A NOTE ON ÉTALE ATLASES FOR ARTIN STACKS, POISSON STRUCTURES AND QUANTISATION

J.P.PRIDHAM

Abstract. We explain how any Artin stack $\mathfrak{X}$ over $\mathbb{Q}$ extends to a functor on non-negatively graded commutative cochain algebras, which we think of as functions on Lie algebroids or stacky affine schemes. There is a notion of étale morphisms for these CDGAs, and Artin stacks admit étale atlases by stacky affines, giving rise to a small étale site of stacky affines over $\mathfrak{X}$. This site has the same quasi-coherent sheaves as $\mathfrak{X}$ and leads to efficient formulations of shifted Poisson structures, differential operators and quantisations for Artin stacks. There are generalisations to higher and derived stacks, and analogues for differentiable and analytic stacks. This note is just a slight elaboration of constructions scattered across several of the author's papers.

Introduction

When attempting to formulate Poisson structures or quantisations for Artin stacks or derived Artin stacks, one immediately encounters the difficulty that such structures on affine schemes are only functorial with respect to étale morphisms. The solution developed in [Pri6, §3.1], globalising [Pri1, Theorem 4.26], is to work with commutative differential graded algebras (CDGAs) of the form

$$B^0 \xrightarrow{\partial} B^1 \xrightarrow{\partial} B^2 \xrightarrow{\partial} \ldots,$$

which we think of as functions on stacky affine schemes. These should not be confused with the CDGAs arising as functions on derived affine schemes, in which the objects are chain algebras, with differentials mapping the other way.

There is a cosimplicial denormalisation functor D from such CDGAs to cosimplicial commutative rings, which can be used to extend any sheaf, stack or simplicial hypersheaf $\mathfrak{X}$ over $\mathbb{Q}$ naturally to a functor $D_*\mathfrak{X}$ on these stacky affine schemes. The power of this construction lies in the fact that when $\mathfrak{X}$ is an Artin n-stack, $D_*\mathfrak{X}$ admits an atlas by stacky affines (Theorem 2.1) which is some sense étale. For example, an étale atlas for $D_*(BG)$ is given by the Chevalley–Eilenberg CDGA of the Lie algebra $\mathfrak{g}$.

Since D_* has good descent properties (Corollary 2.3) for Artin n-stacks, structures such as quasi-coherent sheaves on $\mathfrak{X}$ can be formulated in terms of structures on $D_*\mathfrak{X}$ (§2.2). A further consequence of Theorem 2.1 is that in order to define a new structure on an Artin n-stack $\mathfrak{X}$, it suffices to define it on the small site of stacky affines which are étale over $D_*\mathfrak{X}$. Since tangent modules are functorial with respect to étale morphisms of stacky affines, this allows us efficiently to formulate shifted Poisson structures, differential operators and quantisations for Artin stacks in terms of the small étale site of $D_*\mathfrak{X}$ (§3).

There are generalisations to derived Artin stacks using stacky derived affines, whose rings of functions are the stacky CDGAs of [Pri6, §3.1], given by introducing a second grading and a chain algebra structure. In particular, for a derived Artin n-stack, the associated functor $D_*\mathfrak{X}$ on stacky derived affines admits an étale atlas (Theorem

4.10). There are also analogues for (derived) differentiable stacks, in which setting stacky affines correspond to NQ-manifolds, and for (derived) analytic stacks in both Archimedean and non-Archimedean contexts (§4.2).

Contents

1. Stacky affines

1.1. **Basic definitions.** Fix a commutative $\mathbb{Q}$-algebra R.

Definition 1.1. We write $DG^+\mathrm{CAlg}(R)$ for the category of CDGAs over R concentrated in non-negative cochain degrees. Explicitly, an object A of $DG^+\mathrm{CAlg}(R)$ is a cochain complex

$$A^0 \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^2 \xrightarrow{\partial} \ldots$$

of R-modules equipped with associative graded-commutative multiplications $A^m \otimes_R A^n \to A^{m+n}$, and a unit $1 \in A^0$, such that ∂ is an R-linear derivation in the graded sense that $\partial(ab) = (\partial a)b + (-1)^{\deg a} a(\partial b)$.

For $A \in DG^+\mathrm{CAlg}(R)$, we denote by $\mathrm{Spec}\, A$ the corresponding object of the opposite category $DG^+\mathrm{CAlg}(R)^{\mathrm{opp}}$. We also write $A = O(\mathrm{Spec}\, A)$, and freely make use of the Yoneda embedding to regard $\mathrm{Spec}\, A$ as a set-valued functor on $DG^+\mathrm{CAlg}(R)^{\mathrm{opp}}$.

Beware that in contrast to most instances where cochain complexes crop up, we are primarily interested in these CDGAs up to isomorphism, not up to quasi-isomorphism.

Example 1.2. The key motivating example is given by a finite projective Lie R-algebra $\mathfrak{g}$ acting on a commutative R-algebra B via a Lie R-algebra morphism $\alpha\colon \mathfrak{g} \to \mathrm{Der}_R(A)$. For $Y = \mathrm{Spec}\, B$, this leads to the CDGA $O([Y/\mathfrak{g}])$ of [Pri6, Example 3.6] given by the Chevalley–Eilenberg complex

$$O([Y/\mathfrak{g}]) := (O(Y) \xrightarrow{\partial} O(Y) \otimes \mathfrak{g}^\vee \xrightarrow{\partial} O(Y) \otimes \Lambda^2\mathfrak{g}^\vee \xrightarrow{\partial} \ldots)$$

of $\mathfrak{g}$ with coefficients in the chain $\mathfrak{g}$-module $O(Y)$.

The functor on $DG^+\mathrm{CAlg}(R)$ associated to $[Y/\mathfrak{g}] := \mathrm{Spec}\, O([Y/\mathfrak{g}])$ is then given by the subsets $[Y/\mathfrak{g}](B) \subset Y(B^0) \times (\mathfrak{g} \otimes_R B^1)$ of pairs (y, γ) satisfying the Maurer–Cartan and compatibility conditions

$$\partial_B \gamma + \frac{1}{2}[\gamma, \gamma] = 0 \in \mathfrak{g} \otimes_R B^2$$
$$y \circ \alpha(\gamma) + \partial_B y = 0 \in \mathrm{Der}(O(Y), B^1).$$

Example 1.3. The construction of Example 1.2 naturally generalises to Lie algebroids given by Lie–Rinehart algebras. In particular, for a smooth affine scheme Y, we can consider the CDGA $\Omega^\bullet_{Y/R}$ given by the de Rham algebra of Y over R. As a functor on $DG^+\mathrm{CAlg}(R)$, the associated object $\mathrm{Spec}\, \Omega^\bullet_{Y/R}$ is given by $(\mathrm{Spec}\, \Omega^\bullet_{Y/R})(B) \cong Y(B^0)$.

