Semiregularity as a consequence of Goodwillie's theorem

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Semiregularity

For $Z \subset X$ of codimension p, Bloch (1972) defined

$$\tau\colon\ \mathrm{H}^1(Z, \mathscr{N}_{Z/X}) \to \mathrm{H}^{p+1}(X, \Omega_X^{p-1}).$$

- When Z is a divisor, τ comes from

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(Z) \to \mathscr{N}_{Z/X} \to 0.$$

In detail, Verdier dual

$$\begin{aligned}&\mathrm{H}^{n-p-1}(X, \Omega_X^n \otimes (\Omega_X^{p-1})^\vee)\\ &\to \mathrm{H}^{n-p-1}(Z, \Omega_X^n|_Z \otimes (\Lambda^p \mathscr{N}_{Z/X}) \otimes \mathscr{N}_{Z/X}^\vee)\\ &= \mathrm{H}^{n-p-1}(Z, \Omega_X^n|_Z \otimes (\Lambda^{p-1} \mathscr{N}_{Z/X}))\end{aligned}$$

is Λ^{p-1} applied to $\Omega_X^\vee \to \mathscr{N}_{Z/X}$.

Semiregularity conjectures

For X a smooth proper variety:

- **Theorem (Bloch)**

  τ *annihilates curvilinear obstructions.*

- **Conjecture (Bloch)**

  τ *annihilates all obstructions.*

  - Thus moduli space has virtual dimension

$$\chi(Z, \mathscr{N}_{Z/X}) + h^{p-1,p+1}(X)$$

    when τ surjective.

Smooth proper $f\colon \mathcal{X} \to S$, with $X = \mathcal{X}_s$:

- Conjecture (Bloch)

  $\tau(o_e(Z))$ is the obstruction to deforming $[Z]$ as a Hodge class:

$$\widetilde{[Z]} \in \Gamma(S, \mathbf{R}^{2p} f_* \mathbb{Q}_{\mathcal{X}} \cap \mathbf{R}^{2p} f_* F^p \Omega^\bullet_{\mathcal{X}/S})$$
$$\downarrow$$
$$[Z] \in \mathrm{H}^{2p}(X, \mathbb{Q}) \cap F^p \mathrm{H}^{2p}(X, \mathbb{C}).$$

  - $\rightsquigarrow$ Numerical invariants in Hodge locus.

Sheaves

- $\{$Closed subschemes $Z \subset X\}$ $\rightsquigarrow \{$Coherent $\mathscr{O}_X$-modules $\mathscr{O}_Z\}$.
- Obstruction spaces:

$$\begin{aligned} \mathrm{H}^1(Z, \mathscr{N}_{Z/X}) &\to \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_Z, \mathscr{O}_Z) \\ &\to \mathrm{Ext}^2_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{O}_Z). \end{aligned}$$

Semiregularity for sheaves

Buchweitz and Flenner (2003):

$$\sigma_q \colon \operatorname{Ext}^j_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}) \to \mathrm{H}^{j+q}(X, \Omega^q_X)$$

for perfect complexes $\mathscr{F}$ on X, with

$$\begin{array}{ccc} \mathrm{H}^1(Z, \mathscr{N}_{Z/X}) & \longrightarrow & \operatorname{Ext}^2_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{O}_Z) \\ & \searrow^{\tau} & \downarrow{\scriptstyle \sigma_{p-1}} \\ & & \mathrm{H}^{p+1}(X, \Omega^{p-1}_X). \end{array}$$

Details

- Atiyah class $\mathrm{At}(\mathscr{F}) \in \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$.
- Semiregularity map

$$\sigma_q(\mathscr{F})(\alpha) = \frac{(-1)^q}{q!}\mathrm{tr}(\underbrace{\mathrm{At}(\mathscr{F}) \circ \ldots \circ \mathrm{At}(\mathscr{F})}_{q} \circ \alpha).$$

- Pandharipande–Thomas invariants.

Theorem (P, 2012)

Obstructions of $\mathrm{ch}_{q+1}(\mathscr{F})$ *in the Hodge locus are measured by*

$$\sigma_q\colon\ \mathrm{Ext}^2_{\mathscr{O}_X}(\mathscr{F},\mathscr{F}) \to \mathrm{H}^{2+q}(X,\Omega^q_X).$$

Corollary

Bloch's conjecture holds.

- Reduced obstruction theories.

Standard approach

- Extend Abel–Jacobi to deformations

$$\mathrm{Perf}(X) \to \mathrm{H}^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)) = \mathbb{H}^{2p}(X, \mathbb{Q} \xrightarrow{(2\pi i)^p} \Omega^\bullet_X / F^p).$$

- Obstruction spaces:

$$\mathrm{Ext}^2_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}) \xrightarrow{\sigma_{p-1}} \mathrm{H}^{p+1}(X, \Omega^{p-1}_X) \to \mathbb{H}^{2p}(X, \Omega^\bullet_X / F^p).$$

- **Problem**: $\mathrm{H}^*(X, \mathbb{Q})$ not algebraic.

