

PRESENTING HIGHER STACKS AS SIMPLICIAL SCHEMES

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ABSTRACT. We show that an n -geometric stack may be regarded as a special kind of simplicial scheme, namely a Duskin n -hypergroupoid in affine schemes, where surjectivity is defined in terms of covering maps, yielding Artin n -stacks, Deligne–Mumford n -stacks and n -schemes as the notion of covering varies. This formulation adapts to all HAG contexts, so in particular works for derived n -stacks (replacing rings with simplicial rings). We exploit this to describe quasi-coherent sheaves and complexes on these stacks, and to draw comparisons with Kontsevich’s dg-schemes. As an application, we show how the cotangent complex controls infinitesimal deformations of higher and derived stacks.

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INTRODUCTION

Although the usual approach to defining n -stacks ([TV2] and [Lur1]) undoubtedly yields the correct geometric objects, it has numerous drawbacks. The inductive construction does not lend itself easily to calculations, while the level of abstract homotopy theory involved can make n -stacks seem inaccessible to many. In this paper, we introduce a far more elementary concept, namely a Duskin–Glenn n -hypergroupoid in affine schemes, and show how it is equivalent to the concept of n -geometric stacks introduced in [TV2]. According to [Toë1], this is essentially the formulation of higher stacks originally envisaged by Grothendieck in [Gro2]. It is also closely related to Zhu’s Lie n -groupoids.

In [Gle], Glenn defined an n -dimensional Kan hypergroupoid to be a simplicial set $X \in \mathbb{S}$ for which the horn fillers all exist, and are moreover unique in levels greater than n . Explicitly, the k th horn $\Lambda_k^m \subset \Delta^m$ is defined by deleting the interior and the k th face, and he required that the map

$$\mathrm{Hom}_{\mathbb{S}}(\Delta^m, X) \rightarrow \mathrm{Hom}_{\mathbb{S}}(\Lambda_k^m, X)$$

should be surjective for all k, m , and an isomorphism for $m > n$. When $n = 0$, this gives sets (with constant simplicial structure), and for $n = 1$, the simplicial sets arising in this way are precisely nerves of groupoids. This description is not as complicated as it first seems, since it suffices to truncate X at the $(n + 2)$ th level (Lemma 2.12). The combinatorial properties of these hypergroupoids were extensively studied by Glenn in [Gle].

It is well-known that Artin 1-stacks can be resolved by simplicial schemes, and this idea was exploited in [Ols1], [Ols2], [Aok1] and [Aok2] to study their quasi-coherent sheaves and deformations. It is therefore natural to expect that simplicial resolutions should exist for n -geometric Artin stacks. In Theorem 4.7, we show that every (quasi-compact, quasi-separated, ...) n -geometric Artin stack \mathfrak{X} (as defined in [TV2]) can be resolved by a simplicial affine scheme X , which is an Artin n -hypergroupoid in the sense that the morphism

$$\mathrm{Hom}_{\mathbb{S}}(\Delta^m, X) \rightarrow \mathrm{Hom}_{\mathbb{S}}(\Lambda_k^m, X)$$

of affine schemes is a smooth surjection for all k, m , and an isomorphism for $m > n$. For n -geometric Deligne–Mumford stacks or n -geometric schemes there are similar statements, replacing smooth morphisms with étale morphisms or local isomorphisms. In particular, this means that any functor satisfying the conditions of Lurie’s Representability Theorem ([Lur1] Theorem 7.1.6) gives rise to such a simplicial scheme.

If \mathfrak{X} is a quasi-compact semi-separated scheme, this resolution just corresponds to constructing the Čech nerve of an affine cover. We may therefore think of Artin n -hypergroupoids as being analogues of atlases on manifolds. While they share with atlases an amenability to calculation, they also share the disadvantage of not being canonical. However, there is a notion of trivial relative Artin n -hypergroupoids $X' \rightarrow X$, which is analogous to refinement of an atlas, and we may regard two Artin n -hypergroupoids as being equivalent if they admit a common refinement of this type (Theorem 4.10). In this way, we can recover the entire ∞ -category of n -geometric Artin stacks (Theorem 4.15).

In fact, Theorem 4.15 gives a much more general statement, relating geometric stacks in any HAG context to the corresponding hypergroupoids. In particular, replacing rings with simplicial rings or dg-rings allows us to define derived Artin n -hypergroupoids, which are

then equivalent to the D^- geometric n -stacks of [TV2]. Čech nerves allow us to make close comparisons between derived Artin stacks and Kontsevich's dg-schemes (Remark 6.41.1).

Quasi-coherent sheaves on an n -geometric stack then just correspond to Cartesian quasi-coherent sheaves on the associated n -hypercgroupoid for any HAG context (Corollary 5.12). This facilitates a relatively simple description of the cotangent complex, and our main application of this theory is to describe infinitesimal deformations of n -geometric stacks in terms of the cotangent complex (Theorem 8.8).

The structure of the paper is as follows. Sections 1 and 2 are mostly a recapitulation of background material, with a few results proved in §2 for which the author does not know of any other reference. Section 3 introduces the main objects to be used in the paper, n -hypercgroupoids in model categories (with the Artin n -hypercgroupoids described above as a special case), and establishes their basic properties.

The technical heart of the paper is in Section 4, where we establish an equivalence between the simplicial categories of n -geometric stacks and of affine n -hypercgroupoids. The crucial result is Theorem 4.7, which uses an intricate induction taking $2^n - 1$ steps to construct an n -hypercgroupoid resolving a given n -geometric stack. Readers unfamiliar with the n -geometric stacks of [TV2] can skip most of this section and §1, instead just using Theorem 4.10 to define the stack $|Y|$ associated to an n -hypercgroupoid Y as the functor

$$\mathbf{R}\underline{\mathrm{Hom}}(-, |Y|) : \mathrm{Aff}^{\mathrm{opp}} \rightarrow \mathbb{S},$$

and also to define the simplicial category of geometric stacks. Anyone wishing to make comparisons with [Lur1] should also read Remark 1.27 for differences in terminology.

Section 5 is dedicated to studying quasi-coherent sheaves. The equivalence between Cartesian quasi-coherent sheaves on an n -hypercgroupoid and quasi-coherent sheaves on the associated n -geometric stack amounts to little more than cohomological descent. Inverse images of sheaves are easily understood in terms of this comparison. However, (derived) direct images prove far more complicated, and an explicit description is given in §5.4.3.

In Section 6, various alternative formulations of derived Artin hypercgroupoids are developed. One of these builds on the Quillen equivalence in characteristic zero between simplicial algebras and dg algebras, permitting comparisons with the dg-schemes of [CFK]. The other formulations are based on the observation that in the homotopy category of simplicial rings, every object R can be expressed as a filtered homotopy limit of homotopy nilpotent extensions of the discrete ring $\pi_0 R$. This allows us to replace any cosimplicial affine scheme X with Zariski or étale neighbourhoods X^l or X^h (or sometimes even the formal neighbourhood \hat{X}) of $\pi^0 X$ in X .

Section 7 looks at the results of Section 5 in the context of derived Artin stacks. It then applies them to construct the relative cotangent complex of a morphism in §7.1, compatibly with existing characterisations (Corollary 7.14 and Proposition 7.21).

In §8, we then show how the cotangent complex governs deformations. We first consider deformations of morphisms of derived Artin stacks in Theorem 8.5, generalising the deformations of 1-morphisms of Artin 1-stacks considered in [Aok2]. We then consider deformations of derived Artin stacks in Theorem 8.8, generalising the deformations of Artin 1-stacks considered in [Aok1].

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Notation and conventions. We will denote the category of simplicial sets by \mathbb{S} . A simplicial category is a category enriched in \mathbb{S} . In other words, for all objects $X, Y \in \mathcal{C}$, there is a simplicial set $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$, and these Hom-spaces are equipped

with the usual composition laws for morphisms. The category underlying \mathcal{C} has morphisms $\text{Hom}_{\mathcal{C}}(X, Y) := \underline{\text{Hom}}_{\mathcal{C}}(X, Y)_0$, and the homotopy category $\text{Ho}(\mathcal{C})$ has morphisms $\text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y) := \pi_0 \underline{\text{Hom}}_{\mathcal{C}}(X, Y)$.

A simplicial structure on a category \mathcal{C} is sometimes given by defining $\otimes : \mathbb{S} \times \mathcal{C} \rightarrow \mathcal{C}$ or $(-)^{-} : \mathbb{S}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{C}$. In these cases, $\underline{\text{Hom}}_{\mathcal{C}}$ is determined by the formulae

$$\text{Hom}_{\mathcal{C}}(X \otimes K, Y) = \text{Hom}_{\mathbb{S}}(K, \underline{\text{Hom}}_{\mathcal{C}}(X, Y)) = \text{Hom}_{\mathcal{C}}(X, Y^K).$$

Definition 0.1. Let $\Delta^n \in \mathbb{S}$ be the standard n -simplex, and $\partial\Delta^n \in \mathbb{S}$ its boundary. Given $0 \leq k \leq n$, define the k th horn Λ_k^n of Δ^n to be the simplicial set obtained from Δ^n by removing the interior and the k th face. See [GJ] §I.1 for explicit descriptions.

Definition 0.2. Given a simplicial abelian group A_{\bullet} , we denote the associated normalised chain complex by $N^s A$. Recall that this is given by $N^s(A)_n := \bigcap_{i>0} \ker(\partial_i : A_n \rightarrow A_{n-1})$, with differential ∂_0 . Then $H_*(N^s A) \cong \pi_*(A)$.

Definition 0.3. Given a cosimplicial abelian group A^{\bullet} , we denote the associated conormalised cochain complex by $N_c A$. Recall that this is given by $N_c(A)^n := \bigcap_i \ker(\sigma^i : A^n \rightarrow A^{n-1})$, with differential $\sum (-1)^i \partial^i$.

Definition 0.4. Given an affine scheme X , we will write $O(X) := \Gamma(X, \mathcal{O}_X)$.

Definition 0.5. Given a category \mathcal{C} , write $s\mathcal{C}$ for the category $\mathcal{C}^{\Delta^{\text{opp}}}$ of simplicial objects in \mathcal{C} .

1. BACKGROUND ON n -GEOMETRIC STACKS

We now adapt various definitions from [TV2] to slightly more general settings, and recall several results.

1.1. A general setup. Fix a model category \mathcal{S} in the sense of [Hov] Definition 1.1.4. In particular, we assume that \mathcal{S} contains all small limits and colimits.

1.1.1. Simplicial diagrams.

Definition 1.1. Given a simplicial object X_{\bullet} in a complete category \mathcal{C} , write $\text{Hom}_{\mathbb{S}}(-, X) : \mathbb{S}^{\text{opp}} \rightarrow \mathcal{C}$ for the right Kan extension of X with respect to the Yoneda embedding $\Delta^{\text{opp}} \rightarrow \mathbb{S}$. Explicitly, $\text{Hom}_{\mathbb{S}}(-, X)$ is the unique limit-preserving functor determined by $\text{Hom}_{\mathbb{S}}(\Delta^n, X) = X_n$ together with functoriality for face and boundary maps.

Definition 1.2. Given a simplicial set K and a simplicial object X_{\bullet} in a complete category \mathcal{C} , we follow [GJ, Proposition VII.1.21] in defining the K -matching object in \mathcal{C} by

$$M_K X := \text{Hom}_{\mathbb{S}}(K, X).$$

Thus $X_n = M_{\Delta^n} X$. Note that for finite simplicial sets K , the matching object $M_K X$ still exists even if \mathcal{C} only contains finite limits.

Remark 1.3. The matching object $M_{\partial\Delta^n} X$ is usually denoted $M_n X$. In [Gle], it is called the n th simplicial kernel, and denoted $\Delta^{\bullet}(n)(X)$, while $M_{\Lambda_k^n} X$ is there denoted $\Lambda^k(n)(X)$.

Definition 1.4. Recall from [Hov] Theorem 5.2.5 that for a model category \mathcal{C} , the Reedy model structure on $s\mathcal{C}$ is defined as follows. A morphism $f : X_{\bullet} \rightarrow Y_{\bullet}$ in $s\mathcal{C}$ is said to be a Reedy fibration if the matching maps

$$X_m \rightarrow M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

are fibrations in $s\mathcal{C}$, for all m , and a weak equivalence if each f_n is a weak equivalence in \mathcal{C} .

Definition 1.5. For a model category \mathcal{C} , write $\mathbf{R}\mathrm{Hom}_{\mathbb{S}}(-, X): \mathbb{S}^{\mathrm{opp}} \rightarrow \mathcal{C}$ for the homotopy right Kan extension of X . This is analogous to Definition 1.1, but preserves homotopy limits instead of limits, and can be realised as $\mathrm{Hom}_{\mathbb{S}}(-, \mathbf{R}X)$, where $\mathbf{R}X$ is a Reedy fibrant replacement for X in $s\mathcal{C}$.

Defining the homotopy K -matching object in \mathcal{C} by

$$M_K^h X := \mathbf{R}\mathrm{Hom}_{\mathbb{S}}(K, X).$$

1.1.2. ε -morphisms. We now consider $s\mathcal{S}$ equipped with its Reedy model structure.

Definition 1.6. Define $|-|: s\mathcal{S} \rightarrow \mathcal{S}$ to be realisation functor $\mathrm{holim}_{\rightarrow \Delta^{\mathrm{opp}}}$.

Fix a class ε of morphisms in $\mathrm{Ho}(\mathcal{S})$, containing all weak equivalences and stable under composition and homotopy pullback. Refer to a morphism in \mathcal{S} as an ε -morphism if its image in $\mathrm{Ho}(\mathcal{S})$ is so (the corresponding notion is called epimorphism in [TV2] and covering in [TV1]). We require ε -morphisms to satisfy the following properties:

Properties 1.7. (1) If $X_{\bullet} \in s\mathcal{S}$, then the map $X_0 \rightarrow |X_{\bullet}|$ is an ε -morphism.

(2) Given a morphism $f: X_{\bullet} \rightarrow Y_{\bullet}$ in $s\mathcal{S}$ for which the homotopy matching maps $X_n \rightarrow M_n^h X \times_{M_n^h Y}^h Y_n$ are ε -morphisms for all $n \geq 0$, the map

$$|f|: |X_{\bullet}| \rightarrow |Y_{\bullet}|$$

is a weak equivalence in \mathcal{S} .

(3) Given morphisms $X_{\bullet} \rightarrow Y_{\bullet} \leftarrow Z_{\bullet}$ in $s\mathcal{S}$ for which the homotopy partial matching maps $X_n \rightarrow M_{\Lambda_k^n}^h X \times_{M_{\Lambda_k^n}^h Y}^h Y_n$ are ε -morphisms for all $n \geq 1$ and all k , the map

$$|(X_{\bullet} \times_{Y_{\bullet}}^h Z_{\bullet})| \rightarrow |X_{\bullet}| \times_{|Y_{\bullet}|}^h |Z_{\bullet}|$$

is a weak equivalence in \mathcal{S} .

Fix a full subcategory \mathcal{A} of \mathcal{S} , closed under weak equivalences and finite homotopy limits. In the terminology of [TV1] §4.1, \mathcal{A} is a pseudo-model category.

Fix a class \mathbf{C} of morphisms in $\mathrm{Ho}(\mathcal{A})$, containing all weak equivalences and stable under composition and homotopy pullback. Say that a morphism in \mathcal{A} lies in \mathbf{C} if its image in $\mathrm{Ho}(\mathcal{A})$ does so. We require that \mathcal{A} and \mathbf{C} also satisfy the following properties

Properties 1.8. (1) If $X \rightarrow Y$ is an ε -morphism in $\mathrm{Ho}(\mathcal{S})$ and we have a map $U \rightarrow Y$ for $U \in \mathrm{Ho}(\mathcal{A})$, then there is an ε -morphism $U' \rightarrow U$ in $\mathrm{Ho}(\mathcal{A})$ such that the composite map $U' \rightarrow Y$ in $\mathrm{Ho}(\mathcal{S})$ factors through X .

(2) Every morphism in \mathbf{C} is an ε -morphism.

(3) Morphisms in \mathbf{C} are local with respect to ε -morphisms. In other words, if $f: U \rightarrow V$ in \mathcal{A} has the property that the homotopy pullback $f': U \times_V^h V' \rightarrow V'$ is a \mathbf{C} -morphism for some ε -morphism $V' \rightarrow V$, then f is in \mathbf{C} .

1.2. Geometric stacks.

Definition 1.9. Refer to the objects of \mathcal{S} as *stacks*. Then we follow [TV2] Definition 1.3.3.1 by defining n -geometric stacks (with respect to $(\mathcal{A}, \mathcal{S}, \mathbf{C})$) as follows:

(1) A stack is 0-geometric if it lies in \mathcal{A} .

(2) For $n \geq 0$, a morphism of stacks $F \rightarrow G$ is n -representable if for any $X \in \mathcal{A}$ and any morphism $X \rightarrow G$ in $\mathrm{Ho}(\mathcal{S})$, the homotopy pull-back $F \times_G^h X$ is n -geometric.

(3) A morphism of stacks $f: F \rightarrow G$ is in $0 - \mathbf{C}$ if it is 0-representable, and if for any $X \in \mathcal{A}$ and any morphism $X \rightarrow G$ in $\mathrm{Ho}(\mathcal{S})$, the induced morphism $F \times_G^h X \rightarrow X$ is in \mathbf{C} .

(4) Now let $n > 0$ and let F be any stack. An n -atlas for F is a morphism $U \rightarrow F$ in $\mathrm{Ho}(\mathcal{S})$ such that U is in \mathcal{A} , and $U \rightarrow F$ is in $(n - 1) - \mathbf{C}$.

- (5) For $n > 0$, a stack F is n -geometric if the diagonal morphism $F \rightarrow F \times F$ is $(n - 1)$ -representable and the stack F admits an n -atlas.
- (6) For $n > 0$, a morphism of stacks $F \rightarrow G$ in $\text{Ho}(\mathcal{S})$ is in $n - \mathbf{C}$ if it is n -representable and if for any $X \in \mathcal{A}$ and any morphism $X \rightarrow G$ in $\text{Ho}(\mathcal{S})$, there exists an n -atlas $U \rightarrow F \times_G^h X$, such that $U \rightarrow X$ is in \mathbf{C} .

Moreover, we follow [TV2] Definition 1.3.3.7 in saying that a morphism of stacks is in \mathbf{C} (or a \mathbf{C} -morphism) if it is in $n - \mathbf{C}$ for some integer n .

Lemma 1.10. *Take a morphism $f: X \rightarrow Y$ in $\text{Ho}(\mathcal{S})$, together with an ε -morphism $Y' \rightarrow Y$. Then f is n -representable (resp. in $n - \mathbf{C}$) if and only if the homotopy pullback $f': X \times_Y^h Y'$ is so.*

Proof. The “only if” part is standard (see for instance [TV2] Proposition 1.3.3.3). The “if” part just uses the fact that \mathbf{C} -morphisms are local with respect to ε -morphisms — the argument of [TV2] Proposition 1.3.3.4 adapts to give the required result. \square

Proposition 1.11. *Let $f: F \rightarrow G$ be a \mathbf{C} -morphism and $n \geq 1$. If F is n -geometric and f is in $(n - 1) - \mathbf{C}$, then G is n -geometric.*

Proof. The proof of [TV2] Corollary 1.3.4.5 carries over. \square

1.3. HAG contexts.

1.3.1. Hyperdescent and hypersheaves.

Definition 1.12. Given a cosimplicial object X^\bullet in a model category \mathcal{M} , define $\text{Tot}_{\mathcal{M}} X \in \text{Ho}(\mathcal{M})$ by

$$\text{Tot}_{\mathcal{M}} X := \underset{n \in \Delta}{\text{holim}} X^n.$$

Definition 1.13. Given a model category \mathcal{M} and a site X , an \mathcal{M} -valued presheaf \mathcal{F} on X is said to be a hypersheaf if for all hypercovers $\tilde{U}_\bullet \rightarrow U$, the natural map

$$\mathcal{F}(U) \rightarrow \text{Tot}_{\mathcal{M}} \mathcal{F}(\tilde{U}_\bullet)$$

is a weak equivalence.

Remarks 1.14. If \mathcal{M} has trivial model structure, note that hypersheaves in \mathcal{M} are precisely \mathcal{M} -sheaves. A sheaf \mathcal{F} of modules is a hypersheaf when regarded as a presheaf of non-negatively graded chain complexes, but not a hypersheaf when regarded as a presheaf of unbounded chain complexes unless $H^i(U, \mathcal{F}) = 0$ for all $i > 0$ and all U . Beware that the sheafification of a hypersheaf will not, in general, be a hypersheaf.

Hypersheaves are known as such in [Lur1] §4.1, but are more usually called ∞ -stacks or hypercomplete ∞ -sheaves, and are referred to as stacks in [TV1] Definition 4.6.5. They are also sometimes known as homotopy sheaves, but we avoid this terminology for fear of confusion with homotopy groups of a simplicial sheaf.

Examples 1.15. If \mathcal{M} is a category with trivial model structure (all morphisms are fibrations and cofibrations, isomorphisms are the only weak equivalences), then $\text{Tot}_{\mathcal{M}} X$ is the equaliser in \mathcal{M} of the maps $\partial^0, \partial^1: X^0 \rightarrow X^1$.

$\text{Tot}_{\mathbb{S}}$ is the functor Tot of a cosimplicial space defined in [GJ] §VIII.1. Explicitly,

$$\text{Tot}_{\mathbb{S}} X^\bullet = \left\{ x \in \prod_n (X^n)^{\Delta^n} : \partial_X^i x_n = (\partial_\Delta^i)^* x_{n+1}, \sigma_X^i x_n = (\sigma_\Delta^i)^* x_{n-1} \right\},$$

for X Reedy fibrant. Homotopy groups of the total space are related to a spectral sequence given in [GJ] Proposition VIII.1.15.

In the category $s\text{Ab}$ of simplicial abelian groups, all cosimplicial objects are Reedy fibrant, and $\text{Tot}_{s\text{Ab}} A = \text{Tot}_{\mathbb{S}} A$. Likewise for simplicial modules and simplicial rings. In

all these cases, we may compute homotopy groups by $\pi_n(\mathrm{Tot} {}_s\mathrm{Ab} A) = \mathrm{H}_n(\mathrm{Tot} \mathrm{Ch}_{\geq 0} N^s A)$, for $N^s A$ the normalised cochain complex associated to A , and $\mathrm{Tot} \mathrm{Ch}_{\geq 0}$ defined below. The unnormalised associated chain complex also gives the same result.

For the category Ch of unbounded chain complexes, $\mathrm{Tot} \mathrm{Ch} \simeq \mathrm{Tot}^{\mathrm{II}}$, or the quasi-isomorphic functor $\mathrm{Tot}^{\mathrm{II}} N_c$, where N_c is conormalisation (see Definition 0.3), and $(\mathrm{Tot}^{\mathrm{II}} V)_n = \prod_i V_{n+i}$. Similarly, $\mathrm{Tot} {}_{\mathrm{CoCh}}^{\geq 0} \simeq \mathrm{Tot}^{\mathrm{II}}$ for the category $\mathrm{CoCh}^{\geq 0}$ of non-negatively graded cochain complexes. However, for non-negatively graded chain complexes $\mathrm{Ch}_{\geq 0}$, we have $\mathrm{Tot} \mathrm{Ch}_{\geq 0} \simeq \tau_{\geq 0} \mathrm{Tot}^{\mathrm{II}}$, where τ is good truncation.

For the categories $dg\mathrm{Alg}$ and $dg_{\leq 0}\mathrm{Alg}$ of unbounded and non-positively graded differential graded (chain) algebras, $\mathrm{Tot} dg\mathrm{Alg}$ and $\mathrm{Tot} dg_{\leq 0}\mathrm{Alg}$ are modelled by the functor Th of Thom-Sullivan (or Thom-Whitney) cochains. In the category $dg_{\geq 0}\mathrm{Alg}$ of differential non-negatively graded (chain) algebras, $\mathrm{Tot} dg_{\geq 0}\mathrm{Alg} \simeq \tau_{\geq 0} \mathrm{Th}$.

1.3.2. HA contexts and HAG contexts. By [TV2] Definition 1.1.0.11, a homotopical algebra (or HA) context is a symmetric monoidal model category \mathcal{C} , endowed with some additional structures which will not affect our ability to define n -geometric stacks. We write $\mathrm{Aff}_{\mathcal{C}}$ for the opposite category to the category of commutative monoids in \mathcal{C} .

By [TV2] Definition 1.3.2, a homotopical algebraic geometry (HAG) context is a HA context \mathcal{C} , together with a model pre-topology τ having various good properties summarised in [TV2] Assumption 1.3.2.2, and a class \mathbf{P} of morphisms in $\mathrm{Aff}_{\mathcal{C}}$ satisfying the conditions of [TV2] Assumption 1.3.2.11.

The setting for n -geometric stacks is the model category $\mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$ of stacks (defined as in [TV2] §1.3.2). The underlying category of $\mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$ is the category of simplicial presheaves on $\mathrm{Aff}_{\mathcal{C}}$, and every fibrant object is a hypersheaf with respect for the topology τ . Morphisms between hypersheaves are weak equivalences if and only if they induce isomorphisms on the sheafifications of the homotopy groups.

Definition 1.16. As in [TV2] §1.3.1, we have a Yoneda embedding

$$\mathbf{R}h: \mathrm{Aff}_{\mathcal{C}} \rightarrow \mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$$

given by

$$(\mathbf{R}hX)(U) = \mathrm{map}_{\mathrm{Aff}_{\mathcal{C}}}(U, X),$$

where map is a functorial choice of homotopy function complex (as in [Hir] §17.4). Specifically, [TV2] §1.3.1 takes a left homotopy function complex (denoted by Map_l in [Hov] §5.2). In a simplicial model category, we can just take $\mathrm{map} = \mathbf{R}\underline{\mathrm{Hom}}$, the right-derived functor of the simplicial Hom -functor.

Lemma 1.17. For $X, Y \in \mathrm{Aff}_{\mathcal{C}}$, the canonical map

$$\mathrm{map}_{\mathrm{Aff}_{\mathcal{C}}}(X, Y) \rightarrow \mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}}(\mathbf{R}hX, \mathbf{R}hY)$$

is a weak equivalence.

Proof. By [TV2] Corollary 1.3.2.5, the presheaf $\mathbf{R}hX$ is a hypersheaf, so the result is just [TV1] Proposition 4.6.7. \square

1.3.3. Geometric stacks.

Definition 1.18. The simplicial category of strongly quasi-compact n -geometric stacks for the HAG context $(\mathcal{C}, \tau, \mathbf{P})$ is given by applying the construction of §1.2 to the data $(\mathcal{A}, \mathcal{S}, \varepsilon, \mathbf{C})$ for $\mathcal{S} := \mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$, with \mathcal{A} consisting of objects in the essential image of $\mathbf{R}h$, ε the class of local surjections (epimorphisms in the terminology of [TV2] Remark 1.3.2.6), and \mathbf{C} consisting of those \mathbf{P} -morphisms in \mathcal{A} which are also ε -morphisms.

For the simplicial category of all n -geometric stacks, we leave \mathcal{S} unchanged, but take \mathcal{A} to consist of objects of the form $\coprod_i^{\mathbf{L}} \mathbf{R}hX_i$, for sets $\{X_i\}_i$ of objects of $\mathrm{Aff}_{\mathcal{C}}$. We then let \mathbf{C} consist of ε -morphisms $\coprod_i^{\mathbf{L}} U_i \rightarrow X$ for families $\{U_i \rightarrow X\}_i$ of representable \mathbf{P} -morphisms.

We now verify the conditions of §1.1.

Proposition 1.19. *The setting $(\mathcal{A}, \mathcal{S}, \mathbf{P}, \varepsilon)$ above satisfies Properties 1.7 for the class of ε -morphisms. The class \mathbf{C} satisfies Properties 1.8.*

Proof. Properties 1.8 are immediate, because the topology τ is defined using coverings in $\text{Aff}_{\mathcal{C}}$ and the property \mathbf{P} is, by assumption, local with respect to τ .

Now, let \mathcal{T} be the simplicial localisation of $\text{Aff}_{\mathcal{C}}$ with respect to the weak equivalences (as constructed in [DK]). This has the properties that $\pi_0 \mathcal{T} = \text{Ho}(\text{Aff}_{\mathcal{C}})$, and that $\underline{\text{Hom}}_{\mathcal{T}}(X, Y) \simeq \text{map}_{\text{Aff}_{\mathcal{C}}}(X, Y)$. By [TV1] Theorem 4.7.1, the model category $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ is Quillen equivalent to the model category $s\text{Pr}_{\tau}(\mathcal{T})$ of [TV1] §3.4, and hence to the model category $s\text{Pr}_{\tau, \text{inj}}(\mathcal{T})$ of [TV1] §3.6. The equivalence is given by sending a fibrant object in $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ to its associated simplicial functor. Objects of the model category $\mathcal{I} := s\text{Pr}_{\tau, \text{inj}}(\mathcal{T})$ are simplicial functors $\mathcal{T} \rightarrow \mathbb{S}$, cofibrations are the objectwise monomorphisms, and weak equivalences are the local weak equivalences.

For Property 1.7.1, take $X_{\bullet} \in s\mathcal{S}$. Replacing X_{\bullet} with a weakly equivalent diagram if necessary, we may assume that $X_{\bullet} \in s\mathcal{I}$, with each X_n fibrant (hence a stack). Now, $\text{holim}_{\Delta^{\text{opp}}} : s\mathbb{S} \rightarrow \mathbb{S}$ is given by the diagonal functor $(\text{diag } X)_n = X_{nn}$, so the same is true for the injective model structure on simplicial presheaves. Therefore

$$|X_{\bullet}| \simeq a(\text{diag } X_{\bullet}),$$

where a is the fibrant replacement functor in \mathcal{I} .

Now, by [GJ] Proposition IV.1.7 the functor diag takes levelwise weak equivalences to weak equivalences. Since each X_n is a stack, this means that $\text{diag } X_{\bullet}$ sends weak equivalences in $\text{Aff}_{\mathcal{C}}$ to weak equivalences in \mathbb{S} — in the terminology of [TV2], it is a prestack. Also note that the map $X_0 \rightarrow \text{diag } X_{\bullet}$ induces surjections $\pi_0(X_0(U)) \twoheadrightarrow \pi_0(\text{diag } X_{\bullet}(U))$ for all $U \in \text{Aff}_{\mathcal{C}}$. This necessarily induces surjections on the associated presheaves of sets, so by [TV1] Definition 4.3.7, $X_0 \rightarrow a(\text{diag } X_{\bullet})$ is an ε -morphism, as required.

For Property 1.7.2, we may take a Reedy fibration $f: X_{\bullet} \rightarrow Y_{\bullet}$ between Reedy fibrant objects in $s\mathcal{I}$. Now, a fibration in \mathcal{I} is a fibration objectwise, so a fibration is an ε -morphism if and only if it is locally a surjective map of simplicial sets. Since f is a Reedy fibration, the homotopy matching object $M_n^h X \times_{M_n^h Y} Y_n$ is represented by the ordinary matching object $M_n X \times_{M_n Y} Y_n$, and the condition that the map from X_n to this be an ε -morphism says precisely that f is locally a horizontal trivial Kan fibration. In particular, it is a horizontal local weak equivalence, so $\text{diag } X_{\bullet} \rightarrow \text{diag } Y_{\bullet}$ is a local weak equivalence. Applying a gives $|X_{\bullet}| \simeq |Y_{\bullet}|$, as required.

