*

Remark 2.4. Beware that there are slight differences in terminology between [TV] and [Lur1]. In the former, only affine schemes are 0-representable, so arbitrary schemes might only be 2-geometric, while Artin stacks are 1-geometric stacks if and only if they have affine diagonal. In the latter, algebraic spaces are 0-stacks.

An n-stack $\mathfrak{X}$ in the sense of [Lur1] is called n-truncated in [TV], and can be characterised by the property that that $\pi_i(\mathfrak{X}(A)) = 0$ for all $i > n$ and $A \in \mathrm{Alg}_R$.

It follows easily that every n-geometric stack in [TV] is n-truncated; conversely, any n-truncated stack X is $(n+2)$-geometric. Any Artin stack with affine diagonal (in particular any separated algebraic space) is 1-geometric.

If we used algebraic spaces instead of affine schemes in Definition 2.1, then Proposition 2.3 would adapt to give a characterisation of n-truncated Artin stacks. Our main motivation for using affine schemes as the basic objects is that they will be easy to translate to a derived setting.

3. Morphisms and equivalences

Theorem 2.3 is all very well, but is clearly not the whole story. For a start, it gives us no idea of how to construct the hypersheafification. Thus we have no way of understanding morphisms between n-geometric stacks (as the hypersheafification is clearly not full), or even of knowing when two hypergroupoids will give us equivalent n-geometric stacks. If we think of the hypergroupoid as analogous to the atlas of a manifold, then we need a notion similar to refinement of an open cover.

3.1. **Trivial relative hypergroupoids.**

Definition 3.1. Say that a morphism $f\colon X \to Y$ of simplicial affine schemes is a trivial relative Artin (resp. Deligne–Mumford) n-hypergroupoid if the relative matching maps

$$X_m \to \mathrm{Hom}_{\mathbb{S}}(\partial\Delta^m, X) \times_{\mathrm{Hom}_{\mathbb{S}}(\partial\Delta^m, Y)} Y_m$$

are smooth (resp. étale) surjections for all m, and isomorphisms for all $m \geq n$.

An example of a trivial relative Artin 1-hypergroupoid in stacks is the Čech nerve $\check{Y} \to \mathfrak{Y}$ constructed in the introduction.

Property 3.2. [Pri, Lemma 2.11]: Trivial relative n-hypergroupoids are completely determined by their truncations in levels $< n$. Explicitly, a morphism $f : X \to Y$ is a trivial relative Artin (resp. Deligne–Mumford) n-hypergroupoid if and only if $X = Y \times_{\mathrm{cosk}_{n-1}Y} \mathrm{cosk}_{n-1}X$, and the $(n-1)$-truncated morphism $X_{<n} \to Y_{<n}$ satisfies the conditions of Definition 3.1 (up to level $n-1$).

3.2. **Sheafification and morphisms.**

Lemma 3.3. *If $f\colon X \to Y$ is a trivial relative Artin n-hypergroupoid, then for all rings A, the map $X^\sharp(A) \to Y^\sharp(A)$ is a weak equivalence in $\mathbb{S}$.*

Definition 3.4. Define the simplicial Hom functor on simplicial affine schemes by letting $\underline{\mathrm{Hom}}_{s\mathrm{Aff}}(X, Y)$ be the simplicial set given by

$$\underline{\mathrm{Hom}} s\mathrm{Aff}(X, Y)_n := \mathrm{Hom}_{s\mathrm{Aff}}(\Delta^n \times X, Y).$$

The following is [Pri, Corollary 5.6]:

Theorem 3.5. *If $X \in s\mathrm{Aff}$ and Y is an Artin n-hypergroupoid, then the derived* Hom *functor on hypersheaves is given by*

$$\mathbf{R}\underline{\mathrm{Hom}}(X^\sharp, Y^\sharp) \simeq \varinjlim \underline{\mathrm{Hom}}_{s\mathrm{Aff}}(\tilde{X}, Y),$$

where $\tilde{X} \to X$ runs over all trivial relative Artin n-hypergroupoids (or even just all trivial relative Deligne–Mumford n-hypergroupoids).

Given a ring A, set $X = \mathrm{Spec}\, A$, and note that $Y^\sharp(A) \simeq \mathbf{R}\underline{\mathrm{Hom}}(X^\sharp, Y^\sharp)$, so the theorem gives an explicit description of $Y^\sharp$. In fact, we can take the theorem to be a definition of sheafification, and even as a definition of morphisms in the simplicial category of n-geometric Artin stacks.