The solution

- Find alternative setting where this is automatic.
- *Any* formally étale H^* can replace $\mathrm{H}^*(X, \mathbb{Q})$!
- Use periodic cyclic homology **HP**...
- a.k.a derived de Rham cohomology ...
- a.k.a algebraic de Rham cohomology.

- Algebraic de Rham cohomology (Hartshorne, after Deligne)

$$\mathrm{H}^*_{\mathrm{dR}}(Y) = \mathbb{H}^*(\hat{T}, \varprojlim_n \Omega^\bullet_T / \mathscr{I}_Y^n)$$

  for Y closed in T smooth.
- Independent of T.
- $\mathrm{H}^*_{\mathrm{dR}}(Y) = \mathrm{H}^*_{\mathrm{cris}}(Y, \mathscr{O}_{\mathrm{cris}})$.
- Crystalline Hodge filtration (cf. Feigin–Tsygan, Bhatt):

$$F^p_{\mathrm{cris}} = (\hat{\mathscr{I}}_Y^p \xrightarrow{d} \hat{\mathscr{I}}_Y^{p-1} \hat{\Omega}^1_T \to \ldots \to \hat{\Omega}^p_T \to \ldots)$$

  (equivalently: powers of $\ker(\mathscr{O}_{\mathrm{cris}} \to \mathscr{O}_{\mathrm{Zar}})$).

- Derived de Rham cohomology (Illusie).
- Quasi-free resolution $\tilde{\mathscr{O}}_X$ of $\mathscr{O}_X$.
- Bicomplex

$$\tilde{\mathscr{O}}_X \xrightarrow{d} \Omega^1(\tilde{\mathscr{O}}_X) \to \Omega^2(\tilde{\mathscr{O}}_X) \to \dots.$$

- *Product* total complex gives $\mathbf{L}\Omega^\bullet_X$.
- Derived Hodge filtration $\mathbf{L}F^p\Omega^\bullet_X$.
- Derived crystals (Gaitsgory–Rozenblyum).
- $\mathbf{L}F^p\Omega^\bullet_X \to F^p_{\mathrm{cris}}$, equivalence for LCI.

- Cyclic homology complex

$$\mathbf{HC}(X)^{(p)} = \mathbf{R}\Gamma(X, \mathbf{L}\Omega^{\bullet}_X / \mathbf{L}F^{p+1})^{[2p]}.$$

- Periodic cyclic homology

$$\mathbf{HP}(X)^{(p)} = \mathbf{R}\Gamma(X, \mathbf{L}\Omega^{\bullet}_X)^{[2p]}.$$

- Negative cyclic homology

$$\mathbf{HN}(X)^{(p)} = \mathbf{R}\Gamma(X, \mathbf{L}F^p\Omega^{\bullet}_X)^{[2p]}.$$

(Feigin–Tsygan)

Details of the proof

- Define homotopy pullback

$$\begin{array}{ccc} \mathcal{J}^p_{\mathcal{X}}(A) & \longrightarrow & \mathbf{HN}(\mathcal{X}_A/A)^{(p)} \\ \downarrow & & \downarrow \\ \mathbf{HP}(\mathcal{X}_A/\mathbb{Q})^{(p)} & \longrightarrow & \mathbf{HP}(\mathcal{X}_A/A)^{(p)} \end{array}$$

- Equivalently:

$$\begin{array}{ccc} \mathcal{J}^p_{\mathcal{X}}(A)[2p] & \longrightarrow & \mathbf{R}\Gamma(\mathcal{X}, F^p\Omega^\bullet(\mathcal{X}_A/A)) \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma(\mathcal{X}, \mathbf{L}\Omega^\bullet(\mathcal{X}_A/\mathbb{Q})) & \longrightarrow & \mathbf{R}\Gamma(\mathcal{X}, \Omega^\bullet(\mathcal{X}_A/A)). \end{array}$$

- Goodwillie–Jones Chern character

$$\mathrm{ch}_p^- : \mathbf{K}(\mathcal{X}_A) \to \mathbf{HN}(\mathcal{X}_A/\mathbb{Q})^{(p)}$$

- Hence

$$\begin{array}{ccc} \mathbf{K}(\mathcal{X}_A) & \longrightarrow & \mathbf{HN}(\mathcal{X}_A/A)^{(p)} \\ \downarrow & & \downarrow \\ \mathbf{HP}(\mathcal{X}_A/\mathbb{Q})^{(p)} & \longrightarrow & \mathbf{HP}(\mathcal{X}_A/A)^{(p)}, \end{array}$$

- so $\mathrm{Perf}(\mathcal{X}_A) \to \mathbf{K}(\mathcal{X}_A) \to \mathcal{J}^p_{\mathcal{X}}(A)$.