1.2. Étale morphisms.

Definition 1.4. We say that a morphism $f\colon C' \to C$ in $DG^+\mathrm{CAlg}(R)$ is a square-zero extension if f is surjective in every level and the kernel I is a square-zero ideal. We say that f is a contractible square-zero extension if in addition the cochain complex I admits a C'-linear (or, equivalently, C-linear) contracting homotopy.

Definition 1.5. Given functors F, G from $DG^+\mathrm{CAlg}(R)$ to sets, groupoids or simplicial sets, and a natural transformation $\eta\colon F \to G$, we say that η is formally étale (resp. formally geometric) if the maps

$$F(C') \to F(C) \times^h_{G(C)} G(C')$$

are surjective for all square-zero extensions (resp. all contractible square-zero extensions) $C' \to C$. Here, the symbol $\times^h$ is a homotopy fibre product, which for set-valued functor is is just a plain fibre product, and for groupoid-valued functors is a 2-fibre product. For groupoids, surjectivity here means essential surjectivity, while for simplicial sets it means π_0-surjectivity.

We say that a functor F on $DG^+\mathrm{CAlg}(R)$ is formally étale (resp. formally geometric) if the transformation $F \to *$ to the constant functor is so.

Definition 1.6. We say that a natural transformation $\eta\colon F \to G$ of functors F, G on $DG^+\mathrm{CAlg}(R)$ is l.f.p. if for any filtered colimit $C = \varinjlim_{i \in I} C_i$ in $DG^+\mathrm{CAlg}(R)$, the natural map

$$\varinjlim_{i \in I} F(C_i) \to \varinjlim_{i \in I} G(C_i) \times_{G(C)} F(C)$$

is an equivalence.

We then say that η is étale (resp. geometric) if it is formally étale (resp. formally geometric) and l.f.p.

For $A \in DG^+\mathrm{CAlg}(R)$, we refer to $\mathrm{Spec}\, A$ as a stacky affine if it is geometric.

Since any CDGA $A \in DG^+\mathrm{CAlg}(R)$ gives rise to a set-valued functor $\mathrm{Spec}\, A$, the definitions above give rise to notions of (formally) étale maps between objects of $DG^+\mathrm{CAlg}(R)$. The following lemma is a consequence of standard obstruction arguments. Here, the module $\Omega^1_{B/A}$ of Kähler differentials is a B-module in cochain complexes, spanned by elements db for $b \in B$ subject to the conditions $d(bc) = (-1)^{(\deg b+1)\deg c} c(db) + (-1)^{\deg b} b(dc)$ and $df(A) = 0$.

Lemma 1.7. *A morphism $f\colon A \to B$ is geometric if and only if*

- *the graded algebra $B^{\#}$ underlying B (given by ignoring the differential ∂) is freely generated as a graded-commutative algebra over $A^{\#} \otimes_{A^0} B^0$ by a finite graded projective $(A^{\#} \otimes_{A^0} B^0)$-module.*

The morphism f is étale if and only if it is geometric and

- *$f^0 \colon A^0 \to B^0$ is smooth, and*
- *the cochain complex*
$$\Omega^1_{B/A} \otimes_B B^0$$
of projective B^0-modules is acyclic.

Remark 1.8. The final condition in Lemma 1.7 is the reason for the terminology "étale", and will ensure functoriality of constructions such as tangent modules and differential operators with respect to these morphisms. However, beware that the categorical properties of our étale morphisms are more like those for smooth morphisms of affine schemes. In particular, a section of an étale map will not tend to be smooth, and if we restrict our functors to the subcategory of R-algebras $\mathrm{CAlg}(R) \subset DG^+\mathrm{CAlg}(R)$, then for an étale map $\eta \colon F \to G$, we will only be able to say that $\eta|_{\mathrm{CAlg}(R)} \colon F|_{\mathrm{CAlg}(R)} \to G|_{\mathrm{CAlg}(R)}$ is smooth. This is an essential feature, since it allows smooth maps of affine schemes to have enhancements in stacky affines which are étale in the sense that they preserve cotangent complexes.

Examples 1.9. Given a linear algebraic group G over R with associated Lie algebra $\mathfrak{g}$, it follows from the definitions and Example 1.2 that $[G/\mathfrak{g}]$ is étale over $\mathrm{Spec}\, R$.

Similarly, for a smooth affine scheme Y over R, the stacky affine $\mathrm{Spec}\, \Omega^\bullet_{Y/R}$ of Example 1.3 is étale over $\mathrm{Spec}\, R$.

1.3. **Denormalisation.** To any cosimplicial abelian group V, there is an associated normalised cochain complex NV given by
$$N(V)^n := \bigcap_{i \geq 0} \ker(\sigma^i : V^n \to V^{n-1})$$
with differential $\sum_i (-1)^i \partial^i$. By the Dold–Kan correspondence ([Wei] Theorem 8.4.1, passing to opposite categories and using [Wei] Lemma 8.3.7), this functor gives an equivalence of categories between cosimplicial abelian group and cochain complexes concentrated in non-negative degrees. We will be more concerned with the other half of the equivalence, the denormalisation functor D given in level n by the formal sum
$$D^n A := \bigoplus_{\substack{m+s=n \\ 1 \leq j_1 < \ldots < j_s \leq n}} \partial^{j_s} \ldots \partial^{j_1} A^m.$$
We then define the operations ∂^j and σ^i using the cosimplicial identities

(1) $\partial^j \partial^i = \partial^i \partial^{j-1} \quad i < j$,
(2) $\sigma^j \sigma^i = \sigma^i \sigma^{j+1} \quad i \leq j$,
(3) $\sigma^j \partial^i = \begin{cases} \partial^i \sigma^{j-1} & i < j \\ \mathrm{id} & i = j,\, i = j+1 \\ \partial^{i-1} \sigma^j & i > j+1 \end{cases}$,

subject to the conditions that $\sigma^i v = 0$ and $\partial^0 v = \partial v - \sum_{i=1}^{n+1} (-1)^i \partial^i a$ for all $v \in V^n$.

Definition 1.10. For any $A \in DG^+\mathrm{CAlg}(R)$, the Dold–Kan denormalisation DA is naturally a cosimplicial commutative R-algebra via the Eilenberg–Zilber shuffle product, which reduces to the following description (see for instance [Pri1, Definition 4.20]).

Given a finite set I of strictly positive integers, write $\partial^I = \partial^{i_s} \ldots \partial^{i_1}$, for $I = \{i_1, \ldots i_s\}$, with $1 \le i_1 < \ldots < i_s$. The shuffle product ∇ is then given on the basis by

$$(\partial^I a)\nabla(\partial^J b) := \begin{cases} \partial^{I\cap J}(-1)^{(J\backslash I, I\backslash J)}(a\cdot b) & |a| = |J\backslash I|, |b| = |I\backslash J|, \\ 0 & \text{otherwise,} \end{cases}$$

where for disjoint sets S, T of integers, $(-1)^{(S,T)}$ is the sign of the shuffle permutation of $S \sqcup T$ which sends the first $|S|$ elements to S (in order), and the remaining $|T|$ elements to T (in order).