Even for classical Artin or Deligne–Mumford stacks, this gives a shorter (and arguably more satisfactory) definition, since they are just 1-truncated (see Remark 2.4) geometric stacks. For semi-separated algebraic spaces or schemes (0-truncated étale and Zariski 1-hypergroupoids, respectively), this definition is at least comparable in complexity to the classical one. Note that making use of Properties 1.8.2 and 3.2 allows us characterise all of these concepts in terms of finite diagrams of affine schemes.

3.3. **Eilenberg–Mac Lane spaces and cohomology.** Given any abelian group n, the Dold–Kan correspondence allows us to form a simplicial abelian group $K(A, n)$ given by denormalising the chain complex $A[-n]$. Explicitly, $K(A, n)$ is freely generated under degeneracy operations σ_i by a copy of A in level n, so $K(A, n)_m \cong A^{\binom{m}{n}}$.

Given a smooth commutative affine group scheme A, this construction gives us an Artin n-hypergroupoid $K(A,n)$ (in fact, if $n=1$, A need not be commutative), and then for any Artin stack $\mathfrak{X}$,

$$\mathrm{H}^{n-i}_{\text{ét}}(\mathfrak{X},A) \cong \pi_i \mathbf{R}\underline{\mathrm{Hom}}(X^\sharp, K(A,n)^\sharp).$$

In particular, taking $A=\mathbb{G}_a$ gives us $\mathrm{H}^*(\mathfrak{X},\mathscr{O}_{\mathfrak{X}})$, while taking $A=\mathbb{Z}/\ell^m$ (regarded as a finite scheme) gives $\mathrm{H}^*_{\text{ét}}(\mathfrak{X},\mathbb{Z}/\ell^m)$.

We could generalise this by allowing A to be a smooth commutative algebraic group space, in which case $K(A,n)$ would be an Artin n-hypergroupoid in algebraic spaces, making $K(A,n)^\sharp$ an n-truncated geometric stack (see Remark 2.4), and hence representable by an Artin $(n+2)$-hypergroupoid (in affine schemes).

4. Quasi-coherent sheaves

The following is [Pri, Corollary 6.7]:

Proposition 4.1. *For an Artin n-hypergroupoid X, giving a quasi-coherent module on the n-geometric stack $X^\sharp$ is equivalent to giving*

1. *a quasi-coherent sheaf $\mathscr{F}^n$ on X_n for each n, and*
2. *isomorphisms $\partial^i\colon \partial_i^*\mathscr{F}^{n-1} \to \mathscr{F}^n$ for all i and n, satisfying the usual cosimplicial identities.*

Given a morphism $f\colon X\to Y$ of Artin n-hypergroupoids, inverse images are easy to compute: we just have $(f^*\mathscr{F})^n := f_n^*\mathscr{F}^n$. Direct images are much harder to define, as taking f_* levelwise destroys the Cartesian property. See [Pri, Lemma 6.26] for an explicit description of the derived direct image functor $\mathbf{R}f_*^{\mathrm{cart}}$.

5. Derived stacks

Motivated by the need for good obstruction theory and cotangent complexes, derived algebraic geometry replaces rings with simplicial rings. There is a normalisation functor $N\colon s\mathrm{Alg}_R \to dg_+\mathrm{Alg}_R$ from simplicial R-algebras to commutative chain R-algebras in non-negative degrees. If R is a $\mathbb{Q}$-algebra, then N induces an equivalence on the homotopy categories (and also on the derived Hom-spaces). [When R is not a $\mathbb{Q}$-algebra, $dg_+\mathrm{Alg}_R$ is not even a model category.]

We will write $d\mathrm{Alg}$ for either of the categories $s\mathrm{Alg}_R, dg_+\mathrm{Alg}_R$, and write $d\mathrm{Aff}$ for the opposite category (derived affine schemes over R), denoting objects as $\mathrm{Spec}\, A$. Any object $A\in d\mathrm{Alg}$ can be thought of as essentially an exotic nilpotent thickening of H_0A, so there are equivalent variants of this theory replacing A with its localisation ([Pri, §8.2]), its henselisation ([Pri, §8.3]), or even (in Noetherian cases) its completion ([Pri, Propositions 8.6 and 8.36]) over H_0A.

Remark 5.1. The constructions in this section will work for any model category with a suitable notion of coverings. In particular, they work for symmetric spectra, the basis of a theory known as topological, spectral, brave new or (unfortunately) derived algebraic geometry. Roughly speaking (as explained in the introduction to [Lur2]), simplicial rings serve to apply homotopy theory to algebraic geometry, while symmetric spectra are used to do the opposite. For a detailed discussion, see [Lur1, §2.6].

