SEMIREGULARITY AS A CONSEQUENCE OF GOODWILLIE'S THEOREM

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Abstract. We realise Buchweitz and Flenner's semiregularity map (and hence *a fortiori* Bloch's semiregularity map) for a smooth variety X as the tangent of generalised Abel–Jacobi map on the derived moduli stack of perfect complexes on X. The target of this map is an analogue of Deligne cohomology defined in terms of cyclic homology, and Goodwillie's theorem on nilpotent ideals ensures that it has the desired tangent space (a truncated de Rham complex).

Immediate consequences are the semiregularity conjectures: that the semiregularity maps annihilate all obstructions, and that if X is deformed, semiregularity measures the failure of the Chern character to remain a Hodge class. This gives rise to reduced obstruction theories of the type featuring in the study of reduced Gromov–Witten and Pandharipande–Thomas invariants.

Introduction

In [Blo], Bloch defined a semiregularity map

$$\tau\colon\ \mathrm{H}^1(Z, \mathscr{N}_{Z/X}) \to \mathrm{H}^{p+1}(X, \Omega_X^{p-1})$$

for every local complete intersection Z of codimension p in a smooth proper complex variety X, and showed that curvilinear obstructions lie in the kernel of τ. He conjectured that τ should annihilate all obstructions, and that if X were also deformed, then τ would measure the obstruction to $[Z]$ remaining a Hodge class.

In [BF1], Buchweitz and Flenner extended τ to give maps

$$\sigma_q\colon\ \mathrm{Ext}^2_X(\mathscr{F}, \mathscr{F}) \to \mathrm{H}^{q+2}(X, \Omega_X^q),$$

for any perfect complex $\mathscr{F}$ on X, and showed that curvilinear obstructions to deforming $\mathscr{F}$ lie in the kernel of σ_q. They conjectured that the whole obstruction space lies in the kernel of σ_q for all q.

The philosophy behind these conjectures is that for a deformation $\tilde{\mathscr{F}}$ of $\mathscr{F}$, we must have $\mathrm{ch}(\mathscr{F}) = \mathrm{ch}(\tilde{\mathscr{F}})$ because the cohomological Chern character takes rational values. The homotopy between cycles representing $\mathrm{ch}_{q+1}(\mathscr{F})$ and $\mathrm{ch}_{q+1}(\tilde{\mathscr{F}})$ should then be given by $\sigma_q(\tilde{\mathscr{F}})$.

Buchweitz and Flenner also observed that their conjecture would follow if there existed a generalised Abel–Jacobi map from the deformation groupoid to an intermediate Jacobian or to Deligne cohomology. We modify this idea, observing that the same reasoning holds with any formally étale cohomology theory in place of rational cohomology. In particular it applies to Hartshorne's algebraic de Rham cohomology. Writing $X_A := X \otimes_{\mathbb{C}} A$, we thus set $J^p_X(A, \mathbb{C})[2p]$ to be the cocone of

$$\mathrm{DR}^{\mathrm{alg}}(X_A/\mathbb{C}) \to \mathbf{R}\Gamma(X_A, \Omega^{<p}_{X_A/A}),$$

noting that in general X_A will not be smooth. This definition also adapts to any smooth scheme X over a base ring R containing a field k of characteristic 0, and any R-algebra A.

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To establish existence of the Abel–Jacobi map and functoriality, we reformulate in terms of cyclic homology. Algebraic de Rham cohomology is isomorphic to derived de Rham cohomology, and hence to periodic cyclic homology, giving

$$\prod_p J^p_X(A,k) \simeq \mathrm{cocone}(\mathrm{HP}^k(X_A) \xrightarrow{S} \mathrm{HC}^A(X_A)[-2])$$

for $X_A = X \otimes_R A$. The generalised Abel–Jacobi map Ξ is then induced from the Goodwillie–Jones Chern character

$$\mathrm{ch}^-\colon\ K(X_A) \to \mathrm{HN}^k(X_A) = \mathrm{cocone}(\mathrm{HP}^k(X_A) \xrightarrow{S} \mathrm{HC}^k(X_A)[-2]).$$

Goodwillie's Theorem implies that $\mathrm{HP}^k(X_A)$ is formally étale as a functor in A, so $J^p_X(A,k)$ has the required obstruction space

$$\mathrm{HC}^A_{-2}(X_A)^{(p-1)} = \mathbb{H}^{2p}(X_A, \mathscr{O}_{X_A} \to \Omega^1_{X_A/A} \to \ldots \to \Omega^{p-1}_{X_A/A}).$$