Finally, for Property 1.7.3, we may once again take Reedy fibrations $X_{\bullet} \xrightarrow{f} Y_{\bullet} \leftarrow Z_{\bullet}$ between Reedy fibrant objects in $s\mathcal{I}$. The homotopy partial matching objects are now ordinary matching objects, so the condition that the homotopy partial matching maps of f be ε -morphisms says precisely that f is a locally a horizontal Kan fibration. Then [GJ] Lemma IV.4.8 implies that $\text{diag } f$ is a local fibration, and now [Jar] Lemma 5.16 (adapted to simplicial sites) shows that

$$(\text{diag } X) \times_{\text{diag } Y}^h (\text{diag } Z) \simeq (\text{diag } X) \times_{\text{diag } Y} (\text{diag } Z).$$

The right-hand side is just $\text{diag}(X \times_Y Z)$, and since we started with Reedy fibrations between fibrant objects, this is a model for $\text{diag}(X \times_Y^h Z)$. Thus we have shown

$$|X| \times_{|Y|}^h |Z| \simeq |X \times_Y^h Z|,$$

as required. □

1.3.4. *Higher Artin and Deligne–Mumford stacks.* The first main example of a HA context is to take $\text{Aff}_{\mathcal{C}}$ to be the category of affine schemes with its trivial model structure. In the trivial model structure, all morphisms are both fibrations and cofibrations, and the weak equivalences are just isomorphisms. Then \mathcal{A} is Quillen equivalent to $\text{Aff}_{\mathcal{C}}$ and $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ is the category of simplicial presheaves on $\text{Aff}_{\mathcal{C}}$ with its local projective model structure. Taking τ to be the étale topology, the fibrant objects of $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ are hypersheaves for the étale site. The functor $\mathbf{R}h: \text{Aff}_{\mathcal{C}} \rightarrow \mathcal{S}$ just sends X to its associated sheaf

$$U \mapsto \text{Hom}_{\text{Aff}_{\mathcal{C}}}(U, X).$$

There are two common choices of HAG context associated to this HA context, given by taking \mathbf{P} to be the class either of smooth morphisms or of étale morphisms. In these contexts, the (n, \mathbf{P}) -geometric stacks of §1.2 are known as n -geometric Artin and Deligne–Mumford stacks, respectively. A rarer context is that of n -geometric schemes, which takes \mathbf{P} to be the class of local isomorphisms.

1.3.5. *Derived Artin stacks.* We now recall the geometric D^- -stacks of [TV2] Chapter 2.2. Set $\text{Aff}_{\mathcal{C}}$ to be $c\text{Aff}$, the category of cosimplicial affine schemes, so \mathcal{C} is the category of simplicial algebras. Given a simplicial algebra A , write $\text{Spec } A$ for the corresponding object of $c\text{Aff}$. Let $s\text{Pr}(c\text{Aff})$ denote the category of simplicial presheaves on $c\text{Aff}$.

We give $c\text{Aff}$ its usual simplicial model structure, in which a map $\text{Spec } B \rightarrow \text{Spec } A$ is a weak equivalence if $\pi_n(A) \cong \pi_n(B)$ for all n . The map is cofibration if $A \rightarrow B$ is a Kan fibration, or equivalently if $N_i^s A \rightarrow N_i^s B$ is surjective for all $i > 0$, for N^s the simplicial normalisation of Definition 0.2.

Definition 1.20. The Reedy model category $sc\text{Aff}$ has a natural simplicial structure. If we regard an object $X \in sc\text{Aff}$ as a contravariant functor from $c\text{Aff}$ to \mathbb{S} , then for $K \in \mathbb{S}$, the object $X^K \in sc\text{Aff}$ is given by

$$X^K(U) := X(U)^K,$$

for $U \in c\text{Aff}$. Note that this simplicial structure is independent of the natural simplicial structure on $c\text{Aff}$.

We then define $\underline{\text{Hom}}_{sc\text{Aff}}(X, Y) \in \mathbb{S}$ by

$$\underline{\text{Hom}}_{sc\text{Aff}}(X, Y)_n := \text{Hom}_{sc\text{Aff}}(X, Y^{\Delta^n}).$$

Definition 1.21. A map $\text{Spec } B \rightarrow \text{Spec } A$ in $c\text{Aff}$ is said to be flat (resp. smooth, resp. étale, resp. a local isomorphism) if $\pi_0(A) \rightarrow \pi_0(B)$ is flat (resp. smooth, resp. étale, resp. a local isomorphism), and $\pi_n(B) \cong \pi_n(A) \otimes_{\pi_0(A)} \pi_0(B)$ for all n .

Definition 1.22. Define the functor $\pi^0: c\text{Aff} \rightarrow \text{Aff}$ by $\text{Spec } A \mapsto \text{Spec } \pi_0 A$. A morphism f in $c\text{Aff}$ is said to be surjective if $\pi^0(f)$ is surjective.

Definition 1.23. Given a fibrant object $X \in c\text{Aff}$, define $\underline{X} \in s\text{Pr}(c\text{Aff})$ by

$$\underline{X}(U) := \underline{\text{Hom}}_{c\text{Aff}}(U, X) \in \mathbb{S},$$

for $U \in c\text{Aff}$.

In the notation of §1.3.2, \underline{X} is the stack $\mathbf{R}hX$. Thus an object of $s\text{Pr}(c\text{Aff})$ is a 0-geometric stack if and only if it is weakly equivalent to a disjoint union of such stacks \underline{X} .

For the HAG context given by taking \mathbf{P} to be the class of smooth morphisms (resp. étale morphisms, resp. local isomorphisms) and τ the étale topology, we will refer to the associated geometric stacks as derived Artin (resp. Deligne–Mumford, resp. Zariski) n -geometric stacks. In the terminology of [TV2], the derived Artin n -geometric stacks are called n -geometric D^- -stacks.

1.4. Other examples. Another important example of a HAG context given in [TV2] is that of symmetric spectra, the setting of spectral (or brave new) algebraic geometry. The precise formulation given depends on the choices τ, \mathbf{P} of topology and morphisms.

If our sole aim is to define n -geometric stacks, then HAG contexts might be regarded as unnecessarily restrictive. Rather than taking \mathcal{A} to be a category opposite to commutative monoids, Definition 1.9 will still work for non-commutative rings or for simplicial \mathcal{C}^∞ -rings, giving non-commutative and differential stacks respectively. In both cases, the main difficulty lies in finding a suitable class \mathbf{P} of morphisms.

It is also worth noting that the simplicial category \mathcal{A} in Definition 1.9 need not be the localisation of a model category. This makes it possible to consider localisations of small categories, such as of finitely generated algebras. A better-known example of this type is given by Lurie's n -stacks in [Lur1], where \mathcal{A} is taken to be the category of algebraic spaces and \mathbf{C} the category of smooth surjections.

We now collect some useful lemmas and a consequence for comparing the stacks of [TV2, Lur1].

Definition 1.24. Given $\mathfrak{X} \in \mathcal{S}$ and $K \in \mathbb{S}$, denote by $\mathfrak{X}^{\mathbf{R}K}$ the object of \mathcal{S} defined as the homotopy limit $M_K^h(c\mathfrak{X})$ of Definition 1.5, where $c\mathfrak{X} \in s\mathcal{S}$ is given by $(c\mathfrak{X})_n = \mathfrak{X}$ for all n .

Lemma 1.25. *Given a morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ of stacks, the diagonal of the matching morphism*

$$\mathfrak{X} \simeq \mathfrak{X}^{\mathbf{R}\Delta^{r-1}} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^{r-1}} \times_{\mathfrak{Y}^{\mathbf{R}\partial\Delta^{r-1}}}^h \mathfrak{Y}.$$

is equivalent to

$$\mathfrak{X} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^r} \times_{\mathfrak{Y}^{\mathbf{R}\partial\Delta^r}}^h \mathfrak{Y}$$

Proof. The key observation is that we may express $\partial\Delta^r$ as the pushout

$$\partial\Delta^r = \Lambda_0^r \cup_{\partial^0, \partial\Delta^{r-1}} \Delta^{r-1}.$$

Since this is a pushout of cofibrations, we have

$$\mathfrak{X}^{\mathbf{R}\partial\Delta^r} \simeq \mathfrak{X}^{\mathbf{R}\Lambda_0^r} \times_{\mathfrak{X}^{\mathbf{R}\partial\Delta^{r-1}}}^h \mathfrak{X}^{\mathbf{R}\Delta^{r-1}} \simeq \mathfrak{X} \times_{\mathbf{R}\mathfrak{X}^{\partial\Delta^{r-1}}}^h \mathfrak{X},$$

the second equivalence following because Λ_0^r and Δ^{r-1} are contractible.

This implies that the diagonal

$$\mathfrak{X} \rightarrow (\mathfrak{X} \times_{\mathfrak{X}^{\mathbf{R}\partial\Delta^{r-1}}}^h \mathfrak{X}) \times_{(\mathfrak{Y} \times_{\mathfrak{Y}^{\mathbf{R}\partial\Delta^{r-1}}}^h \mathfrak{Y})}^h \mathfrak{Y}$$

of $\mathfrak{X} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^{r-1}} \times_{\mathfrak{Y}^{\mathbf{R}\partial\Delta^{r-1}}}^h \mathfrak{Y}$ is just

$$\mathfrak{X} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^r} \times_{\mathfrak{Y}^{\mathbf{R}\partial\Delta^r}}^h \mathfrak{Y}.$$

□

Lemma 1.26. *Given an n -representable morphism $\mathfrak{X} \rightarrow \mathfrak{Y}$ of stacks, the matching map*

$$\mathfrak{X} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^r} \times_{\mathfrak{Y}^{\mathbf{R}\partial\Delta^r}}^h \mathfrak{Y}$$

is $(n-r)$ -representable for all $0 \leq r \leq n$, a 0-representable closed immersion for $r = n+1$, and an equivalence for $r > n+1$.

Proof. We prove this by induction on r . When $r = 0$, $\partial\Delta^0 = \emptyset$, so the statement is trivially true. Now assume the statement for $r-1$, with $r \leq n$. Since $\mathfrak{X} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^{r-1}} \times_{\mathfrak{Y}^{\mathbf{R}\partial\Delta^{r-1}}}^h \mathfrak{Y}$ is $(n-r+1)$ -representable, its diagonal must be $(n-r)$ -representable, giving the required result by Lemma 1.25. The statement for $r = n+1$ follows because the diagonal of an affine morphism is a closed immersion. For $r > n+1$, observe that the diagonal of a closed immersion is an isomorphism. □

Remark 1.27. When we take \mathbf{P} to be the class of smooth morphisms, there are slight differences in terminology between [TV2] and [Lur1] in relation to higher stacks. In the former, only disjoint unions of affine schemes are 0-representable, so arbitrary schemes are 2-geometric stacks, and Artin stacks are 1-geometric stacks if and only if they have affine diagonal. In the latter, algebraic spaces are 0-stacks. An n -stack in the sense of [Lur1] is called n -truncated in [TV2], and it follows easily that every n -geometric stack in [TV2] is n -truncated.

Conversely, for any n -truncated stack X , the map $X \rightarrow X^{\mathbf{R}\partial\Delta^{n+2}}$ is an isomorphism (by Lemma 1.26), hence 0-representable. Thus X must be $(n+2)$ -geometric.

We can summarise this by saying that for a geometric stack \mathfrak{X} to be n -truncated means that $\mathfrak{X} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^{n+2}}$ is an equivalence, or equivalently that $\mathfrak{X} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^n}$ is representable by algebraic spaces. For \mathfrak{X} to be n -geometric means that $\mathfrak{X} \rightarrow \mathfrak{X}^{\mathbf{R}\partial\Delta^n}$ is representable by disjoint unions of affine schemes.

As an example of this difference, consider the scheme X obtained by gluing two copies of \mathbb{A}^2 , with intersection $\mathbb{A}^2 - \{0\}$. This is certainly an algebraic space, so is 0-truncated. However, the diagonal $X \rightarrow X \times X$ is not an affine morphism, so X is not even 1-geometric, but merely 2-geometric.

Nevertheless, any Artin stack with affine diagonal (in particular any separated algebraic space) is 1-geometric.

2. n -HYPERGROUPOIDS

Definition 2.1. An n -hypergroupoid is an object $X \in \mathbb{S}$ for which the partial matching maps

$$X_m \rightarrow M_{\Lambda_k^m} X$$

are surjective for all k, m (i.e. X is a Kan complex), and are isomorphisms for all $m > n$.

Remark 2.2. Terminology for n -hypergroupoids varies considerably. Other names include “ n -dimensional Kan hypergroupoids” ([Gle, §3]), “weak n -groupoids” ([Get, Definition 2.3]) and “exact n -types” ([Bek, Definition 3.3]). The relative notion of Definition 2.4 below is called a “fibration exact in $\dim \geq n$ ” in [Dus1, Definition 1.5]).

Remarks 2.3. (1) A 0-hypergroupoid is just a set $X = X_0$. A 1-hypergroupoid is the nerve of a groupoid G , which can be recovered by taking objects X_0 , morphisms X_1 , source and target $\partial_0, \partial_1 : X_1 \rightarrow X_0$, identity $\sigma_0 : X_0 \rightarrow X_1$ and multiplication

$$X_1 \times_{\partial_0, X_0, \partial_1} X_1 \xrightarrow{(\partial_2, \partial_0)^{-1}} X_2 \xrightarrow{\partial_1} X_1.$$

Equivalently, G is the fundamental groupoid $\pi_f X$ of X .

- (2) For an n -hypergroupoid X , the factorisation $\Lambda_k^m \rightarrow \partial\Delta^m \rightarrow \Delta^m$ ensures that for all $m > n$, $X_m \rightarrow M_{\partial\Delta^m}$ has a section, so is surjective. This implies that $\pi_m X = 0$ for all $m > n$.
- (3) Giving an n -hypergroupoid X is equivalent to giving the sets X_i for $i \leq n$, together with the operations between them, and an operation on certain $(n+1)$ -tuples in X_n , satisfying the hyper-associativity and hyper-unit laws of [Gle] §3.2. A related result will be given in Lemma 2.12.
- (4) Under the Dold–Kan correspondence between \mathbb{N}_0 -graded chain complexes and abelian groups, n -hypergroupoids in abelian groups correspond to chain complexes concentrated in degrees $\leq n$. This is essentially because the normalisation functor is given by $N_n A = \ker(A_n \rightarrow M_{\Lambda_n^n} A)$.

We also have a relative version of this definition.

Definition 2.4. Given $Y \in \mathbb{S}$, define a (relative) n -hypergroupoid over Y to be a morphism $f : X \rightarrow Y$ in \mathbb{S} , such that the relative partial matching maps

$$X_m \rightarrow M_{\Lambda_k^n}(X) \times_{M_{\Lambda_k^m} Y} Y_m$$

are surjective for all k, m (i.e. f is a Kan fibration), and isomorphisms for all $m > n$. In the terminology of [Gle], this says that f is a Kan fibration which is an exact fibration in all dimensions $> n$.

Examples 2.5. (1) A morphism $f : X \rightarrow Y$ between 1-hypergroupoids X, Y makes X a 1-hypergroupoid over Y if and only if it satisfies the path-lifting property that for all objects $x \in X_0$, and all morphisms m in $\pi_f Y$ with source fx , there exists a unique morphism \tilde{m} in $\pi_f X$ with source x and $f(\tilde{m}) = m$.
(2) A 0-hypergroupoid $f : X \rightarrow Y$ is just a space which is Cartesian over Y , in the sense that the maps

$$X_n \xrightarrow{(\partial_i, f)} X_{n-1} \times_{Y_{n-1}, \partial_i} Y_n$$

are all isomorphisms.

Lemma 2.6. *Given a Kan fibration $f : X \rightarrow Y$ and a morphism $g : Y \rightarrow Z$ such that $gf : X \rightarrow Z$ is an n -hypergroupoid over Z , the morphism f must be an n -hypergroupoid.*

Proof. Existence of the lifts follows from f being a fibration, and uniqueness follows from uniqueness for gf . \square

Definition 2.7. Let $\Delta_n \subset \Delta$ be the full subcategory on objects \mathbf{i} , for $i \leq n$. The category of n -truncated simplicial sets is $\text{Set}^{\Delta_n^{\text{opp}}}$. The forgetful functor i_{n*} from \mathbb{S} to $\text{Set}^{\Delta_n^{\text{opp}}}$ has a left adjoint i_n^* , and a right adjoint $i_n^!$. The composites are denoted $\text{sk}_n := i_n^* i_{n*}$ and $\text{cosk}_n := i_n^! i_{n*}$, giving $\text{Hom}(K, \text{cosk}_n X) = \text{Hom}(\text{sk}_n K, X)$. We sometimes denote $i_n^* X$ by $X_{\leq n}$.

Definition 2.8. Given $Y \in \mathbb{S}$, define a trivial n -hypergroupoid over Y to be a morphism $f : X \rightarrow Y$ in \mathbb{S} , such that the relative matching maps

$$X_m \rightarrow M_{\partial \Delta^m}(X) \times_{M_{\partial \Delta^m} Y} Y_m$$

are surjective for all m (i.e. f is a trivial Kan fibration), and isomorphisms for all $m \geq n$.

Note that this gives the following lemma immediately.

Lemma 2.9. *A morphism $f : X \rightarrow Y$ is a trivial n -hypergroupoid over Y if and only if $X = Y \times_{\text{cosk}_{n-1} Y} \text{cosk}_{n-1} X$, and the $(n-1)$ -truncated morphism $X_{\leq n-1} \rightarrow Y_{\leq n-1}$ satisfies the conditions of Definition 2.8 (up to level $n-1$).*

Lemma 2.10. *A morphism f is a trivial n -hypergroupoid if and only if it is an n -hypergroupoid and a weak equivalence.*

Proof. The ‘‘only if’’ part is immediate. For the converse, note that f is a trivial fibration, so has the right lifting property (RLP) with respect to all cofibrations, and consider the maps $\Lambda_0^i \rightarrow \partial \Delta^i \rightarrow \Delta^i$. Since f has exact RLP with respect to $\Lambda_0^i \rightarrow \Delta^i$ for $i > n$, and RLP with respect to $\partial \Delta^i \rightarrow \Delta^i$ for all $i \geq n$, it follows that f has exact RLP with respect to $\partial \Delta^i \rightarrow \Delta^i$ for all $i > n$. It therefore remains only to prove uniqueness of the lift for $i = n$.

Consider the pushout $\partial \Delta^{n+1} = \Lambda_0^{n+1} \cup_{\partial \Delta^n} \Delta^n$. Since $\partial \Delta^n \rightarrow \Lambda_0^{n+1}$ is a cofibration, exact RLP with respect to $\partial \Delta^n \rightarrow \Delta^n$ will follow from exact RLP with respect to $\Lambda_0^{n+1} \rightarrow \partial \Delta^{n+1}$. This now follows from exact RLP with respect to $\Lambda_0^{n+1} \rightarrow \Delta^{n+1}$, since both $\Lambda_0^{n+1} \rightarrow \partial \Delta^{n+1}$ and $\partial \Delta^{n+1} \rightarrow \Delta^{n+1}$ are cofibrations. \square

Definition 2.11. Say that a morphism of l -truncated simplicial sets is an n -hypergroupoid if it satisfies the conditions of Definition 2.4 (up to level l).

Lemma 2.12. *A morphism $f : X \rightarrow Y$ is an n -hypergroupoid if and only if $X = Y \times_{\text{cosk}_{n+1}Y} \text{cosk}_{n+1}X$, and the $(n+2)$ -truncated morphism $X_{\leq n+2} \rightarrow Y_{\leq n+2}$ is an n -hypergroupoid.*

Proof. For the first part, note that for $i \geq n+1$, the factorisation $\Lambda_0^i \rightarrow \partial\Delta^i \rightarrow \Delta^i$ implies that the map

$$X_i \rightarrow Y_i \times_{M_i Y} M_i X$$

has a retraction ρ , so is injective. Since $\Lambda_0^i \rightarrow \partial\Delta^i$ is the pushout of $\partial\Delta^{i-1} \rightarrow \Delta^i$ along $\partial^0 : \partial\Delta^{i-1} \rightarrow \Lambda_0^i$, for $i \geq n+2$ the map

$$M_i X \rightarrow M_i Y \times_{M_{\Lambda_0^i} Y} M_{\Lambda_0^i} X$$

also has a retraction, so

$$\rho : Y_i \times_{M_i Y} M_i X \rightarrow Y_i \times_{M_{\Lambda_0^i} Y} M_{\Lambda_0^i} X \cong Y_i \times_{Y_i} X_i = X_i$$

has a retraction, and so must be an isomorphism.

Thus the matching maps $X_i \rightarrow Y_i \times_{M_i Y} M_i X$ are isomorphisms for all $i \geq n+1$, so

$$X = Y \times_{\text{cosk}_{n+1}Y} \text{cosk}_{n+1}X.$$

Finally, note that $\text{sk}_{n+1}\Lambda_k^i \rightarrow \text{sk}_{n+1}\Delta^i$ is an isomorphism for all $i \geq n+3$, so automatically has the exact LLP with respect to $X = Y \times_{\text{cosk}_{n+1}Y} \text{cosk}_{n+1}X$. This only leaves the lifting conditions from Definition 2.4 up to level $n+2$. \square

Remarks 2.13. (1) When $n=1$ and Y is a point, we can compare this with the data required to define a groupoid. Levels 0 and 1 determine the objects, morphisms and identities, but the face maps from level 2 are needed to define the multiplication, and hence to construct the groupoid. Only once the face maps from level 3 have been defined can we ensure that the multiplication is associative.

(2) When $n=0$, a 0-hypergroupoid $X \rightarrow Y$ is a Cartesian morphism. Thus level 0 gives the fibres, level 1 the descent datum (thereby determining the fibration), and level 2 gives the cocycle condition, ensuring that the descent datum is effective.

(3) In Lemma 2.9, only the $(n-1)$ -truncation was required to describe a trivial n -hypergroupoid over a base. In the 1-dimensional case, this says that a contractible groupoid is just a set, with a unique morphism between any pair of points.

Observe that an n -hypergroupoid X is determined by X_m for $m \leq n+1$, while a trivial n -hypergroupoid X is determined by X_m for $m < n$. The next Lemma accounts for this apparent discrepancy.

Lemma 2.14. *If $X \rightarrow Y$ is a relative n -hypergroupoid (resp. a trivial n -hypergroupoid), and $K \rightarrow L$ is a trivial cofibration (resp. a cofibration) giving an isomorphism $K_{\leq n-1} \rightarrow L_{\leq n-1}$ on $(n-1)$ -truncations, then the maps*

$$M_L X \rightarrow M_L Y \times_{M_K Y} M_K X$$

are isomorphisms.

Proof. This is essentially the same as the proof that the maps $\Lambda_k^m \rightarrow \Delta^m$ (resp. $\partial\Delta^m \rightarrow \Delta^m$) generate the trivial cofibrations (resp. all cofibrations) in \mathbb{S} . The same proof adapts to show that every map $K \rightarrow L$ satisfying the conditions above is a retract of a transfinite composition of pushouts of maps $\Lambda_k^m \rightarrow \Delta^m$ for $m > n$ (resp. maps $\partial\Delta^m \rightarrow \Delta^m$ for $m \geq n$). \square

Now, for a 1-hypergroupoid X , the characterisation of X as the nerve of a groupoid means that we can describe each set X_m in terms of X_0, X_1 by

$$X_m \cong \overbrace{X_1 \times_{\partial_0, X_0, \partial_1} X_1 \times_{\partial_0, X_0, \partial_1} \cdots \times_{\partial_0, X_0, \partial_1} X_1}^m.$$

The following lemma gives an analogue of this result for n -hypergroupoids.

Definition 2.15. Define Δ_* to be the subcategory of the ordinal number category Δ containing only those morphisms f with $f(0) = 0$. Given a category \mathcal{C} , define the category $s_+\mathcal{C}$ of almost simplicial diagrams over \mathcal{C} to consist of functors $\Delta_*^{\text{opp}} \rightarrow \mathcal{C}$. Thus an almost simplicial object X_* consists of objects $X_n \in \mathcal{C}$, with all of the operations ∂_i, σ_i of a simplicial diagram except ∂_0 , and satisfying the usual relations.

Lemma 2.16. *If X is a relative n -hypergroupoid over Y , then the underlying almost simplicial set over Y is determined by the truncation $X_{\leq n}$.*

Proof. We can write Δ^m as the left cone $(\Delta^{m-1})^\triangleleft$, i.e. as the join $\Delta^0 \star \Delta^{m-1}$. Consider the subspaces $K^m := (\text{sk}_{n-1}\Delta^{m-1})^\triangleleft \subset (\Delta^{m-1})^\triangleleft = \Delta^m$; the map $K^m \rightarrow \Delta^m$ is a trivial cofibration inducing an isomorphism on $(n-1)$ -skeleta. Thus Lemma 2.14 gives $X_n \cong M_{K^n}X \times_{M_{K^n}Y} Y_n$.

Now, $K^m = \text{sk}_n(K^m)$, so $M_{K^m}X$ is determined by $X_{\leq n}$. For the degeneracy maps $\sigma^i: \Delta^m \rightarrow \Delta^{m-1}$ we have $\sigma^i(K^m) \subset K^{m+1}$, and for the face maps $\partial^i: \Delta^m \rightarrow \Delta^{m+1}$ we have $\partial^i(K^m) \subset K^{m+1}$ provided $i > 0$. Thus the operations σ_j and ∂_i for $i > 0$ on X can be recovered from the corresponding maps $\sigma_j: M_{K^m}X \rightarrow M_{K^{m+1}}X$, $\partial_i: M_{K^m}X \rightarrow M_{K^{m-1}}X$. \square

Definition 2.17. Given a simplicial set K and a simplicial object X_\bullet in a category \mathcal{C} , define the object $X^K \in s\mathcal{C}$ by

$$(X^K)_n := \text{Hom}_{\mathbb{S}}(K \times \Delta^n, X),$$

for $\text{Hom}_{\mathbb{S}}$ defined as in Definition 1.1. Thus $M_K X = (X^K)_0$.

Lemma 2.18. *If $f: X \rightarrow Y$ is an n -hypergroupoid, and $g: K \rightarrow L$ a cofibration in \mathbb{S} , then the map*

$$F: X^L \rightarrow Y^L \times_{Y^K} X^L$$

is an n -hypergroupoid, which is trivial if either f or g is trivial. Moreover, if $g_{\leq n-1}: K_{\leq n-1} \rightarrow L_{\leq n-1}$ is an isomorphism, then F is a 0-hypergroupoid.

Proof. This amounts to describing when a diagram

$$\begin{array}{ccc} (A \times L) \cup_{(A \times K)} (B \times K) & \twoheadrightarrow & X \\ \downarrow & \dashrightarrow & \downarrow f \\ B \times L & \longrightarrow & Y \end{array}$$

admits the lifting shown, and determining when the lift is unique. For all cofibrations $h: A \rightarrow B$, the map $h': (A \times L) \cup_{(A \times K)} (B \times K) \rightarrow B \times L$ is a cofibration, so the lift must exist if f is trivial. If either g or h is trivial, then h' is trivial, so the lift exists.

Now,

$$((A \times L) \cup_{(A \times K)} (B \times K))_{\leq n-1} \rightarrow (B \times L)_{\leq n-1}$$

is an isomorphism whenever either of the maps $A_{\leq n-1} \rightarrow B_{\leq n-1}$ or $K_{\leq n-1} \rightarrow L_{\leq n-1}$ is an isomorphism, so Lemma 2.14 then implies that the lift is unique in this case. \square

Corollary 2.19. *If $f: X \rightarrow Y$ is an n -hypergroupoid, then*

$$X^{\Delta^n} \rightarrow Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\Delta^n}$$

is a 0-hypergroupoid.

The following definition is taken from [Ill, Remarks VI.1.4.1]:

Definition 2.20. Define the décalage functor $\text{Dec}_+ : \mathbb{S} \rightarrow \mathbb{S}$ by $\text{Dec}_+(X)_n = X_{n+1}$, with $\partial_i^{\text{Dec}_+X} = \partial_i^X$ and $\sigma_i^{\text{Dec}_+X} = \sigma_i^X$. This is a comonad, with counit $\partial_{n+1}^X : \text{Dec}_+(X)_n \rightarrow X_n$, and comultiplication $\sigma_{n+1}^X : \text{Dec}_+(X)_n \rightarrow \text{Dec}_+^2(X)_n$.

Remark 2.21. Dec_+ is the functor denoted by DEC in [Gle], and known to homotopy theorists as the simplicial path space P .

Lemma 2.22. *The maps $X_0 \xrightarrow{\sigma_0^X} \text{Dec}_+(X) \xrightarrow{\partial_0^X} X_0$ form a deformation retract, for all $X \in \mathbb{S}$.*

Proof. See [Dus2] §2.6. □

The following is immediate:

Lemma 2.23. *A left adjoint to Dec_+ is given by the right cone $(-)^{\triangleright}$, defined on simplices by $(\Delta^n)^{\triangleright} = \Delta^{n+1}$, with $(\Delta^{n-1} \xrightarrow{\partial^i} \Delta^n)^{\triangleright} = (\Delta^n \xrightarrow{\partial^i} \Delta^{n+1})$ and $(\Delta^{n+1} \xrightarrow{\sigma^i} \Delta^n)^{\triangleright} = (\Delta^{n+2} \xrightarrow{\sigma^i} \Delta^{n+1})$.*

Corollary 2.24. *If $f : X \rightarrow Y$ is an n -hypergroupoid, then $\text{Dec}_+(X) \rightarrow \text{Dec}_+(Y) \times_Y X$ is an $(n-1)$ -hypergroupoid.*

Proof. It suffices to describe the maps $\alpha_k^m : (\Lambda_k^m)^{\triangleright} \cup_{\Lambda_k^m} \Delta^m \rightarrow (\Delta^m)^{\triangleright}$, where the natural transformation $\text{id} \rightarrow (-)^{\triangleright}$ is adjoint to the co-unit $\text{Dec}_+ \rightarrow \text{id}$ of the adjunction. Now, $(\Lambda_k^m)^{\triangleright}$ is the union of all faces of Δ^{m+1} except the k th and the $(m+1)$ th, so $(\Lambda_k^m)^{\triangleright} \cup_{\Lambda_k^m} \Delta^m = \Lambda_k^{m+1}$, and α_k^m is just the natural inclusion $\Lambda_k^{m+1} \hookrightarrow \Delta^{m+1}$.

The result now follows immediately since α_k^m lifts with respect to $(-)^{\triangleright}$ for all m , and lifts uniquely for $m > n-1$. □

Remark 2.25. Given an n -hypergroupoid X , this gives us a way to regard the homotopy fibre product $X_0 \times_X^h X_0$ as an $(n-1)$ -hypergroupoid, since a model for it is given by the $(n-1)$ -hypergroupoid

$$\text{Dec}_+(X) \times_X X_0,$$

as $\text{Dec}_+(X) \rightarrow X$ is a fibration. This object is usually called the path-homotopy complex ([Dus2] §2.6).

Lemma 2.26. *If $f : X \rightarrow Y$ is a relative n -hypergroupoid, then $\text{Dec}_+(X) \rightarrow \text{Dec}_+(Y) \times_{Y_0} X_0$ is a trivial relative n -hypergroupoid.*

Proof. The retraction $\text{Dec}_+X \rightarrow X_0$ corresponds to a section $\pi_0 K \rightarrow K^{\triangleright}$, functorial in K . It thus suffices to describe the maps $(\partial \Delta^m)^{\triangleright} \cup_{\pi_0 \partial \Delta^m} \pi_0(\Delta^m) \rightarrow (\Delta^m)^{\triangleright}$. Calculation gives this map as $\Lambda_{m+1}^{m+1} \rightarrow \Delta^{m+1}$. Therefore the m th relative matching map of $\text{Dec}_+(X) \rightarrow \text{Dec}_+(Y) \times_{Y_0} X_0$ is surjective for $m < n$, and an isomorphism for $m \geq n$, as required. □

3. HOMOTOPY n -HYPERGROUPOIDS IN MODEL CATEGORIES

Fix a pseudo-model category \mathcal{A} (as in §1.1.2), and a class \mathbf{C} of morphisms in $\text{Ho}(\mathcal{A})$, containing all isomorphisms and stable under composition and homotopy pullback. Say that a morphism in \mathcal{A} is a \mathbf{C} -morphism if its image in $\text{Ho}(\mathcal{A})$ is so. Refer to fibrations in \mathbf{C} as \mathbf{C} -fibrations.

3.1. Definitions and basic properties.

Definition 3.1. Given $Y \in s\mathcal{A}$, define a (relative) homotopy (n, \mathbf{C}) -hypergroupoid over Y to be a morphism $X_\bullet \rightarrow Y_\bullet$ in $s\mathcal{A}$, for which the homotopy partial matching maps

$$X_m \rightarrow M_{\Lambda_k^m}^h(X) \times_{M_{\Lambda_k^m}^h(Y)} Y_m$$

of Definition 1.5 are \mathbf{C} -morphisms for all k, m , and are weak equivalences for all $m > n$ and all k . When Y is the final object, we will simply refer to X as a homotopy (n, \mathbf{C}) -hypergroupoid.

Define a (relative) (n, \mathbf{C}) -hypergroupoid over Y to be a homotopy (n, \mathbf{C}) -hypergroupoid over Y which is also a Reedy fibration (as in Definition 1.4).