5.1. **Derived hypergroupoids.**

Definition 5.2. Say that a morphism in $s\mathrm{Alg}_R$ is quasi-free if it is freely generated in each level, with the generators closed under the degeneracy operations σ_i. Say that a morphism in $dg_+\mathrm{Alg}_R$ is quasi-free if the underlying morphism of skew-commutative graded algebras is freely generated.

Say that a morphism in $d\mathrm{Alg}$ is a cofibration if it is a retract of a quasi-free map, and that a morphism in $d\mathrm{Aff}$ is a fibration if it is Spec of a cofibration.

Write $sd\mathrm{Aff}$ for the category of simplicial derived affine schemes, i.e. simplicial diagrams in $d\mathrm{Aff}$. Weak equivalences in this category are maps $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ inducing isomorphisms $\mathrm{H}_*(B) \cong \mathrm{H}_*(A)$.

Definition 5.3. An object $X_\bullet \in sd\mathrm{Aff}$ is said to be Reedy fibrant if the matching maps

$$X_n \to \mathrm{Hom}_{\mathbb{S}}(\partial\Delta^n, X)$$

are fibrations in $d\mathrm{Aff}$ for all n.

Example 5.4. Given a cofibration $R \to A$ in $s\mathrm{Alg}_R$, write $X = \mathrm{Spec}\, A$. We may form an object $A \otimes \Delta^n \in s\mathrm{Alg}_R$ by

$$(A \otimes \Delta^n)_i := \overbrace{A_i \otimes_R A_i \otimes_R \ldots \otimes_R A_i}^{\Delta^n_i}.$$

Then the simplicial derived affine scheme $\underline{X}$ given by $\underline{X}_n := \mathrm{Spec}\,(A \otimes \Delta^n)$ is Reedy fibrant, and $X \to \underline{X}$ is a weak equivalence levelwise.

$\underline{X}$ will be familiar to some readers from the construction of simplicial Hom, since by definition

$$\underline{\mathrm{Hom}}_{s\mathrm{Alg}_R}(A, B) = \underline{X}(B) \in \mathbb{S}$$

for $A \in s\mathrm{Alg}_R$. More generally, for any $A \in s\mathrm{Alg}_R$, we can always take a quasi-isomorphism $A' \to A$ for A' cofibrant, and then set

$$\mathbf{R}\underline{\mathrm{Hom}}_{s\mathrm{Alg}_R}(A, B) := \underline{\mathrm{Hom}}(A', B).$$

There are analogous constructions in $dg_+\mathrm{Alg}_R$, but they are not so easily described.

Definition 5.5. We say that a morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ in $d\mathrm{Aff}$ is a smooth (resp. étale) surjection if $\mathrm{Spec}\, \mathrm{H}_0 B \to \mathrm{Spec}\, \mathrm{H}_0 A$ is so, and the maps

$$\mathrm{H}_i(A) \otimes_{\mathrm{H}_0(A)} \mathrm{H}_0(B) \to \mathrm{H}_i(B)$$

are all isomorphisms.

Definition 5.6. A derived Artin n-hypergroupoid is a Reedy fibrant object $X_\bullet \in sd\mathrm{Aff}$ for which the partial matching maps

$$X_m \to \mathrm{Hom}_{\mathbb{S}}(\Lambda^{m,k} X)$$

are smooth surjections for all $m \geq 1$, and weak equivalences for all $m > n$.

Remark 5.7. Given any derived Artin n-hypergroupoid X, we may form a simplicial scheme $\pi^0 X$ by setting

$$\pi^0 X_n := \mathrm{Spec}\,(\mathrm{H}_0 O(X_n)) \in \mathrm{Aff}.$$

Then observe that $\pi^0 X$ is an Artin n-hypergroupoid, equipped with a map $\pi^0 X \to X$. We call this the underived part of X.

5.2. **Derived Artin stacks.** The following is [Pri, Proposition 7.6 and Theorem 7.7]:

Theorem 5.8. *If X is a derived Artin n-hypergroupoid X over R, then its hypersheafification $X^\sharp\colon d\mathrm{Alg}_R \to \mathbb{S}$ is an n-geometric derived Artin stack in the sense of* [TV, Definition 1.3.3.1]. *Every n-geometric derived Artin stack arises in this way.*

Remarks 5.9. As with Remarks 2.4, there is a difference in terminology between [Lur1] and [TV]. A geometric derived Artin ∞-stack $\mathfrak{X}$ is called an n-stack (Lurie) or n-truncated (Toën–Vezzosi) if the associated underived Artin ∞-stack $\pi^0\mathfrak{X}$ is n-truncated. This implies that for $A \in d\mathrm{Alg}$ with $\mathrm{H}_i(A) = 0$ for all $i > m$, we have $\pi_i(\mathfrak{X}(A)) = 0$ for all $i > m + n$.