Beware that this description cannot be used to calculate $(\partial^I a)\nabla\partial^J w$ when $0 \in I \cup J$ (for the obvious generalisation of ∂^I to finite sets I of distinct non-negative integers).

Note that the map $\overbrace{\sigma^0 \circ \sigma^0 \circ \ldots \circ \sigma^0}^{n}\colon D^n A \to A^0$ is surjective, with n-nilpotent kernel. Combining these for all n gives a morphism $DA \to A^0$ of cosimplicial algebras (constant cosimplicial structure on the right) which is a nilpotent extension in every level.

1.4. **Stacks give functors on CDGAs.**

Definition 1.11. Given simplicial sets K, X, we denote by X^K the simplicial set given in level i by

$$(X^k)_i := \mathrm{Hom}_{s\mathrm{Set}}(K \times \Delta^i, X).$$

Definition 1.12. Given a functor F from R-algebras to sets, groupoids or simplicial sets, we define a functor D_*F on $DG^+\mathrm{CAlg}(R)$ as the homotopy limit

$$D_*F(B) := \mathrm{ho}\varprojlim_{n\in\Delta} F(D^n B).$$

For set-valued functors, this just gives $D_*F(B)$ as the equaliser of $\partial^0, \partial^1\colon F(B^0) \to F(D^1 B)$.

For groupoid-valued functors, $D_*F(B)$ is equivalent to the groupoid whose objects are pairs (x, g) with $x \in F(B^0)$ and $g\colon \partial^0 x \to \partial^1 x$ an isomorphism $F(D^1 B)$ satisfying the cocycle condition

$$\partial^1 g = (\partial^2 g) \circ (\partial^0 g)\colon \partial^1\partial^0 x \to \partial^2\partial^1 x$$

in $F(D^2 B)$; an isomorphism between (x, g) and (x', g') is then given by an isomorphism $h\colon x \to x'$ in $F(B^0)$ satisfying $g' \circ \partial^0 h = (\partial^1 h) \circ g\colon \partial^0 x \to \partial^1 x'$.

For simplicial set-valued functors, a model for D_*F is the derived total space

$$\mathbf{R}\mathrm{Tot}\, F(D^\bullet B) = \{x \in \prod_n \mathbf{R}F(D^n B)^{\Delta^n} \ : \partial^i x_n = \partial_i^\Delta x_{n+1}, \sigma^i x_n = \sigma_i^\Delta x_{n-1}\},$$

of [GJ, §VIII.1], where $\mathbf{R}F(D^\bullet B)$ is a Reedy fibrant replacement of the cosimplicial space $F(D^\bullet B)$, and $\partial i^\Delta, \sigma_i^\Delta$ are defined in terms of the face and degeneracy maps between the simplices Δ^n.

Example 1.13. Given a linear algebraic group G acting on an affine scheme Y, the calculation of [Pri6, Example 3.6] implies that for the associated groupoid R-scheme $(Y \times G \rightrightarrows Y)$, we have

$$D_*(G \times Y \rightrightarrows Y) = ([(Y \times G)/(\mathfrak{g} \oplus \mathfrak{g})] \rightrightarrows [Y/\mathfrak{g}]),$$

where $[Y/\mathfrak{g}]$ is defined in Example 1.2, $\mathfrak{g}$ is the Lie algebra of G, and the action of $\mathfrak{g} \oplus \mathfrak{g}$ on $Y \times G$ is given by the first factor combining the action on Y with the left action on G, while the second factor acts via the right action on G. One of the maps $[(Y \times G)/(\mathfrak{g} \oplus \mathfrak{g})] \to [Y/\mathfrak{g}]$ is projection onto the first factors, while the other combines the action $Y \times G \to Y$ with projection $\mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ onto the second factor.

Example 1.14. Given a smooth R-scheme Y, Simpson's de Rham stack Y_{dR} from [Sim2] is the functor on R-algebras defined by $Y_{\mathrm{dR}}(B) := Y(B^{\mathrm{red}})$, for B^{red} the quotient of B by the ideal of nilpotent elements. Since the natural map $DA \to A^0$ is a nilpotent extension on each level, we get

$$(D_* Y_{\mathrm{dR}})(B) = Y((B^0)^{\mathrm{red}}).$$

For Y affine, the natural maps $\mathrm{H}^0 B \to B^0 \to (B^0)^{\mathrm{red}}$ thus give natural transformations

$$D_* Y \to \mathrm{Spec}\, \Omega^\bullet_Y \to D_* Y_{\mathrm{dR}},$$

for $\mathrm{Spec}\, \Omega^\bullet_Y$ as in Example 1.3. It should not be surprising that $\mathrm{Spec}\, \Omega^\bullet_Y$ differs somewhat from $D_* Y_{\mathrm{dR}}$, since $\Omega^\bullet_Y$ can recover the Hodge filtration while Y_{dR} cannot.

Definition 1.15. Given a functor F from $DG^+\mathrm{CAlg}(R)$ to sets, groupoids or simplicial sets, denote by $F^\sharp$ the sheafification (resp. stackification, resp. hypersheafification) of F with respect to those étale maps (cf. Definition 1.6, Lemma 1.7) $A \to B$ in $DG^+\mathrm{CAlg}(R)$ for which $f^0\colon A^0 \to B^0$ is faithfully flat.

Example 1.16. Associated to the algebraic group GL_n, there is a stack $B\mathrm{GL}_n$, and we may combine Definitions 1.12, 1.15 to give a stack $(D_* B\mathrm{GL}_n)^\sharp$ on $DG^+\mathrm{CAlg}(R)$ with respect to the stacky affine étale covers. Using Example 1.13, we may characterise this as the stackification of the groupoid functor $([\mathrm{GL}_n/(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)] \rightrightarrows [*/\mathfrak{gl}_n])$.

It follows that $(D_* B\mathrm{GL}_n)^\sharp(B)$ is equivalent to the groupoid of pairs (M, ∇) for M a projective B^0-module of rank n and $\nabla\colon M \to B^1 \otimes_{B^0} M$ a flat ∂_B-connection, i.e. a degree 1 operator ∇ on $B^\# \otimes_{B^0} M$ satisfying $\nabla(bm) = \partial(b)m + (-1)^{\deg b}\nabla(m)$ for $b \in B^\#, m \in M$ and $\nabla \circ \nabla = 0$.