Note that for any homotopy (n, \mathbf{C}) -hypergroupoid $X \rightarrow Y$, the Reedy model structure on $s\mathcal{A}$ gives a factorisation $X \rightarrow \hat{X} \rightarrow Y$, with $X \rightarrow \hat{X}$ a levelwise weak equivalence, and $\hat{X} \rightarrow Y$ an (n, \mathbf{C}) -hypergroupoid.

Definition 3.2. Given $Y \in s\mathcal{A}$, define a trivial homotopy (n, \mathbf{C}) -hypergroupoid over Y to be a morphism $X_\bullet \rightarrow Y_\bullet$ in $s\mathcal{A}$, for which the homotopy matching maps

$$X_m \rightarrow M_{\partial\Delta^m}^h(X) \times_{M_{\partial\Delta^m}^h(Y)} Y_m$$

are \mathbf{C} -morphisms for all m , and are weak equivalences for all $m \geq n$.

Define a trivial (n, \mathbf{C}) -hypergroupoid over Y to be a homotopy (n, \mathbf{C}) -hypergroupoid over Y which is also a Reedy fibration.

Remark 3.3. Note that trivial relative (n, \mathbf{C}) -hypergroupoids of affine schemes correspond to the truncated hypercovers considered in [Bek].

The following is a straightforward uncoiling of the definitions:

Lemma 3.4. *A morphism $X \rightarrow Y$ in $s\mathcal{A}$ is a relative (n, \mathbf{C}) -hypergroupoid if and only if*

- (1) *the matching maps*

$$X_m \rightarrow M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

of Definition 1.2 are fibrations for all $m \geq 0$;

- (2) *the partial matching maps*

$$X_m \rightarrow M_{\Lambda_k^m}(X) \times_{M_{\Lambda_k^m}^h(Y)} Y_m$$

are \mathbf{C} -fibrations for all k, m , and are trivial fibrations for all $m > n$ and all k .

A morphism $X \rightarrow Y$ in $s\mathcal{A}$ is a trivial (n, \mathbf{C}) -hypergroupoid if and only if the matching maps

$$X_m \rightarrow M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

of Definition 1.2 are \mathbf{C} -fibrations for all m , and are trivial fibrations for all $m \geq n$.

Examples 3.5. The simplest examples of homotopy hypergroupoids arise when the category \mathcal{A} has trivial model structure (so all morphisms are fibrations, and weak equivalences are isomorphisms). In these cases, the Reedy fibration conditions are vacuous, and $M_K^h = M_K$.

If $\mathcal{A} = \text{Set}$ and \mathbf{C} is the class of surjective morphisms, then (n, \mathbf{C}) -hypergroupoids are just the n -hypergroupoids of §2. Other important examples with trivial model structure arise when we take \mathcal{A} to be the category of affine schemes. Taking \mathbf{C} to be the class of smooth surjections (resp. étale surjections, resp. surjective local isomorphisms), we refer to (n, \mathbf{C}) -hypergroupoids as Artin (resp. Deligne–Mumford, resp. Zariski) n -hypergroupoids. We will see in Theorem 4.15 that these model strongly quasi-compact Artin (resp. Deligne–Mumford, resp. Zariski) n -geometric stacks.

Our main motivating example of a non-trivial model category is to take \mathcal{A} to be the category of cosimplicial affine schemes with the model structure of §1.3.5. Taking \mathbf{C} to be the

class of smooth surjections (resp. étale surjections, resp. surjective local isomorphisms), we refer to (n, \mathbf{C}) -hypergroupoids as derived Artin (resp. derived Deligne–Mumford, resp. derived Zariski) n -hypergroupoids. Again, Theorem 4.15 will show that these model the corresponding n -geometric stacks.

For an infinitesimal version, we could take \mathcal{A} to be the category of left-exact set-valued functors on Artinian simplicial rings (so $\mathcal{A} = \mathbf{cSp}$ in the notation of [Pri2]), and \mathbf{C} the class of formally smooth morphisms. Then the key notions in [Pri2] of quasi-smooth (resp. trivially smooth) morphisms correspond to (∞, \mathbf{C}) -hypergroupoids (resp. trivial (∞, \mathbf{C}) -hypergroupoids), for the obvious generalisation of the definitions above to $n = \infty$.

Finally, note that the definitions make sense if we only require \mathcal{A} to contain pullbacks by morphisms in \mathbf{C} , rather than all pullbacks. Taking \mathcal{A} to be the category of differential manifolds, and taking \mathbf{C} the class of surjective submersions, (n, \mathbf{C}) -hypergroupoids are just Zhu’s Lie n -groupoids from [Zhu].

Definition 3.6. The functor $i_n^! : \mathcal{A}^{\Delta_n^{\text{opp}}} \rightarrow s\mathcal{A}$ of Definition 2.7 is a right Quillen functor of Reedy model categories. Write $\mathbf{R}i_n^!$ for the associated derived functor, and set $\text{cosk}_n^h := \mathbf{R}i_n^! i_{n*}$, giving $\text{map}_{s\mathcal{A}}(K, \text{cosk}_n^h X) \simeq \text{map}_{\mathcal{A}^{\Delta_n^{\text{opp}}}}(i_{n*} K, X)$.

The following is immediate:

Lemma 3.7. *A morphism $f : X \rightarrow Y$ in $s\mathcal{A}$ is a trivial homotopy (n, \mathbf{C}) -hypergroupoid over Y if and only if $X \simeq Y \times_{\text{cosk}_{n-1}^h Y}^h \text{cosk}_{n-1}^h X$, and the $(n-1)$ -truncated morphism $i_{n-1,*} X \rightarrow i_{n-1,*} Y$ satisfies the conditions of Definition 3.2 (up to level $n-1$).*

Lemma 3.8. *Any composition of (trivial) homotopy (n, \mathbf{C}) -hypergroupoids is a (trivial) homotopy (n, \mathbf{C}) -hypergroupoid.*

Proof. This follows immediately by verifying the axioms. \square

Now observe that the functor Dec_+ of Definition 2.20 extends to any simplicial category; in particular, it gives $\text{Dec}_+ : s\mathcal{A} \rightarrow s\mathcal{A}$.

Lemma 3.9. *If $f : X \rightarrow Y$ is a homotopy (n, \mathbf{C}) -hypergroupoid, then $\text{Dec}_+(X) \rightarrow \text{Dec}_+(Y) \times_Y^h X$ is a homotopy $(n-1, \mathbf{C})$ -hypergroupoid, and $\text{Dec}_+(X) \rightarrow \text{Dec}_+(Y) \times_{Y_0}^h X_0$ is a trivial homotopy (n, \mathbf{C}) -hypergroupoid.*

Proof. The proofs of Corollary 2.24 and Lemma 2.26 carry over to this context. \square

Lemma 3.10. *If X is a homotopy (n, \mathbf{C}) -hypergroupoid over Y , then the underlying almost simplicial object over Y in $s_+\mathcal{A}$ is determined by up to equivalence by the truncation $X_{\leq n}$.*

Proof. The proof of Lemma 2.16 carries over immediately to this generality. \square

Remark 3.11. Beware that the natural analogue of our other truncation result, Lemma 2.12, does not hold for homotopy (n, \mathbf{C}) -hypergroupoids in general. Say that a map in $\text{Ho}(\mathcal{A})$ is a section if it has a left inverse. Then the proof only carries over contexts where the homotopy pullback of a section in $\text{Ho}(\mathcal{A})$ must be a weak equivalence if it has a right inverse. It will thus apply when \mathcal{A} is the category of affine schemes with trivial model structure, since the pullback of a section is a closed immersion.

Lemma 3.12. *If $f : X \rightarrow Y$ is a homotopy (n, \mathbf{C}) -hypergroupoid, and $g : K \rightarrow L$ a cofibration of finite simplicial sets, then the map*

$$F : X^{\mathbf{RL}} \rightarrow Y^{\mathbf{RL}} \times_{Y^{\mathbf{RK}}}^h X^{\mathbf{RL}}$$

is a homotopy (n, \mathbf{C}) -hypergroupoid, which is trivial if either f or g is trivial. Moreover, if $\text{sk}_{n-1} g : \text{sk}_{n-1} K \rightarrow \text{sk}_{n-1} L$ is an isomorphism, then F is a homotopy $(0, \mathbf{C})$ -hypergroupoid.

Proof. The proof of Lemma 2.18 carries over. \square

Lemma 3.13. *If $f : X \rightarrow Y$ is a homotopy (n, \mathbf{C}) -hypergroupoid, then $\text{Dec}_+(X) \rightarrow \text{Dec}_+(Y) \times_{Y_0} X_0$ is a trivial homotopy (n, \mathbf{C}) -hypergroupoid.*

3.2. Hom-spaces.

Definition 3.14. Given objects X, Y over $S \in s\mathcal{A}$, let $\underline{\text{Hom}}_{s\mathcal{A}|S}(X, Y) \in \mathbb{S}$ be given by

$$\underline{\text{Hom}}_{s\mathcal{A}|S}(X, Y)_n := \text{Hom}_{s\mathcal{A}|S}(X, Y^{\Delta^n} \times_{S^{\Delta^n}} S) = \text{Hom}_{s\mathcal{A}|S}(X \times \Delta^n, Y),$$

where $(X \times \Delta^n)_i$ is given by the coproduct of Δ_i^n copies of X_i .

However, the category $s\mathcal{A}$ does not have enough morphisms to model the theory of stacks:

Example 3.15. If we take X to be an affine scheme, and $Y = BG$, for G a smooth affine group scheme, we have

$$\text{Hom}_{s\text{Aff}}(X, BG) = \text{Hom}_{\text{Aff}}(X, (BG)_0) = \text{Hom}_{\text{Aff}}(X, \text{Spec } \mathbb{Z}) = \bullet,$$

the one-point set, whereas in the category of Artin stacks, $\text{Hom}(X, BG)$ is given by isomorphism classes in the groupoid of G -torsors on X .

Now, a trivial relative Artin 1-hypergroupoid $\tilde{X} \rightarrow X$ is just given by $\tilde{X} = \text{cosk}_0(X'/X)$, for some smooth surjection $X' \rightarrow X$, with $X' := \tilde{X}_0$. Then an element of $\text{Hom}(\tilde{X}, BG)$ is a G -torsor P on X equipped with a trivialisation $\theta : P \times_X X' \cong G \times X'$, while an element of $\text{Hom}(\tilde{X}, (BG)^{\Delta^1})$ consists of two such pairs $(P_1, \theta_1), (P_2, \theta_2)$, and an isomorphism $g \in G(X')$ between them.

Thus isomorphism classes of G -torsors on X will be given by the colimit over all \tilde{X} of $\pi_0 \underline{\text{Hom}}(\tilde{X}, BG)$.

Our solution is thus to take a form of localisation with respect to trivial (n, \mathbf{C}) -hypergroupoids.

3.3. Morphisms via pro-objects.

Definition 3.16. Given a category \mathcal{C} , recall that the category $\text{pro}(\mathcal{C})$ has objects consisting of inverse systems $\{A_\alpha \in \mathcal{C}\}$, with

$$\text{Hom}_{\text{pro}(\mathcal{C})}(\{A_\alpha\}, \{B_\beta\}) = \lim_{\leftarrow \beta} \lim_{\rightarrow \alpha} \text{Hom}_{\mathcal{C}}(A_\alpha, B_\beta).$$

We regard \mathcal{C} as a full subcategory of $\text{pro}(\mathcal{C})$ by sending A to the singleton inverse system $\{A\}$.

Definition 3.17. Let TC_n be the subcategory of $s\mathcal{A}$ containing all objects, with morphisms consisting of trivial (n, \mathbf{C}) -hypergroupoids.

Definition 3.18. Say that an object $X = \{X_\alpha\}_{\alpha \in I}$ of $\text{pro}(s\mathcal{A})$ is TC_n -projective if for every $\alpha \in I$ and every trivial (n, \mathbf{C}) -hypergroupoid $Y \rightarrow X_\alpha$, there exists $\beta \geq \alpha$ and a morphism $X_\beta \rightarrow Y$ compatible with the structure map $X_\beta \rightarrow X_\alpha$.

Lemma 3.19. *If $f : Y \rightarrow S$ is a (trivial) (n, \mathbf{C}) -hypergroupoid, then for all TC_n -projectives $X \in \text{pro}(s\mathcal{A})$, the morphism*

$$f_* : \underline{\text{Hom}}_{\text{pro}(s\mathcal{A})}(X, Y) \rightarrow \underline{\text{Hom}}_{\text{pro}(s\mathcal{A})}(X, S)$$

is a (trivial) Kan fibration.

Moreover, if the model structure on \mathcal{A} is trivial then f_ is a (trivial) n -hypergroupoid.*

Proof. Take a cofibration $i : K \rightarrow L$ of finite simplicial sets, with either i or f being trivial. Thus the map $Y^L \times_{S^L} S \rightarrow Y^K \times_{S^K} S$ is a relative (n, \mathbf{C}) -hypergroupoid. For finite simplicial sets K , matching objects are given by

$$M_K \underline{\text{Hom}}_{\text{pro}(s\mathcal{A})}(X, Y) = \text{Hom}_{\text{pro}(s\mathcal{A})}(X, Y^K),$$

so the TC_n -projective property of X ensures that f_* is a (trivial) Kan fibration.

Moreover, if $K_{<n-1} \cong L_{<n-1}$, then the map $Y^L \times_{S^L} S \rightarrow Y^K \times_{S^K} S$ is a weak equivalence, and hence an isomorphism when \mathcal{A} has trivial model structure. In that case, the matching maps on $\underline{\text{Hom}}(X, -)$ are also isomorphisms, so f_* is an n -hypergroupoid. \square

We now establish existence of TC_n -projectives.

Assumption 3.20. From now on, assume that there exists a small full subcategory $\mathcal{A}^s \subset \mathcal{A}$ with the property that for every trivial (n, \mathbf{C}) -hypergroupoid $X \rightarrow Y$ in $s\mathcal{A}$, there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \bar{Y}, \end{array}$$

with $\bar{X} \rightarrow \bar{Y}$ a trivial (n, \mathbf{C}) -hypergroupoid in $s\mathcal{A}^s$, and $X \rightarrow Y \times_{\bar{Y}} \bar{X}$ a Reedy trivial fibration.

Examples 3.21. For our main motivating examples (Artin and Deligne–Mumford hypergroupoids, together with their derived analogues), Assumption 3.20 is satisfied by taking \mathcal{A}^s to consist of objects of finite type. This works because every smooth (resp. étale) morphism $A \rightarrow B$ of rings is the pullback of a smooth (resp. étale) morphism $A' \rightarrow B'$ for some finitely generated subring $A' \subset A$.

The subring A' is constructed by choosing generators and relations for B over A , then taking A' to be generated by the coefficients of the relations together with various discriminants. We know by Lemma 3.7 that a trivial (n, \mathbf{C}) -hypergroupoid $f: X \rightarrow Y$ is determined by the morphism $f_{<n}: X_{<n} \rightarrow Y_{<n}$ of finite diagrams, and the discussion above shows that $f_{<n}$ must be the pullback of a trivial (n, \mathbf{C}) -hypergroupoid in $s_{n-1}\mathcal{A}^s$, from which it follows that Assumption 3.20 is satisfied.

If we wish to work in the full generality of [TV2] or of §1.1, then we can follow [TV2] in picking two universes $\mathbb{U} \subset \mathbb{V}$, with $\text{Aff}_{\mathcal{C}}$ being a \mathbb{V} -small category with \mathbb{U} -small Hom-sets, and containing only \mathbb{U} -small limits. We can then take \mathcal{A}^s to be the image of $\text{Aff}_{\mathcal{C}}$ under $\mathbf{R}h$, and Assumption 3.20 applies provided we interpret “small” as “ \mathbb{V} -small”. We must then permit \mathbb{V} -small inverse systems in Definition 3.16. The corresponding application of this assumption in [TV2] lies in the construction of $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ via Bousfield localisation of a \mathbb{V} -small set. In practice, the only interesting example known to the author of a HAG context requiring the introduction of a larger universe is when we take \mathbf{C} to be the class of fpqc morphisms of affine schemes.

We now give two definitions adapted from [Hov] §2.1.

Definition 3.22. Given an ordinal λ , define a λ -cosequence in a category \mathcal{C} to be a limit-preserving functor $X: \lambda^{\text{opp}} \rightarrow \mathcal{C}$. We say that the *composition* of X is the map $\varprojlim_{\beta < \lambda} X_{\beta} \rightarrow X_0$.

Definition 3.23. Given a class P of morphisms in a category \mathcal{C} , we say that a morphism f in \mathcal{C} is a relative P -cocell if it can be written as a transfinite composition of pullbacks of elements of P . In other words, for some ordinal λ , f can be written as the composition of a λ -cosequence $X: \lambda^{\text{opp}} \rightarrow \mathcal{C}$ with the property that for all β with $\beta + 1 < \lambda$, there is a pullback diagram

$$\begin{array}{ccc} X_{\beta+1} & \longrightarrow & X_{\beta} \\ \downarrow & & \downarrow \\ D_{\beta} & \xrightarrow{g_{\beta}} & C_{\beta} \end{array}$$

with g_{β} in P .

Proposition 3.24. *There is a functor $Q: \text{pro}(s\mathcal{A}) \rightarrow \text{pro}(s\mathcal{A})$ equipped with a natural transformation $Q \rightarrow \text{id}$ such that for all $X \in \mathcal{A}$, the morphism $QX \rightarrow X$ is a relative TC_n -cocell, with QX a TC_n -projective object of $\text{pro}(s\mathcal{A})$.*

Proof. The factorial factorisation provided by the Reedy model structure on $s\mathcal{A}$ gives us a functor $Q_R: s\mathcal{A} \rightarrow s\mathcal{A}$ for which each Q_RX is Reedy cofibrant, together with a natural transformation $Q_R \rightarrow \text{id}$ with $Q_RX \rightarrow X$ a Reedy trivial fibration.

Applying the small object argument ([Hov] Theorem 2.1.14) to the set $TC_n^s := \mathcal{A}^s \cap TC_n$ yields a functor $Q^s: \text{pro}(\mathcal{A}) \rightarrow \text{pro}(\mathcal{A})$ and a natural transformation $Q^s \rightarrow \text{id}$ with each $Q^s X \rightarrow X$ a relative TC_n -cocell with $Q^s X$ being TC_n -projective.

Now set $QX := \varprojlim_n (Q^s Q_R)^n$, the limit taken in $\text{pro}(s\mathcal{A})$. This is automatically both TC_n^s -projective and Reedy cofibrant. By assumption 3.20, every element of TC_n can be written as the composition of a Reedy trivial fibration with the pullback of a morphism in TC_n^s , so it follows that QX is TC_n -projective. \square

Definition 3.25. Given $S \in s\mathcal{A}$ Reedy fibrant, define $\mathcal{G}_n(\mathcal{A}, \mathbf{C}, S)$, the *localised simplicial category of (n, \mathbf{C}) -hypergroupoids over S* as follows. Objects of $\mathcal{G}_n(\mathcal{A}, \mathbf{C}, S)$ are (n, \mathbf{C}) -hypergroupoids $X \rightarrow S$, while morphisms are given by

$$\underline{\text{Hom}}_{\mathcal{G}_n(\mathcal{A}, \mathbf{C}, S)}(X, Y) := \underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(QX, QY),$$

for the TC_n -projective replacement functor $Q: s\mathcal{A} \rightarrow \text{pro}(s\mathcal{A})$ of Proposition 3.24.

The choice of TC_n -projective replacement does not matter:

Lemma 3.26. *Take $\tilde{X} \rightarrow X$ a relative TC_n -cocell in $\text{pro}(s\mathcal{A})$ with \tilde{X} a TC_n -projective. Then for all (n, \mathbf{C}) -hypergroupoids $Y \rightarrow S$, the spaces $\underline{\text{Hom}}_{\mathcal{G}_n(\mathcal{A}, \mathbf{C}, S)}(X, Y)$ and $\underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}, Y)$ are weakly equivalent.*

Proof. By Lemma 3.19, the map

$$\underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(QX, QY) \rightarrow \underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(QX, Y)$$

is a trivial fibration. It therefore suffices to show that for any two choices \tilde{X}_1, \tilde{X}_2 of \tilde{X} above, we have $\underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}_1, Y) \simeq \underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}_2, Y)$

Now, let $\tilde{X}' = Q(\tilde{X}_1 \times_X \tilde{X}_2)$; this is a TC_n -projective, and the canonical maps $p: \tilde{X}' \rightarrow \tilde{X}_i$, are both relative TC_n -cocells. We will show that $p^*: \underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}_i, Y) \rightarrow \underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}', Y)$ is a homotopy equivalence.

Since \tilde{X}_i is TC_n -projective, the map p has a section s . Then we have $(\text{id}, sp): \tilde{X}' \rightarrow \tilde{X}' \times_{\tilde{X}_i} \tilde{X}'$, but since \tilde{X}' is TC_n -projective, this lifts to a map $h: \tilde{X}' \rightarrow (\tilde{X}')^{\Delta^1} \times_{\tilde{X}_i^{\Delta^1}} \tilde{X}_i$. For any $Z \in \text{pro}(s\mathcal{A})$, this induces a map

$$h^*: \text{Hom}_{\text{pro}(s\mathcal{A})}(\tilde{X}', Z) \rightarrow \text{Hom}_{\text{pro}(s\mathcal{A})}(\tilde{X}', Z^{\Delta^1}),$$

given by $f \mapsto (f^{\Delta^1}) \circ h$. Considering Z of the form $Y^{\Delta^1}, S^{\Delta^1}$, this gives us a homotopy

$$h^*: \underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}', Y) \rightarrow \underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}', Y)^{\Delta^1}$$

between $p^* s^*$ and id , so p^* is a deformation retract. \square

Remark 3.27. Considering [BK2] XI.9.2, it is natural to ask whether the diagram \tilde{X} is cofinal in some more natural category I over X , in which case $\underline{\text{Hom}}_{\text{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}, Y) \simeq \text{holim}_{X' \in I} \underline{\text{Hom}}_{s\mathcal{A} \downarrow S}(X', Y)$. Unfortunately, \tilde{X} is not cofinal in the category $\mathcal{T}_n(X)$ of trivial (n, \mathbf{C}) -hypergroupoids over X , in general. However, if we regard $\mathcal{T}_n(X)$ as a full simplicial

subcategory of $s\mathcal{A}\downarrow X$, then \tilde{X} is cofinal in the homotopy category $\pi_0\mathcal{T}_n(X)$. This means that

$$\begin{aligned}\pi_0\mathbf{Hom}_{\mathrm{pro}(s\mathcal{A})\downarrow S}(\tilde{X}, Y) &\simeq \varinjlim_{X' \in \pi_0\mathcal{T}_n(X)} \pi_0\mathbf{Hom}_{s\mathcal{A}\downarrow S}(X', Y), \\ \pi_n(\mathbf{Hom}_{\mathrm{pro}(s\mathcal{A})\downarrow S}(\tilde{X}, Y), f) &\simeq \varinjlim_{X' \in \pi_0\mathcal{T}_n(X)} \pi_n(\mathbf{Hom}_{s\mathcal{A}\downarrow S}(X', Y), f).\end{aligned}$$

More generally, if we write $\tilde{X} = \{\tilde{X}_i\}_i$, then for any $T \in \mathcal{T}_n(X)$, any $m \geq 0$ and any element $f \in M_m\mathbf{Hom}_{s\mathcal{A}\downarrow X}(\tilde{X}_i, T)$, there is a $j \geq i$ for which f lifts from $M_m\mathbf{Hom}_{s\mathcal{A}\downarrow X}(\tilde{X}_j, T)$ to $\mathbf{Hom}_{s\mathcal{A}\downarrow X}(\tilde{X}_j, T)_m$. This suggests that we should be working with a notion of homotopy colimits indexed by the *simplicial* category $\mathcal{T}_n(X)$, possibly by generalising the formulae of [BK2] Ch. XII to involve nerves of simplicial categories.

Remark 3.28. The full subcategory of $s\mathcal{A}\downarrow S$ consisting of the (n, \mathbf{C}) -hypergroupoids over S has the structure of a category of fibrant objects in the sense of [Bro] §1. We can say that $f : X \rightarrow Y$ is a weak equivalence if $\mathbf{Hom}_{\mathrm{pro}(s\mathcal{A})\downarrow S}(QU, X) \rightarrow \mathbf{Hom}_{\mathrm{pro}(s\mathcal{A})\downarrow S}(QU, Y)$ is a weak equivalence in \mathbb{S} for all $U \in \mathcal{A}$, and fibrations are relative (n, \mathbf{C}) -hypergroupoids. The path object of X is given by X^{Δ^1} . We also obtain a category of fibrant objects if we take the union over all n .

This raises the question of whether (n, \mathbf{C}) -hypergroupoids can be realised as fibrant objects of some simplicial model structure on $s\mathcal{A}$. Since a model structure is determined by the fibrant objects and the trivial fibrations, this is very unlikely to be true in general. On the other hand, $\mathcal{G}_n(\mathcal{A}, \mathbf{C}, S)$ can be regarded as a full simplicial subcategory of the right Bousfield localisation $R_{TC_n}(\mathrm{pro}(s\mathcal{A}))$ of $\mathrm{pro}(s\mathcal{A})$ with respect to trivial (n, \mathbf{C}) -hypergroupoids, with Q becoming a cofibrant replacement functor. However, this model category has many more fibrant objects than just the (n, \mathbf{C}) -hypergroupoids.

3.4. Artin n -hypergroupoids and étale hypercoverings. We now specialise to the case where $\mathcal{A} = \mathrm{Aff}$, the category of affine schemes, with trivial model structure. If \mathbf{C} is the class of smooth surjections, then Proposition 3.24 seems inadequate, since we would expect to be able to define the sheafification by considering just étale hypercoverings, rather than smooth hypercoverings. We will now show how to establish this in a slightly more general setting.

For a class \mathbf{C} as in §3.1, let $\mathbf{E} \subset \mathbf{C}$ be a class of morphisms of affine schemes containing all isomorphisms and stable under composition and homotopy pullback, together with the following condition:

- (E) for all diagrams $Z \xrightarrow{f} X \xrightarrow{g} Y$, with g in \mathbf{C} and gf a closed immersion, there exists a factorisation $Z \xrightarrow{\alpha} W \xrightarrow{\beta} X$ of f , for which $g\beta$ is in \mathbf{E} .

Lemma 3.29. *If \mathbf{C} is the class of smooth surjections, then the class of étale surjections satisfies condition (E).*

Proof. If $Z = \emptyset$, then this follows from [Gro1] 17.16.3, giving $W_0 \rightarrow X$ for which the composition $W_0 \rightarrow Y$ is an étale surjection.

In general, since $Z \rightarrow Y$ is a closed immersion, we may form the henselisation $Z \rightarrow Y^h \xrightarrow{f'} Y$, as in [Gre2]; then (Z, Y^h) is a henselian pair (as in [Laf]), and $Y^h \rightarrow Y$ is a filtered inverse limit of étale morphisms. Now, [Gru] Theorem I.8 shows that the map $Z \rightarrow X$ extends to $Y^h \rightarrow X$ over Y , since g is smooth. Finally, since g is finitely presented, and f' is pro-étale, the map $Y^h \rightarrow X$ factors through some quotient W' with $W' \rightarrow Y$ étale. Setting $W = W_0 \amalg W'$ completes the proof. \square

Definition 3.30. Given a simplicial diagram X_\bullet in a cocomplete category \mathcal{C} , recall from [GJ] §VII.1 that the n th latching object $L_n X_\bullet$ is defined to be $(\mathrm{sk}_{n-1} X)_n$. Explicitly, this

is the coequaliser

$$\coprod_{i=0}^{n-1} \coprod_{j=0}^{i-1} X_{n-2} \xrightarrow[\beta]{\alpha} \coprod_{i=0}^{n-1} X_{n-1} \longrightarrow L_n X,$$

where for $x \in X_{n-2}^{(i,j)}$, we define $\alpha(x) = \sigma_j x \in X_{n-1}^{(i)}$, and $\beta(x) = \sigma_{i-1} x \in X_{n-1}^j$.

Note that there is a map $L_n X \rightarrow X$ given by σ_i on $X_{n-1}^{(i)}$ — this comes from the counit $\text{sk}_{n-1} X \rightarrow X$ of the adjunction $i_{n-1}^* \dashv i_{n-1,*}$ of Definition 2.7.

Proposition 3.31. *Given a diagram $Z \xrightarrow{f} X \xrightarrow{g} Y$ in $s\text{Aff}$, with g a trivial relative (n, \mathbf{C}) -hypergroupoid and gf a levelwise closed immersion, there exists a factorisation $Z \xrightarrow{\alpha} W \xrightarrow{\beta} X$ of f , for which $g\beta$ is a trivial relative (n, \mathbf{E}) -hypergroupoid.*

Proof. The construction of W is inductive. For $r < n$, assume that we have constructed an $(r-1)$ -truncated simplicial affine scheme $W_{<r}$, with $Z_{<r} \rightarrow W_{<r} \rightarrow X_{<r}$ satisfying the required properties up to level $r-1$.

Since we are working with affine schemes, all colimits exist, including latching objects, and we now seek W_r fitting into the diagram

$$L_r W \cup_{L_r Z} Z_r \rightarrow W_r \rightarrow M_r W \times_{M_r X} X_r,$$

with the relative matching map $W_r \rightarrow M_r W \times_{M_r Y} Y_r$ an \mathbf{E} -morphism.

Since $X_r \rightarrow M_r X \times_{M_r Y} Y_r$ is a \mathbf{C} -morphism, pulling back along $M_r W \rightarrow M_r X$ gives a \mathbf{C} -morphism $M_r W \times_{M_r X} X_r \rightarrow M_r W \times_{M_r Y} Y_r$. Since the relative matching map of a surjection of cosimplicial abelian groups is abelian, the same is true for cosimplicial rings. Thus the latching maps of a levelwise closed immersion of simplicial affine schemes are all closed immersions. In particular, $L_r W \cup_{L_r Z} Z_r \rightarrow M_r W \times_{M_r Y} Y_r$ is a closed immersion.

Applying condition (E) to $L_r W \cup_{L_r Z} Z_r \rightarrow M_r W \times_{M_r X} X_r \rightarrow M_r W \times_{M_r Y} Y_r$ provides a factorisation $L_r W \cup_{L_r Z} Z_r \rightarrow W_r \rightarrow M_r W \times_{M_r X} X_r$, with the composition $W_r \rightarrow M_r W \times_{M_r Y} Y_r$ an \mathbf{E} -morphism. This completes the inductive step. Having constructed $W_{\leq n-1}$, we set $W = (i_{n-1}^! W_{\leq n-1}) \times_{\text{cosk}_{n-1} Y} Y$. \square

Corollary 3.32. *There is a functor $Q: \text{pro}(s\text{Aff}) \rightarrow \text{pro}(s\text{Aff})$ equipped with a natural transformation $Q \rightarrow \text{id}$ such that for all $X \in \mathcal{A}$, the morphism $QX \rightarrow X$ is a relative $T\mathbf{E}_n$ -cocell, with QX a $T\mathbf{C}_n$ -projective object of $\text{pro}(s\mathcal{A})$.*

Proof. Proposition 3.24 applied to the class \mathbf{E} provides us with a functor Q for which $QX \rightarrow X$ is a relative $T\mathbf{E}_n$ -cocell, with QX a $T\mathbf{E}_n$ -projective object of $\text{pro}(s\mathcal{A})$. Now just observe that Proposition 3.31 ensures that every $T\mathbf{E}_n$ -projective is $T\mathbf{C}_n$ -projective. \square

4. HYPERGROUPOIDS VS. n -STACKS

4.1. Passage to n -stacks. We now assume that our pseudo-model category $\mathcal{A} \subset \mathcal{S}$ is equipped with all the additional structure of §1.1.

Proposition 4.1. *If $f: X \rightarrow Y$ is a relative (n, \mathbf{C}) -hypergroupoid, then the associated morphism*

$$|f|: |X| \rightarrow |Y|$$

in \mathcal{S} an n -representable morphism in the sense of Definition 1.9.

If $f_0: X_0 \rightarrow Y_0$ is also in \mathbf{C} , then $|f|$ is an (n, \mathbf{C}) -morphism.