Now, Ξ_p restricts from K-theory to a map on the nerve $\mathrm{Perf}_X(A)$ of the ∞-category of perfect complexes on X_A, thereby giving us a morphism of derived stacks (where we now allow A to be a simplicial R-algebra). On tangent spaces, the map Ξ_p becomes

$$\xi_p\colon\ \mathrm{Ext}^r_{X_A}(\mathscr{F}, \mathscr{F} \otimes_A M) \to \mathrm{H}^{2p-2+r}(X, (\mathscr{O}_X \to \ldots \to \Omega^{p-1}_X) \otimes_R M),$$

for A-modules M, when X is smooth over R. In Proposition 2.12, we show that this map is just the Lefschetz map of [BNT], and hence (Remark 2.13) equivalent to the semiregularity map of [BF1].

The hypersheaves Perf_X and $J^p_X(-,k)$ satisfy homotopy-homogeneity, a left-exactness property analogous to Schlessinger's conditions, which in particular gives a functorial identification of tangent spaces with obstruction spaces. Given a square-zero extension $A \to B$ of simplicial algebras with kernel I, and a perfect complex $\mathscr{F}$ on $X \otimes_R B$, the obstruction $o(\mathscr{F})$ to lifting $\mathscr{F}$ to $X \otimes_R A$ lies in $\mathrm{Ext}^2_X(\mathscr{F}, \mathscr{F} \otimes_B I)$. Homotopy-homogeneity ensures that the obstruction to lifting $\Xi_p(\mathscr{F})$ from $\mathrm{H}_0 J^p_X(B,k)$ to $\mathrm{H}_0 J^p_X(A,k)$ is just $\xi_p(o(\mathscr{F}))$.

Whenever $A \to B$ admits a splitting in the derived category of R-modules, this obstruction is zero, giving (Corollary 2.15):

$$o(\mathscr{F}) \in \bigcap_{p \geq 1} \ker(\xi_p).$$

In cases where such a splitting does not exist but A is Artinian, we can interpret $\xi_p(o(\mathscr{F}))$ as the obstruction to lifting $\mathrm{ch}_p(\mathscr{F})$ as a horizontal section lying in $F^p\mathrm{H}^{2p}\mathrm{DR}(X \otimes_R A/A)$ (Corollary 2.20 and Remark 2.24). In particular, this produces a reduced obstruction theory for the stable pairs featuring in the study of Pandharipande–Thomas invariants.

By considering perfect complexes of the form $\mathbf{R}f_*\mathscr{O}_Z$, this also gives rise to a reduced obstruction theory for proper morphisms $f\colon Z \to X$, and in particular for stable curves (Remark 2.25). It thus establishes Bloch's semiregularity conjectures in the generality envisaged (Remark 2.26).

Attacking these conjectures has been the purpose of much recent research, most notably by Iacono and Manetti, who in [IM] proved Bloch's first semiregularity conjecture in the case where Z is smooth. Their approach was to construct an explicit infinitesimal Abel–Jacobi map by L_∞ methods, having also identified $\mathbb{H}^{2p}(X, \Omega^{<p}_X)$ as a more natural target for the semiregularity map than $\mathrm{H}^{p+1}(X, \Omega^{p-1}_X)$. Other recent work such as [STV, KT] focuses on the case $p = 1$, where this discrepancy does not arise.

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Contents

Notation and conventions. Throughout this paper, k will denote a field of characteristic zero.

The Dold–Kan correspondence gives an equivalence of categories between simplicial abelian groups and non-negatively graded chain complexes with homotopy groups corresponding to homology groups, and we will pass between these categories without further comment. Under this equivalence, homotopy groups correspond to homology groups.

Given a chain complex V, we will write $V[n]$ for the chain complex given by $V[n]_i = V_{n+i}$. We also write $\tau_{\geq 0}V$ for the good truncation of V in non-negative chain degrees (which we can then regard as a simplicial abelian group by the Dold–Kan correspondence above).