Explicitly, we can see this by first characterising elements of $[*/\mathfrak{g}](B)$ as Maurer–Cartan elements

$$\{\omega \in \mathfrak{g} \otimes B^1 \ : \ \partial_B \omega + \frac{1}{2}[\omega, \omega] = 0 \in \mathfrak{g} \otimes B^2\} =: \mathrm{MC}(\mathfrak{g} \otimes B),$$

and elements of $[G/(\mathfrak{g} \oplus \mathfrak{g})]$ as triples $(g, \omega_1, \omega_2) \in G(B^0) \times \mathrm{MC}(\mathfrak{g} \otimes B)^2$ such that $\omega_2 = g\omega_1 g^{-1} + dg.g^{-1}$, meaning that g is a gauge transformation.

Once we note that the groupoid of pairs (M, ∇) satisfies étale descent, the equivalence is given by mapping each element ω of $[*/\mathfrak{gl}_n](B)$ to a connection $\nabla_\omega := \partial_B + \omega$ on $(B^0)^{\oplus n}$, mapping each $g \in \mathrm{GL}_n(B^0)$ to the corresponding automorphism of $(B^0)^{\oplus n}$ intertwining $\nabla_{\omega_1}, \nabla_{\omega_2}$, and observing that this map of groupoid-valued functors is a local isomorphism with respect to étale covers.

Example 1.17. For an arbitrary linear algebraic group G, a similar argument to the previous example gives $(D_* BG)^\sharp(B)$ as the groupoid of pairs (P, ∇), for P a G-torsor over B^0, and $\nabla\colon O(P) \to O(P) \otimes_{B^0} B^1$ a flat connection acting as a differential with respect to the multiplicative structure on the ring of functions $O(P)$, required to be G-equivariant in the sense that $\mu \circ \nabla = (\nabla \otimes \mathrm{id}) \circ \mu$, for $\mu\colon O(P) \to O(P) \otimes O(G)$ the co-action.

2. Étale atlases and quasi-coherent sheaves

2.1. **Étale atlases.** Given a sheaf or stack F on $DG^+\mathrm{CAlg}(R)$ with respect to the étale covers of Definition 1.15, we can define étale atlases or 1-atlases exactly as we do for algebraic spaces or Deligne–Mumford stacks [LMB], in terms of étale covers $U \to F$ satisfying relative representability conditions. Likewise for simplicial hypersheaves on $DG^+\mathrm{CAlg}(R)$, we have a notion of $(n, \text{ét})$-geometricity by following the yoga of [Sim1, TV] (the terminology of [Lur] is slightly different — see [Pri3, Remark 1.27] for a comparison). This leads to the following key theorem:

Theorem 2.1. *Given an Artin stack $\mathfrak{X}$ over R, the stack $(D_*\mathfrak{X})^\sharp$ on $DG^+\mathrm{CAlg}(R)$ admits an étale atlas by a cover of stacky affines. More generally, for any n-geometric Artin stack $\mathfrak{X}$ over R, the hypersheaf $(D_*\mathfrak{X})^\sharp$ is $(n, \text{ét})$-geometric.*

Proof (sketch). We follow the construction of [Pri6, §3.1]. The essential step is given by [Pri3, Theorem 4.7] (see also [Pri2, Theorem 2.3]), which provides an Artin $(n+1)$-hypergroupoid $X_\bullet$ resolving $\mathfrak{X}$ — this is a simplicial scheme with all the partial matching maps (and hence all the face maps ∂_i) being smooth surjections, and each X_n being a disjoint union of affine schemes. When $\mathfrak{X}$ has affine diagonal (corresponding to 0-geometric in the later versions of [TV]), then this resolution is straightforward to construct as a Čech nerve, but beware that the higher cases are fairly non-trivial: if $\mathfrak{X}$ is n-geometric (so $\mathfrak{X} \to \mathfrak{X}^{hS^n}$ is affine), then the construction has $2^{n+1} - 1$ steps.

We first address the case where $\mathfrak{X}$ is strongly quasi-compact in the sense of [TV], meaning that all the higher diagonals $\mathfrak{X} \to \mathfrak{X}^{hS^n}$ are quasi-compact for $n \geq -1$ (where $S^{-1} = \emptyset$). Then we may assume that each scheme X_n is affine, giving us a cosimplicial R-algebra $O(X)$. The denormalisation functor D has a left adjoint D^* (for an explicit description, see [Pri10, Definition 4.14]), and unwinding Definition 1.12 it follows that $(D_*\mathfrak{X})^\sharp$ is equivalent to the hypersheafification of the simplicial object $\mathrm{Spec}\, D^*O(X^{\Delta^\bullet})$ given by

$$n \mapsto \mathrm{Spec}\, D^*O(X^{\Delta^n}),$$

for X^{Δ^n} as in Definition 1.11.

Looking at the effect of the functor D on square-zero extensions and on contractible square-zero extensions, together with the Artin hypergroupoid conditions from [Pri3, Definition 3.2] or [Pri2, Definition 2.1] on the resolution X, it follows that the partial matching maps of $\mathrm{Spec}\, D^*O(X^{\Delta^\bullet})$ are étale while the matching maps are geometric, in the sense of Definition 1.6. In particular, this means that each $\mathrm{Spec}\, D^*O(X^{\Delta^n})$ is a stacky affine, and in the terminology of [Pri3], the simplicial object $\mathrm{Spec}\, D^*O(X^{\Delta^\bullet})$ is an $(n+1, \text{ét})$-hypergroupoid in stacky affines via [Pri3, Lemma 2.18]. By [Pri3, Proposition 4.1], this implies that $(D_*\mathfrak{X})^\sharp$ is $(n, \text{ét})$-geometric, with $\mathrm{Spec}\, D^*O(X) \to \mathfrak{X}$ being an étale n-atlas.

If $\mathfrak{X}$ is not strongly quasi-compact, we take the simplicial diagram $X_\bullet$ of disjoint unions of affine schemes, and write $X_0 = \coprod_\alpha U_\alpha$. For $B \in DG^+\mathrm{CAlg}(R)$, a morphism $\mathrm{Spec}\, DB \to X_\bullet$ of simplicial schemes always factors through the completion of $X_\bullet$ along the iterated degeneracy map $\underline{\sigma}\colon X_0 \to X_\bullet$, because the natural map $DB \to B^0$ is a nilpotent extension on each level. In particular, each such morphism factors through the localisation at X_0, and the constructions above adapt if we replace each instance of $\mathrm{Spec}\, D^*O(X)$ with the coproduct $\coprod_\alpha \mathrm{Spec}\, D^*\Gamma(U_\alpha, \sigma_\bullet^{-1}\mathscr{O}_X)$, with a similar replacement for each $\mathrm{Spec}\, D^*O(X^{\Delta^n})$ based on affine covers of each X_n. □

Example 2.2. For a linear algebraic group G acting on an affine scheme Y, we can apply the theorem to the quotient stack $[Y/G]$, in which case we just recover the description of Example 1.13. Explicitly, an étale atlas for $(D_*[Y/G])^\sharp$ is given by the stacky affine $[Y/\mathfrak{g}]$, and the associated simplicial resolution is the nerve of the groupoid $([(Y\times G)/(\mathfrak{g}\oplus\mathfrak{g})] \Rightarrow [Y/\mathfrak{g}])$ in stacky affines.