Proof. First, we reduce this to the case when $Y \in \mathcal{A}$. By Property 1.7.1, the map $Y_0 \rightarrow |Y|$ is an ε -morphism. By Lemma 1.10, it therefore suffices to show that $|X| \times_{|Y|}^h Y_0 \rightarrow Y_0$ is n -representable, and that it lies in $n - \mathbf{C}$ when f_0 is in \mathbf{C} .

Since f satisfies the conditions of Property 1.7.3, we have

$$|X| \times_{|Y|}^h Y_0 \simeq |X \times_Y^h Y_0|.$$

By homotopy pullback, $X \times_Y^h Y_0$ is an (n, \mathbf{C}) -hypergroupoid over Y_0 , and $(X \times_Y^h Y_0)_0 \simeq X_0$, allowing us to replace Y with Y_0 for both statements.

We now proceed by induction on n . For $n = 0$, the statements are immediate.

Now assume that the inductive hypothesis (over arbitrary base) holds for $n - 1$, and take a relative (n, \mathbf{C}) -hypergroupoid $X \rightarrow Y_0$. Since X is then an (n, \mathbf{C}) -hypergroupoid, Lemma 3.9 implies that $\text{Dec}_+(X) \rightarrow X$ is a relative $(n - 1, \mathbf{C})$ -hypergroupoid, with $\text{Dec}_+(X)_0 \rightarrow X_0$ in \mathbf{C} , so by induction $|\text{Dec}_+(X)| \rightarrow |X|$ is an $(n - 1, \mathbf{C})$ -morphism. Moreover, Lemma 3.13 and Property 1.7.2 imply that $X_0 \simeq |\text{Dec}_+(X)|$, which means that we have an $(n - 1, \mathbf{C})$ -morphism $X_0 \rightarrow |X|$. Since X_0 is 0-geometric, Proposition 1.11 now implies that $|X|$ is n -geometric. As Y_0 is in \mathcal{A} , this implies that $|f| : |X| \rightarrow Y_0$ is n -representable.

If $f_0 : X_0 \rightarrow Y_0$ is in \mathbf{C} , then the n -atlas X_0 is a \mathbf{C} -morphism over Y_0 , so $|X| \rightarrow Y_0$ is in $n\text{-}\mathbf{C}$. This completes the induction. \square

4.2. Resolutions. In this section, we will establish the converse to Proposition 4.1. By way of motivation, we begin with the cases of low degree. Given a 1-geometric stack \mathfrak{X} , take a 1-atlas $X_0 \rightarrow \mathfrak{X}$, and observe that $\text{cosk}_0^h(X_0/\mathfrak{X})$ gives a $(1, \mathbf{C})$ -hypergroupoid resolving \mathfrak{X} .

Now for the construction of a $(2, \mathbf{C})$ -hypergroupoid X resolving a 2-geometric stack \mathfrak{X} . Take a 2-atlas $Z_0 \rightarrow \mathfrak{X}$; since $Z_0 \times_{\mathfrak{X}}^h Z_0$ is 1-geometric, it admits a 1-atlas Z_1 . Since an atlas is a \mathbf{C} -morphism, the diagonal map $Z_0 \rightarrow Z_0 \times_{\mathfrak{X}}^h Z_0$ must admit a lift $X_0 \rightarrow Z_1$ for some \mathbf{C} -morphism $X_0 \rightarrow Z_0$. Setting $X_1 := Z_1 \times_{(Z_0 \times Z_0)} (X_0 \times X_0)$ gives a 1-truncated simplicial scheme $X_{\leq 1}$ over \mathfrak{X} , so we can take X to be the coskeleton

$$X := \text{cosk}_1^h(X_{\leq 1}/\mathfrak{X}),$$

and X will be a $(2, \mathbf{C})$ -hypergroupoid.

Note that we cannot adapt the approach of [SD] to construct X as a split simplicial resolution, since that would entail taking $X_0 = Z_0$ and $X_1 = Z_1 \sqcup Z_0$. It fails because the diagonal map $(Z_0 \times_{\mathfrak{X}} Z_0) \sqcup Z_0 \rightarrow Z_0 \times_{\mathfrak{X}} Z_0$ will seldom be a \mathbf{C} -morphism.

Definition 4.2. Given an object $S \in \mathcal{S}$, define $S^{\Delta_r} \in s\mathcal{S}$ to be given by $(S^{\Delta_r})_i = S^{\Delta_r^i}$.

Lemma 4.3. For $K \in \mathbb{S}$, there is a natural equivalence

$$M_K^h(S^{\Delta_r}) \simeq S^{K_r},$$

where M_K^h is the homotopy matching object of Definition 1.5.

Thus, for a morphism $f : S \rightarrow T$ in \mathcal{S} , the i th homotopy matching map $(S^{\Delta_r})_i \rightarrow M_i^h(S^{\Delta_r}) \times_{M_i^h(T^{\Delta_r})} (T^{\Delta_r})_i$ is

- (1) a weak equivalence when $i > r$;
- (2) a homotopy pullback of f when $i = r$;
- (3) a homotopy pullback of finitely many copies of f when $i < r$.

Proof. The first statement follows because the functors $K \mapsto M_K^h(S^{\Delta_r})$ and $K \mapsto S^{K_r}$ both send limits in \mathbb{S}^{opp} to homotopy limits in \mathcal{S} , and the functors agree on the objects Δ^n .

Thus we have canonical equivalences

$$(S^{\Delta_r})_i \simeq S^{\Delta_r^i} \simeq S^{\partial \Delta_r^i} \times S^{\Delta_r^i - \partial \Delta_r^i} \simeq M_i(S^{\Delta_r}) \times S^{\Delta_r^i - \partial \Delta_r^i},$$

giving the required isomorphisms and pullbacks. \square

Definition 4.4. Given $Z_{\bullet} \in s\mathcal{S}$ (i.e. a simplicial diagram in \mathcal{S}), let $a_r(Z) \in \mathbb{N}_0$ be the smallest number for which the homotopy matching morphism

$$Z_r \rightarrow M_r^h Z$$

is $a_r(Z)$ -representable, whenever this is defined.

If the numbers $a_r(Z)$ are all defined and are eventually 0, define

$$\nu(Z) := \sum_{\substack{i \geq 0 \\ a_i(Z) \neq 0}} 2^i.$$

Proposition 4.5. *Given $Z \in s\mathcal{S}$ for which $\nu(Z)$ is defined and $\nu(Z) < 2^n$, there exists $\tilde{Z} \in s\mathcal{A}$ and a morphism $\tilde{Z} \rightarrow Z$ in $\text{Ho}(s\mathcal{S})$ with the property that the relative homotopy matching maps*

$$\tilde{Z}_r \rightarrow (M_r^h \tilde{Z}) \times_{(M_r^h Z)}^h Z_r$$

are \mathbf{C} -morphisms for all $r \geq 0$, and equivalences for all $r \geq n$.

Proof. Begin by replacing Z with a Reedy fibrant object in $s\mathcal{S}$ — this will allow us to choose atlases in \mathcal{S} rather than just in $\text{Ho}(\mathcal{S})$. We work by induction on $\nu(Z)$. If $\nu(Z) = 0$, then the matching maps are all 0-representable. Since 0-representability of the objects Z_i for $i < r$ implies 0-representability of $M_r^h Z$, we deduce that each Z_r is 0-geometric, giving $Z \in s\mathcal{A}$.

Now assume that the result holds for all Y with $\nu(Y) < \nu(Z)$. Let r be the smallest number for which $a_r(Z) \neq 0$; since $\nu(Z) < 2^n$, we know that $r < n$. The stacks Z_i are all 0-representable for $i < r$, so $M_r^h Z$ is 0-representable. As $Z_r \rightarrow M_r^h Z$ is $a_r(Z)$ -representable, we know that Z_r is an $a_r(Z)$ -geometric stack.

Let $f : S \rightarrow Z_r$ be an $a_r(Z)$ -atlas for Z_r and a fibration, observe that Lemma 4.3 gives a canonical map $S \rightarrow Z_r^{\Delta_r}$, and let

$$Z' := Z \times_{Z_r^{\Delta_r}}^h S^{\Delta_r} = Z \times_{Z_r^{\Delta_r}} S^{\Delta_r}.$$

Since f is in $(a_r(Z) - 1) - \mathbf{C}$, Lemma 4.3 now implies that the i th relative homotopy matching map of $Z' \rightarrow Z$ is a \mathbf{C} -morphism and is

- (1) an equivalence when $i > r$;
- (2) a pullback of f when $i = r$.

As the homotopy matching maps of Z' are simply the composition of these relative homotopy matching maps with a pullback of the homotopy matching maps for Z , the r th homotopy matching map is given by pulling back the composition

$$S \rightarrow Z_r \rightarrow M_r^h Z = M_r Z$$

along $M_r Z' \rightarrow M_r Z$. Since S and $M_r^h Z$ are both 0-representable, it follows that this map is 0-representable, so $a_r(Z') = 0$. Likewise, $a_i(Z') \leq a_i(Z)$ for all $i > r$. Moreover, $a_i(Z')$ is defined for all i , since the relative matching homotopy maps are \mathbf{C} -morphisms, so k -geometric for some k .

Thus $\nu(Z') < \nu(Z)$ (as $2^r > \sum_{i < r} 2^i$), so satisfies the inductive hypothesis, giving $\tilde{Z} \rightarrow Z'$ satisfying the conditions of the proposition. Now observe that the composition $\tilde{Z} \rightarrow Z' \rightarrow Z$ also satisfies these conditions, which completes the proof of the inductive step. \square

Remark 4.6. Here is a further explanation of why the induction has $2^n - 1$ steps. If we let $f(r)$ be the number of steps required to set a_i to 0 for all $i < r$, then one further step sets a_r to 0, at the expense of the a_i becoming non-zero for $i < r$. We therefore require $f(r)$ further steps to get $a_i = 0$ for all $i < r + 1$. Thus we have the recurrence relation $f(r + 1) = 2f(r) + 1$, with initial condition $f(0) = 0$, so $f(n) = 2^n - 1$.

Theorem 4.7. *Given $Y \in s\mathcal{A}$, and an n -representable morphism $f : \mathfrak{X} \rightarrow |Y|$ in $\text{Ho}(\mathcal{S})$, there exists a relative (n, \mathbf{C}) -hypergroupoid $X \rightarrow Y$ in $\text{Ho}(s\mathcal{A})$ whose realisation $|X|$ is equivalent to \mathfrak{X} over $|Y|$. Moreover, if f is a \mathbf{C} -morphism, then $X_0 \rightarrow Y_0$ is in \mathbf{C} .*

Proof. We first let $\mathfrak{Y} := |Y|$, and form the simplicial stack $Z \in s\mathcal{S}$ given by $Z_r := \mathfrak{X} \times_{\mathfrak{Y}}^h Y_r$. The r th relative homotopy matching map of $Z \rightarrow Y$ is just the pullback of that for $\mathfrak{X} \rightarrow \mathfrak{Y}$, so by Lemma 1.26, it is $(n-r)$ -representable for $r \leq n$, and 0-representable for $r > n$. Since the matching maps of Y are 0-representable (being in \mathcal{A}), it follows that $a_r(Z) \leq n-r$ for $r \leq n$, and $a_r(Z) = 0$ for $r \geq n$. Thus $\nu(Z)$ is defined and $\nu(Z) \leq 2^n - 1$.

Proposition 4.5 now gives a morphism $X \rightarrow Z$ in $\text{Ho}(s\mathcal{A})$ for which the relative homotopy matching maps

$$X_r \rightarrow (M_r X) \times_{M_r^h Z}^h Z_r$$

are \mathbf{C} -morphisms for all r , and equivalences for $r \geq n$. Since the relative homotopy matching maps are ε -morphisms, Property 1.7.2 ensures that $|X| \rightarrow |Z|$ is an equivalence in \mathcal{S} .

Now, Property 1.7.3 gives

$$|Z| \simeq \mathfrak{X} \times_{\mathfrak{Y}}^h |Y| \simeq \mathfrak{X},$$

so we have a weak equivalence $|X| \rightarrow \mathfrak{X}$.

For any cofibration $K \hookrightarrow L$ of finite simplicial sets, the map

$$M_L^h X \rightarrow M_K^h X \times_{M_K^h Z}^h M_L^h Z = (M_K^h X \times_{M_K^h Y}^h M_L^h Y) \times_{(\mathfrak{X}^{\mathbf{R}K} \times_{\mathfrak{Y}^{\mathbf{R}K}}^h \mathfrak{Y}^{\mathbf{R}L})}^h \mathfrak{X}^{\mathbf{R}L}$$

is a \mathbf{C} -morphism, and moreover an equivalence if $\text{sk}_{n-1}K \cong \text{sk}_{n-1}L$. When K and L are contractible, this map just becomes

$$M_L^h X \rightarrow M_K^h X \times_{M_K^h Y}^h M_L^h Y.$$

Thus X is indeed an (n, \mathbf{C}) -hypergroupoid over Y .

Finally, if f is a \mathbf{C} -morphism, then

$$|X| \times_{|Y|}^h Y_0 \rightarrow Y_0$$

is also a \mathbf{C} -morphism. Since $X_0 \rightarrow |X| \times_{|Y|}^h |Y_0|$ is an n -atlas, it is a \mathbf{C} -morphism too, so $X_0 \rightarrow Y_0$ is in \mathbf{C} . \square

4.3. Morphisms. As we saw in §3.3, the realisation functor $|-|$ from $s\mathcal{A}$ to stacks is not full, even if we restrict to (n, \mathbf{C}) -hypergroupoids. We will therefore now have to localise over trivial (n, \mathbf{C}) -hypergroupoids.

Lemma 4.8. *Given an (n, \mathbf{C}) -hypergroupoid $Y \rightarrow S$, the i th relative homotopy matching map of*

$$Y \rightarrow S \times_{|S|}^h |Y|$$

is a \mathbf{C} -morphism for all i , and an equivalence for $i \geq n$. If $i \leq n$, it is $(n-i)$ -representable.

Proof. Let $\mathfrak{S} = |S|, \mathfrak{Y} = |Y|$. The matching map is

$$\mu_i : Y_i \rightarrow (M_i^h Y \times_{M_i^h S}^h S_i) \times_{(\mathfrak{Y}^{\mathbf{R}\partial\Delta^i} \times_{\mathfrak{S}^{\mathbf{R}\partial\Delta^i}}^h \mathfrak{S})}^h \mathfrak{Y},$$

which by Property 1.7.3 is given by applying $|-|$ to the map

$$Y_i \rightarrow (M_i Y \times_{M_i S}^h S_i) \times_{(Y^{\partial\Delta^i} \times_{S^{\partial\Delta^i}}^h S^{\Delta^i})}^h Y^{\Delta^i},$$

where we have used the fact that $Y \rightarrow S$ is a Reedy fibration to replace homotopy matching objects with strict matching objects.

The map is given in simplicial level 0 by

$$Y_i \rightarrow (M_i Y \times_{M_i S} S_i) \times_{(M_i Y \times_{M_i S} S_i)} Y_i = Y_i;$$

this is certainly a \mathbf{C} -morphism, so μ_i is also a \mathbf{C} -morphism by Proposition 4.1.

If $i \geq n$, then Lemma 1.26 implies that μ_i is 0-representable, and if $i \leq n$, then it is $(n-i)$ -representable. \square

Proposition 4.9. *Take a morphism $X \rightarrow S$ in $s\mathcal{A}$ with S Reedy fibrant, a homotopy (n, \mathbf{C}) -hypergroupoid $Y \rightarrow S$, and a morphism*

$$f : |X| \rightarrow |Y|$$

in the homotopy category $\mathrm{Ho}(\mathcal{S} \downarrow |S|)$.

Then there exists a trivial homotopy (n, \mathbf{C}) -hypergroupoid $\pi : \tilde{X} \rightarrow X$ and a morphism $\tilde{f} : \tilde{X} \rightarrow Y$ in $\mathrm{Ho}(s\mathcal{A})$, such that $f \circ |\pi| = |f|$. Moreover, the map $(\pi, \tilde{f}) : \tilde{X} \rightarrow X \times_S^h Y$ is a homotopy (n, \mathbf{C}) -hypergroupoid.

Proof. Without loss of generality, we may assume that Y is a Reedy fibration over S . Write $\mathfrak{S} := |S|$, $\mathfrak{Y} := |Y|$ and $\mathfrak{X} := |X|$, and define the simplicial stack $Z \in s\mathcal{S}$ by

$$Z_r := X_r \times_{(\mathfrak{Y} \times_{\mathfrak{S}}^h S_r)}^h Y_r.$$

Observe that the relative homotopy matching maps of $Z \rightarrow X$ are obtained by pulling back the relative homotopy matching maps of $Y \rightarrow \mathfrak{Y} \times_{\mathfrak{S}}^h S$ along $X \rightarrow \mathfrak{Y} \times_{\mathfrak{S}}^h S$.

Now by Lemma 4.8, the i th such map is a \mathbf{C} -morphism, which is an equivalence for $i \geq n$, and $(n-i)$ -representable for $i \leq n$. Since the matching maps of X are 0-geometric, we may apply Proposition 4.5 to Z , obtaining $\tilde{X} \in s\mathcal{A}$, with the i th relative homotopy matching map of $\tilde{X} \rightarrow Z$ being a \mathbf{C} -morphism for all i , and an equivalence for $i \geq n$.

We therefore conclude that the i th matching map of $\pi : \tilde{X} \rightarrow X$ is a \mathbf{C} -morphism for all i , and an equivalence for $i \geq n$, so π is a trivial homotopy (n, \mathbf{C}) -hypergroupoid. Projection $Z \rightarrow Y$ on the second factor gives the map $\tilde{f} : \tilde{X} \rightarrow Y$.

Finally, observe that $Z \rightarrow X \times_S Y$ is a pullback of the $(n-1)$ -geometric map $\mathfrak{Y} \rightarrow \mathfrak{Y} \times_{\mathfrak{S}}^h \mathfrak{Y}$, so the relative partial matching maps of $\tilde{X} \rightarrow X \times_S Y$ are \mathbf{C} -morphisms, and are weak equivalences in levels above n , as required. \square

Theorem 4.10. *Take $S \in s\mathcal{A}$ Reedy fibrant, and take a morphism $X \rightarrow S$ in $s\mathcal{A}$. Take a relative cocell $\tilde{X} \rightarrow X$ in trivial (n, \mathbf{C}) -hypergroupoids in $\mathrm{pro}(s\mathcal{A})$, with \tilde{X} a $T\mathbf{C}_n$ -projective object in $\mathrm{pro}(s\mathcal{A})$ (see Definition 3.18). Then for all (n, \mathbf{C}) -hypergroupoids $f : Y \rightarrow S$ in $s\mathcal{A}$, the natural map*

$$\underline{\mathrm{Hom}}_{\mathrm{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}, Y) \rightarrow \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{S} \downarrow |S|}(|X|, |Y|)$$

is a weak equivalence.

Proof. Writing $L := \underline{\mathrm{Hom}}_{\mathrm{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}, Y)$ and $R := \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{S} \downarrow |S|}(|X|, |Y|)$, it suffices to show that the maps

$$L_m \rightarrow M_m L \times_{R^{\mathbf{R}\partial\Delta^m}}^h R^{\mathbf{R}\Delta^m}$$

are surjective on π_0 for all $m \geq 0$.

Now, for $K \in \mathbb{S}$, we have

$$\begin{aligned} M_K L &= \underline{\mathrm{Hom}}_{\mathrm{pro}(s\mathcal{A}) \downarrow S}(\tilde{X}, Y^K \times_{S^K} S), \\ R^{\mathbf{R}K} &\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{S} \downarrow |S|}(|X|, |Y^K \times_{S^K} S|). \end{aligned}$$

Fix a morphism $h : \tilde{X} \rightarrow Y^{\partial\Delta^m} \times_{S^{\partial\Delta^m}} S$, and note that this factors through some $\tilde{X}_\alpha \in s\mathcal{A} \downarrow X$, where $\tilde{X} = \{\tilde{X}_\alpha\}_\alpha$. Also fix an element g of $\pi_0(\{h\} \times_{R^{\mathbf{R}\partial\Delta^m}}^h R^{\mathbf{R}\Delta^m})$.

We now just apply Proposition 4.9, replacing X with \tilde{X}_α , Y with Y^{Δ^m} and S with $Y^{\partial\Delta^m} \times_{S^{\partial\Delta^m}} S$. This gives a trivial (n, \mathbf{C}) -hypergroupoid $X' \rightarrow \tilde{X}_\alpha$, with an element $f \in \mathrm{Hom}_{s\mathcal{A} \downarrow (Y^{\partial\Delta^m} \times_{S^{\partial\Delta^m}} S)}(X', Y^{\Delta^m})$ mapping to g . Since \tilde{X} is $T\mathbf{C}_n$ -projective, we have a section $\tilde{X} \rightarrow X'$, so f lifts to an element of $L_m \times_{M_m L} \{h\}$ over g , which gives the required surjectivity. \square

Remark 4.11. Following Remark 3.27, note that the homotopy groups thus have the more natural characterisation

$$\begin{aligned}\pi_0 \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{S}\parallel\mathbb{S}}(|X|, |Y|) &\simeq \varinjlim_{X' \in \pi_0 \mathcal{T}_n(X)} \pi_0 \underline{\mathrm{Hom}}_{s\mathcal{A}\parallel\mathbb{S}}(X', Y), \\ \pi_n(\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{S}\parallel\mathbb{S}}(|X|, |Y|), f) &\simeq \varinjlim_{X' \in \pi_0 \mathcal{T}_n(X)} \pi_n(\underline{\mathrm{Hom}}_{s\mathcal{A}\parallel\mathbb{S}}(X', Y), f),\end{aligned}$$

where the colimits are taken over the category of trivial (n, \mathbf{C}) -hypergroupoids X' over X . In the case of Artin stacks, the results of §3.4 show that it suffices to take the colimit over trivial Deligne–Mumford hypergroupoids over X .

4.4. From representables to affines. In the case of strongly quasi-compact n -geometric stacks, recall that Definition 1.18 takes \mathcal{A} to be the essential image of the functor $\mathbf{R}h: \mathrm{Aff}_{\mathcal{C}} \rightarrow \mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$. While Theorems 4.7 and 4.10 characterise n -geometric stacks as n -hypergroupoids in \mathcal{A} , a description as n -hypergroupoids in $\mathrm{Aff}_{\mathcal{C}}$ would be far more satisfactory. We now show that these two theories are indeed equivalent.

Lemma 4.12. *When $\mathcal{A} \subset \mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$ is the pseudo-model category of representable stacks, the functor $\mathbf{R}h: s\mathrm{Aff}_{\mathcal{C}} \rightarrow s\mathcal{A}$ induces a weak equivalence on the simplicial subcategories of fibrant cofibrant objects.*

Proof. In the proof of [TV2] Lemma 1.3.2.9, it is shown that the essential image of $\mathbf{R}h: \mathrm{Ho}(s\mathrm{Aff}_{\mathcal{C}}) \rightarrow \mathrm{Ho}(s\mathrm{Aff}_{\mathcal{C}}^{\sim, \tau})$ consists of simplicial stacks F_{\bullet} for which each F_n is representable, i.e. strongly quasi-compact and 0-geometric. In other words, the essential image is $\mathrm{Ho}(s\mathcal{A})$.

Since $\mathbf{R}\underline{\mathrm{Hom}}_{s\mathcal{A}}(\mathbf{R}hX, \mathbf{R}hY)$ and $\mathbf{R}\underline{\mathrm{Hom}}_{s\mathrm{Aff}_{\mathcal{C}}}(X, Y)$ can be expressed as homotopy limits of $\mathrm{map}_{\mathcal{A}}(\mathbf{R}hX_m, \mathbf{R}hY_n) \simeq \mathrm{map}_{\mathrm{Aff}_{\mathcal{C}}}(X_m, Y_n)$, we also have weak equivalences

$$\mathbf{R}\underline{\mathrm{Hom}}_{s\mathcal{A}}(\mathbf{R}hX, \mathbf{R}hY) \rightarrow \mathbf{R}\underline{\mathrm{Hom}}_{s\mathrm{Aff}_{\mathcal{C}}}(X, Y),$$

giving equivalence of simplicial Hom-spaces. \square

Proposition 4.13. *For $S \in s\mathrm{Aff}_{\mathcal{C}}$ Reedy fibrant, the functor $\mathbf{R}h$ induces a equivalence*

$$\mathcal{G}_n(\mathrm{Aff}_{\mathcal{C}}, \mathbf{C}, S) \rightarrow \mathcal{G}_n(\mathcal{A}, \mathbf{C}, \mathbf{R}hS)$$

of simplicial categories, for \mathcal{G}_n the localised category of (n, \mathbf{C}) -hypergroupoids from Definition 3.25.

Proof. This is an immediate consequence of Lemma 4.12 once we note that the functor Q of Proposition 3.24 sends fibrant objects to inverse systems of fibrant cofibrant objects. \square

Remark 4.14. In particular, when Y is an (n, \mathbf{C}) -hypergroupoid in $s\mathrm{Aff}_{\mathcal{C}}$, note that the simplicial presheaf $Y: \mathrm{Aff}_{\mathcal{C}} \rightarrow \mathbb{S}$ is determined by $Y(U) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathbf{R}hU, |Y|)$. By Theorem 4.10, this can be expressed as a direct limit of spaces $\underline{\mathrm{Hom}}_{s\mathcal{A}}(U', Y)$ for certain families of trivial (n, \mathbf{C}) -hypergroupoids $U' \rightarrow U$. This gives us a characterisation of the sheafification functor $\mathbf{R}h$ depending only on \mathbf{C} (without reference to the topology τ).

Summarising the main results so far gives the following.

Theorem 4.15. *The simplicial category of (n, \mathbf{C}) -geometric stacks is weakly equivalent to the simplicial category $\mathcal{G}_n(\mathcal{A}, \mathbf{C})$ whose objects are (n, \mathbf{C}) -hypergroupoids in the category \mathcal{A} of 0-geometric stacks, with morphisms given by*

$$\underline{\mathrm{Hom}}_{\mathcal{G}_n}(X, Y) = \underline{\mathrm{Hom}}_{\mathrm{pro}(s\mathcal{A})}(QX, QY),$$

for Q the $T\mathbf{C}_n$ -projective replacement functor of Proposition 3.24.

The simplicial category of strongly quasi-compact (n, \mathbf{C}) -geometric stacks is weakly equivalent to the simplicial category $\mathcal{G}_n(\text{Aff}_{\mathcal{C}}, \mathbf{C})$ whose objects are (n, \mathbf{C}) -hypergroupoids in $\text{Aff}_{\mathcal{C}}$, with morphisms given by

$$\underline{\text{Hom}}_{\mathcal{G}_n}(X, Y) = \underline{\text{Hom}}_{\text{pro}(s\text{Aff}_{\mathcal{C}})}(QX, QY).$$

Proof. The functor from $\mathcal{G}_n(\mathcal{A}, \mathbf{C}, S)$ to (n, \mathbf{C}) -geometric stacks is given by geometric realisation $X \mapsto |X|$. Essential surjectivity of this functor is given by Theorem 4.7, while full faithfulness is Theorem 4.10 combined with Lemma 3.26. The $\text{Aff}_{\mathcal{C}}$ statements then follow from Proposition 4.13. \square

Remarks 4.16. Observe that this categorisation of n -geometric stacks gives the same category for any topology τ' with respect to which the class of \mathbf{C} -morphisms is local.

Also note that in the case of n -geometric Artin stacks, we can apply Corollary 3.32 and construct Q using just étale (rather than smooth) covers.

For a simpler description of morphisms $\pi_0 \underline{\text{Hom}}$ in the homotopy category, see Remark 4.11.

5. QUASI-COHERENT MODULES

Theorem 4.7 allows us to replace (n, \mathbf{C}) -geometric stacks with (n, \mathbf{C}) -hypergroupoids. This means that we can apply the results of [Ols1] concerning sheaves on simplicial algebraic spaces, thus extending their consequences from Artin 1-stacks to higher Artin stacks, and possibly beyond.

However, it is easier to define quasi-coherent modules as presheaves than as sheaves. This becomes especially important when we work with quasi-coherent complexes, which involve hyperdescent rather than descent.

5.1. Left Quillen presheaves and Cartesian sections. The following is taken from [HS2] §17:

Definition 5.1. Given a category \mathbb{I} , define a *left Quillen presheaf* M on \mathbb{I} to consist of model categories $M(i)$ for all objects $i \in \mathbb{I}$, together with left Quillen functors $f^*: M(j) \rightarrow M(i)$ for all morphisms $f: i \rightarrow j$ in \mathbb{I} , satisfying associativity. Denote the right adjoint to f^* by f_* .

Note that we can regard M as a functor from \mathbb{I}^{opp} to the category of categories.

Definition 5.2. Given a left Quillen presheaf M on a category \mathbb{I} , define the category $M^{\mathbb{I}^{\text{opp}}}$ of sections to consist of natural transformations $\text{id}_{\mathbb{I}^{\text{opp}}} \rightarrow M$. In other words, an object of $M^{\mathbb{I}^{\text{opp}}}$ consists of objects $m(i) \in M(i)$ for all $i \in \mathbb{I}$ and morphisms $\eta(f): f^*m(j) \rightarrow m(i)$ for all morphisms $f: i \rightarrow j$ in \mathbb{I} , with η satisfying associativity. A morphism $g: (m, \eta) \rightarrow (n, \zeta)$ consists of morphisms $g(i): m(i) \rightarrow n(i)$ for all $i \in \mathbb{I}$, satisfying $\zeta(f) \circ f^*g(j) = g(i) \circ \eta(f)$ for all $f: i \rightarrow j$ in \mathbb{I} .

A morphism g in $M^{\mathbb{I}^{\text{opp}}}$ is said to be a *weak equivalence* if each $g(i)$ is a weak equivalence in $M(i)$.

Definition 5.3. An object of $(m, \eta) M^{\mathbb{I}^{\text{opp}}}$ is said to be *homotopy-Cartesian* if the transformations $\eta(f): f^*m(j) \rightarrow m(i)$ induce weak equivalences $\mathbf{L}f^*m(j) \rightarrow m(i)$ for all morphisms f in \mathbb{I} . When the model structures on the model categories are all trivial, we refer to homotopy-Cartesian sections as *Cartesian*.

Write $M_{\text{cart}}^{\mathbb{I}^{\text{opp}}}$ for the full subcategory of $M^{\mathbb{I}^{\text{opp}}}$ consisting of homotopy-Cartesian sections, and refer to a morphism in $M_{\text{cart}}^{\mathbb{I}^{\text{opp}}}$ as a weak equivalence if it is so in $M^{\mathbb{I}^{\text{opp}}}$.

Definition 5.4. Given a category \mathcal{C} and a subcategory \mathcal{W} (the *weak equivalences*) containing all objects, we follow [Rez] 3.3 in defining the classification diagram $\mathbf{N}(\mathcal{C}) := \mathbf{N}(\mathcal{C}, \mathcal{W})$ to be the following simplicial space. $N(\mathcal{C}, \mathcal{W})_n$ is the nerve of the category $w(\mathcal{C}^{[n]})$ whose

objects are strings of morphisms of length n , with morphisms in $w(\mathcal{C}^{[n]})$ given by weak equivalences.

Lemma 5.5. *Given a left Quillen presheaf M on a category \mathbb{I} , there is a canonical weak equivalence*

$$\mathbf{N}(M_{\text{cart}}^{\mathbb{I}^{\text{opp}}}) \simeq \mathop{\text{holim}}_{n \in \mathbb{I}^{\text{opp}}} \mathbf{N}(M^n)$$

of bisimplicial sets (with weak equivalences defined levelwise).

Proof. By [Rez] Theorem 8.3, any Reedy fibrant replacement of $N(\mathcal{C}, \mathcal{W})$ is a complete Segal space. Thus we may consider homotopy limits and weak equivalences in the CSS model structure, so the result reduces to [Ber2] Theorem 4.1. \square

Definition 5.6. Given a left Quillen presheaf M on \mathcal{A} and $X \in s\mathcal{A}$, write $cM(X) := (X^*M)^\Delta$ and $cM(X)_{\text{cart}} := (X^*M)_{\text{cart}}^\Delta$.

Remarks 5.7. Note that in order to establish that an object $\mathcal{F} \in cM(X)$ lies in $cM(X)_{\text{cart}}$, it suffices to verify that the maps $\partial^i: \mathbf{L}\partial_i^* \mathcal{F}^n \rightarrow \mathcal{F}^{n+1}$ are weak equivalences, since $\sigma^i: \mathbf{L}\sigma_i^* \mathcal{F}^{n+1} \rightarrow \mathcal{F}^n$ has right inverse $\mathbf{L}\sigma_{i*} \partial^i$.