For a morphism $f\colon V \to W$ of chain complexes, $\mathrm{cocone}(f)$ will denote the homotopy kernel of f, which fits into an exact triangle

$$\mathrm{cocone}(f) \to V \to W \to \mathrm{cocone}(f)[-1].$$

Definition 0.1. Given a morphism $A \to R$ of rings, with A commutative, write $\mathrm{HC}^A(R)$ (resp. $\mathrm{HN}^A(R)$, resp. $\mathrm{HP}^A(R)$, resp. $\mathrm{HH}^A(R)$) for the chain complex associated to cyclic (resp. negative cyclic, resp. periodic cyclic, resp. Hochschild) homology of R over A.

Given a morphism $A \to R$ of simplicial rings, with A commutative, and a homology theory E as in the previous paragraph, define the complex $E^A(R)$ by first forming the simplicial chain complex given by $E^{A_n}(R_n)$ in level n, then taking the product total complex.

Remark 0.2. When working with cyclic homology, it is usual to fix a base ring and to omit it from the notation. Since varying the base will be crucial to our constructions, we have introduced the superscript A above. Also beware that the cohomology theories HN and HP are frequently denoted by HC^- and $\mathrm{HC}^{\mathrm{per}}$ in the literature, and that our complexes are related to cyclic homology groups by

$$\mathrm{HC}^A_i(R, M) := \mathrm{H}_i\mathrm{HC}^A(R, M)$$

etc. In the notation of [Wei2, Ch. 9], the complexes $\mathrm{HH}, \mathrm{HC}, \mathrm{HN}, \mathrm{HP}$ are denoted by $\mathrm{CC}^h_*, \mathrm{Tot}\,\mathrm{CC}_{**}, \mathrm{Tot}^{\Pi}\mathrm{CC}^N_{**}, \mathrm{Tot}^{\Pi}\mathrm{CC}^P_{**}$.

When A is a discrete ring, note that the complexes above are those studied in [Goo2].

Definition 0.3. Each of the homology theories E above admits a Hodge decomposition, which we denote by

$$E^A(R) = \prod_{p\in\mathbb{Z}} E^A(R)^{(p)}.$$

Recall from [Wei2, §9.6.1] that there are exact triangles (the SBI sequences)

$$\mathrm{HN}^A(R)^{(p)} \xrightarrow{I} \mathrm{HP}^A(R)^{(p)} \xrightarrow{S} \mathrm{HC}^A(R)^{(p-1)}[-2] \xrightarrow{B} \mathrm{HN}^A(R)^{(p)}[-1]$$
$$\mathrm{HH}^A(R)^{(p)} \xrightarrow{I} \mathrm{HC}^A(R)^{(p)} \xrightarrow{S} \mathrm{HC}^A(R)^{(p-1)}[-2] \xrightarrow{B} \mathrm{HH}^A(R)^{(p)}[-1],$$

compatible with the projection map $\mathrm{HN}^A(R)^{(p)} \to \mathrm{HH}^A(R)^{(p)}$ and $S\colon \mathrm{HP}^A(R)^{(p)} \to \mathrm{HC}^A(R)^{(p)}$.

Definition 0.4. Given a simplicial ring R, we follow [Wal] in writing $K(R)$ for the K-theory space of R (the 0th part of the K-theory spectrum). This is an infinite loop space with $\pi_i K(R) = K_i(R)$.

1. The Abel–Jacobi map for rings

Fix a simplicial commutative k-algebra R. Take a simplicial R-algebra $O(X)$, which need not be commutative, with the property that each $O(X)_n$ is flat as an R_n-module. We will write $F(X) := F(O(X))$ when F is a functor such as $K, \mathrm{HP}, \mathrm{HC}, \mathrm{HN}, \mathrm{HH}$. Write $s\mathrm{CAlg}_R$ for the category of simplicial commutative R-algebras.

1.1. The Abel–Jacobi map.

Definition 1.1. Define a functor $J^p_X(-,k)$ from $s\mathrm{CAlg}_R$ to chain complexes by setting

$$J^p_X(A,k) := \mathrm{cocone}(\mathrm{HP}^k(X_A)^{(p)} \xrightarrow{S} \mathrm{HC}^A(X_A)^{(p-1)}[-2]),$$

where $O(X_A) := O(X)\otimes_R A$. Write $J_X(A,k) := \prod_p J^p_X(A,k)$.