Corollary 2.3. *For any simplicial hypersheaf F on affine R-schemes and any Artin n-stack $\mathfrak{X}$, the canonical natural transformation*

$$\mathrm{map}(\mathfrak{X}, F) \to \mathrm{map}((D_*\mathfrak{X})^\sharp, (D_*F)^\sharp)$$

of simplicial mapping spaces is a weak equivalence. This space is moreover equivalent to the homotopy limit of D_*F evaluated on the simplicial category of stacky affines étale over $(D_*\mathfrak{X})^\sharp$.

Proof. As in the proof of Theorem 2.1, $\mathfrak{X}$ admits an Artin $(n+1)$-hypergroupoid resolution $X_\bullet$, and $(D_*\mathfrak{X})^\sharp$ is equivalent to the hypersheafification of a simplicial object $\mathrm{Spec}\, D^*O(X^{\Delta^\bullet})$ (with a slight variation in the non-strongly quasi-compact case). As in [Pri6, Corollary 3.14], both of our assertions then reduce to the statement that the map

$$X \to \mathrm{diag}\,\mathrm{Spec}\, DD^*O(X^{\Delta^\bullet})$$

is a weak equivalence of simplicial presheaves.

This last statement follows from [Pri6, Proposition 3.13] (or rather its proof, the fibrant hypotheses not being strictly necessary). The key idea is that D and D^* descend to functors between graded-commutative algebras and almost cosimplicial commutative algebras, giving rise for each j to contracting homotopies

$$D^j(D^*O((X^{\Delta^\bullet})_i)^\#) \to D^j(D^*O((X^{\Delta^\bullet})_i)^\#)^{\Delta^1_i}$$

making the natural map $D^j(D^*O((X^{\Delta^\bullet})_i)^\#) \to O(X_i)$ a deformation retract, functorial in i with respect to the simplicial operations. □

2.2. **Quasi-coherent sheaves and torsors.** We can apply Corollary 2.3 to stacks F such as $B\mathrm{GL}_n$, BG, the stack of quasi-coherent sheaves or the higher stack of perfect complexes. We now give the resulting characterisations of the groupoids of vector bundles, of G-torsors, of quasi-coherent sheaves and of perfect complexes on $\mathfrak{X}$, all in terms of data on the site of stacky affines étale over $(D_*\mathfrak{X})^\sharp$.

Example 2.4 (Vector bundles). If we define a vector bundle on a stacky affine $\mathrm{Spec}\, B$ to be a pair (M, ∇) for M a projective B^0-module of rank n and $\nabla\colon M \to B^1 \otimes_{B^0} M$ a flat ∂_B-connection as in Example 1.16, then it follows from Corollary 2.3 applied to $\mathrm{map}(\mathfrak{X}, B\mathrm{GL}_n)$ that the groupoid of vector bundles on an Artin stack $\mathfrak{X}$ is equivalent to the groupoid of vector bundles on the site of stacky affines étale over $(D_*\mathfrak{X})^\sharp$.

Example 2.5 (Torsors). For an arbitrary linear algebraic group G, if we define a G-torsor on on a stacky affine $\mathrm{Spec}\, B$ to be a pair (P, ∇) for P a G-torsor over B^0 and $\nabla\colon O(P) \to O(P) \otimes_{B^0} B^1$ a G-equivariant flat connection as in Example 1.17, then it follows from Corollary 2.3 applied to $\mathrm{map}(\mathfrak{X}, BG)$ that the groupoid of G-torsors on an Artin stack $\mathfrak{X}$ is equivalent to the groupoid of G-torsors on the site of stacky affines étale over $(D_*\mathfrak{X})^\sharp$.

Definition 2.6. Given a stacky affine $\mathrm{Spec}\, B$, define a Cartesian B-module to be a B-module M in cochain complexes for which the natural map

$$M^0 \otimes_{B^0} B^\# \to M^\#$$

of graded $B^{\#}$-modules is an isomorphism.

Equivalently, we could characterise a Cartesian B-module as a pair (M^0, ∇), for a B^0-module M and a flat ∂_B-connection $\nabla\colon\, M^0 \to M^0 \otimes_{B^0} B^1$. Given M, the associated pair is $(M^0, \partial_M\colon\, M^0 \to M^1)$.

Example 2.7 (Quasi-coherent sheaves). Applying Corollary 2.3 to the stack of quasi-coherent sheaves, the reasoning of Example 2.4 adapts to show that the groupoid of quasi-coherent sheaves on an Artin stack $\mathfrak{X}$ is equivalent to the groupoid of Cartesian modules on the site of stacky affines étale over $(D_*\mathfrak{X})^\sharp$. Looking at stacks of morphisms of quasi-coherent sheaves, we can then conclude the the category of quasi-coherent sheaves on an Artin stack $\mathfrak{X}$ is equivalent to the category of quasi-coherent Cartesian $\mathscr{O}$-modules on the site of stacky affines étale over $(D_*\mathfrak{X})^\sharp$.

Definition 2.8. Given a stacky affine $\operatorname{Spec} B$, define a quasi-Cartesian B-complex to be a B-module $M = \bigoplus_{i,n} M^n_i$ in chain cochain complexes for which the natural map

$$M^0 \otimes_{B^0} B^{\#} \to M^{\#}$$

of graded $B^{\#}$-modules is an quasi-isomorphism of graded chain complexes.

Example 2.9 (Quasi-coherent complexes). Following [Pri3, §5.4.2], we can define a quasi-coherent complex on an Artin stack $\mathfrak{X}$ to be a presheaf $\mathscr{F}_\bullet$ of $\mathscr{O}_{\mathfrak{X}}$-modules whose homology *presheaves* $\mathrm{H}_i(\mathscr{F}_\bullet)$ are all quasi-coherent. Then Example 2.7 generalises to the statement that the ∞-category of quasi-coherent complexes on $\mathfrak{X}$ is equivalent to the category of quasi-coherent quasi-Cartesian $\mathscr{O}$-modules on the site of stacky affines étale over $(D_*\mathfrak{X})^\sharp$, localised at levelwise quasi-isomorphisms.

A similar statement holds for perfect complexes; as observed in [Pri8, Proposition 3.11], it follows by combining [Pri3, Proposition 5.12] with [Pri6, Lemma 3.9 and Corollary 3.14].