When each category $M(Z)$ has trivial model structure, Lemma 2.12 adapts to show that the category $cM(X)_{\text{cart}}$ is equivalent to the category of pairs (\mathcal{G}, ω) , with $\mathcal{G} \in M(X_0)$ and $\omega: \partial_0^* \mathcal{G} \rightarrow \partial_1^* \mathcal{G}$ satisfying the cocycle conditions $\sigma_0^* \omega = \text{id}$ and $(\partial_2^* \omega) \circ (\partial_0^* \omega) = \partial_1^* \omega$.

When each $M(Z)$ is some category of sheaves on Z , Definition 5.6 is essentially the same definition as [Del] 5.1.6. Thus we may also describe $cM(X_\bullet)$ as the category of sheaves on the site of pairs (\mathbf{n}, U) , for $\mathbf{n} \in \Delta^{\text{opp}}$ and $U \rightarrow X_n$.

5.2. Comparison with sheaves on stacks. Fix a homotopical algebra context in the sense of [TV2] Definition 1.1.0.11. Thus we have a symmetric monoidal model category \mathcal{C} , and as in [TV2] §1.3.2 let $\text{Aff}_{\mathcal{C}}$ be opposite to the category of commutative monoids in \mathcal{C} . Now fix a topology τ on $\text{Aff}_{\mathcal{C}}$ as in §1.3.

Definition 5.8. Say that a left Quillen presheaf M on $\text{Aff}_{\mathcal{C}}$ is a left Quillen hypersheaf if for every $X \in \mathcal{A}$ and every τ -hypercov $\pi: \tilde{X}_\bullet \rightarrow X$ in $s\mathcal{A}$, the natural transformations

$$\begin{aligned} \mathcal{F} &\rightarrow \mathop{\text{holim}}_{n \in \Delta} \mathbf{R}\pi_{n*} \pi_n^* \mathcal{F} \\ \mathbf{L}\pi_n^* \mathop{\varprojlim}_{m \in \Delta} \pi_{m*} \mathcal{G}^m &\rightarrow \mathcal{G}^n \end{aligned}$$

are weak equivalences for all cofibrant objects $\mathcal{F} \in M(X)$ and all Reedy fibrant objects $\mathcal{G} \in cM(\tilde{X})_{\text{cart}}$.

Proposition 5.9. *Given a left Quillen hypersheaf M on $\text{Aff}_{\mathcal{C}}$ and $X \in s\text{Aff}_{\mathcal{C}}$, there are compatible equivalences*

$$\mathbf{N}(cM(X)_{\text{cart}})_i \simeq \mathbf{R}\underline{\text{Hom}}_{\text{Aff}_{\mathcal{C}}^{\sim, \tau}}(|X|, \mathbf{N}(M)_i)$$

of simplicial sets for all i .

Proof. This is based on [TV2] Theorem 1.3.7.2. The left Quillen hypersheaf condition ensures that each $\mathbf{N}(M)_i$ is a hypersheaf. Since $|X| = \mathop{\text{holim}}_n X_n$, this implies

$$\mathbf{R}\underline{\text{Hom}}_{\text{Aff}_{\mathcal{C}}^{\sim, \tau}}(|X|, \mathbf{N}(M)_i) \simeq \mathop{\text{holim}}_{n \in \Delta} \mathbf{N}(M(X_n))_i,$$

and the result reduces to Lemma 5.5. \square

Remark 5.10. In particular, note that Proposition 5.9 combines with Theorem 4.15 to show that for any trivial (n, \mathbf{C}) -hypergroupoid $\pi: \tilde{X} \rightarrow X$, the functor

$$\mathbf{L}\pi^*: cM(X)_{\text{cart}} \rightarrow cM(\tilde{X})_{\text{cart}}$$

induces weak equivalences of classifying diagrams. This makes it a weak equivalence of relative categories in the sense of [BK1].

Definition 5.11. For any object $\text{Spec } A \in \text{Aff}_{\mathcal{C}}$, there is a model category $\text{Mod}(A)$ of A -modules in \mathcal{C} , so Mod defines a left Quillen presheaf on $\text{Aff}_{\mathcal{C}}^{\text{opp}}$ as in [TV2] §1.3.7.

Corollary 5.12. For all $X \in s\text{Aff}_{\mathcal{C}}$, there are compatible equivalences

$$\mathbf{N}(\text{cMod}(X)_{\text{cart}})_i \simeq \mathbf{R}\underline{\text{Hom}}_{\text{Aff}_{\mathcal{C}}^{\sim, \tau}}(|X|, \mathbf{N}(\text{Mod})_i)$$

of simplicial sets for all i .

Remark 5.13. In [TV2], $\mathbf{N}(\text{Mod})_0$ is denoted by \mathbf{QCoh} , and $\mathbf{R}\underline{\text{Hom}}_{\text{Aff}_{\mathcal{C}}^{\sim, \tau}}(|X|, \mathbf{N}(\text{Mod})_0)$ is regarded as nerve of the ∞ -groupoid of quasi-coherent sheaves on $|X|$. Using the equivalences of [Ber1], we should thus regard the complete Segal space $i \mapsto \mathbf{R}\underline{\text{Hom}}_{\text{Aff}_{\mathcal{C}}^{\sim, \tau}}(|X|, \mathbf{N}(\text{Mod})_i)$ as the ∞ -category of quasi-coherent sheaves on $|X|$, and the proposition above shows us that this is just $\mathbf{N}(\text{cMod}(X)_{\text{cart}})$.

5.3. Cartesian replacement and derived direct images. By applying Remark 5.10 to left Quillen hypersheaves with additive structure, we can define Ext groups and hence cohomology. However, the existence of (derived) direct images is more subtle in general. Although a morphism $f: X \rightarrow Y$ in $s\mathcal{A}$ will induce functors $\mathbf{R}f_*: cM(X) \rightarrow cM(Y)$ for left Quillen hypersheaves M , these functors seldom preserve homotopy-Cartesian objects. We will therefore now construct a homotopy retraction of the inclusion $cM(X)_{\text{cart}} \hookrightarrow cM(X)$.

Definition 5.14. Given $X_{\bullet} \in s\mathcal{A}$, let $(\text{Dec}_+)_{\bullet} X_{\bullet}$ be the bisimplicial scheme given in level n by $(\text{Dec}_+)^{n+1} X_{\bullet}$, for Dec_+ as in Definition 2.20. The simplicial structure is defined as in [Wei] 8.6.4, using the comonadic structure of Dec_+ . This admits an augmentation $\partial_{\top}: (\text{Dec}_+)_0 X_{\bullet} \rightarrow X_{\bullet}$, given by the co-unit.

This also has an augmentation α in the orthogonal direction, given by $(\partial_0)^{\bullet+1}: (\text{Dec}_+)^{n+1} X_{\bullet} \rightarrow ((\text{Dec}_+)^n X_{\bullet})_0 = X_n$, corresponding to the retraction of Lemma 2.22.

Definition 5.15. Given a left Quillen presheaf M on \mathcal{A} and $X \in s\mathcal{A}$, define

$$\mathbf{R}\text{cart}_*: cM(X) \rightarrow cM(X)$$

to be the composition

$$(X^* M)^{\Delta} \xrightarrow{\mathbf{L}\partial_{\top}^*} (((\text{Dec}_+)_{\bullet} X_{\bullet})^* M)^{\Delta \times \Delta} \xrightarrow{\mathbf{R}\alpha_*} (cM(X))^{\Delta} \xrightarrow{\text{holim}_{\Delta}} cM(X).$$

Note that when X has constant simplicial structure (i.e. $X \in \mathcal{A}$), then $(\mathbf{L}\partial_{\top}^* \mathcal{F})^{mn} = \mathcal{F}^m$ and $\mathbf{R}\alpha_*$ is the identity on $M(X)^{\Delta \times \Delta}$, so $\mathbf{R}\text{cart}_*$ is just given by the constant diagram $(\mathbf{R}\text{cart}_* \mathcal{F})^n = \text{holim}_{m \in \Delta} \mathcal{F}^m$.

We now take a HAG context $(\mathcal{C}, \mathbf{P}, \tau)$, and consider $\mathcal{A} = \text{Aff}_{\mathcal{C}}$, letting $\mathbf{C} \subset \mathbf{P}$ consist of \mathbf{P} -morphisms which are local τ -surjections.

Definition 5.16. Say that a left Quillen hypersheaf M on $\text{Aff}_{\mathcal{C}}$ satisfies (n, \mathbf{C}) -base change if for all $X \in \text{Aff}_{\mathcal{C}}$, all trivial (n, \mathbf{C}) -hypergroupoids $\pi: Y \rightarrow X$ and all \mathbf{C} -morphisms $f: Y \rightarrow X$, the natural map

$$f^* (\text{holim}_{\Delta} \mathbf{R}\pi_*) \mathcal{E} \rightarrow (\text{holim}_{\Delta} \mathbf{R}\pi'_*) f^* \mathcal{E}$$

is a weak equivalence in $M(X')$ for all $\mathcal{E} \in cM(Y)$, where $\pi': Y' \rightarrow X'$ is the pullback along f of π .

The following is immediate:

Lemma 5.17. Assume that a left Quillen hypersheaf M on $\text{Aff}_{\mathcal{C}}$ satisfies the conditions:

(1) for all \mathbf{C} -morphisms $\pi: X \rightarrow Y$ and $f: X' \rightarrow X$ in $\text{Aff}_{\mathbf{C}}$, the natural map

$$f^* \mathbf{R}\pi_* \mathcal{E} \rightarrow \mathbf{R}\pi'_* f^* \mathcal{E}$$

is a weak equivalence in $M(X')$ for all $\mathcal{E} \in M(Y)$, where $\pi': Y' \rightarrow X'$ is the pullback along f of π ;

(2) for all \mathbf{C} -morphisms $f: X' \rightarrow X$ in $\text{Aff}_{\mathbf{C}}$ and all $\mathcal{E} \in cM(X)$, the natural map

$$f^* (\text{holim}_{\Delta} \mathcal{E}) \rightarrow \text{holim}_{\Delta} f^* \mathcal{E}$$

is a weak equivalence in $M(X')$.

Then M satisfies (n, \mathbf{C}) -base change for all n .

Proposition 5.18. *Assume that M is a left Quillen hypersheaf satisfying (l, \mathbf{C}) -base change. Then for every (l, \mathbf{C}) -hypergroupoid X and every $\mathcal{F} \in cM(X)$, the object $\mathbf{R}\text{cart}_* \mathcal{F}$ of $cM(X)$ is homotopy-Cartesian.*

Proof. Consider the maps $\partial_i(\text{Dec}_+) : \text{Dec}_+^{n+1} X \rightarrow \text{Dec}_+^n X$ associated to the comonad Dec_+ . If we write $\varepsilon_Y : \text{Dec}_+ Y \rightarrow Y$ for the counit of the adjunction, then $\partial_i(\text{Dec}_+) = (\text{Dec}_+)^i \varepsilon_{\text{Dec}_+^{n-i} Y}$.

Since $\partial_i : X_n \rightarrow X_{n-1}$ is a \mathbf{C} -fibration, (l, \mathbf{C}) -base change implies that

$$\mathbf{L}\partial_i^* \text{holim}_{\Delta} \mathbf{R}\alpha_{n-1,*} (\mathbf{L}\partial_+^*)^n \mathcal{F} \simeq \text{holim}_{\Delta} \mathbf{R}\text{pr}_{2*} \text{pr}_1^* (\mathbf{L}\partial_+^*)^n \mathcal{F},$$

for $\text{pr}_1 : \text{Dec}_+^n X \times_{X_{n-1}} X_n \rightarrow \text{Dec}_+^n X$ and $\text{pr}_2 : \text{Dec}_+^n X \times_{X_{n-1}} X_n \rightarrow X_n$.

From Lemma 3.9, it follows that the morphisms

$$(\partial_i(\text{Dec}_+), \partial_0) : \text{Dec}_+^{n+1} X \rightarrow \text{Dec}_+^n X \times_{X_{n-1}} X_n$$

are trivial (l, \mathbf{C}) -hypergroupoids. In particular, they are τ -hypercoverings, so Remark 5.10 gives that

$$\mathbf{L}(\partial_i(\text{Dec}_+), \partial_0)^* : cM(\text{Dec}_+^n X \times_{X_{n-1}} X_n)_{\text{cart}} \rightarrow cM(\text{Dec}_+^{n+1} X)_{\text{cart}}$$

is a weak equivalence.

Now, $\text{holim}_{\Delta} \mathbf{R}\alpha_{n*} : cM(\text{Dec}_+^{n+1} X)_{\text{cart}} \rightarrow M(X_n)$ is homotopy right adjoint to $\mathbf{L}\alpha_n^*$, so the weak equivalence above gives $\text{holim}_{\Delta} \mathbf{R}\alpha_{n*} \circ \mathbf{L}(\partial_i(\text{Dec}_+), \partial_0)^*$ as homotopy right adjoint to $\mathbf{L}\text{pr}_2^* : \text{Mod}(X_n) \rightarrow cM(\text{Dec}_+^n X \times_{X_{n-1}} X_n)_{\text{cart}}$. Another homotopy right adjoint is given by $\text{holim}_{\Delta} \mathbf{R}\text{pr}_{2*}$, so

$$\text{holim}_{\Delta} \mathbf{R}\text{pr}_{2*} \simeq \text{holim}_{\Delta} \mathbf{R}\alpha_{n*} \circ \mathbf{L}(\partial_i(\text{Dec}_+), \partial_0)^*.$$

Combining the equivalences above, we have

$$\begin{aligned} \mathbf{L}\partial_i^* \text{holim}_{\Delta} \mathbf{R}\alpha_{n-1,*} (\mathbf{L}\partial_+^*)^n \mathcal{F} &\simeq \text{holim}_{\Delta} \mathbf{R}\alpha_{n*} \circ \mathbf{L}(\partial_i(\text{Dec}_+), \partial_0)^* \text{pr}_1^* (\mathbf{L}\partial_+^*)^n \mathcal{F} \\ &= \text{holim}_{\Delta} \mathbf{R}\alpha_{n*} (\mathbf{L}\partial_+^*)^{n+1} \mathcal{F}, \end{aligned}$$

so we have shown that

$$\partial^i : \mathbf{L}\partial_i^* (\mathbf{R}\text{cart}_*(\mathcal{F})^{n-1}) \rightarrow \mathbf{R}\text{cart}_*(\mathcal{F})^n$$

is a quasi-isomorphism, as required. \square

Proposition 5.19. *For X and M as above and $\mathcal{F} \in cM(X)$, there is a natural transformation $\varepsilon_{\text{cart}} : \mathbf{R}\text{cart}_* \mathcal{F} \rightarrow \mathcal{F}$, which is a weak equivalence whenever \mathcal{F} is homotopy-Cartesian.*

Proof. Given a module $\mathcal{G} \in cM(Y)$ for $Y \in s\text{Aff}_{\mathbf{C}}$, we may define $\text{Dec}^+(\mathcal{G}) \in cM(\text{Dec}_+ Y)$ by $(\text{Dec}^+(\mathcal{G}))^m = \mathcal{G}^{m+1}$, with $\partial_{\text{Dec}^+(\mathcal{G})}^i = \partial_{\mathcal{G}}^i$ and $\sigma_{\text{Dec}^+(\mathcal{G})}^i = \sigma_{\mathcal{G}}^i$. Setting

$$((\text{Dec}^+)^{\bullet} \mathcal{F})^n := (\text{Dec}^+)^{n+1} \mathcal{F}$$

for any $\mathcal{F} \in cM(X)$, this gives us $(\text{Dec}^+)^{\bullet} \mathcal{F} \in M((\text{Dec}_+)^{\bullet} X)^{\Delta \times \Delta}$, and there is a natural map $\partial^{\top} : \mathbf{L}\partial_{\top}^* \mathcal{F} \rightarrow (\text{Dec}^+)^{\bullet} \mathcal{F}$, which is a weak equivalence whenever \mathcal{F} is homotopy-Cartesian. Applying the functor $\text{holim}_{\Delta} \mathbf{R}\alpha_*$ gives a map $\partial^{\top} : \mathbf{R}\text{cart}_* \mathcal{F} \rightarrow \text{holim}_{\Delta} \mathbf{R}\alpha_*(\text{Dec}^+)^{\bullet} \mathcal{F}$.

Now, the deformation retraction $\partial_0 : \text{Dec}_+ Y \rightarrow Y_0$ corresponds to a deformation retraction

$$\mathcal{G}^0 \xrightarrow{\partial^0} \text{holim}_{\Delta} \mathbf{R}(\partial_0^*)_* \text{Dec}^+(\mathcal{G}) \xrightarrow{\mathbf{R}\partial_0^*(\sigma^0)^{\bullet}} \mathcal{G}^0$$

in $cM(Y_0)$. Applying this to $Y = (\text{Dec}_+)^n X$ and $\mathcal{G} = (\text{Dec}^+)^n \mathcal{F}$ for all n gives a deformation retraction

$$\mathcal{F} \xrightarrow{\partial^0} \mathbf{R}\alpha_*(\text{Dec}^+)^{\bullet} \mathcal{F} \xrightarrow{\mathbf{R}\alpha_*(\sigma^0)^{\bullet}} \mathcal{F}.$$

We now define $\varepsilon_{\text{cart}}$ to be $\mathbf{R}\alpha_*(\sigma^0)^{\bullet} \circ \partial^{\top} : \mathbf{R}\text{cart}_* \mathcal{F} \rightarrow \mathcal{F}$. \square

Definition 5.20. For any morphism $f : X \rightarrow Y$ in $s\text{Aff}_{\mathbf{C}}$ with Y an (n, \mathbf{C}) -hypergroupoid, and for any left Quillen hypersheaf M satisfying (n, \mathbf{C}) -base change, we may now define $\mathbf{R}f_*^{\text{cart}} : cM(X)_{\text{cart}} \rightarrow cM(Y)_{\text{cart}}$ by

$$\mathbf{R}f_*^{\text{cart}} := \mathbf{R}\text{cart}_* \circ \mathbf{R}f_*,$$

where $\mathbf{R}f_* : cM(X) \rightarrow cM(Y)$ is given on X_n by $\mathbf{R}f_{n*}$.

When $n = 0$, note that we just have $\mathbf{R}f_*^{\text{cart}} = \text{holim}_{\Delta} \mathbf{R}f_*$.

5.4. Sheaves on Artin n -stacks. We now specialise to the context of strongly quasi-compact Artin n -stacks. If we let \mathbf{C} be the class of smooth surjections of affine schemes, then Theorem 4.7 allows us to work instead with Artin n -hypergroupoids (i.e. (n, \mathbf{C}) -hypergroupoids in affine schemes — see Examples 3.5). Write Aff for the category of affine schemes.

5.4.1. Quasi-coherent modules. As in Definition 5.11, let $\text{Mod} : \text{Aff}^{\text{opp}} \rightarrow \text{Cat}$ send $\text{Spec } A$ to the category of A -modules. The following is now a special case of Remarks 5.7:

Lemma 5.21. *Given an Artin n -hypergroupoid X , the category $c\text{Mod}(X)_{\text{cart}}$ is equivalent to the category of pairs (\mathcal{G}, ω) , where \mathcal{G} is a quasi-coherent sheaf on X_0 and $\omega : \partial_0^* \mathcal{G} \rightarrow \partial_1^* \mathcal{G}$ is a morphism of quasi-coherent sheaves on X_1 satisfying the cocycle conditions $\sigma_0^* \omega = \text{id}$ on X_0 , and $(\partial_2^* \omega) \circ (\partial_0^* \omega) = \partial_1^* \omega$ on X_2 .*

When X is an Artin 1-hypergroupoid and $\mathfrak{X} = |X|$, we thus have that $c\text{Mod}(X)_{\text{cart}}$ is equivalent to the category of Cartesian $\mathcal{O}_{\mathfrak{X}}$ -modules in the sense of [Ols1] Lemma 4.5.

5.4.2. Quasi-coherent complexes. Now consider the functors $dg\text{Mod}$ (resp. $dg_+\text{Mod}$, resp. $dg_-\text{Mod}$) from Aff^{opp} to Cat which send $\text{Spec } A$ to the model category of chain complexes (resp. chain complexes in non-negative degrees, resp. chain complexes in non-positive degrees) of A -modules. Weak equivalences in all these categories are quasi-isomorphisms. Fibrations in $dg\text{Mod}(A)$ and $dg_-\text{Mod}(A)$ are surjections, while fibrations in $dg_+\text{Mod}(A)$ are surjective in non-zero degrees.

The following is an immediate consequence of the definitions and flat base change:

Lemma 5.22. *Given an Artin n -hypergroupoid X , an object $\mathcal{F}_{\bullet} \in cdg\text{Mod}(X)$ is homotopy-Cartesian if and only if the homology presheaves $H_i(\mathcal{F}_{\bullet}) \in c\text{Mod}(X)$ are Cartesian for all i .*

Lemma 5.23. *The functors $dg\text{Mod}$, $dg_+\text{Mod}$, $dg_-\text{Mod}$ are left Quillen hypersheaves on Aff with respect to the fpqc topology.*

Proof. We first consider the case of $dg\text{Mod}$. Taking an fpqc hypercover $p: \tilde{X}_\bullet \rightarrow X$, we need to show that the maps

$$\begin{aligned} \mathcal{F} &\xrightarrow{\eta} \underset{n \in \Delta}{\text{holim}} \mathbf{R}p_{n*} p_n^* \mathcal{F} \\ \mathbf{L}p_n^* \underset{m \in \Delta}{\text{lim}} p_{m*} \mathcal{G}^m &\xrightarrow{\varepsilon} \mathcal{G}^n \end{aligned}$$

are weak equivalences for all cofibrant objects $\mathcal{F} \in dg\text{Mod}(X)$ and all fibrant objects $\mathcal{G} \in cdg\text{Mod}(\tilde{X})_{\text{cart}}$.

All objects of $dg\text{Mod}(\tilde{X}_n)$ are fibrant, so $p_{n*} = \mathbf{R}p_{n*}$. Since the morphisms $p_n: \tilde{X}_n \rightarrow X$ are all flat, p_n^* also preserves weak equivalences, so $\mathbf{L}p_n^* = p_n^*$.

As in Examples 1.15, a model for the functor $\underset{n \in \Delta}{\text{holim}}$ is given by the product total complex. Writing H_c^* for cohomology of a cosimplicial complex, and taking $\mathcal{G} \in cdg\text{Mod}(\tilde{X})_{\text{cart}}$, there is spectral sequence

$$H_c^i p_* H_j \mathcal{G} \implies H_{j-i} \text{Tot}^\Pi p_* \mathcal{G},$$

which converges weakly, by [Wei] §5.6, since $H_c^i p^* \mathcal{F}_j = 0$ for $i < 0$ (and hence for $i < 0, j < 0$).

Since $H_j \mathcal{G} \in \text{Mod}(\tilde{X})_{\text{cart}}$, by faithfully flat descent there exists $\mathcal{E}_j = H_c^0(p_* H_j \mathcal{G}) \in dg\text{Mod}(X)$ with $p^* \mathcal{E}_j \cong H_j \mathcal{G}$. Thus cohomological descent gives $H_c^i p_* H_j \mathcal{G} = 0$ for all $i > 0$, and the spectral sequence converges, giving

$$H_n \text{Tot}^\Pi p_* \mathcal{G} \cong H_c^0(p_* H_n \mathcal{G}).$$

the isomorphism $p^* \mathcal{E}_j \cong H_j \mathcal{G}$ thus ensures that

$$\varepsilon_{\mathcal{G}}: p^* H_c^0(p_* \mathcal{G}) \rightarrow \mathcal{G}$$

is a quasi-isomorphism.

For $\mathcal{F} \in dg\text{Mod}(X)$, this implies that $\varepsilon_{p^* \mathcal{F}}$ is a quasi-isomorphism. Thus $p^* \eta_{\mathcal{F}}$ is also a quasi-isomorphism, since $\varepsilon_{p^* \mathcal{F}} \circ p^* \eta_{\mathcal{F}} = \text{id}_{\mathcal{F}}$. In particular, $p_0^* \eta_{\mathcal{F}}$ is a quasi-isomorphism, so $\eta_{\mathcal{F}}$ must also be so, p_0 being faithfully flat.

This completes the proof for $dg\text{Mod}$. The same proof applies to $dg_-\text{Mod}$, in which case the product total complex is just a direct sum. For $dg_+\text{Mod}$, $\underset{\Delta}{\text{holim}}$ is given by applying the good truncation $\tau_{\geq 0}$ to Tot^Π , so the proof above still applies, except that we need only consider H_n for $n \geq 0$. \square

Remarks 5.24. Note that $cdg\text{Mod}(X)_{\text{cart}}$ corresponds to the category of quasi-coherent complexes on $|X|$ given in [Toë2] 3.7, which anticipated the simplicial characterisation of n -geometric stacks given in Theorem 4.7.

However, this differs crucially from the construction of $D_{\text{cart}}(\mathcal{O}_{X_\bullet})$ in [Ols1] 4.6. The latter takes complexes \mathcal{F}_\bullet of sheaves of \mathcal{O}_X -modules whose homology sheaves $\mathcal{H}_n(\mathcal{F})$ are Cartesian. Thus the sheafification \mathcal{F}^\sharp of any homotopy-Cartesian \mathcal{O}_X -module \mathcal{F} lies in $D_{\text{cart}}(\mathcal{O}_{X_\bullet})$, but \mathcal{F}^\sharp might not be homotopy-Cartesian. One consequence is that whereas projective resolutions exist in the category of quasi-coherent complexes, they do not exist in $D_{\text{cart}}(\mathcal{O}_{X_\bullet})$. This means that derived pullbacks $\mathbf{L}f^*$ exist automatically for quasi-coherent complexes (as will be exploited in Corollary 7.14), but are constructed only with considerable difficulty in [Ols1].

Objects of $cdg_+\text{Mod}(X)_{\text{cart}}$ correspond to the definition of homotopy-Cartesian modules in [TV2] Definition 1.2.12.1. These are called quasi-coherent complexes in [Lur1] §5.2. Again, this differs slightly from the construction of $D_{\text{cart}}(\mathfrak{X})$ in [Ols1] 3.10, which takes complexes of sheaves of $\mathcal{O}_{\mathfrak{X}}$ -modules whose homology sheaves are quasi-coherent. The

differences arise because quasi-coherent complexes are hypersheaves (as explained in §1.3.1) rather than sheaves.

For a quasi-compact semi-separated scheme X , it is observed in the introduction of [Hüt] that the proof of [Hüt] Theorem 4.5.1 adapts to show that the homotopy category of quasi-coherent complexes X is equivalent to the homotopy category of complexes of quasi-coherent sheaves on X . By [BN], this in turn is equivalent to $D_{\text{cart}}(\mathcal{O}_X)$ under the same hypotheses.

5.4.3. *Derived direct images and cohomology.* Observe that the left Quillen hypersheaves $dg_- \text{Mod}$, $dg_+ \text{Mod}$, $dg \text{Mod}$ on Aff all satisfy the first condition of Lemma 5.17, which is just flat base change. The second condition, however, is only satisfied by $dg_- \text{Mod}$ in general, since Tot^Π does not commute with pullbacks. Thus, for any morphism $f: X \rightarrow Y$ in $s\text{Aff}$ with Y an Artin n -hypergroupoid, Definition 5.20 gives

$$\mathbf{R}f_*^{\text{cart}}: cdg_- \text{Mod}(X)_{\text{cart}} \rightarrow cdg_- \text{Mod}(Y)_{\text{cart}},$$

homotopy right adjoint to $\mathbf{L}f^*$.

When Y is an affine scheme, or even the Čech resolution of a quasi-compact semi-separated scheme, the total product Tot^Π becomes a finite sum, giving

$$\mathbf{R}f_*^{\text{cart}}: cdg \text{Mod}(X)_{\text{cart}} \rightarrow cdg \text{Mod}(Y)_{\text{cart}},$$

and similarly for $dg_+ \text{Mod}$.

Taking Y to be affine, we can write $\mathbf{R}\Gamma(X, -)$ for $\mathbf{R}f_*^{\text{cart}}$, giving

$$\mathbf{R}\Gamma(X, \mathcal{F}) = \text{Tot}^\Pi \mathcal{F}.$$

We also have the following generalisation of the vanishing of higher direct images of an affine morphism of schemes, allowing us to regard f_* as $\mathbf{R}f_*^{\text{cart}}$.

Lemma 5.25. *If $f: X \rightarrow Y$ is a relative Artin 0-hypergroupoid over an Artin n -hypergroupoid Y , then*

$$\mathbf{R}f_* \simeq f_*: cdg \text{Mod}(X) \rightarrow cdg \text{Mod}(Y)$$

sends $cdg \text{Mod}(X)_{\text{cart}}$ to $cdg \text{Mod}(Y)_{\text{cart}}$.

Proof. Since f is a relative Artin 0-hypergroupoid, it is Cartesian, so f_*M is Cartesian for all $M \in cdg \text{Mod}(X)_{\text{cart}}$. As all objects of $dg \text{Mod}(A)$ are fibrant, we also have $\mathbf{R}f_* = f_*$, which completes the proof. \square

6. ALTERNATIVE FORMULATIONS OF DERIVED ARTIN STACKS

In this section, we will restrict to the strongly quasi-compact case in order to simplify the exposition.

Intuitively, a derived Artin n -stack X should be a small neighbourhood of the associated Artin n -stack $\pi^0 X$ given in Definition 1.22. We will see how any homotopy derived Artin n -hypergroupoid X is equivalent to the Zariski neighbourhood X^l of $\pi^0 X$ in X , and even the étale neighbourhood X^h of $\pi^0 X$ in X . Indeed, under fairly strong Noetherian hypotheses, X is equivalent to the formal neighbourhood \hat{X} .

6.1. Approximation and completion. We use the terms *formally étale* and *formally smooth* in the sense of [TV2], meaning that a morphism $f: A \rightarrow B$ of simplicial rings is formally étale (resp. formally smooth) if the cotangent complex $\mathbb{L}_{\bullet}^{B/A} \simeq 0$ (resp. $\mathbb{L}_{\bullet}^{B/A}$ is equivalent to a retract of a direct sum of copies of B). Note that when A and B are discrete rings, these notions are stronger than those used classically (e.g. in [Mil]).

Definition 6.1. Recall from [TV2] Definition 2.2.2.1 that a morphism $f: A \rightarrow B$ in $s\text{Ring}$ is said to be strong if the maps $\pi_n(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_n(B)$ are isomorphisms for all n .

Lemma 6.2. *A morphism $f : A \rightarrow B$ in $s\text{Ring}$ is formally étale if and only if the relative cotangent complex $\mathbb{L}^{B/A} \otimes_B \pi_0 B$ is contractible as a simplicial $\pi_0(B)$ -module, i.e. $\pi_*(\mathbb{L}^{B/A} \otimes_B \pi_0 B) = 0$.*

Proof. The argument from the proof of [TV2] Lemma 2.2.2.8 shows that contractibility of $\mathbb{L}^{B/A} \otimes_B \pi_0 B$ implies contractibility of the simplicial B -module $\mathbb{L}^{B/A}$, which in turn means that f is formally étale. \square

Lemma 6.3. *A morphism $f : A \rightarrow B$ in $s\text{Ring}$ is a weak equivalence if and only if f is formally étale and $\pi_0(f)$ is an isomorphism.*

Proof. [TV2] Proposition 2.2.2.4 implies that a formally étale map f is étale if and only if $\pi_0 f$ is finitely presented. Since $\pi_0 f$ is an isomorphism, this holds, so f is étale. Finally, [TV2] Theorem 2.2.2.6 states that étale morphisms f in $s\text{Ring}$ are precisely strong morphisms for which $\pi_0 f$ is étale. This implies that f is a weak equivalence. \square

Lemma 6.4. *If $f : A \rightarrow B$ is a morphism in $s\text{Ring}$, for which each map $f_n : A_n \rightarrow B_n$ is formally étale (in the sense of [TV2]), then f is formally étale (i.e. the cotangent complex $\mathbb{L}^{B/A}$ is contractible). In fact, we only need $\mathbb{L}^{B_n/A_n} \otimes_{B_n} \pi_0 B$ to be contractible for all n .*

Proof. Let $\tilde{B}_{n\bullet}$ be the cotriple resolution of B_n as an A_n -algebra. Thus $\tilde{B}_{\bullet\bullet}$ is a bisimplicial ring, and $A \rightarrow \text{diag } \tilde{B}_{\bullet\bullet}$ is a cofibrant approximation to B in $A \downarrow s\text{Ring}$.