The Goodwillie–Jones Chern character

$$\mathrm{ch}^-\colon\ K(X_A) \to \mathrm{HN}^k(X_A) = \mathrm{cocone}(\mathrm{HP}^k(X_A) \xrightarrow{S} \mathrm{HC}^k(X_A)[-2])$$

of [Goo2] then combines with the natural map $\mathrm{HC}^k(X_A) \to \mathrm{HC}^A(X_A)$ to give a map

$$\Xi\colon\ K(X_A) \to \prod_p J^p_X(A,k),$$

which we call the (generalised) Abel–Jacobi map.

Definition 1.2. Given a simplicial ring S, define $\mathrm{Perf}(S)$ to be the simplicial set given by the nerve of the core of the category of perfect S-modules. This becomes a simplicial semiring with addition given by block sum and multiplication by tensor product.

Definition 1.3. By [Wal, Theorem 2.3.2], there is a natural map $\mathrm{Perf}(S) \to K(S)$. Composing this with the Abel–Jacobi map above gives

$$\Xi_p\colon\ \mathrm{Perf}(X_A) \to \tau_{\geq 0} J^p_X(A,k).$$

1.2. **Homogeneity and obstructions.** Say that a map $A \to B$ in $s\mathrm{CAlg}_R$ is a nilpotent extension if it is levelwise surjective, with the kernel I satisfying $I^n = 0$ for some n.

Definition 1.4. We say that a homotopy-preserving functor F from $s\mathrm{CAlg}_R$ to a model category $\mathcal{C}$ is homotopy-homogeneous if for $A \to B$ a nilpotent extension in $s\mathrm{CAlg}_R$ and $C \to B$ any morphism, the map

$$F(A \times_B C) \to F(A) \times^h_{F(B)} F(C)$$

(to the homotopy fibre product) is a weak equivalence. When $\mathcal{C}$ is the category of cochain complexes, this is equivalent to saying that we have an exact triangle

$$F(B)[-1] \to F(A \times_B C) \to F(A) \oplus F(C) \to F(B).$$

Definition 1.5. Define the functor Perf_X on $s\mathrm{CAlg}_R$ by $\mathrm{Perf}_X(A) := \mathrm{Perf}(O(X) \otimes_R A)$.

Definition 1.6. Given a homotopy-homogeneous functor $F\colon s\mathrm{CAlg}_R \to \mathcal{C}$, an object $x \in F(A)$ and an A-module M in non-positively graded cochain complexes, define the tangent space

$$T_x(F, M)$$

to be the homotopy fibre of $F(A \oplus M) \to F(A)$ over x.

Note that homotopy-homogeneity of F ensures that $T_x(F, M)$ has a homotopy-abelian group structure in $\mathcal{C}$. We thus define tangent cohomology by $\mathrm{D}^{n-i}_x(F, M) := \pi_i T_x(F, M[n])$, which is well-defined by [Pri2, Lemma 1.12].

Lemma 1.7. *Take a homotopy-preserving and homotopy-homogeneous simplicial set-valued functor F on $s\mathrm{CAlg}_R$. For any square-zero extension $e : I \to A \xrightarrow{f} B$ in $s\mathrm{CAlg}_R$, and any $x \in FB$, there is then a functorial obstruction*

$$o_e(x) \in \mathrm{D}^1_x(F, I),$$

which is zero if and only if $[x]$ lies in the image of

$$f_*\colon \pi_0(FA) \to \pi_0(FB).$$

Proof. This is contained in [Pri2, Lemma 1.17], in which the obstruction maps given here are just one term in a long exact sequence of homotopy groups. □

The following is well-known (see for instance [Pri1, Theorem 4.12] when $O(X)$ is commutative):

Lemma 1.8. *The functor Perf_X is homotopy-preserving and homotopy-homogeneous. The tangent space $T_{\mathscr{E}}(\mathrm{Perf}_X, M)$ is $\tau_{\ge 0}(\mathbf{R}\underline{\mathrm{Hom}}_{O(X)}(\mathscr{E}, \mathscr{E} \otimes M)[-1])$, so the tangent cohomology groups are*

$$\mathrm{D}^i_{\mathscr{E}}(\mathrm{Perf}_X, M) \cong \mathbb{E}\mathrm{xt}^{i+1}_{O(X)}(\mathscr{E}, \mathscr{E} \otimes M).$$

1.3. **Goodwillie's theorem.** The following is [Goo2, Lemma I.3.3], a reformulation of Goodwillie's theorem on nilpotent ideals ([Goo1, Theorems II.5.1 and IV.2.6]):

Theorem 1.9. *If $S \to T$ is a map of simplicial k-algebras such that $\pi_0 S \to \pi_0 T$ is a nilpotent extension, then the map*

$$\mathrm{HP}^k(S) \to \mathrm{HP}^k(T)$$

is a quasi-isomorphism of chain complexes.