3. Tangent modules, Poisson structures, differential operators and quantisations

3.1. Cotangent and tangent complexes.

3.1.1. *Cotangent complexes and symplectic structures.* On a smooth Artin n-stack $\mathfrak{X}$, the cotangent complex $\mathbb{L}_{\mathfrak{X}}$ is a perfect complex of amplitude n.

From the constructions of [Pri3, §7.1] and [Pri6, Proposition 3.19], it follows that the complex of derived global sections $\mathbf{R}\Gamma(\mathfrak{X}, \Lambda^p\mathbb{L}_{\mathfrak{X}})$ is quasi-isomorphic to the complex of derived global sections of the presheaf $B \mapsto \Omega^p_B$ on the site $(D_*\mathfrak{X})^\sharp_{\text{ét}}$ of stacky affines $\operatorname{Spec} B$ étale over $(D_*\mathfrak{X})^\sharp$.

We can also apply this to the de Rham complex and its filtered pieces, so as in [Pri6, Lemma 3.25],

$$F^p\mathrm{DR}(\mathfrak{X}) \simeq \mathbf{R}\Gamma((D_*\mathfrak{X})^\sharp_{\text{ét}}, F^p\Omega^\bullet_{\mathscr{O}}),$$

while the space $\mathcal{A}^{p,cl}_R(\mathfrak{X}, n)$ of closed p-forms of degree n from [PTVV] is just the Dold-Kan denormalisation of the good truncation in non-positive degrees of

$$\mathbf{R}\Gamma((D_*\mathfrak{X})^\sharp_{\text{ét}}, F^p\Omega^\bullet_{\mathscr{O}})[n+p].$$

In particular, for $p = 2$, the space of n-shifted symplectic structures [PTVV] then consists of the non-degenerate elements in this space.

3.1.2. *Tangent complexes.* On a stacky affine $\operatorname{Spec} B$, we can use the internal Hom-functor $\mathcal{H}om_B$ for B-modules in complexes to give a complex $\mathcal{H}om_B(\Omega^1_B, B)$ of B-modules. When B^0 is smooth, and $B \to C$ is étale in the sense of Definition 1.6, note that the canonical map

$$\mathcal{H}om_B(\Omega^1_B, B) \otimes_B C \to \mathcal{H}om_C(\Omega^1_C, C)$$

is a quasi-isomorphism. By [Pri6, Proposition 3.19], we then have a quasi-isomorphism

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathfrak{X}}}(\mathbb{L}_{\mathfrak{X}}, \mathscr{O}_{\mathfrak{X}}) \simeq \mathbf{R}\Gamma((D_*\mathfrak{X})^{\sharp}_{\text{ét}}, \mathcal{H}om_{\mathscr{O}}(\Omega^1_{\mathscr{O}}, \mathscr{O})),$$

and indeed

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathfrak{X}}}(\mathbb{L}_{\mathfrak{X}}^{\otimes p}, \mathscr{O}_{\mathfrak{X}}) \simeq \mathbf{R}\Gamma((D_*\mathfrak{X})^{\sharp}_{\text{ét}}, \mathcal{H}om_{\mathscr{O}}((\Omega^1_{\mathscr{O}})^{\otimes p}, \mathscr{O})).$$

3.2. **Poisson structures and quantisations.** The expressions in §3.1.2 lead to well-behaved complexes of polyvectors and shifted polyvectors on $\mathfrak{X}$, which in particular carry Schouten–Nijenhuis brackets, permitting the formulation of shifted Poisson structures [Pri6, §3.3]. In brief, an n-shifted Poisson structure on a stacky affine $\operatorname{Spec} B$ is an enrichment of the CDGA structure on B to a strong homotopy P_{n+1}-algebra structure, and an n-shifted Poisson structure on $\mathfrak{X}$ is an ∞-functorial choice of n-shifted Poisson structure for each stacky affine étale over $(D_*\mathfrak{X})^{\sharp}$. The equivalence between n-shifted symplectic structures and non-degenerate n-shifted Poisson structures in this sense is then given by [Pri6, Theorem 3.33], by solving a tower of obstruction problems. Over a Noetherian base ring, shifted Poisson structures in our sense are expected to coincide with the more involved formulation of [CPT$^+$]; the correspondence between symplectic and non-degenerate Poisson structures is also established in [CPT$^+$], using a less direct method.

Hochschild complexes and rings of differential operators admit canonical filtrations, whose associated gradeds are complexes of (shifted) polyvectors. They also therefore satisfy functoriality with respect to the étale morphisms of Definition 1.6, and deformation quantisations can be formulated in terms of almost commutative deformations of the structure sheaf $\mathscr{O}$ on $(D_*\mathfrak{X})^{\sharp}_{\text{ét}}$. Via Example 2.9, 0-shifted quantisations then lead to deformations of the dg category of perfect complexes on $\mathfrak{X}$.

As observed in [CPT$^+$], for positively shifted structures, existence of quantisations follows immediately from formality of the E_{n+1}-operads, leading to deformations of perfect complexes as an n-tuply monoidal dg category. Our smoothness hypotheses above do not lead to many interesting negatively shifted structures; in particular, n-shifted symplectic structures on smooth Artin N-stacks only exist for $n \geq 0$. There are also many quantisation results for negatively shifted structures on singular stacks when regarded as derived stacks; see §4.1.5.

A shifted coisotropic structure on a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of stacky affines can be formulated in terms of an n-shifted Poisson structure on A, an $n-1$-shifted Poisson structure on B and a homotopy P_{n+1}-algebra morphism from A to twisted shifted polyvectors on B (cf. [CPT$^+$, MS1]). For $n \geq 2$, quantisations of such structures are established in [MS2], while for $n = 0$, quantisations are established in non-degenerate cases (i.e. 0-shifted Lagrangians) in [Pri4]. It is expected that the approach of [Pri4] should adapt to 1-shifted Lagrangians; if the Swiss cheese operad has the good properties conjectured in [MS2], then [Pri5] should adapt to give quantisations of general 1-shifted coisotropic structures.

4. Generalisations: derived and analytic structures

4.1. **Derived Artin stacks.** In order to simplify the exposition, we have restricted our attention so far to higher stacks and assumed smoothness, avoiding any derived structure. The various results cited above are all formulated in the more general setting of derived stacks, and we now briefly explain what extra structure has to be introduced and where the subtleties lie.

4.1.1. *Stacky CDGAs.* We will be working systematically with chain cochain complexes V, which are bigraded abelian groups $V = \bigoplus_{i,j} V^i_j$, equipped with square-zero linear maps $\partial\colon V^i_j \to V^{i+1}_j$ and $\delta\colon V^i_j \to V^i_{j-1}$ such that $\partial\delta + \delta\partial = 0$.

The following is taken from [Pri6, Definition 3.2]:

Definition 4.1. We define a stacky CDGA to be a chain cochain complex A of $\mathbb{Q}$-vector spaces equipped with a commutative product $A \otimes A \to A$ and unit $\mathbb{Q} \to A$.