5.3. Morphisms and equivalences.

Definition 5.10. A trivial relative derived Artin n-hypergroupoid is a morphism $X_\bullet \to Y_\bullet$ in $sd\mathrm{Aff}$ for which the matching maps

$$X_m \to \mathrm{Hom}_{\mathbb{S}}(\partial\Delta^m, X) \times_{\mathrm{Hom}_{\mathbb{S}}(\partial\Delta^m, Y)} Y_m$$

are smooth surjective fibrations for all $m > 1$ and weak equivalences for $m \geq n$.

The results of §3.2 all now carry over. In particular, we can define Hom-spaces $\underline{\mathrm{Hom}}_{sd\mathrm{Aff}}$, and then the following is [Pri, Theorem 7.10]:

Theorem 5.11. *If $X \in sd\mathrm{Aff}$ and Y is a derived Artin n-hypergroupoid, then the derived* Hom *functor on hypersheaves is given by*

$$\mathbf{R}\underline{\mathrm{Hom}}(X^\sharp, Y^\sharp) \simeq \varinjlim \underline{\mathrm{Hom}}_{sd\mathrm{Aff}}(\tilde{X}, Y),$$

where $\tilde{X} \to X$ runs over all trivial relative derived Artin n-hypergroupoids (or even just all trivial relative derived Deligne–Mumford n-hypergroupoids).

Given a derived R-algebra A, set $X = \mathrm{Spec}\, A$, and note that $Y^\sharp(A) \simeq \mathbf{R}\underline{\mathrm{Hom}}(X^\sharp, Y^\sharp)$, so the theorem gives an explicit description of $Y^\sharp$. In fact, we can take the theorem to be a definition of sheafification, and even as a definition of the simplicial category of n-geometric derived Artin stacks.

5.4. **Quasi-coherent complexes.** Since the basic building blocks for derived algebraic geometry are simplicial rings or chain algebras, the correct analogue of quasi-coherent sheaves has to involve complexes.

Definition 5.12. Given a chain algebra A, an A-module M in complexes is a (possibly unbounded) chain complex M equipped with a distributive chain morphism $A \otimes M \to M$. Given a simplicial ring A, we just define an A-module in complexes to be an NA-module in complexes, where NA is the chain algebra given by Dold–Kan normalisation.

The following is [Pri, Proposition 9.3]:

Proposition 5.13. *For a derived Artin n-hypergroupoid X, giving a quasi-coherent complex (in the sense of* [Lur1, §5.2]*) on the n-geometric derived stack $X^\sharp$ is equivalent (up to quasi-isomorphism) to giving*

1. *an $O(X_n)$-module $\mathscr{F}^n$ in complexes for each n, and*
2. *quasi-isomorphisms $\partial^i \colon \partial_i^* \mathscr{F}^{n-1} \to \mathscr{F}^n$ for all i and n, satisfying the usual cosimplicial identities.*

In broad terms, quasi-coherent complexes correspond to complexes $\mathscr{F}_\bullet$ of *presheaves* of $\mathscr{O}_X$-modules whose homology *presheaves* $\mathscr{H}_n(\mathscr{F}_\bullet)$ are quasi-coherent. To understand how these are related to complexes of quasi-coherent sheaves on schemes, see [Pri, Remarks 6.11].

As in §4, inverse images of quasi-coherent complexes are easy to compute, while derived direct images are more complicated — see [Pri, Definition 9.10].

References

[Gro] A. Grothendieck. Pursuing stacks. unpublished manuscript, 1983.

[Lur1] J. Lurie. *Derived Algebraic Geometry.* PhD thesis, M.I.T., 2004. www.math.harvard.edu/~lurie/papers/DAG.pdf or http://hdl.handle.net/1721.1/30144.

[Lur2] Jacob Lurie. Derived algebraic geometry V: Structured spaces. arXiv:0905.0459v1 [math.CT], 2009.

[Man] Marco Manetti. Extended deformation functors. *Int. Math. Res. Not.*, (14):719–756, 2002.

[Ols] Martin Olsson. Sheaves on Artin stacks. *J. Reine Angew. Math.*, 603:55–112, 2007.

[Pri] J. P. Pridham. Presenting higher stacks as simplicial schemes. arXiv:0905.4044v2 [math.AG], submitted, 2009.

[TV] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008. arXiv math.AG/0404373 v7.