Therefore

$$\mathbb{L}^{B/A} \otimes_B \pi_0 B \simeq \Omega(\text{diag } \tilde{B}_{\bullet\bullet}/A_\bullet) \otimes_{\text{diag } \tilde{B}_{\bullet\bullet}} \pi_0 B,$$

and this is the diagonal of the bisimplicial diagram

$$M_{nm} := \Omega(\tilde{B}_{nm}/A_n) \otimes_{\tilde{B}_{nm}} \pi_0 B$$

Since f_n is formally étale,

$$L_n := \pi_* M_{n\bullet} = \pi_*(\mathbb{L}^{B_n/A_n} \otimes_{B_n} \pi_0 B) = 0,$$

for all n . Since $\pi_*(\text{diag } M_{\bullet\bullet}) \cong H_*(\text{Tot } M_{\bullet\bullet})$ (Eilenberg-Zilber), we have a convergent spectral sequence

$$0 = \pi_*(0) = \pi_*(L_\bullet) \implies \pi_*(\text{diag } M_{\bullet\bullet}),$$

so f is formally étale by Lemma 6.2. \square

Definition 6.5. Given a cosimplicial affine scheme X , define $\hat{X} \in c\text{Aff}$ by setting \hat{X}^n to be the formal neighbourhood of $\pi^0 X$ in X^n , so $O(\hat{X}^n) = \varprojlim_m O(X^n)/(I_n)^m$, where $I_n = \ker(O(X^n) \rightarrow \pi_0 O(X))$.

Proposition 6.6. *For $X \in c\text{Aff}$ levelwise Noetherian, the morphism $f : \hat{X} \rightarrow X$ is a weak equivalence.*

Proof. Write $X^n = \text{Spec } A_n$ and $\hat{X}^n = \text{Spec } \hat{A}_n$. We first describe the cotangent complex of $f_n : A_n \rightarrow \hat{A}_n$. Since A_n is Noetherian, [Mat] Theorem 8.8 implies that f_n is flat. Thus flat base change shows that

$$\mathbb{L}^{\hat{A}_n/A_n} \otimes_{A_n} \pi_0 A \simeq \mathbb{L}^{(\hat{A}_n \otimes_{A_n} \pi_0 A)/\pi_0 A},$$

and note that since $\mathbb{L}^{\hat{A}_n/A_n} \otimes_{A_n} \pi_0 A = \mathbb{L}^{\hat{A}_n/A_n} \otimes_{\hat{A}_n} (\hat{A}_n \otimes_{A_n} \pi_0 A)$, we then have

$$\mathbb{L}^{\hat{A}_n/A_n} \otimes_{\hat{A}_n} \pi_0 A = \mathbb{L}^{(\hat{A}_n \otimes_{A_n} \pi_0 A)/\pi_0 A} \otimes_{(\hat{A}_n \otimes_{A_n} \pi_0 A)} \pi_0 A.$$

[Mat] Theorem 8.7 implies that $\hat{A}_n \otimes_{A_n} \pi_0 A = \pi_0 A$, so we have shown that

$$\mathbb{L}^{\hat{A}_n/A_n} \otimes_{\hat{A}_n} \pi_0 A \simeq 0,$$

and hence that f is formally étale, by Lemma 6.4.

In order to satisfy the conditions of Lemma 6.3, it only remains to prove that $\pi_0 \hat{A} = \pi_0 A$. This follows by letting $I = \ker(A \rightarrow \pi_0 A)$, and observing that the surjections $I^{\otimes n} \rightarrow I^n$ imply that $\pi_0(I^n) = 0$ for all $n > 0$, so $\pi_0(\varprojlim A/I^n) = \pi_0 A$. \square

Note that [Mat] Theorem 8.12 implies that \hat{X} is then also levelwise Noetherian. See [Nis] for more properties preserved by I -adic completion.

Corollary 6.7. *Any levelwise Noetherian homotopy derived Artin n -hypergroupoid X is weakly equivalent to the homotopy derived Artin n -hypergroupoid \hat{X} , defined by setting $(\hat{X})_n := \widehat{(X_n)}$.*

6.2. Local thickenings.

Definition 6.8. Given a morphism $f : X \rightarrow Y$ of affine schemes, define $(X/Y)^{\text{loc}}$ to be the localisation of Y at X . By [Ane] Proposition 52, this is given by

$$(X/Y)^{\text{loc}} = \text{Spec} \varinjlim_{\substack{f(X) \subset U \subset Y \\ U \text{ open}}} \Gamma(U, \mathcal{O}_Y) = \text{Spec} \Gamma(X, f_{\text{Pr}}^{-1} \mathcal{O}_{Y, \text{Zar}}),$$

where f_{Pr}^{-1} is the presheaf inverse image functor.

Lemma 6.9. *If $X_0 \rightarrow X$ is a morphism of affine schemes, and $X^l := (X_0/X)^{\text{loc}}$, then $X_0 \times_X X^l = X_0$.*

Proof. Given any affine scheme Y under X_0 , write Y^l for the localisation $(X_0/Y)^{\text{loc}}$. Properties of localisation from [Ane] give

$$(X_0 \times_X X^l)^l = X_0^l \times_{X^l} X^l = X_0,$$

so the morphism $X_0 \rightarrow X_0 \times_X X^l$ is conservative (i.e. an inverse limit of open immersions), and hence a monomorphism. Since $X^l \rightarrow X$ is also conservative, the projection $X_0 \times_X X^l \rightarrow X_0$ is also a monomorphism, so must in fact be an isomorphism. \square

Definition 6.10. Given a cosimplicial affine scheme X , define $X^l \in c\text{Aff}$ by $(X^l)^n := (\pi^0 X/X^n)^{\text{loc}}$.

Proposition 6.11. *For $X \in c\text{Aff}$, the morphism $f : X^l \rightarrow X$ is a weak equivalence.*

Proof. Given any affine scheme Y under $\pi^0 X$, write Y^l for the localisation $(\pi^0 X/Y)^{\text{loc}}$.

By construction, the maps $f^n : (X^l)^n \rightarrow X^n$ have trivial cotangent complexes (being filtered limits of open immersions), so Lemma 6.4 implies that f is formally étale. By Lemma 6.3, it only remains to prove that $\pi^0 f$ is an isomorphism.

Observe that the functor $Y \mapsto Y^l$ from affine schemes under $\pi^0 X$ to localisations preserves limits, since it has a right adjoint given by inclusion. Thus $\pi^0(X^l)$, being the equaliser of the maps $\partial^0, \partial^1 : (X^l)^0 \rightarrow (X^l)^1$, is just the localisation of the equaliser $\pi^0 X$ of $\partial^0, \partial^1 : X^0 \rightarrow X^1$. Thus

$$\pi^0(X^l) = (\pi^0 X)^l = \pi^0 X.$$

\square

Corollary 6.12. *Any homotopy derived Artin n -hypergroupoid X is weakly equivalent to the homotopy derived Artin n -hypergroupoid X^l , defined by setting $(X^l)_n := (X_n)^l$.*

Remark 6.13. In fact, $s\text{Ring}^l$ has a simplicial model structure, in which a morphism is a weak equivalence or fibration whenever the underlying map in $s\text{Ring}$ is so.

Theorem 6.14. *Fix a scheme Z over a ring R , and fix $m \geq 2$ (or $m \geq 1$ if Z is semi-separated). Then the homotopy category of m -geometric derived schemes X over R with $\pi^0 X \simeq Z$ is weakly equivalent to the homotopy category $\mathcal{T}_{\text{Zar}}(Z)$ of presheaves \mathcal{A}_\bullet of simplicial R -algebras on the category $\text{Aff}_{\text{Zar}}(Z)$ of affine open subschemes of Z , satisfying the following:*

- (1) $\pi_0(\mathcal{A}_\bullet) = \mathcal{O}_Z$;
- (2) for all i , the presheaf $\pi_i(\mathcal{A}_\bullet)$ is a quasi-coherent \mathcal{O}_Z -module.

Proof. The key idea behind this proof is that given a local morphism $B \rightarrow A$, we can define a presheaf $(B/\mathcal{O}_Y)^{\text{loc}}$ on $Y := (\text{Spec } A)_{\text{Zar}}$, by $(B/\mathcal{O}_Y)^{\text{loc}}(\text{Spec } C) = (B/C)^{\text{loc}}$.

Recall from Remark 1.27 that all schemes are 2-geometric, and that a quasi-separated scheme is 1-geometric if and only if it is semi-separated. Theorem 4.7 therefore gives a Zariski m -hypergroupoid \tilde{Z}_\bullet in disjoint unions of affine schemes (see Examples 3.5), with $a: \tilde{Z} \rightarrow Z$ such that $|\tilde{Z}| \simeq Z$.

Now, the proof of Lemma 5.23 adapts to show that \mathcal{A}_\bullet is a Zariski hypersheaf. Thus a^{-1} gives a weak equivalence from $\mathcal{T}_{\text{Zar}}(Z)$ to the homotopy category $\mathcal{T}_{\text{Zar}}(\tilde{Z}_\bullet)$ of homotopy-Cartesian presheaves \mathcal{B} of those simplicial algebras on the simplicial site $\text{Aff}_{\text{Zar}}(\tilde{Z}_\bullet)$ for which $\pi_0 \mathcal{B}_\bullet \cong \mathcal{O}_{\tilde{Z}}$ and $\pi_n \mathcal{B}_\bullet$ is a Cartesian $\mathcal{O}_{\tilde{Z}}$ -module.

We then define the simplicial cosimplicial scheme \tilde{X}_\bullet by letting \tilde{X}_n be the cosimplicial scheme $\mathbf{Spec}_{\tilde{Z}_n} \mathcal{B}_\bullet^n$. It follows immediately from the properties of \mathcal{B}_\bullet that \tilde{X}_\bullet is a homotopy derived Zariski n -hypergroupoid, and that $\pi^0 \tilde{X} = \tilde{Z}$. Therefore $X(Z, \mathcal{A}_\bullet) := \tilde{X}^\sharp$ has all the required properties.

If we replace \tilde{Z} by a trivial relative Zariski m -hypergroupoid $\tilde{Z}' \rightarrow \tilde{Z}$ in the above construction, we get a homotopy trivial derived Zariski n -hypergroupoid $\tilde{X}' \rightarrow \tilde{X}$, so $(\tilde{X}')^\sharp \simeq \tilde{X}^\sharp$, and the functor $X(-)$ is independent of the choice of \tilde{Z}_\bullet .

For the inverse construction, take a derived m -geometric scheme X with $\pi^0 X \simeq Z$, and apply Theorem 4.7 to form a derived Zariski m -hypergroupoid \tilde{X} with $\tilde{X}^\sharp \simeq X$. Let $\tilde{Z} := \pi^0 \tilde{X}$, and for $\iota: \tilde{Z} \rightarrow \tilde{X}$, consider the presheaf $\mathcal{B}_\bullet := \iota_{\text{Pr}}^{-1} \mathcal{O}_{\tilde{X}}$ on the simplicial site $\text{Aff}_{\text{Zar}}(\tilde{Z}_\bullet)$ — this is a Zariski presheaf of simplicial algebras. Since ι_{Pr}^{-1} commutes with colimits, $\pi_0(\mathcal{B}_\bullet) = \mathcal{O}_{\tilde{Z}}$. Properties of localisation imply that $(\mathcal{B}_n)(U) = (O(\tilde{X}^n)/O(U))^{\text{loc}}$, which is étale (as in the proof of Proposition 6.11), so

$$\pi_i(\mathcal{B}_\bullet(U)) \cong \pi_0(\mathcal{B}_\bullet(U)) \otimes_{\pi_0(\mathcal{B}_\bullet(\tilde{Z}))} \pi_i(\mathcal{B}_\bullet(\tilde{Z})) = \mathcal{O}_{\tilde{Z}}(U) \otimes_{\mathcal{O}_{\tilde{Z}}(\tilde{Z})} \pi_i(\mathcal{B}_\bullet(\tilde{Z})),$$

which means that $\pi_i(\mathcal{B}_\bullet)$ is homotopy-Cartesian and quasi-coherent.

This allows us to define a Zariski presheaf $\mathcal{A}_\bullet(X) := \mathbf{R}a_* \mathcal{B}_\bullet$, for $a: \tilde{Z}_\bullet \rightarrow Z$. If we replace \tilde{X} by a homotopy trivial relative Zariski n -hypergroupoid $\tilde{X}' \rightarrow \tilde{X}$ in the above construction, we get a weakly equivalent presheaf, so $\mathcal{A}_\bullet(X)$ is well-defined.

Moreover, $a^{-1} \mathcal{A}_\bullet(X) \simeq \mathcal{B}_\bullet$, and $\mathbf{Spec}_{\tilde{Z}_n} \mathcal{B}_\bullet^n = \tilde{X}_n^l$, so

$$X(Z, \mathcal{A}_\bullet(X)) \simeq (\tilde{X}^l)^\sharp \simeq X,$$

by Proposition 6.11.

Conversely, to compute $\mathcal{A}_\bullet(X(\mathcal{A}_\bullet))$, we may choose the resolution \tilde{X}_\bullet of X used in the construction of $X(\mathcal{A}_\bullet)$, and then we have

$$\mathcal{A}_\bullet(X(\mathcal{A}_\bullet)) = \mathbf{R}a_*(\Gamma(\tilde{Z}, a^{-1} \mathcal{A}_\bullet)/\mathcal{O}_{\tilde{Z}})^{\text{loc}};$$

which is weakly equivalent to $\mathbf{R}a_* a^{-1} \mathcal{A}_\bullet$ by Proposition 6.11. This in turn is weakly equivalent to \mathcal{A}_\bullet , as \mathcal{A}_\bullet is a hypersheaf. \square

Remark 6.15. In fact, since \mathcal{A}_\bullet is a hypersheaf, we can replace the category $\text{Aff}_{\text{Zar}}(Z)$ with any subcategory \mathcal{U} generating the Zariski topology on Z . This means that the objects of \mathcal{U} must cover Z , and that for any $U, V \in \mathcal{U}$, the scheme $U \cap V$ can be covered by objects of \mathcal{U} .

6.3. Henselian thickenings.

Definition 6.16. Given a morphism $f : X \rightarrow Y$ of affine schemes, define $(X/Y)^{\text{hen}}$ to be the Henselisation of Y at X . It follows from [Ane] Proposition 64 that this is given by

$$(X/Y)^{\text{hen}} = \text{Spec} \varinjlim_{\substack{X \rightarrow U \xrightarrow{e} Y \\ e \text{ étale}}} \Gamma(U, \mathcal{O}_Y) = \text{Spec} \Gamma(X, f_{\text{Pr}}^{-1} \mathcal{O}_{Y, \text{ét}}),$$

where f_{Pr}^{-1} is the presheaf inverse image functor.

Remark 6.17. Note that if f is also a closed immersion, then (X, Y) is a Henselian pair (as in [Laf] or [Ray] §11).

Lemma 6.18. *If $X_0 \rightarrow X$ is a closed immersion of affine schemes, and $X^h := (X_0/X)^{\text{hen}}$, then $X_0 \times_X X^h = X_0$.*

Proof. Given any affine scheme Y under X_0 , write Y^h for the localisation $(X_0/Y)^{\text{hen}}$. Properties of henselisation from [Ane] give

$$(X_0 \times_X X^h)^h = X_0^h \times_{X^h} X^h = X_0,$$

so the morphism $X_0 \rightarrow X_0 \times_X X^h$ is pro-étale. It is automatically a closed immersion, so we must have $X_0 \times_X X^l = X_0 \sqcup T$, for some T . However, by [Gre1] Theorem 4.1, components of $X^h \times_X X_0 \rightarrow X^h$ correspond to components of $X_0 \times_X X_0 \rightarrow X_0$. The latter is an isomorphism, so $X_0 \rightarrow X_0 \times_X X^h$ must be an isomorphism on the set of components, hence an isomorphism. \square

Definition 6.19. Given a cosimplicial affine scheme X , define $X^h \in \text{cAff}$ by $(X^h)^n := (\pi^0 X / X^n)^{\text{hen}}$.

Proposition 6.20. *For $X \in \text{cAff}$, the morphism $f : X^h \rightarrow X$ is a weak equivalence.*

Proof. The proof of Proposition 6.11 carries over. \square

Corollary 6.21. *For any simplicial cosimplicial affine scheme X , there is a (levelwise) weak equivalence $X^h \rightarrow X$ in scAff , where X^h is defined by setting $(X^h)_i := (X_i)^h$. In particular, this applies when X is any homotopy derived Artin n -hypergroupoid (in which case X^h is also).*

Remarks 6.22. The maps $\pi^0(X) \rightarrow X^h$ are levelwise closed immersions, so $\pi^0(X) \rightarrow X_i^h$ is a Henselian pair. By [Gru] Theorem I.8, Henselian pairs are lifted by all smooth morphisms of affine schemes. This means that many techniques involving infinitesimal extensions can be adapted for Henselian pairs. Henselian pairs are also the largest class of maps with this lifting property, since lifting with respect to all standard étale morphisms forces a map to be Henselian, while lifting with respect to \mathbb{A}^1 forces a map to be a closed immersion. There is, in fact, a simplicial model structure on sRing^h , for which a morphism is a weak equivalence or fibration whenever the underlying map in sRing is so. For this model structure, smooth rings are cofibrant.

Recall from Remark 1.27 that all algebraic spaces are 2-geometric, and that a quasi-separated algebraic space is 1-geometric if and only if it is semi-separated.

Theorem 6.23. *Fix an algebraic space Z over a ring R , and fix $m \geq 2$ (or $m \geq 1$ if Z is semi-separated). Then the homotopy category of m -geometric derived Deligne–Mumford stacks X over R with $\pi^0 X \simeq Z$ is weakly equivalent to the homotopy category of presheaves \mathcal{A}_\bullet of simplicial R -algebras on the category of affine schemes étale over Z , satisfying the following:*

- (1) $\pi_0(\mathcal{A}_\bullet) = \mathcal{O}_Z$;
- (2) for all i , the presheaf $\pi_i(\mathcal{A}_\bullet)$ is a quasi-coherent \mathcal{O}_Z -module.

Proof. The proof of Theorem 6.14 carries over to this context, replacing Proposition 6.11 with Proposition 6.20. \square

Remark 6.24. As in Remark 6.15, we can replace the category $\text{Aff}_{\text{ét}}(Z)$ with any subcategory \mathcal{U} generating the étale topology on Z .

Remark 6.25. Take a homotopy derived Artin n -hypergroupoid Y and let $\mathfrak{Y} = Y^\sharp$; since $Y^h \rightarrow Y$ is a weak equivalence and $\pi^0 Y \rightarrow Y^h$ is a Henselian pair on each level, the étale sites of \mathfrak{Y} and $\pi^0 \mathfrak{Y}$ are weakly equivalent.

[Another way to look at this phenomenon is that taking the Postnikov tower of $O(Y)$ (as in [GJ] Definition VI.3.4) allows us to describe Y as the colimit of a sequence $\pi^0 Y \hookrightarrow \pi^{\leq 1} Y \hookrightarrow \pi^{\leq 2} Y \hookrightarrow \dots$ of homotopy derived Artin n -hypergroupoids, with the i th closed immersion defined by an ideal sheaf weakly equivalent to $\pi_i O(Y)[-i]$. Thus \mathfrak{Y} can be expressed as the colimit of the sequence $\pi^0 \mathfrak{Y} \rightarrow \pi^{\leq 1} \mathfrak{Y} \rightarrow \pi^{\leq 2} \mathfrak{Y} \rightarrow \dots$ of homotopy square-zero extensions, so the étale sites of \mathfrak{Y} and $\pi^0 \mathfrak{Y}$ are isomorphic.]

Thus the corresponding ∞ -topoi are equivalent, so giving the étale sheaf $\mathcal{O}_{\mathfrak{Y}}$ on \mathfrak{Y} is equivalent to giving the étale sheaf $\iota^{-1} \mathcal{O}_{\mathfrak{Y}}$ on $\pi^0 \mathfrak{Y}$. The same is not true for smooth morphisms, where the ∞ -topoi differ (although the homotopy categories are the same), which suggests that we cannot expect an exact analogue for derived Artin stacks. However, Corollary 8.11 will provide a strong substitute.

6.4. DG stacks and dg-schemes. Using the Eilenberg-Zilber shuffle product, the normalisation functor N^s from Definition 0.2 extends to a functor

$$N^s : s\text{Ring} \rightarrow dg_+ \text{Ring}$$

from simplicial rings to (graded-commutative) algebras in non-negatively graded chain complexes. Where no confusion is likely, we will denote this functor by N , and its left adjoint by N^* . There is a model structure on $dg_+ \text{Alg}_{\mathbb{Q}} := \mathbb{Q} \downarrow dg_+ \text{Ring}$, defined by setting weak equivalences to be quasi-isomorphisms, and fibrations to be maps which are surjective in strictly positive degrees.

Now, the functor

$$N : s\text{Alg}_{\mathbb{Q}} \rightarrow dg_+ \text{Alg}_{\mathbb{Q}}$$

is a right Quillen equivalence by [Qui]. In particular, this implies that it gives a weak equivalence of the associated ∞ -categories, and hence that

$$N : \text{Ho}(s\text{Alg}_A) \rightarrow \text{Ho}(dg_+ \text{Alg}_A)$$

is a weak equivalence for any \mathbb{Q} -algebra A .

We may now apply Theorems 4.7 and 4.10 to the HAG context given by smooth morphisms and $dg_+ \text{Alg}_A$. Note that this is not the same as the HAG context of D -stacks in [TV2], since our chain complexes are non-negatively graded. However, since N is a Quillen equivalence, this HAG context is equivalent to that of D^- -stacks. Explicitly:

Definition 6.26. A morphism $f : A \rightarrow B$ in $dg_+ \text{Ring}$ is said to be smooth if $H_0(f) : H_0 A \rightarrow H_0 B$ is smooth, and the maps $H_n(A) \otimes_{H_0(A)} H_0(B) \rightarrow H_n(B)$ are isomorphisms for all n .

Definition 6.27. Let $DG^+ \text{Aff}_{\mathbb{Q}}$ be the opposite category to $dg_+ \text{Alg}_{\mathbb{Q}}$. Define the denormalisation functor $D : DG^+ \text{Aff}_{\mathbb{Q}} \rightarrow c\text{Aff}_{\mathbb{Q}}$ to be opposite to N^* , so $DX(A) = X(NA)$, for $X \in DG^+ \text{Aff}_{\mathbb{Q}}, A \in s\text{Alg}_{\mathbb{Q}}$. This has left adjoint D^* , given by $D^* \text{Spec } A = \text{Spec } NA$.

Definition 6.28. Let $\pi^0 : DG^+ \text{Aff}_{\mathbb{Q}} \rightarrow \text{Aff}_{\mathbb{Q}}$ be the functor $\text{Spec } A \mapsto \text{Spec } H_0 A$, and say that a morphism in $DG^+ \text{Aff}_{\mathbb{Q}}$ is *surjective* if its image under π^0 is so.

Definition 6.29. Taking \mathbf{C} to be the class of smooth surjections in $DG^+ \text{Aff}$, we will (by analogy with Examples 3.5) refer to (trivial) (n, \mathbf{C}) -hypergroupoids as DG Artin n -hypergroupoids.

The following is an immediate consequence of the Quillen equivalence between simplicial and dg algebras:

Proposition 6.30. *If $f : X \rightarrow S$ is a DG Artin n -hypergroupoid, then $Df : DX \rightarrow DS$ is a derived Artin n -hypergroupoid, which is trivial whenever f is so. If $S = D^*Z$ for some $Z \in \text{scAff}$, then the map*

$$D^*DX \times_{D^*DS} S \rightarrow X$$

is a weak equivalence.

*If $g : Y \rightarrow T$ is a homotopy derived Artin n -hypergroupoid, then $D^*g : D^*Y \rightarrow D^*T$ is a homotopy DG Artin n -hypergroupoid, which is trivial whenever g is so. If $\widehat{D^*Y}$ is a Reedy fibrant approximation to D^*Y over D^*T , then the map*

$$Y \rightarrow D\widehat{D^*Y} \times_{DD^*T} T$$

is a weak equivalence.

Note that the functor D only behaves well when applied to Reedy fibrations, so the proposition does not apply if f is only a homotopy DG Artin n -hypergroupoid. However, we will now define a homotopy inverse to N , similar to the Thom–Sullivan functor, which does preserve weak equivalences.

Definition 6.31. Let Ω_n be the cochain algebra

$$\mathbb{Q}[t_0, t_1, \dots, t_n, dt_0, dt_1, \dots, dt_n] / (\sum t_i - 1, \sum dt_i)$$

of rational differential forms on the n -simplex Δ^n . These fit together to form a simplicial diagram Ω_\bullet of DG-algebras.

Definition 6.32. Given a simplicial module S_\bullet and a cosimplicial module C^\bullet , define $S \otimes_{\leftarrow} C$ by

$$S \otimes_{\leftarrow} C := \{x \in \prod_n S_n \otimes C^m : (1 \otimes \partial^i)x_n = (\partial_i \otimes 1)x_{n+1}, (1 \otimes \sigma^i)x = (\sigma_i \otimes 1)x_{n-1}\}.$$

Similarly, given a chain complex S_\bullet and a cochain complex C^\bullet , define $S \otimes_{\leftarrow} C$ by

$$S \otimes_{\leftarrow} C := \{x \in \prod_n S_n \otimes C^m : (1 \otimes d)x_n = (d \otimes 1)x_{n+1}\}.$$

Definition 6.33. Define the functor $T : dg_+ \text{Alg}_{\mathbb{Q}} \rightarrow s\text{Alg}_{\mathbb{Q}}$ by $T(B)_n := \Omega_n \otimes_{\leftarrow} B$.

Define $T : DG^+ \text{Aff}_{\mathbb{Q}} \rightarrow c\text{Aff}_{\mathbb{Q}}$ by $T(\text{Spec } A) := \text{Spec } TA$.

Proposition 6.34. *For a \mathbb{Q} -algebra A , the functor T is quasi-inverse to normalisation $N : \text{Ho}(s\text{Alg}_A) \rightarrow \text{Ho}(dg_+ \text{Alg}_A)$.*

Proof. Let \mathbb{Q}^Δ be the simplicial cosimplicial \mathbb{Q} -algebra given by $(\mathbb{Q}^\Delta)_n^m = \mathbb{Q}^{\Delta_n^m}$. As in [HS1], there are cochain quasi-isomorphisms $\int : \Omega_m \rightarrow N_c(\mathbb{Q}^\Delta)_m$ for all m , where N_c denotes cosimplicial normalisation. Explicitly,

$$\left(\int \omega\right)(f) = \int_{|\Delta^n|} f^* \omega,$$

for $\omega \in \Omega_m^n, f \in \Delta_n^m$. Denormalisation gives a quasi-isomorphism $\int : N_c^{-1}\Omega_m \rightarrow (\mathbb{Q}^\Delta)_m$ of cosimplicial complexes, and analysis of the shuffle product shows that this is a quasi-isomorphism of cosimplicial algebras.

Now, for $A \in s\text{Alg}_A$,

$$T(N^s A) = (N^s A) \otimes_{\leftarrow, \mathbb{Q}} \Omega \cong A \otimes_{\leftarrow, \mathbb{Q}} (N_c \Omega),$$

and \int gives a quasi-isomorphism

$$TN^s A \rightarrow A \otimes_{\leftarrow, \mathbb{Q}} (\mathbb{Q}^\Delta) = A$$

of simplicial algebras, so T is a quasi-inverse to N^s on the homotopy categories. \square

Corollary 6.35. *If $f : X \rightarrow S$ is a homotopy DG Artin n -hypergroupoid, then $Tf : TX \rightarrow TS$ is a homotopy derived Artin n -hypergroupoid, which is trivial whenever f is so. If $S = D^*Z$ for some $Z \in \text{scAff}$, then the map*

$$D^*TX \times_{D^*TS} S \rightarrow X$$

is a weak equivalence.

We now introduce an analogue of the results of §§6.1–6.3.

Definition 6.36. Given $A \in \text{dg}_+\text{Alg}_{\mathbb{Q}}$, define localisation and henselisation of A by

$$A^l := (A_0/H_0A)^{\text{loc}} \otimes_{A_0} A, \quad A^h := (A_0/H_0A)^{\text{hen}} \otimes_{A_0} A.$$

Define completion by letting $I := \ker(A_0 \rightarrow H_0A)$ and setting

$$\hat{A} := \varprojlim_n A/I^n A.$$

Lemma 6.37. *The maps $A \rightarrow A^l$ and $A \rightarrow A^h$ are weak equivalences. If A_0 is Noetherian and each A_n is a finite A_0 -module, then $A \rightarrow \hat{A}$ is also a weak equivalence.*

Proof. We begin by noting that the maps $A_0 \rightarrow A_0^l$ and $A_0 \rightarrow A_0^h$ are flat, so

$$H_*(A^l) \cong H_*(A) \otimes_{A_0} A_0^l, \quad H_*(A^h) \cong H_*(A) \otimes_{A_0} A_0^h.$$

Now, Lemmas 6.9 and 6.18 imply that $H_0(A) \otimes_{A_0} A_0^l \cong H_0(A)$ and $H_0(A) \otimes_{A_0} A_0^h \cong H_0(A)$. Since $H_*(A)$ is automatically an $H_0(A)$ -module, this completes the proof in the local and Henselian cases.

If A_0 is Noetherian, then [Mat] Theorem 8.8 implies that $A_0 \rightarrow \hat{A}_0$ is flat. If A_n is a finite A_0 -module, then [Mat] Theorem 8.7 implies that $\hat{A}_n = \hat{A}_0 \otimes_{A_0} A_n$. Thus

$$H_*(\hat{A}) \cong H_*(A) \otimes_{A_0} \hat{A}_0,$$

and applying [Mat] Theorem 8.7 to the A_0 -module H_0A gives that $H_0(A) \otimes_{A_0} \hat{A}_0 \cong H_0(A)$, so $H_*(\hat{A}) \cong H_*(A)$, as required. \square

Corollary 6.38. *For any simplicial diagram X in $DG^+\text{Aff}$, there are (levelwise) weak equivalences $X^l \rightarrow X$ and $X^h \rightarrow X$ in $sDG^+\text{Aff}$, where X^l is defined by setting $(X^l)_i := (X_i)^l$ and similarly for X^h . In particular, this applies when X is any homotopy DG Artin n -hypergroupoid (in which case X^l and X^h are also). If X^0 is levelwise Noetherian, with each $\mathcal{O}(X^n)$ levelwise coherent on X^0 , the same is true of the levelwise completion $\hat{X} \rightarrow X$.*

We are now in a position to describe the D^- -stack associated to a dg-scheme.

Definition 6.39. Recall from [CFK] Definition 2.2.1 that a dg-scheme $X = (X^0, \mathcal{O}_{X,\bullet})$ consists of a scheme X^0 , together with a sheaf $\mathcal{O}_{X,\bullet}$ of quasi-coherent non-negatively graded (chain) dg-algebras over \mathcal{O}_{X^0} , with $\mathcal{O}_{X,0} = \mathcal{O}_{X^0}$.

Definition 6.40. The degree 0 truncation $\pi^0 X$ of a dg-scheme X is defined to be the closed subscheme

$$\pi^0 X := \mathbf{Spec} \mathcal{H}_0(\mathcal{O}_{X,\bullet}) \subset X^0.$$

This is denoted by $\pi_0 X$ in [CFK].

A morphism $f : X \rightarrow Y$ of dg-schemes is said to be a quasi-isomorphism if it induces an isomorphism $(\pi^0 X, \mathcal{H}_*(\mathcal{O}_{X,\bullet})) \rightarrow (\pi^0 Y, \mathcal{H}_*(\mathcal{O}_{Y,\bullet}))$.

Note that Lemma 6.37 implies that for a dg-scheme $X = (X^0, \mathcal{O}_X)$, the maps

$$((\pi^0 X/X)^{\text{hen}}, a^* b^* \mathcal{O}_X) \xrightarrow{a} ((\pi^0 X/X)^{\text{loc}}, b^* \mathcal{O}_X) \xrightarrow{b} X$$

are quasi-isomorphisms. If X^0 is locally Noetherian and each $\mathcal{O}_{X,n}$ coherent, then the map

$$((\widehat{\pi^0 X/X}), c^* a^* b^* \mathcal{O}_X) \xrightarrow{c} ((\pi^0 X/X)^{\text{hen}}, a^* b^* \mathcal{O}_X)$$

is also a quasi-isomorphism, where $(\widehat{\pi^0 X/X})$ is the formal completion of X along $\pi^0 X$.