We regard all chain complexes as chain cochain complexes $V = V^0_\bullet$. Given a chain CDGA R, a stacky CDGA over R is then a morphism $R \to A$ of stacky CDGAs. We write $DG^+dg_+\mathrm{CAlg}(R)$ for the category consisting of stacky CDGAs A over R concentrated in non-negative bidegrees (i.e. $A^i_j = 0$ unless $i, j \geq 0$).

For $A \in DG^+dg_+\mathrm{CAlg}(R)$, we denote by $\operatorname{Spec} A$ the corresponding object of the opposite category $DG^+dg_+\mathrm{CAlg}(R)^{\mathrm{opp}}$; we also write $A = O(\operatorname{Spec} A)$.

There are obvious analogues of Examples 1.2 and 1.3, giving stacky CDGAs $O([Y/\mathfrak{g}])$ and $\Omega^\bullet_{Y/R}$ for non-negatively graded chain CDGAs $O(Y)$; see [Pri6, Example 3.6] for details of $[Y/\mathfrak{g}]$. These behave well when $O(Y)$ is semi-smooth in the sense that $O(Y)_0$ is smooth and $O(Y)_\#$ is freely generated over $O(Y)_0$ by a graded projective module.

Definition 4.2. Say that a morphism $U \to V$ of chain cochain complexes is a levelwise quasi-isomorphism if the map $U^i \to V^i$ of chain complexes is a quasi-isomorphism for all i. Say that a morphism of stacky CDGAs is a levelwise quasi-isomorphism if the underlying morphism of chain cochain complexes is so.

The following adapts [Pri6, Lemma 3.4]

Lemma 4.3. *There is a cofibrantly generated model structure on* $DG^+dg_+\mathrm{CAlg}(R)$ *in which fibrations are surjective in strictly positive chain degrees and weak equivalences are levelwise quasi-isomorphisms.*

The following appear as [Pri6, Definitions 3.7 and 3.8]. The functor $\hat{\mathrm{Tot}}$ corresponds to the Tate realisation of [CPT+].

Definition 4.4. Given a chain cochain complex V, define the cochain complex $\hat{\mathrm{Tot}}\, V \subset \mathrm{Tot}^{\Pi} V$ by

$$(\hat{\mathrm{Tot}}\, V)^m := (\bigoplus_{i<0} V^i_{i-m}) \oplus (\prod_{i\geq 0} V^i_{i-m})$$

with differential $\partial \pm \delta$.

Definition 4.5. Given A-modules M, N in chain cochain complexes, we define internal Hom spaces $\mathcal{H}om_A(M, N)$ by

$$\mathcal{H}om_A(M, N)^i_j = \mathrm{Hom}_{A^\#_\#}(M^\#_\#, N^{\#[i]}_{\#[j]}),$$

with differentials $\partial f := \partial_N \circ f \pm f \circ \partial_M$ and $\delta f := \delta_N \circ f \pm f \circ \delta_M$, where $V^{\#}_{\#}$ denotes the bigraded vector space underlying a chain cochain complex V.

We then define the Hom complex $\underline{\hat{\mathrm{Hom}}}_A(M,N)$ by

$$\underline{\hat{\mathrm{Hom}}}_A(M,N) := \hat{\mathrm{Tot}}\,\mathcal{H}om_A(M,N).$$

The reason for working with the functors $\hat{\mathrm{Tot}}$ and $\underline{\hat{\mathrm{Hom}}}$ is that they send levelwise quasi-isomorphisms to quasi-isomorphisms. They also behave well with respect to tensor products.

4.1.2. *Étale morphisms.* The definitions of 1.2 now adapt to stacky CDGAs:

Definition 4.6. We say that a morphism $f\colon C' \to C$ in $DG^+dg_+\mathrm{CAlg}(R)$ is a square-zero extension if f is surjective in every level and the kernel I is a square-zero ideal.

We say that f is a contractible square-zero extension if in addition I admits a C'-linear (or, equivalently, C-linear) contracting cochain homotopy compatible with the chain maps. Explicitly, this means that we have maps $h\colon I^n_j \to I^{n+1}_j$ for all n, j, satisfying $h \circ h = 0$, $h \circ \partial + \partial \circ h = \mathrm{id}$ and $h \circ \delta + \delta \circ h = 0$.

We will refer to functors as *homotopy-preserving* if they preserve weak equivalences.

Definition 4.7. Given homotopy-preserving functors F, G from $dg_+DG^+\mathrm{CAlg}(R)$ to simplicial sets, and a natural transformation $\eta\colon F \to G$, we say that η is homotopy formally étale (resp. homotopy formally geometric) if the maps

$$F(C') \to F(C) \times^h_{G(C)} G(C')$$

are π_0-surjective for all square-zero extensions (resp. all contractible square-zero extensions) $C' \to C$.

We say that a functor F on $dg_+DG^+\mathrm{CAlg}(R)$ is homotopy formally étale (resp. homotopy formally geometric) if the transformation $F \to *$ to the constant functor is so.

Definition 4.8. We say that a natural transformation $\eta\colon F \to G$ of functors F, G on $dg_+DG^+\mathrm{CAlg}(R)$ is l.f.p. if for any filtered colimit $C = \varinjlim_{i\in I} C_i$ in $DG^+\mathrm{CAlg}(R)$, the natural map

$$\varinjlim_{i\in I} F(C_i) \to \varinjlim_{i\in I} G(C_i) \times_{G(C)} F(C)$$

is a weak equivalence.

We then say that η is homotopy étale (resp. homotopy geometric) if it is homotopy formally étale (resp. homotopy formally geometric) and l.f.p.

For $A \in dg_+DG^+\mathrm{CAlg}(R)$, we refer to $\mathrm{Spec}\,A$ as a stacky affine if the functor $\mathbf{R}\mathrm{Spec}\,A := \mathrm{map}_{dg_+DG^+\mathrm{CAlg}(R)}(A, -)$ is geometric.

Since any stacky CDGA $A \in dg_+DG^+\mathrm{CAlg}(R)$ gives rise to a simplicial set-valued functor $\mathbf{R}\mathrm{Spec}$, the definitions above give rise to notions of homotopy (formally) étale maps between objects of $dg_+DG^+\mathrm{CAlg}(R)$. The following lemma is a consequence of standard obstruction arguments.

Considering obstructions classes in $\mathrm{H}_{-1}\mathrm{Z}^0\mathcal{H}om_B(\Omega^1_{B/A}, M)$ leads to the following lemma, thus tying in with the homotopy formally étale morphisms of [Pri6, §3.4.2].