Proposition 1.10. *The functor $J_X(-, k)$ from $s\mathrm{CAlg}_R$ to chain complexes is homotopy-homogeneous.*

Proof. The chain complex $\mathrm{HC}^R(X)$ is bounded below, so $\mathrm{HC}^A(X_A) \simeq \mathrm{HC}^R(X) \otimes^{\mathbf{L}}_R A$ for all $A \in s\mathrm{CAlg}_R$, which ensures that the functor $A \mapsto \mathrm{HC}^A(X_A)$ is homotopy-homogeneous.

Take a morphism $C \to B$ and a nilpotent extension $A \to B$ in $s\mathrm{CAlg}_R$. Now, since $A \times_B C \to C$ is a nilpotent extension, Proposition 1.9 gives quasi-isomorphisms

$$\mathrm{HP}^k(X_{A\times_B C}) \to \mathrm{HP}^k(X_C), \quad \mathrm{HP}^k(X_A) \to \mathrm{HP}^k(X_B).$$

Thus $\mathrm{HP}^k(X_{A\times_B C})$ is trivially quasi-isomorphic to the cocone of

$$\mathrm{HP}^k(X_C) \oplus \mathrm{HP}^k(X_A) \to \mathrm{HP}^k(X_B),$$

and the result follows by taking homotopy pullbacks. □

The following is a straightforward calculation:

Lemma 1.11. *For $f \in \mathrm{H}_0(J_X(A,k))$, the tangent space $T_f(\tau_{\geq 0}(J^p_X(-,k), M)$ is canonically quasi-isomorphic to $\tau_{\geq 0}(\mathrm{HC}^R(X)^{(p-1)} \otimes^{\mathbf{L}}_R M[-1])$.*

1.4. **The semiregularity map.**

Definition 1.12. Given a simplicial A-algebra S with ideal J, write $\mathrm{HC}^A(S \to S/J) := \mathrm{cocone}(\mathrm{HC}^A(S) \to \mathrm{HC}^A(S/J))$, and define $\mathrm{HH}^A(S \to S/J)$ similarly.

Lemma 1.13. *Take a simplicial commutative ring A, a simplicial A-module M, and a (possibly non-commutative) simplicial A-algebra $O(Y)$. Then for $C = A \oplus M$, the map*

$$\mathrm{HC}^A(O(Y)\otimes^{\mathbf{L}}_A C \to O(Y)) \to \mathrm{cocone}(\mathrm{HC}^C(O(Y)\otimes^{\mathbf{L}}_A C) \to \mathrm{HC}^A(O(Y))) \simeq \mathrm{HC}^A(O(Y))\otimes^{\mathbf{L}}_A M$$

is naturally homotopic to the composition

$$\begin{array}{rcl} \mathrm{HC}^A(O(Y)\otimes^{\mathbf{L}}_A C \to O(Y)) & \xrightarrow{B} & \mathrm{HH}^A(O(Y)\otimes^{\mathbf{L}}_A C \to O(Y))[1] \\ & \simeq & \mathrm{HH}^A(O(Y))\otimes^{\mathbf{L}}_A \mathrm{HH}^A(C \to A)[1] \\ & \xrightarrow{I\otimes\delta} & \mathrm{HC}^A(O(Y))\otimes^{\mathbf{L}}_A M. \end{array}$$

where $\delta\colon \mathrm{HH}^A(C \to A)[1] \to M$ is the map induced by the canonical derivation $\Omega^1_{C/A} \cong M$.

Proof. By taking diagonals of bisimplicial abelian groups and replacing M with an equivalent levelwise flat A-module, we reduce to the case where $A, O(Y), M$ have constant simplicial structure, with M flat over A. Write $\otimes$ for $\otimes^{\mathbf{L}}_A$ and $\mathrm{HC} := \mathrm{HC}^A$, $\mathrm{HH} := \mathrm{HH}^A$.