Given a dg-scheme $X = (X^0, \mathcal{O}_X)$, with X^0 semi-separated, we may take an affine cover $U = \coprod_i U_i$ of X^0 , and define the simplicial scheme Y^0 by $Y^0 := \text{cosk}_0(U/X^0)$. Hence

$$\begin{aligned} Y_n^0 &= \overbrace{U \times_{X^0} U \times_{X^0} \dots \times_{X^0} U}^{n+1}, \\ &= \coprod_{i_0, \dots, i_n} U_{i_0} \cap \dots \cap U_{i_n}, \end{aligned}$$

so Y_n^0 is a disjoint union of affine schemes (as X^0 is semi-separated, i.e. $X^0 \rightarrow X^0 \times X^0$ is affine). Thus Y^0 is a Zariski 1-hypergroupoid and $a : Y^0 \rightarrow X^0$ a resolution, and we may define a homotopy DG (1, open)-hypergroupoid $Y = \text{Gpd}(X)$ by

$$\text{Gpd}(X) := \mathbf{Spec}(a^{-1} \mathcal{O}_X).$$

A fortiori, this is a homotopy DG Artin 1-hypergroupoid.

Remarks 6.41. (1) Note that Y^0 is a variant of the unnormalised Čech resolution for X^0 . The standard normalised Čech resolution is given by $\coprod_{i_0 < \dots < i_n} U_{i_0} \cap \dots \cap U_{i_n}$ in level n , so the standard unnormalised Čech resolution $(X^0)'$ is given by $(X^0)'_n = \coprod_{i_0 \leq \dots \leq i_n} U_{i_0} \cap \dots \cap U_{i_n}$. There is a canonical morphism $(X^0)' \rightarrow Y$, but the partial matching map

$$(X^0)'_2 \rightarrow M_{\Lambda^2}(X^0)'$$

is not surjective, so $(X^0)'$ is not a relative Zariski n -hypergroupoid for any n .

- (2) If X^0 is not semi-separated, we could instead apply Theorem 4.7 to obtain a Zariski 2-hypergroupoid Y^0 , and then a homotopy DG (2, open)-hypergroupoid $\text{Gpd}(X)$ by the formula above.
- (3) If X is a dg-manifold in the sense of [CFK], then Y^h will be Reedy fibrant in the model category $sDG^+ \text{Aff}^h$ (defined analogously to Remarks 6.22). [CFK] Theorem 2.6.1 shows that if X^0 is quasi-projective, then there exists a dg-manifold X' quasi-isomorphic to X , which can be thought of as a form of fibrant approximation. In fact, the construction is such that $(X')^l$ is Reedy fibrant in the model category $sDG^+ \text{Aff}^l$, since $\mathcal{O}_{X'}$ is locally free for the Zariski topology.

Theorem 6.42. *Fix an algebraic space Z over a \mathbb{Q} -algebra R , and fix $m \geq 2$ (or $m \geq 1$ if Z is semi-separated). Then the homotopy category of m -geometric derived schemes (resp. m -geometric derived Deligne–Mumford stacks) X over R with $\pi^0 X \simeq Z$ is weakly equivalent to the homotopy category of presheaves \mathcal{A}_\bullet of non-negatively graded dg R -algebras on the category $\text{Aff}_{\text{Zar}}(Z)$ (resp. $\text{Aff}_{\text{ét}}(Z)$), satisfying the following:*

- (1) $\pi_0(\mathcal{A}_\bullet) = \mathcal{O}_Z$;
- (2) for all i , the presheaf $H_i(\mathcal{A}_\bullet)$ is a quasi-coherent \mathcal{O}_Z -module.

In particular, the corresponding homotopy categories are equivalent.

Proof. The proofs of Theorems 6.14 and 6.23 carry over to this context, replacing $\iota_{\text{Pr}}^{-1} \mathcal{O}_{\tilde{X}}$ with the presheaves

$$(O(\tilde{X}^0)/\mathcal{O}_{\tilde{Z}})^{\text{loc}} \otimes_{O(\tilde{X}^0)} O(\tilde{X})_\bullet \text{ or } (O(\tilde{X}^0)/\mathcal{O}_{\tilde{Z}})^{\text{hen}} \otimes_{O(\tilde{X}^0)} O(\tilde{X})_\bullet.$$

□

Corollary 6.43. *If \mathfrak{X} is an n -geometric derived Deligne–Mumford stack with $\pi^0 \mathfrak{X}$ a quasi-affine scheme, then there exists a dg-scheme X with $\text{Gpd}(X)^\# \simeq \mathfrak{X}$.*

Proof. Let $Y = \text{Spec} \Gamma(\pi^0 \mathfrak{X}, \mathcal{O}_{\pi^0 \mathfrak{X}})$; since $\pi^0 \mathfrak{X} \rightarrow Y$ is an open immersion, the complement Z is closed. Take a presheaf \mathcal{A}_\bullet as in Theorem 6.42, and let $W := \text{Spec} \Gamma(\pi^0 \mathfrak{X}, \mathcal{A}_0)$, noting that $Y \rightarrow W$ is a closed immersion. Now set X^0 to be the quasi-affine scheme $W - Z$,

and define $\mathcal{O}_{X,n}$ to be the quasi-coherent sheaf associated to the module $\Gamma(\pi^0 \mathfrak{X}, \mathcal{A}_n)$ on the affine W . \square

7. DERIVED QUASI-COHERENT SHEAVES AND THE COTANGENT COMPLEX

We now return to the derived Artin hypergroupoids of §1.3.5 and Examples 3.5, so $c\text{Aff}$ will denote the model category of cosimplicial affine schemes. For a derived Artin n -hypergroupoid X , we write X^\sharp for the associated n -geometric derived Artin stack $|\mathbf{R}hX|$ given by combining Propositions 4.1 and 4.13.

For a simplicial ring A_\bullet , the simplicial normalisation functor N^s induces an equivalence between the categories $s\text{Mod}(A)$ of simplicial A_\bullet -modules, and $dg_+\text{Mod}(N^s A)$ of $N^s A$ -modules in non-negatively graded chain complexes, where $N^s A$ is given a graded-commutative multiplication by the Eilenberg–Zilber shuffle product. As observed in [TV2] §2.2.1, this extends to give a weak equivalence between the ∞ -category of stable A -modules, and the ∞ -category $dg\text{Mod}(N^s A)$ of $N^s A$ -modules in \mathbb{Z} -graded chain complexes, and hence an equivalence between the corresponding homotopy categories.

Definition 7.1. Define left Quillen presheaves $dg\text{Mod}$ and $dg_+\text{Mod}$ on $c\text{Aff}$ by $dg\text{Mod}(X) := dg\text{Mod}(N^s O(X))$ and $dg_+\text{Mod}(X) := dg_+\text{Mod}(N^s O(X))$. Note that under the Dold–Kan equivalence, $dg_+\text{Mod}$ is equivalent to the left Quillen presheaf Mod of Definition 5.11.

Morphisms are said to be weak equivalences if they induce isomorphisms on homology. Fibrations in $dg\text{Mod}(X)$ are surjections, while fibrations in $dg_+\text{Mod}(X)$ are surjective in non-zero degrees.

Lemma 7.2. *The functors $dg\text{Mod}, dg_+\text{Mod}$ are left Quillen hypersheaves on $c\text{Aff}$ with respect to flat surjections.*

Proof. The proof of Lemma 5.23 carries over, since [TV2] Proposition 2.2.2.5 shows that for any flat morphism f in $c\text{Aff}$, we have $\mathbf{L}f^* \simeq f^*$. This means that for a faithfully flat hypercover $p: \tilde{X}_\bullet \rightarrow X$ in $c\text{Aff}$, and $\mathcal{F} \in cdg\text{Mod}(\tilde{X}_\bullet)_{\text{cart}}$, we have $H_i \mathcal{F} \in c\text{Mod}(\pi^0 \tilde{X}_\bullet)_{\text{cart}}$ for all i , for π^0 as in Definition 1.22. \square

Note that Remark 5.13 specialises to give that for any derived Artin n -hypergroupoid X , $cdg_+\text{Mod}(X)_{\text{cart}}$ is equivalent to the relative category of quasi-coherent modules on the derived n -geometric stack X^\sharp . Quasi-coherent complexes on X^\sharp in the sense of [Lur1] §5.2 correspond to $cdg\text{Mod}(X)_{\text{cart}}$.

7.1. Cotangent complexes. Fix a derived Artin m -hypergroupoid $X \rightarrow S$.

Definition 7.3. We make $cdg\text{Mod}(X)$ into a simplicial category by setting (for $K \in \mathbb{S}$)

$$(M^K)^n := (M^n)^{K_n},$$

as an $N^s O(X)^n$ -module in chain complexes. This has a left adjoint $M \mapsto M \otimes K$. Given a cofibration $K \hookrightarrow L$, we write $M \otimes (L/K) := (M \otimes L)/(M \otimes K)$.

Given $M \in cdg\text{Mod}(X)$, define $lineM \in (cdg\text{Mod}(X))^\Delta$ to be the cosimplicial complex given in cosimplicial level n by $M \otimes \Delta^n$.

Lemma 7.4. *If $N^s \Omega(X/S) \in cdg\text{Mod}(X)$ denotes the chain complex given on X_n by $N^s \Omega_{X_n/S_n}$, then*

$$N^s \Omega(X/S) \otimes K = \eta^* N^s \Omega(X^K/S^K),$$

for $\eta: X \rightarrow X^K$ corresponding to the constant map $K \rightarrow \bullet$

Proof. This follows immediately from the definitions of X^K and $M \otimes K$. \square

Definition 7.5. Given $M \in \text{cdg}_+ \text{Mod}(X)$, define the derived deformation space by

$$\underline{\text{Der}}(X/S, M) := \underline{\text{Hom}}_{(X \downarrow \text{scAff} \downarrow S)}(\mathbf{Spec}(\mathcal{O}_X \oplus N_s^{-1}M), X) \in \mathbb{S},$$

for the simplicial structure of Definition 1.20. Since $\mathbf{Spec}(\mathcal{O}_X \oplus N_s^{-1}M)$ is an abelian cogroup object in $(X \downarrow \text{scAff} \downarrow S)$, $\underline{\text{Der}}(X/S, M)$ has the natural structure of a simplicial abelian group.

The following is immediate:

Lemma 7.6. For $M \in \text{cdg}_+ \text{Mod}(X)$, the simplicial group $\underline{\text{Der}}(X/S, M)$ is given in level n by $\text{Hom}_{\text{cdgMod}(X)}(\Omega(X/S)^n, M)$.

Definition 7.7. Define the cotangent complex $\mathbb{L}^{X/S} \in \text{cdgMod}(X)$ by $\mathbb{L}^{X/S} := \text{Tot } N_c N^s \Omega(X/S)$, where N_c denotes cosimplicial conormalisation (Definition 0.3), and Tot is the direct sum total functor from cochain chain complexes to chain complexes. Note that Lemma 7.4 implies that $N_c^i N^s \Omega(X/S) \cong \eta^* N^s \Omega(X^{\Delta^i} / X^{\Lambda_0^i} \times_{S^{\Lambda_0^i}} S^{\Delta^i})$.

Lemma 7.8. If we write $\iota : \pi^0 X \rightarrow X$, then $\iota^* \mathbb{L}^{X/S}$ is equivalent to the complex

$$\iota^* N^s \Omega(X/S) \rightarrow \Omega(\pi^0 X / \pi^0 S) \otimes (\Delta^1 / \Lambda_0^1) \rightarrow \dots \rightarrow \Omega(\pi^0 X / \pi^0 S) \otimes (\Delta^m / \Lambda_0^m),$$

in (chain) degrees $[-m, \infty)$. If moreover $f : X \rightarrow S$ is smooth, then $\iota^* \mathbb{L}^{X/S}$ is equivalent to the complex

$$\Omega(\pi^0 X / \pi^0 S) \rightarrow \Omega(\pi^0 X / \pi^0 S) \otimes (\Delta^1 / \Lambda_0^1) \rightarrow \dots \rightarrow \Omega(\pi^0 X / \pi^0 S) \otimes (\Delta^m / \Lambda_0^m),$$

which is locally projective and concentrated in degrees $[-m, 0]$.

Proof. The definition of cosimplicial normalisation shows that

$$N_c^i N^s \Omega(X/S) \cong \eta^* N^s \Omega(X^{\Delta^i} / X^{\Lambda_0^i} \times_{S^{\Lambda_0^i}} S^{\Delta^i}),$$

and 3.12 implies that for $i \geq m$, the maps $\theta_i(f) : X^{\Delta^i} \rightarrow X^{\Lambda_0^i} \times_{Y^{\Lambda_0^i}} Y^{\Delta^i}$ are trivial relative 0-hypergroupoids, hence levelwise weak equivalences, so $N_c^i N^s \Omega(X/S)$ is contractible for $i > m$. For all $i > 0$, the maps $\theta_i(f)$ are smooth relative m -hypergroupoids, hence levelwise smooth, so

$$\iota^* N_c^i N^s \Omega(X/S) \cong N_c^i N^s \Omega(\pi^0 X / \pi^0 S) = N_c^i \Omega(\pi^0 X / \pi^0 S),$$

using the fact that $\Omega(\pi^0 X / \pi^0 S)$ is a module rather than a complex.

Finally, if $X \rightarrow S$ is smooth, then it is levelwise smooth, so $\iota^* N^s \Omega(X/S) \simeq \Omega(\pi^0 X / \pi^0 S)$. This is locally projective in the sense that each $\Omega(\pi^0 X_n / \pi^0 S_n)$ is projective on X_n . Note that in general it is not, however, projective in $\text{cdgMod}(X)$, since it is not Reedy cofibrant. \square

Remark 7.9. To understand the distinction between locally projective and projective objects P in $\text{Mod}(X)$, note that projectivity implies vanishing of the higher Ext groups $\text{Ext}_X^i(P, \mathcal{F})$, while local projectivity merely implies vanishing of the higher Ext-presheaves $\mathcal{E}xt_{\mathcal{O}_X}^i(P, \mathcal{F})$.

Definition 7.10. Let HOM be \mathbb{Z} -graded derived Hom for complexes. Explicitly, for levelwise projective U , $\text{HOM}(U, V)_n$ is the space of degree n graded homomorphisms from U to V (ignoring the differentials), and the chain complex structure on HOM is given by $df = d_V \circ f \pm f \circ d_U$. Note that this is related to simplicial enrichments by $N^s \underline{\text{Hom}}(U, V) = \tau_{\geq 0} \text{HOM}(U, V)$, where N^s is simplicial normalisation and τ is good truncation.

Proposition 7.11. For all $M \in \text{cdg}_+ \text{Mod}(X)$, there is a weak equivalence

$$\underline{\text{Hom}}_{\text{cdgMod}(X)}(\mathbb{L}^{X/S}, M) \simeq \underline{\text{Der}}(X/S, M).$$

Proof. Given a cochain chain complex L , observe that $\mathrm{HOM}(\mathrm{Tot} L, M) \cong \mathrm{Tot}^{\mathrm{II}}\mathrm{HOM}(L, M)$, noting that $\mathrm{HOM}(L, M)$ is a bichain complex. Thus

$$N^s\mathrm{Hom}(\mathbb{L}^{X/S}, M) \cong \tau_{\geq 0}\mathrm{Tot}^{\mathrm{II}}\mathrm{HOM}(N_c^\bullet\Omega(X/S), M).$$

Now, $N_c^n\Omega(X/S) = \eta_n^*N^s\Omega(X^{\Delta^n}/X^{\Lambda_0^n} \times_{S^{\Lambda_0^n}} S^{\Delta^n})$. Since $X^{\Delta^n} \rightarrow X^{\Lambda_0^n} \times_{S^{\Lambda_0^n}} S^{\Delta^n}$ is a trivial derived Artin m -hypergroupoid, we know that $N_c^n\Omega(X/S)$ is projective in the category $cdg_+\mathrm{Mod}(X)$ (since the smoothness conditions imply that the latching maps are monomorphisms with projective cokernel).

For $M \in cdg_+\mathrm{Mod}(X)$, the map $M[-i] \oplus M[1-i] \xrightarrow{\mathrm{id}, d} M[-i]$ is a surjection in $cdg_+\mathrm{Mod}(X)$ for $i > 0$; this implies that

$$\mathrm{HOM}_{1-i}(N_c^n\Omega(X/S), M) \xrightarrow{d} \mathrm{Z}_i\mathrm{HOM}(N_c^n\Omega(X/S), M)$$

is surjective, so $\tau_{\geq 0}\mathrm{HOM}(N_c^n\Omega(X/S), M) \simeq \mathrm{HOM}(N_c^n\Omega(X/S), M)$ for all $n > 0$. We have therefore shown that

$$N^s\mathrm{Hom}(\mathbb{L}^{X/S}, M) \simeq \mathrm{Tot} \tau_{\geq 0}\mathrm{HOM}(N_c^\bullet\Omega(X/S), M).$$

Now, the Reedy fibration conditions on $X \rightarrow S$ imply that for all n , the map

$$N_c^n\Omega(X/S)/dN_c^{n-1}\Omega(X/S) \rightarrow N_c^{n+1}\Omega(X/S)$$

is a monomorphism, with projective cokernel in the category $g\mathrm{Mod}(X)$ of $N^sO(X)$ -modules in graded abelian groups (ignoring the simplicial differential d^s). This implies that the sequence

$$\dots \xrightarrow{d_\Omega} \mathrm{HOM}(N_c^{i+1}\Omega(X/S), M) \xrightarrow{d_\Omega} \mathrm{HOM}(N_c^i\Omega(X/S), M) \xrightarrow{d_\Omega} \dots$$

is exact. Since $\tau_{\geq 0}\mathrm{HOM}_n = \mathrm{HOM}_n$ for $n > 0$, and $(\tau_{\geq 0}\mathrm{HOM})_0 = \mathrm{Hom}$, this implies that

$$\mathrm{Tot} \tau_{\geq 0}\mathrm{HOM}(N_c^\bullet\Omega(X/S), M) \simeq \mathrm{Hom}(N_c^\bullet\Omega(X/S), M) \simeq N^s\mathrm{Der}(X/S, M),$$

by Lemma 7.6. □

Corollary 7.12. *If $f : Y \rightarrow S$ is a trivial derived Artin m -hypergroupoid, then $\mathbb{L}^{Y/S} \simeq 0$.*

Proof. Since f takes small extensions to trivial fibrations, $\mathrm{Der}(Y/S, M)$ is trivially fibrant, so $\mathrm{Der}(Y/S, M) \simeq 0$. Therefore $\mathbb{L}^{Y/S} \simeq 0$. □

Corollary 7.13.

$$\mathrm{Der}(X/S, M) \simeq \mathbf{R}\mathrm{Hom}_{(X^\sharp \downarrow_{(c\mathrm{Aff})} \sim, \tau \downarrow_{S^\sharp})}(\mathbf{Spec}(\mathcal{O}_X \oplus (N^s)^{-1}M)^\sharp, X^\sharp).$$

Proof. Setting $Y := \mathbf{Spec}(\mathcal{O}_X \oplus (N^s)^{-1}M)$, Theorem 4.10 implies that the space $\mathbf{R}\mathrm{Hom}_{(X^\sharp \downarrow_{(c\mathrm{Aff})} \sim, \tau \downarrow_{S^\sharp})}(Y^\sharp, X^\sharp)$ is a colimit of spaces

$$\mathrm{Hom}_{(X' \downarrow_{sc\mathrm{Aff}[S]})}(Y', X),$$

for a suitable system of trivial derived Artin m -hypergroupoids $Y' \rightarrow Y$, with $X' = X \times_Y Y'$.

By Proposition 7.11, this space is given by $\mathrm{Hom}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}, \pi_*\pi^*M)$, for $\pi : X' \rightarrow X$, which by adjunction is just $\mathrm{Hom}_{cdg\mathrm{Mod}(X')}(\pi^*\mathbb{L}^{X/S}, \pi^*M)$. Since $dg\mathrm{Mod}$ is a left Quillen hypersheaf by Lemma 7.2, the functor $\pi^* : cdg\mathrm{Mod}(X) \rightarrow cdg\mathrm{Mod}(X')$ gives an equivalence of relative categories, so

$$\mathrm{Hom}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}, \pi_*\pi^*M) \simeq \mathrm{Hom}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}, M),$$

which completes the proof. □

Corollary 7.14. $\mathbb{L}^{X/S}$ gives rise to a cotangent complex in the sense of [TV2] Definition 1.4.1.7 (or equivalently [Lur1] §3.2) by associating to any morphism $f : U \rightarrow X^\sharp$ with $U \in c\text{Aff}$, the complex $f^*\mathbb{L}^{X/S} \in \text{cdgMod}(U)_{\text{cart}}$. In other words,

$$\underline{\text{Hom}}_{\text{dgMod}(U)}(f^*\mathbb{L}^{X/S}, N) \simeq \underline{\text{Hom}}_{(U \downarrow (c\text{Aff}) \sim, \tau \downarrow S^\sharp)}(\mathbf{Spec}(\mathcal{O}_U \oplus N_s^{-1}N)^\sharp, X^\sharp),$$

functorially in $N \in \text{dg}_+\text{Mod}(U)$ and in f .

Proof. Passing to a suitable étale cover, we may assume that f factors as $f : U \rightarrow X$. If we set $M := f_*N \in \text{cdg}_+\text{Mod}(X)$ (which need not be Cartesian and is not to be confused with $f_*^{\text{cart}}N$), then

$$\underline{\text{Hom}}_{(U \downarrow (c\text{Aff}) \sim, \tau \downarrow S^\sharp)}(\mathbf{Spec}(\mathcal{O}_U \oplus N_s^{-1}N)^\sharp, X^\sharp) \simeq \underline{\text{Der}}(X/S, f_*N)$$

by Corollary 7.13, so the result follows from Proposition 7.11. \square

Remark 7.15. Beware that the cotangent complex of [Lur1] is not the same as the better-known cotangent complex of [Lur2]. The difference is that the former is based on simplicial rings, while the latter uses symmetric spectra. Thus they correspond locally to André–Quillen and topological André–Quillen homology, respectively. Roughly speaking, simplicial rings serve to apply homotopy theory to algebraic geometry, while symmetric spectra are used to do the opposite. As an example of the differences, some higher topological André–Quillen homology groups of $\mathbb{Z}[t]$ over \mathbb{Z} are non-zero — in other words, $\mathbb{Z} \rightarrow \mathbb{Z}[t]$ is not formally smooth as a morphism of symmetric spectra.

Definition 7.16. Given a morphism $\mathfrak{X} \rightarrow \mathfrak{S}$ of m -geometric derived Artin stacks, use Theorem 4.7 to form a derived Artin m -hypergroupoid $X \rightarrow S$ to a derived Artin m -hypergroupoid, with $X^\sharp \simeq \mathfrak{X}, S^\sharp \simeq \mathfrak{S}$, and define the quasi-coherent complex $\mathbb{L}^{\mathfrak{X}/\mathfrak{S}}$ on \mathfrak{X} by $a^*\mathbb{L}^{\mathfrak{X}/\mathfrak{S}} \simeq \mathbb{L}^{X/S}$, for $a : X \rightarrow \mathfrak{X}$, using the equivalence of Corollary 5.12. The characterisation in Corollary 7.14 ensures that this is well defined.

Lemma 7.17. $\mathbb{L}^{X/S}$ is homotopy-Cartesian.

Proof. We need to show that for each map $\partial_i : X_{n+1} \rightarrow X_n$, the map

$$\partial^i : \partial_i^*(\mathbb{L}^{X/S})^n \rightarrow (\mathbb{L}^{X/S})^{n+1}$$

is a quasi-isomorphism of chain complexes of quasi-coherent sheaves on X_{n+1} .

It follows from Lemma 7.4 that $\mathbb{L}(X/S)^n = \text{Tot } N_c N^s \eta^* \Omega(X^{\Delta^n}/S^{\Delta^n})$ in $\text{dgMod}(X_n)$. Thus ∂^i is a monomorphism, and its cokernel is

$$C := \text{Tot } N_c N^s \eta^* \Omega(X^{\Delta^{n+1}}/X^{\Delta^n} \times_{S^{\Delta^n}} S^{\Delta^{n+1}});$$

we wish to show that C is contractible.

Now, observe that C is $\mathbb{L}(X^{\Delta^{n+1}}/X^{\Delta^n} \times_{S^{\Delta^n}} S^{\Delta^{n+1}})^0$ on $X_n = (X^{\Delta^{n+1}})_0$. Since $\partial^i : \Delta^n \rightarrow \Delta^{n+1}$ is a trivial cofibration, $X^{\Delta^{n+1}} \rightarrow X^{\Delta^n} \times_{S^{\Delta^n}} S^{\Delta^{n+1}}$ is a trivial relative m -hypergroupoid by Lemma 3.12, so Corollary 7.12 implies that $C \simeq 0$, as required. \square

Lemma 7.18. If $f : X \rightarrow S$ is a derived Artin 0-hypergroupoid, then $\mathbb{L}^{X/S} \simeq N^s \Omega(X/S)$.

Proof. It suffices to show that each map $\partial_\Delta^i : N^s \Omega(X/S) \otimes \Delta^n \rightarrow \Omega(X/S) \otimes \Delta^{n+1}$ is a weak equivalence in $\text{cdg}_+\text{Mod}(X)$. This map is a monomorphism, with cokernel

$$\eta^* \Omega(X^{\Delta^{n+1}}/X^{\Delta^n} \times_{S^{\Delta^n}} S^{\Delta^{n+1}}).$$

By Lemma 3.12, $X^{\Delta^{n+1}} \rightarrow X^{\Delta^n} \times_{S^{\Delta^n}} S^{\Delta^{n+1}}$ is a trivial relative Artin 0-hypergroupoid, which is equivalent to saying that it is a levelwise weak equivalence, so $\Omega(X^{\Delta^{n+1}}/X^{\Delta^n} \times_{S^{\Delta^n}} S^{\Delta^{n+1}})$ is contractible, as required. \square

Corollary 7.19. *If $X \rightarrow S$ is a derived Artin m -hypergroupoid, then*

$$\mathbb{L}^{X/S} \simeq N^s\Omega(X/S) \otimes (\Delta^m/\partial\Delta^m)[m].$$

In particular, this implies that $H_i\mathbb{L}^{X/S} = 0$ for $i < m$.

Proof. We use the characterisation of the cotangent complex from the proof of Corollary 7.14, namely that

$$\underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}, M) \simeq \underline{\mathrm{Der}}(X/S, M),$$

for all $M \in cdg_+\mathrm{Mod}(X)$.

Now, simplicial considerations show that

$$\underline{\mathrm{Der}}(X^{\Delta^m}/X^{\partial\Delta^m} \times_{S^{\partial\Delta^m}} S^{\Delta^m}, \eta_*M) = \ker(\underline{\mathrm{Der}}(X/S, M)^{\Delta^m} \rightarrow \underline{\mathrm{Der}}(X/S, M)^{\partial\Delta^m}),$$

which is a model for the iterated loop space $\Omega^m\underline{\mathrm{Der}}(X/S, M)$, so

$$\Omega^m\underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}, M) \simeq \underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(\eta^*\mathbb{L}^{X^{\Delta^m}/X^{\partial\Delta^m} \times_{S^{\partial\Delta^m}} S^{\Delta^m}}, M).$$

Since $X^{\Delta^m} \rightarrow X^{\partial\Delta^m} \times_{S^{\partial\Delta^m}} S^{\Delta^m}$ is a derived Artin 0-hypergroupoid (by Lemma 3.12), Lemma 7.18 then implies that

$$\Omega^m\underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}, M) \simeq \underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(N^s\Omega(X/S) \otimes (\Delta^m/\partial\Delta^m), M),$$

for all $M \in cdg_+\mathrm{Mod}(X)$.

Moreover,

$$\begin{aligned} N^s\Omega^m\underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}, M) &\simeq \tau_{\geq 0}(N^s\underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}, M)[m]) \\ &= N^s\underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}[-m], M). \end{aligned}$$

Thus, for $M \in cdg_+\mathrm{Mod}(X)$, we have

$$\underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(\mathbb{L}^{X/S}[-m], M) \simeq \underline{\mathrm{Hom}}_{cdg\mathrm{Mod}(X)}(N^s\Omega(X/S) \otimes (\Delta^m/\partial\Delta^m), M),$$

from which we deduce that

$$\mathbb{L}^{X/S}[-m] \simeq N^s\Omega(X/S) \otimes (\Delta^m/\partial\Delta^m).$$

□

Definition 7.20. Given a homotopy derived Artin m -hypergroupoid $X \rightarrow S$, define $\mathbb{L}^{X/S} \in cdg\mathrm{Mod}(X)_{\mathrm{cart}}$ by taking a Reedy fibrant approximation $u : X \rightarrow \hat{X}$ over S , and setting $\mathbb{L}^{X/S} := u^*\mathbb{L}^{\hat{X}/S}$. In particular, we may apply this when $X \rightarrow S$ is an Artin m -hypergroupoid (with trivial derived structure), and then exploit the description of $\mathbb{L}^{X/S}$ in Lemma 7.8.

7.1.1. *Comparison with Olsson.* To compare our definition of the cotangent complex with Olsson's (in [Ols1]), we could simply note that Corollary 7.14 ensures that his must be the sheafification of ours. However, a more direct comparison is possible.

If \mathfrak{X} is a quasi-compact Artin 1-stack, we may take a presentation $X_0 \rightarrow \mathfrak{X}$, for X_0 affine. If \mathfrak{X} has affine diagonal, then \mathfrak{X} is 1-geometric, and $X := \mathrm{cosk}_0(X_0/\mathfrak{X})$ will be a simplicial affine scheme.

Given a strongly quasi-compact relative Artin 1-hypergroupoid $f : X \rightarrow Y$, for Y a strongly quasi-compact Artin 1-hypergroupoid, let $\mathfrak{X} := X^\sharp$ and $\mathfrak{Y} := Y^\sharp$ be the associated stacks, and $u : X \rightarrow \hat{X}$ a Reedy fibrant approximation over Y in the category of simplicial cosimplicial affine schemes. Then [Ols1] 8.2.3 defines the cotangent complex to be the sheafification of the complex

$$\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}^{X/Y} := \mathrm{Tot}(u^*N^s\Omega(\hat{X}/Y) \rightarrow \Omega(X/Y \times_{\mathfrak{Y}}^h \mathfrak{X}))$$

in $cdg\mathrm{Mod}(X)$.

Proposition 7.21. *For $X, Y, \mathfrak{X}, \mathfrak{Y}$ as above, $\mathbb{L}_{\mathfrak{X}/\mathfrak{Y}}^{X/Y}$ is the sheafification of the cotangent complex $\mathbb{L}^{X/Y}$.*

Proof. The characterisation in Definition 7.20 reduces to

$$\mathbb{L}^{X/Y} = \text{Tot}(u^*\Omega(\hat{X}/Y) \rightarrow \eta^*\Omega(X^{\Delta^1}/Y^{\Delta^1} \times_{\partial_0, Y} X)),$$

so it will suffice to show that $\Omega(X/Y \times_{\mathfrak{Y}}^h \mathfrak{X}) \cong \eta^*\Omega(X^{\Delta^1}/Y^{\Delta^1} \times_{\partial_0, Y} X)$.

Now, if we write $\delta : X \rightarrow X \times_{\mathfrak{X}}^h X$, then

$$\Omega(X/Y \times_{\mathfrak{Y}}^h \mathfrak{X}) = \delta^*\Omega((X \times_{\mathfrak{X}}^h X)/(X \times_{\mathfrak{Y}}^h Y)).$$

However, $X^{\Delta^1} \simeq X^{\Delta^1} \times_{\mathfrak{X}^{\Delta^1}}^h \mathfrak{X}$, so (since $X = \text{cosk}_0(X_0/\mathfrak{X})$)

$$X^{\Delta^1} \simeq X^{\partial\Delta^1} \times_{\mathfrak{X}^{\partial\Delta^1}}^h \mathfrak{X} \simeq (X \times X) \times_{\mathfrak{X} \times \mathfrak{X}}^h \mathfrak{X} \simeq X \times_{\mathfrak{X}}^h X,$$

and similarly for Y . This gives the required isomorphism. \square

8. DEFORMATIONS OF DERIVED ARTIN STACKS

8.1. Deforming morphisms of derived stacks.

Definition 8.1. Given a category \mathcal{C} , define the category $c_+\mathcal{C}$ of almost cosimplicial diagrams in \mathcal{C} to consist of functors $\Delta_* \rightarrow \mathcal{C}$, for Δ_* as in Definition 2.15. Thus an almost cosimplicial object X^* consists of objects $X^n \in \mathcal{C}$, with all of the operations ∂^i, σ^i of a cosimplicial diagram except ∂^0 , satisfying the usual relations.