Lemma 4.9. *A cofibration $f\colon A \to B$ is homotopy formally geometric if and only if*

- *for all $i, n > 0$, we have $\mathrm{H}_i((\Omega^1_{B/A} \otimes_B \mathrm{H}_0 B^0)^n) = 0$, and*

- *for all* $n > 0$, *the* $\mathrm{H}_0 B^0$*-module* $\mathrm{H}_0((\Omega^1_{B/A} \otimes_B \mathrm{H}_0 B^0)^n)$ *is projective.*

The morphism f *is homotopy formally étale if and only if it is homotopy formally geometric and*

- *the inverse system* $\{\mathrm{Tot}\, \sigma^{\le q}(\Omega^1_{B/A} \otimes_B \mathrm{H}_0 B^0)\}_q$ *of complexes of projective* $\mathrm{H}_0 B^0$*-modules is pro-quasi-isomorphic to* 0*, where* σ *denotes brutal truncation in the cochain direction.*

Note that when f is l.f.p., the final condition reduces to saying that the complexes $\{\mathrm{Tot}\, \sigma^{\le q}(\Omega^1_{B/A} \otimes_B \mathrm{H}_0 B^0)\}_q$ of projective $\mathrm{H}_0 B^0$-modules are acyclic for $q \gg 0$, because the inverse system stabilises.

4.1.3. *Étale atlases.* The denormalisation functor D extends naturally to give a functor from stacky CDGAs to cosimplicial chain algebras. Given a functor F from chain algebras $dg_+\mathrm{CAlg}(R)$ to sets simplicial sets, we may adapt Definition 1.12 to define a functor D_*F on $DG^+dg_+\mathrm{CAlg}(R)$ as the homotopy limit

$$D_*F(B) := \mathrm{ho}\varprojlim_{n \in \Delta} F(D^n B).$$

This leads to the following generalisation of Theorem 2.1 to the derived setting. This has the same proof as Theorem 2.1, and similar consequences for quasi-coherent complexes.

Theorem 4.10. *Given an* n*-geometric derived Artin stack* $\mathfrak{X}$ *over* R*, the hypersheaf* $(D_*\mathfrak{X})^\sharp$ *is* $(n, \text{ét})$*-geometric with respect to the étale morphisms of Definition 4.8.*

4.1.4. *Tangent complexes and Poisson structures.* Analogues of the expressions in §3.1.2 hold for stacky CDGAs, with the same references, so that in particular

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathfrak{X}}}(\mathbb{L}_{\mathfrak{X}}^{\otimes p}, \mathscr{O}_{\mathfrak{X}}) \simeq \mathbf{R}\Gamma((D_*\mathfrak{X})^\sharp_{\text{ét}}, \underline{\hat{\mathrm{Hom}}}_{\mathscr{O}}((\Omega^1_{\mathscr{O}})^{\otimes p}, \mathscr{O})),$$

for $(D_*\mathfrak{X})^\sharp_{\text{ét}}$ the site of étale maps $\mathrm{Spec}\, A \to (D_*\mathfrak{X})^\sharp_{\text{ét}}$ (∞-localised at levelwise quasi-isomorphisms) for cofibrant geometric stacky CDGAs A.

This immediately leads to a formulation of shifted Poisson structures [Pri6, §3.3]. An n-shifted Poisson structure on a stacky derived affine $\mathrm{Spec}\, B$ is an enrichment of the CDGA structure on $\hat{\mathrm{Tot}}\, B$ to a strong homotopy P_{n+1}-algebra structure, but with some boundedness restrictions on the (higher) Poisson brackets, which are required to lie in $\underline{\hat{\mathrm{Hom}}}_B((\Omega^1_B)^{\otimes p}, B)$. An n-shifted Poisson structure on a derived Artin stack $\mathfrak{X}$ is then an ∞-functorial choice of n-shifted Poisson structure for each stacky derived affine étale over $(D_*\mathfrak{X})^\sharp$.

4.1.5. *Quantisation.* All of the quantisations described in §3.2 extend to the derived setting, once the relevant Hochschild complexes are defined by using $\underline{\hat{\mathrm{Hom}}}$ in the appropriate places. There are also several quantisation results which are only really interesting or new in derived or singular cases, which we now describe.

Existence of curved A_∞ quantisation of 0-shifted Poisson structures, formulated in terms of the Hochschild complex and allowing curvature, are given in [Pri8, Pri5].

Existence of (-1)-shifted quantisations, as BD-algebras formulated in terms of differential operators, are given in [Pri11]. For (-2)-shifted structures, the Beilinson–Drinfeld hierarchy of operads breaks down, but formulations are sometimes possible in terms of solutions of a quantum master equation [Pri7].

4.2. **Differentiable and analytic stacks.** We now explain how to extend all of the results and formulations described so far to differentiable and analytic stacks.

4.2.1. *Differentiable stacks.* In the differentiable setting, the analogue of a smooth stacky affine is just an NQ-manifold. In other words, we have a CDGA

$$A^0 \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^2 \xrightarrow{\partial} \ldots$$

where A^0 is the ring of $\mathcal{C}^\infty$ functions on a some differentiable manifold X_0 and $\partial\colon A^0 \to A^1$ is a $\mathcal{C}^\infty$-derivation, with $A^\#$ freely generated over A^0 by a graded projective module (equivalently, functions on a graded vector bundle over X_0).

Making use of the theory of $\mathcal{C}^\infty$-rings [Dub, MR] and their homotopy theory as in [CR], all of the results of §§1–3 adapt to differentiable stacks. Most of the constructions are described in [Pri10], but in brief, every differentiable n-stack admits an étale n-atlas of NQ-manifolds. These atlases allow Poisson structures and deformation quantisations to be formulated in terms of $\mathcal{C}^\infty$-multiderivations and differential operators, and the algebraic existence and classification proofs adapt verbatim.

Introducing a second grading and a chain differential leads to derived structures as in [Nui], with the results of §4.1 then adapting to derived differentiable stacks — again, see [Pri10].

4.2.2. *Analytic stacks.* In the complex analytic setting, the analogue of a smooth stacky affine is a complex Stein NQ-manifold. In other words, we have a CDGA

$$A^0 \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^2 \xrightarrow{\partial} \ldots$$

where A^0 is the ring of holomorphic functions on a some Stein manifold X_0 and $\partial\colon A^0 \to A^1$ is an analytic derivation, with $A^\#$ freely generated over A^0 by a graded projective module (equivalently, functions on a graded vector bundle over X_0).

Again, introducing a second grading and a chain differential leads to derived structures, which are formulated in detail in [Pri9], using the theory of rings with entire functional calculus. All of the results from the differentiable setting of [Pri10] adapt to the analytic setting, and indeed to any Fermat theory.

In non-Archimedean settings, [Pri9] also gives a formulation of derived analytic geometry based on enriched Stein manifolds and Stein spaces, to which the results of [Pri10] all adapt. The setup is based on overconvergent functions, and pro-objects crop up more than in other geometries, since we have to regard open polydiscs as inverse limits of the closed polydiscs containing them.

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