Theorem (Goodwillie)

For $A \twoheadrightarrow B$ nilpotent,

$$\mathbf{HP}(A/\mathbb{Q}) \xrightarrow{\sim} \mathbf{HP}(B/\mathbb{Q}).$$

Thus:

- $\mathbf{L}\Omega^\bullet(A/\mathbb{Q}) \xrightarrow{\sim} \mathbf{L}\Omega^\bullet(B/\mathbb{Q})$.
- cohomology theory is formally étale.
- Hypersheafify: $A \rightsquigarrow \mathbf{HP}(\mathcal{X}_A/\mathbb{Q})^{(p)}$ formally étale (horizontal sections).

Since

$$\begin{array}{ccc} \mathcal{J}^p_{\mathcal{X}}(A) & \longrightarrow & \mathbf{HN}(\mathcal{X}_A/A)^{(p)} \\ \downarrow & & \downarrow \\ \mathbf{HP}(\mathcal{X}_A/\mathbb{Q})^{(p)} & \longrightarrow & \mathbf{HP}(\mathcal{X}_A/A)^{(p)}, \end{array}$$

$\mathcal{J}^p_{\mathcal{X}}(A)$ has tangent space

$$\begin{array}{ccc} T\mathcal{J}^p_{\mathcal{X}}(A) & \longrightarrow & \mathbf{HN}(\mathcal{X}_A/A)^{(p)} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{HP}(\mathcal{X}_A/A)^{(p)} \end{array},$$

- Thus

$$\begin{aligned} & T\mathcal{J}^p_{\mathcal{X}}(A) \\ = \; & \mathbf{HC}(\mathcal{X}_A/A)^{(p-1)}[1] \\ = \; & \mathbf{R}\Gamma(\mathcal{X}, \mathbf{L}\Omega^\bullet(\mathcal{X}_A/A)/F^p)^{[2p-1]}. \end{aligned}$$

- Hence obstruction space (DDT)

$$\mathrm{H}^{2p}(\mathcal{X}, \mathbf{L}\Omega^\bullet(\mathcal{X}_A/A)/F^p).$$

- Tangent map $T\mathrm{Perf}_{\mathcal{X}} \to T\mathcal{J}^p_{\mathcal{X}}$ is Bressler–Nest–Tsygan's Lefschetz $\mathcal{L}$.
- ... which is Buchweitz–Flenner's σ.
- Sketch proof: $\mathrm{ch}_p(\mathscr{F})$ is

$$\frac{(-1)^p}{p!}\mathrm{tr}(\underbrace{\mathrm{At}(\mathscr{F}) \circ \ldots \circ \mathrm{At}(\mathscr{F})}_{p}),$$

so tangent map $(q = p - 1)$

$$\alpha \mapsto \frac{(-1)^q}{q}\mathrm{tr}(\underbrace{\mathrm{At}(\mathscr{F}) \circ \ldots \circ \mathrm{At}(\mathscr{F})}_{q} \circ \alpha).$$

The theorem revisited

Theorem (P)

The composition

$$\mathrm{Ext}^2_{\mathscr{O}_X}(\mathscr{F},\mathscr{F}) \xrightarrow{\mathcal{L}} \mathbb{H}\mathrm{H}_{-2}(X) \xrightarrow{I} \mathbb{H}\mathrm{C}_{-2}(X)$$

measures obstructions to deforming $\mathrm{ch}(\mathscr{F})$ *in the Hodge locus:* $o_e(\mathrm{ch}(\mathscr{F})) = I\mathcal{L}(o_e(\mathscr{F}))$.

- Includes relative case $\mathcal{X} \to S$.

Equivalently:

- Obstructions of $\mathrm{ch}_{q+1}(\mathscr{F})$ in the Hodge locus are given by $I\sigma_q(o_e(\mathscr{F}))$:

$$\mathrm{Ext}^2_{\mathscr{O}_X}(\mathscr{F},\mathscr{F}) \xrightarrow{\sigma_q} \mathrm{H}^{q+2}(X,\Omega^q_X)$$
$$\xrightarrow{I} \mathrm{H}^{2q+2}(X,\Omega^\bullet_X/F^{q+1})$$

- I is injective when X proper.
- Derived moduli interpretation:

$$\mathrm{Perf}_X \times^h_{\mathcal{J}^p_X} \mathrm{Hodge}.$$