Definition 8.2. Taking the dual construction to $\text{Dec}_+^{\text{opp}}$, we may define, for any almost cosimplicial object Y , an almost cosimplicial object $\text{Dec}_{\text{opp}}^+ Y$ given by $(\text{Dec}_{\text{opp}}^+ Y)^n = Y^{n+1}$, with $\partial_{\text{Dec}_{\text{opp}}^+ Y}^i = \partial_Y^{i+1}$. The operations $\partial_Y^0 : Y \rightarrow \text{Dec}_{\text{opp}}^+ Y$ define an almost cosimplicial map, which has a retraction σ_Y^0 . $\text{Dec}_{\text{opp}}^+$ is a comonad on almost cosimplicial complexes, with coproduct given by $\partial_Y^1 : \text{Dec}_{\text{opp}}^+ Y \rightarrow \text{Dec}_{\text{opp}}^+ \text{Dec}_{\text{opp}}^+ Y$, the co-unit being σ_Y^0 .

Proposition 8.3. *Take a diagram $Z \xrightarrow{g} X \xrightarrow{f} S$ of simplicial cosimplicial affine schemes, with f a Reedy fibration. If $Z \hookrightarrow \tilde{Z}$ is a levelwise closed immersion over S defined by a square-zero ideal \mathcal{I} , then the obstruction to extending g to a morphism $\tilde{g} : \tilde{Z} \rightarrow X$ over S lies in*

$$\text{H}_{-1}\text{HOM}_{\text{cdgMod}(X)}(N^s\Omega^{X/S}, g_*N^s\mathcal{I}).$$

If the obstruction is zero, then the Hom-space of possible extensions is given by

$$N^s\text{Hom}_{\text{Z}_{\text{scAff}}S}(\tilde{Z}, X) \simeq \tau_{\geq 0}\text{HOM}_{\text{cdgMod}(X)}(\mathbb{L}^{X/S}, N^s g_*\mathcal{I}).$$

Proof. Since f is a Reedy fibration, the simplicial matching maps

$$X_n \rightarrow M_n X \times_{M_n S} S_n$$

are fibrations of cosimplicial affine schemes. For a morphism $V \rightarrow W$ of cosimplicial affine schemes to be a fibration implies that the cosimplicial matching maps $V^j \rightarrow W^j \times_{M^{j-1}W} M^{j-1}V$ have the right lifting property (RLP) with respect to all closed immersions of affine schemes, so in particular are formally smooth.

Now, $\Delta \times \Delta^{\text{opp}}$ is a Reedy category (as in [Hov] §5.2), and we may regard f as a morphism of $\Delta \times \Delta^{\text{opp}}$ -diagrams in affine schemes. The results above imply that the $\Delta \times \Delta^{\text{opp}}$ matching maps

$$X_n^j \rightarrow (M_n X^j \times_{M^{j-1}X_n} X_n) \times_{(M_n S^j \times_{M_n M^{j-1}S} M^{j-1}S_n)} S_n^j$$

have the RLP for all closed immersions.

Likewise, the simplicial latching maps

$$L_n \tilde{Z} \cup_{L_n Z} Z_n \rightarrow \tilde{Z}_n$$

are levelwise closed immersions. However, the cosimplicial latching maps of a levelwise closed immersion $g : V \rightarrow W$ of cosimplicial affine schemes will only be closed immersions if g is a weak equivalence. Thus we instead consider the underlying diagram of almost cosimplicial affine schemes. For a simplicial ring A_\bullet , the almost cosimplicial latching object $L_*^i \text{Spec}(A_*)$ is given by $\text{Spec}(M_{\Lambda_0^i} A)$, so the almost cosimplicial matching maps of a levelwise closed immersion are closed immersions.

Letting \mathbb{I} be the Reedy category $\Delta_* \times \Delta^{\text{opp}}$, this implies that the latching maps

$$L(\mathbb{I})_n^j \tilde{Z} \cup_{L(\mathbb{I})_n^j Z} Z_n^j \rightarrow \tilde{Z}_n^j$$

are closed immersions, which allows us inductively to construct a lift

$$\lambda : \tilde{Z}_\bullet \rightarrow X_\bullet$$

of simplicial almost cosimplicial affine schemes.

Now, the obstruction to λ being a cosimplicial morphism is the map

$$\partial_0 \lambda^\sharp - \lambda^\sharp \partial_0 : O(X)_n \rightarrow O(\tilde{Z})_{n-1},$$

whose image is contained in $\ker(O(\tilde{Z}) \rightarrow O(Z)) = \mathcal{I}$. Thus we have a map

$$O(X)_n \rightarrow \mathcal{I}_{n-1};$$

since \mathcal{I} is a square-zero ideal, this map is an $O(S)$ -linear $g^\sharp \partial_0$ -derivation, so corresponds to an $(O(X), g^\sharp \partial_0)$ -linear map

$$\delta : \Omega(X/S)_n \rightarrow \mathcal{I}_{n-1}.$$

Furthermore, $\partial_i \delta = \delta \partial_{i+1}$ for all $i > 0$, so δ descends to a map $N^s \delta$ on the normalised complexes, which will be 0 if and only if $\delta = 0$. Moreover, $\delta \sigma_i = \sigma_{i-1} \delta$ for $i \geq 1$, $\delta \sigma_0 = 0$, and $\delta \partial_0 = 0$.

Thus for $m \in \Omega(X/S)_n$ and $a \in O(X)_n$,

$$\delta(m \nabla a) = \delta \left(\sum \pm \sigma_I m \cdot \sigma_J a \right) = \sum \pm \delta(\sigma_I m) \cdot g^\sharp \partial_0 \sigma_J a,$$

where the sum is over shuffle permutations (I, J) of $[0, n-1]$, \pm is the sign of the permutation, and $\sigma_I = \sigma_{i_r} \dots \sigma_{i_2} \sigma_{i_1}$ for I the sequence $i_1 < i_2 < \dots < i_r$.

Now, $\sigma_I m = 0$ if I contains 0, so the only non-zero terms in the sum have $0 \in J$, in which case $\partial_0 \sigma_{K \cup \{0\}} a = \sigma_{K-1} a$. We also have $\delta(\sigma_I m) = \sigma_{I-1} m$ in this case, so renumbering I and K gives

$$\delta(m \nabla a) = \delta(m) \nabla g^\sharp a;$$

in other words, $N^s \delta$ is an $(N^s O(X), g^\sharp)$ -linear map.

The conditions on ∂_i ensure that $N^s \delta$ is also a chain map, so we have

$$N^s \delta \in Z_{-1} \text{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s g_* \mathcal{I}).$$

However, the choice of λ was not unique. Another choice λ' gives rise to $\beta := \lambda^* - (\lambda')^*$. Similar reasoning shows that this is a g^\sharp -derivation $\beta : \Omega(X/S) \rightarrow g_* \mathcal{I}$, and corresponds to an $N^s O(X)$ -linear map on normalised complexes. The operation δ' is given by $\delta - \partial_0 \beta + \beta \partial_0$, so $N^s \delta' = N^s \delta - [d, N^s \beta]$. Thus the obstruction lies in

$$\text{H}_{-1} \text{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s g_* \mathcal{I}).$$

Finally, $\tilde{Z} \cup_Z \tilde{Z} \cong \mathbf{Spec}(\mathcal{O}_{\tilde{Z}} \oplus \mathcal{I} \epsilon)$, for $\epsilon^2 = 0$, so the choice of \tilde{g} gives

$$\underline{\text{Hom}}_{Z|_{\text{scAff}S}}(\tilde{Z}, X) \cong \underline{\text{Hom}}_{\tilde{Z}|_{\text{scAff}S}}(\tilde{Z} \cup_Z \tilde{Z}, X) \cong \underline{\text{Hom}}_{\text{cdgMod}(X)}(N^s \Omega(X/S), N^s g_* \mathcal{I}),$$

and Corollary 7.13 completes the proof. \square

Lemma 8.4. *If $f : X \rightarrow S$ is a derived Artin m -hypergroupoid, and $M \in \text{cdg}_+\text{Mod}(X)$, then*

$$\mathbf{H}_i\text{HOM}_{\text{cdgMod}(X)}(\mathbb{L}^{X/S}, M) \cong \mathbf{H}_i\text{HOM}_{\text{cdgMod}(X)}(\Omega(X/S), M)$$

for $i \leq 0$.

Proof. As in Lemma 7.8, the map $\mathbb{L}^{X/S} \rightarrow N^s\Omega(X/S)$ is surjective, and the kernel is the total complex of

$$0 \rightarrow N^s\Omega(X^{\Delta^1}/X^{\Lambda_0^1} \times_{S^{\Lambda_0^1}} S^{\Delta^1}) \rightarrow \dots \rightarrow N^s\Omega(X^{\Delta^i}/X^{\Lambda_0^i} \times_{S^{\Lambda_0^i}} S^{\Delta^i}) \rightarrow \dots$$

Since $X^{\Delta^i} \rightarrow X^{\Lambda_0^i} \times_{S^{\Lambda_0^i}} S^{\Delta^i}$ is a trivial relative derived Artin m -hypergroupoid, its simplicial matching maps are smooth. By [TV2] Definition 1.2.7.1, this means that the cotangent complexes associated to the matching maps are projective, so the chain complex $N^s\Omega(X/S) \otimes (\Delta^i/\Lambda_0^i)$ is a projective object of $\text{cdg}_+\text{Mod}(X)$ (but not of $\text{cdgMod}(X)$). Hence

$$\mathbf{H}_j\text{HOM}_{\text{cdgMod}(X)}(N^s\Omega(X/S) \otimes (\Delta^i/\Lambda_0^i), M) = 0$$

for all $i > 0, j < 0$ and $M \in \text{cdg}_+\text{Mod}(X)$.

If we set $\Lambda_0^0 := \emptyset$, then we have a spectral sequence

$$\mathbf{H}_j\text{HOM}_{\text{cdgMod}(X)}(N^s\Omega(X/S) \otimes (\Delta^i/\Lambda_0^i), M) \implies \mathbf{H}_{j+i}\text{HOM}_{\text{cdgMod}(X)}(\mathbb{L}^{X/S}, M).$$

When $M \in \text{cdg}_+\text{Mod}(X)$, this combines with the calculation above to show that

$$\mathbf{H}_j\text{HOM}_{\text{cdgMod}(X)}(\mathbb{L}^{X/S}, M) \cong \mathbf{H}_j\text{HOM}_{\text{cdgMod}(X)}(N^s\Omega(X/S), M)$$

for $j < 0$, as required. \square

Theorem 8.5. *Take a diagram $\mathfrak{Z} \xrightarrow{g} \mathfrak{X} \xrightarrow{f} \mathfrak{S}$ of derived m -geometric Artin stacks. If $\mathfrak{Z} \hookrightarrow \tilde{\mathfrak{Z}}$ is a closed immersion over \mathfrak{S} defined by a square-zero quasi-coherent complex \mathcal{I} , then the obstruction to extending g to a morphism $\tilde{g} : \tilde{\mathfrak{Z}} \rightarrow \mathfrak{X}$ over \mathfrak{S} lies in*

$$\text{Ext}_{\mathcal{O}_{\tilde{\mathfrak{Z}}}}^1(\mathbf{L}g^*\mathbb{L}^{\mathfrak{X}/\mathfrak{S}}, \mathcal{I}).$$

If the obstruction is zero, then the Hom-space of possible extensions is given by

$$\pi_i\text{Hom}_{\mathbb{Z}\text{-scaff}S}(\tilde{Z}, X) \simeq \text{Ext}_{\mathcal{O}_{\tilde{\mathfrak{Z}}}}^{-i}(\mathbf{L}g^*\mathbb{L}^{\mathfrak{X}/\mathfrak{S}}, \mathcal{I}).$$

Proof. Apply Theorem 4.7 to obtain a derived Artin m -hypergroupoid S with $S^\sharp \simeq \mathfrak{S}$, and relative derived Artin m -hypergroupoids $Z \rightarrow X \rightarrow S$ with $X^\sharp \simeq \mathfrak{X}$ and $Z^\sharp \simeq \mathfrak{Z}$. It will follow from the proof of Theorem 8.8 that square-zero deformations of Z correspond to square-zero deformations of \mathfrak{Z} , so there exists an extension $Z \rightarrow \tilde{Z}$ of simplicial cosimplicial affine schemes with $\tilde{Z}^\sharp \simeq \tilde{\mathfrak{Z}}$. Thus the extension is defined by the square-zero ideal $a^*\mathcal{I}$, for $a : \tilde{Z} \rightarrow \tilde{\mathfrak{Z}}$.

We are now in the scenario of Proposition 8.3, and sheafification defines a functor from lifts $\tilde{Z} \rightarrow X$ to lifts $\tilde{\mathfrak{Z}} \rightarrow \mathfrak{X}$. Applying Lemma 8.4, we see that these lifts are governed by

$$H_*\text{HOM}_{\text{cdgMod}(Z)}(g^*\mathbb{L}^{X/S}, \mathcal{I}) \cong H_*\text{HOM}_{\text{cdgMod}(Z)}(g^*\mathbb{L}^{X/S}, \mathcal{I}) \cong \text{Ext}_{\mathcal{O}_{\tilde{\mathfrak{Z}}}}^{-*}(\mathbf{L}g^*\mathbb{L}^{\mathfrak{X}/\mathfrak{S}}, \mathcal{I}).$$

Since this final description is independent of the choice $Z \rightarrow X \rightarrow S$ of resolutions, we deduce that another choice $Z' \rightarrow X' \rightarrow S'$ of resolutions would give an equivalent space of lifts. Explicitly, we could take a third sequence $Z'' \rightarrow X'' \rightarrow S''$, with trivial relative derived Artin hypergroupoids $\pi_Z : Z'' \rightarrow Z$, $Z'' \rightarrow Z'$ and similarly for X'', S'' , compatible with the other morphisms. Since $\pi_X^*\mathbb{L}^{X/S} \simeq \mathbb{L}^{X''/S''}$, we get $\pi_Z^*g^*\mathbb{L}^{X/S} \simeq (g'')^*\mathbb{L}^{X''/S''}$, so pulling back along $X'' \rightarrow X$ gives an equivalence of lifting spaces, and similarly for $X'' \rightarrow X'$. \square

8.2. Deformations of derived stacks.

Proposition 8.6. *Take a Reedy fibration $f : X \rightarrow S$ of simplicial cosimplicial affine schemes. If $S \hookrightarrow \tilde{S}$ is a levelwise closed immersion defined by a square-zero ideal \mathcal{I} , then the obstruction to lifting f to a levelwise flat morphism $\tilde{f} : \tilde{X} \rightarrow \tilde{S}$, with $\tilde{X} \times_{\tilde{S}} S = X$, lies in*

$$\mathbb{H}_{-2}\mathrm{HOM}_{\mathrm{cdgMod}(X)}(N^s\Omega^{X/S}, f^*N^s\mathcal{I}).$$

If the obstruction is zero, then the isomorphism class of liftings is (non-canonically) isomorphic to

$$\mathbb{H}_{-1}\mathrm{HOM}_{\mathrm{cdgMod}(X)}(N^s\Omega^{X/S}, f^*N^s\mathcal{I}).$$

Proof. We adapt the proof of Proposition 8.3.

Letting \mathbb{I} be the Reedy category $\Delta_* \times \Delta^{\mathrm{opp}}$, we have seen that the matching maps

$$X_n^j \rightarrow M(\mathbb{I})_n^j X \times_{M(\mathbb{I})_n^j S} S_n^j$$

lift closed immersions (so are pro-smooth), and that the latching maps

$$L(\mathbb{I})_n^j X \rightarrow X_n^j$$

are closed immersions. Using the fact that deformations of pro-smooth morphisms of affine schemes are unobstructed and unique, we may inductively construct a lift

$$\tilde{f} : \tilde{X}_\bullet \rightarrow \tilde{S}_\bullet$$

in the category of simplicial almost cosimplicial affine schemes, with \tilde{f} \mathbb{I} -Reedy fibrant, and note that this lift is unique up to isomorphism. It therefore remains only to understand how the operation ∂^0 on X can deform.

Using the formal smoothness of \tilde{f} , we may lift $\partial_X^0 : X \rightarrow \mathrm{Dec}_{\mathrm{opp}}^+ X$ to an almost cosimplicial map

$$\lambda : \tilde{X} \rightarrow \mathrm{Dec}_{\mathrm{opp}}^+ \tilde{X},$$

satisfying $\tilde{f}\lambda = \partial^0 \tilde{f}$ and $\sigma^0 \lambda = \mathrm{id}$. This will make \tilde{X} into a cosimplicial scheme if and only if $\lambda^2 = \partial^1 \lambda$, since all the other conditions are automatic.

Thus the obstruction to $(\tilde{X}_\bullet, \lambda)$ being a cosimplicial scheme is the map

$$(\lambda^\#)^2 - \lambda^\# \partial_1 : O(\tilde{X})_n \rightarrow O(\tilde{X})_{n-2},$$

whose image is contained in $\ker(O(\tilde{X}) \rightarrow O(X)) = f^*\mathcal{I}$. Since \mathcal{I} is a square-zero ideal, we know that $f^*\mathcal{I}$ must also map to 0. Thus we have a map

$$O(X)_n \rightarrow (f^*\mathcal{I})_{n-2}$$

This map is an $O(S)$ -linear $(\partial_0)^2$ -derivation, so corresponds to an $(O(X), (\partial_0)^2)$ -linear map

$$\delta : \Omega(X/S)_n \rightarrow (f^*\mathcal{I})_{n-2}.$$

Moreover, $\partial_i \delta = \delta \partial_{i+2}$ for all $i > 0$, so δ descends to a map $N^s \delta$ on the normalised complexes, which will be 0 if and only if $\delta = 0$. Moreover, $\delta \sigma_i = \sigma_{i-2} \delta$ for $i \geq 2$, and 0 for $i = 0, 1$. A calculation similar to that in Proposition 8.3 shows that $N^s \delta$ is an $N^s O(X)$ -linear map.

We need to show that $\delta(\lambda^\# - \partial_1 + \partial_2) = \lambda^\# \delta$ to ensure that $N^s \delta$ is a chain map.

$$\delta(\lambda^\# - \partial_1 + \partial_2) = [(\lambda^\#)^3 - (\lambda^\#)^2 \partial_1 + \lambda^\# \partial_2] - [\lambda^\# \partial_1 \lambda^\#] = (\lambda^\#)^3 - (\lambda^\#)^2 \partial_1 = \lambda^\# \delta,$$

as required, so $N^s \delta \in \mathbb{Z}_{-2}\mathrm{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s f^* \mathcal{I})$.

However, the choice of λ was not unique. Another choice λ' gives rise to $\beta := \lambda^* - (\lambda')^*$. Similar reasoning shows that this is a ∂^0 -derivation $\beta : \Omega(X/S) \rightarrow f^*\mathcal{I}$, and corresponds to an $N^s O(X)$ -linear map on normalised complexes. The operation δ' is given by $\delta - \partial_0 \beta + \beta(\partial_1 - \partial_0)$, so $N^s \delta' = \delta - [d, N^s \beta]$. Thus the obstruction lies in

$$\mathbb{H}_{-2}\mathrm{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s (f^* \mathcal{I})).$$

To understand the isomorphism class of the deformations, fix a deformation $(\tilde{X}_\bullet^*, \lambda)$. Another choice of lift is given by $(\lambda')^* = \lambda^* - \beta$, and for this to be unobstructed, we need $[d, N^s\beta] = 0$, so $N^s\beta \in Z_{-1}\mathrm{HOM}_{N^s\mathcal{O}(X)}(N^s\Omega(X/S), N^s f^* \mathcal{I})$. Now, an isomorphism of \tilde{X}_\bullet^* is equivalent to a map $\alpha \in \mathrm{HOM}_{N^s\mathcal{O}(X)}(N^s\Omega(X/S), N^s f^* \mathcal{I})^0$, and transformation by α sends $N^s\beta$ to $N^s\beta + [d, \alpha]$. Thus the isomorphism class is

$$\mathrm{H}_{-1}\mathrm{HOM}_{N^s\mathcal{O}(X)}(N^s\Omega(X/S), N^s f^* \mathcal{I}).$$

□

Lemma 8.7. *If the morphism f in Proposition 8.6 is a relative derived Artin m -hypergroupoid, the simplicial automorphism group $\underline{\mathrm{Aut}}_{\tilde{\mathfrak{S}}}(\tilde{X})_X$ of a deformation \tilde{X} is given by*

$$N^s \underline{\mathrm{Aut}}_{\tilde{\mathfrak{S}}}(\tilde{X})_X \simeq \tau_{\geq 0} \mathrm{HOM}_{\mathrm{cdgMod}(X)}(\mathbb{L}^{X/S}, f^* N^s \mathcal{I}).$$

Proof. This follows immediately from Corollary 7.14. □

Theorem 8.8. *Take a morphism $f : \mathfrak{X} \rightarrow \mathfrak{S}$ of derived m -geometric Artin stacks. If $\mathfrak{S} \hookrightarrow \tilde{\mathfrak{S}}$ is a 0-representable closed immersion defined by a square-zero quasi-coherent complex \mathcal{I} , then the obstruction to lifting f to a morphism $\tilde{f} : \tilde{\mathfrak{X}} \rightarrow \tilde{\mathfrak{S}}$ with $\tilde{\mathfrak{X}} \times_{\tilde{\mathfrak{S}}}^h \mathfrak{S} \simeq \mathfrak{X}$ lies in*

$$\mathrm{Ext}_{\mathcal{O}_{\tilde{\mathfrak{X}}}}^2(\mathbb{L}^{\tilde{\mathfrak{X}}/\tilde{\mathfrak{S}}}, f^* \mathcal{I}).$$

If this obstruction is 0, then the equivalence class of deformations is

$$\mathrm{Ext}_{\mathcal{O}_{\tilde{\mathfrak{X}}}}^1(\mathbb{L}^{\tilde{\mathfrak{X}}/\tilde{\mathfrak{S}}}, f^* \mathcal{I}).$$

and the simplicial automorphism group $\underline{\mathrm{Aut}}_{\tilde{\mathfrak{S}}}(\tilde{X})_{\tilde{\mathfrak{X}}}$ of any deformation $\tilde{\mathfrak{X}}$ is given by

$$\pi_i \underline{\mathrm{Aut}}_{\tilde{\mathfrak{S}}}(\tilde{\mathfrak{X}})_{\tilde{\mathfrak{X}}} \cong \mathrm{Ext}_{\mathcal{O}_{\tilde{\mathfrak{X}}}}^{-i}(\mathbb{L}^{\tilde{\mathfrak{X}}/\tilde{\mathfrak{S}}}, f^* \mathcal{I}).$$

Proof. Apply Theorem 4.7 to obtain a derived Artin m -hypergroupoid \tilde{S} with $\tilde{S}^\sharp \simeq \tilde{\mathfrak{S}}$, and pull back \mathcal{I} to give a levelwise closed immersion $S \hookrightarrow \tilde{S}$. Similarly, there exists a relative derived Artin m -hypergroupoid $X \rightarrow S$, with $X^\sharp \simeq \mathfrak{X}$. Proposition 8.6, Lemma 8.4 and Lemma 8.7 now imply that deformations of X are governed by

$$\mathrm{Ext}_{\mathcal{O}_X}^*(\mathbb{L}^{X/S}, f^* \mathcal{I}),$$

and sheafification defines a functor from the ∞ -groupoid of deformations of X to the ∞ -groupoid of deformations of \mathfrak{X} , so we just need to show that this is an equivalence.

Given a deformation $\tilde{\mathfrak{X}}$ of \mathfrak{X} , there exists a relative derived Artin m -hypergroupoid $\tilde{X}' \rightarrow \tilde{S}$, with $(\tilde{X}')^\sharp \simeq \tilde{\mathfrak{X}}$. Setting $X' := \tilde{X}' \times_{\tilde{S}} S$ gives $(X')^\sharp \simeq \mathfrak{X}$. By Theorem 4.10, there exists a derived Artin m -hypergroupoid X'' equipped with morphisms $X'' \rightarrow X'$, $X'' \rightarrow X$ which are trivial relative derived Artin m -hypergroupoids.

It therefore suffices to show that the simplicial groupoid Γ of deformations of a trivial relative derived Artin m -hypergroupoid $T \rightarrow Y$ is contractible. By Corollary 7.12, $\mathbb{L}^{T/Y} \simeq 0$, so Proposition 8.6, Lemma 8.4 and Lemma 8.7 imply that Γ has one equivalence class of objects, with contractible automorphism group. □

Remark 8.9. Theorem 8.8 has an alternative proof, as a Corollary of Theorem 8.5, which we now sketch.

Given a quasi-coherent complex \mathcal{F} on \mathfrak{S} , let $\mathfrak{S} \oplus \mathcal{F} := \mathbf{Spec}(\mathcal{O}_{\mathfrak{S}} \oplus (N^s)^{-1} \mathcal{F} \epsilon)$, with $\epsilon^2 = 0$. Then there exists a morphism $t : \mathfrak{S} \oplus \mathcal{I}[-1] \rightarrow \mathfrak{S}$ such that $\tilde{\mathfrak{S}}$ is the homotopy cofibre product $\tilde{\mathfrak{S}} := \mathfrak{S} \cup_{0, \mathfrak{S} \oplus \mathcal{I}[-1], t}^h \mathfrak{S}$. The category of deformations $\tilde{\mathfrak{X}}$ over $\tilde{\mathfrak{S}}$ is therefore equivalent to the simplicial category of equivalences $\alpha : t^* \mathfrak{X} \rightarrow 0^* \mathfrak{X}$ over $\mathfrak{S} \oplus \mathcal{I}[-1]$, fixing the fibre \mathfrak{X} over \mathfrak{S} . This is the same as the simplicial category of morphisms $t^* \mathfrak{X} \rightarrow \mathfrak{X}$ over \mathfrak{S} , fixing \mathfrak{X} . Since $\mathfrak{X} \rightarrow t^* \mathfrak{X}$ is a square-zero extension, Theorem 8.5 describes these morphisms in terms of the cotangent complex.

Remark 8.10. If \mathfrak{X} and \mathfrak{S} are (non-derived) geometric Artin stacks, then flatness of $\tilde{\mathfrak{X}} \rightarrow \tilde{\mathfrak{S}}$ is sufficient to ensure that $\tilde{\mathfrak{X}} \times_{\tilde{\mathfrak{S}}}^h \mathfrak{S} \simeq \tilde{\mathfrak{X}} \times_{\mathfrak{S}} \mathfrak{S}$. In this case, Theorems 8.5 and 8.8 have alternative proofs, by considering the almost simplicial schemes underlying the associated simplicial schemes. It is then necessary to study deformations of X_0 and of ∂_0^X , rather than of ∂_X^0 . This is the approach taken in [Pri1] §5.

Corollary 8.11. *Given a derived m -geometric Artin stack \mathfrak{X} , and an Artin m -hypergroupoid Z with $Z^\sharp \simeq \pi^0 \mathfrak{X}$, there exists a derived Artin m -hypergroupoid X with $X^\sharp \simeq \mathfrak{X}$ and $\pi^0 X = Z$.*

Proof. It suffices to find a homotopy derived Artin hypergroupoid X with these properties, since we may then take a Reedy fibrant approximation. Writing \mathfrak{X} as the colimit $\pi^0 \mathfrak{X} \rightarrow \pi^{\leq 1} \mathfrak{X} \rightarrow \pi^{\leq 2} \mathfrak{X} \rightarrow \dots$ of homotopy square-zero extensions as in Remark 6.25, we will inductively construct $a : X(i) \rightarrow \mathfrak{X}$ with $X(i)^\sharp \simeq \mathfrak{X}(i) := \pi^{\leq i} \mathfrak{X}$ and $\pi^0 X(i) = Z$.

Assume that we have constructed $X(i-1)$. We will now construct the schemes $X(i)_n$ inductively, so assume that we have done this compatibly for all $j < n$, giving latching and matching objects $L_n X(i), M_n X(i)$. We seek a diagram

$$L_n X(i) \xrightarrow{\alpha_i} X(i)_n \xrightarrow{\beta_i} M_n X(i) \times_{\mathfrak{X}(i) \partial \Delta^n}^h \mathfrak{X}(i)$$

lifting

$$L_n X(i-1) \xrightarrow{\alpha_{i-1}} X(i-1)_n \xrightarrow{\beta_{i-1}} M_n X(i-1) \times_{\mathfrak{X}(i-1) \partial \Delta^n}^h \mathfrak{X}(i-1).$$

Since the embedding $\mathfrak{X}(i-1) \rightarrow \mathfrak{X}(i)$ is defined by the square-zero sheaf $\pi_i \mathcal{O}_{\mathfrak{X}}[-i]$, deformations of β_{i-1} governed by the complex

$$\begin{aligned} C_\bullet &:= \mathrm{HOM}_{\mathrm{dgMod}(X(i-1)_n)}(\mathbb{L}^{X(i-1)_n/M_n X(i-1) \times_{\mathfrak{X}(i-1) \partial \Delta^n}^h \mathfrak{X}(i-1)}, a_n^* \pi_i \mathcal{O}_{\mathfrak{X}}[-i]) \\ &\cong \mathrm{HOM}_{\mathrm{dgMod}(Z_n)}(\mathbb{L}^{Z_n/M_n Z \times_{\pi^0 \mathfrak{X} \partial \Delta^n}^h \pi^0 \mathfrak{X}}, a_n^* \pi_i \mathcal{O}_{\mathfrak{X}}[-i]). \end{aligned}$$

Since $a : Z \rightarrow \pi^0 \mathfrak{X}$ is a resolution, the map $Z_n \rightarrow M_n Z \times_{\pi^0 \mathfrak{X} \partial \Delta^n}^h \pi^0 \mathfrak{X}$ is a smooth m -representable morphism, so Lemma 7.8 implies that the cotangent complex $\mathbb{L}_{\bullet}^{Z_n/M_n Z \times_{\pi^0 \mathfrak{X} \partial \Delta^n}^h \pi^0 \mathfrak{X}}$ is equivalent to a complex of locally projective modules concentrated in degrees $[-m, 0]$. Since Z_n is affine, locally projective modules on Z_n are projective, so $H_j(C_\bullet) = 0$ for all $j < i$, and in particular for $j = 0, -1, -2$.

Since $i > 0$, Theorem 8.8 then implies that $\beta_i : X(i)_n \rightarrow M_n X(i) \times_{\mathfrak{X}(i) \partial \Delta^n}^h \mathfrak{X}(i)$ exists, and is unique up to unique isomorphism in the homotopy category, so we may choose a Reedy fibrant representative. In particular, β_i is smooth, and since $L_n X(i-1) \rightarrow L_n X(i)$ is a homotopy square-zero extension, Theorem 8.5 implies that deformations of α_{i-1} are governed by the complex

$$\begin{aligned} D_\bullet &:= \mathrm{HOM}_{\mathrm{dgMod}(L_n X(i-1))}(\mathbf{L}\alpha^* \mathbb{L}^{X(i-1)_n/M_n X(i-1) \times_{\mathfrak{X}(i-1) \partial \Delta^n}^h \mathfrak{X}(i-1)}, \pi_i \mathcal{O}_{L_n X}[-i]) \\ &\cong \mathrm{HOM}_{\mathrm{dgMod}(L_n Z)}(\mathbf{L}\alpha^* \mathbb{L}^{Z_n/M_n Z \times_{\pi^0 \mathfrak{X} \partial \Delta^n}^h \pi^0 \mathfrak{X}}, \pi_i \mathcal{O}_{L_n X}[-i]). \end{aligned}$$

As for C_\bullet , we deduce that $H_j(D_\bullet) = 0$ for all $j < i$. In particular, this holds for $j = 0, -1$, so α_i exists, and is unique up to weak equivalence.

This completes the inductive step for constructing $X(i)$, and we set $X = \varinjlim X(i)$. \square

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