The Künneth formula for cyclic homology in [Lod, Corollary 4.3.12] gives $\mathrm{HC}(O(Y)\otimes C)$ as the cone of

$$\mathrm{HC}(O(Y)) \otimes \mathrm{HC}(C)[1] \xrightarrow{S\otimes 1 - 1\otimes S} \mathrm{HC}(O(Y)) \otimes \mathrm{HC}(C)[-1].$$

The natural map $\phi\colon \mathrm{HC}(O(Y)\otimes C) \to \mathrm{HC}(O(Y)) \otimes C$ is then given by projection on the left-hand factor $\mathrm{HC}(O(Y)) \otimes \mathrm{HC}(C)$ composed with the map $\mathrm{HC}(C) \to C$.

We may put a grading on C by setting A to have weight 0 and M to have weight 1. This induces gradings on all the complexes above, and since $\mathrm{HC}^A(O(Y)) \otimes_A M$ has weight 1, we may restrict to the weight 1 components, which we denote by $\mathcal{W}_1$. By Goodwillie's Theorem ([Wei2, Theorem 9.9.1]), S is homotopic to 0 on $\mathcal{W}_1\mathrm{HC}^A(C)$, so $\mathcal{W}_1\mathrm{HC}(O(Y)\otimes C)$ is quasi-isomorphic to the cone of

$$\mathrm{HC}(O(Y)) \otimes \mathcal{W}_1\mathrm{HC}(C)[1] \xrightarrow{S\otimes 1} \mathrm{HC}(O(Y)) \otimes \mathcal{W}_1\mathrm{HC}(C)[-1],$$

making $(B\otimes 1)\colon \mathcal{W}_1\mathrm{HC}(O(Y)\otimes C) \to \mathrm{HH}(O(Y))\otimes \mathcal{W}_1\mathrm{HC}(C)$ a quasi-isomorphism. For the map ϕ above, $\mathcal{W}_1\phi$ can thus be interpreted as the composition of $(IB\otimes 1)\colon \mathcal{W}_1\mathrm{HC}(O(Y)\otimes C) \to \mathrm{HC}(O(Y)) \otimes \mathcal{W}_1\mathrm{HC}(C)$ with $\mathrm{HC}(C) \to C$.

Now, [Wei2, Exercise 9.9.1] gives a quasi-isomorphism $\mathcal{W}_1\mathrm{HC}(C) \simeq M$. Since $S \sim 0$, the sequence $\mathcal{W}_1\mathrm{HC}(C)[-1] \xrightarrow{B} \mathcal{W}_1\mathrm{HH}(C) \xrightarrow{I} \mathcal{W}_1\mathrm{HC}(C)$ splits, giving $\mathcal{W}_1\mathrm{HH}(C) \simeq M \oplus M[-1]$. Moreover, the map $\mathcal{W}_1\mathrm{HC}(C) \to \mathcal{W}_1 C = M$ is now just $\delta \circ B$. Therefore

$$\mathcal{W}_1\phi = (I \circ B) \otimes (\delta \circ B) = (I \otimes \delta) \circ B,$$

since $B\colon \mathrm{HC}[-1] \to \mathrm{HH}$ respects products. □

Definition 1.14. Recall from [BNT, §4.3] that for a perfect B-complex $\mathscr{E}$, the *Lefschetz map* $\mathcal{L}_{B/A}\colon \mathbf{R}\underline{\mathrm{Hom}}_B(\mathscr{E},\mathscr{E}) \to \mathrm{HH}^A(B)$ is given by

$$\mathbf{R}\underline{\mathrm{Hom}}_B(\mathscr{E},\mathscr{E}) \simeq (\mathscr{E} \otimes^{\mathbf{L}}_A \mathbf{R}\underline{\mathrm{Hom}}_B(\mathscr{E},B)) \otimes^{\mathbf{L}}_{B\otimes^{\mathbf{L}}_A B^{\mathrm{opp}}} B \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} B \otimes^{\mathbf{L}}_{B\otimes^{\mathbf{L}}_A B^{\mathrm{opp}}} B.$$

Proposition 1.15. *The tangent map* $T_{\mathscr{E}}(\Xi, M)\colon \tau_{\geq 0}(\mathbf{R}\underline{\mathrm{Hom}}_{O(X)}(\mathscr{E},\mathscr{E} \otimes_R M)[-1]) \to \tau_{\geq 0}(\mathrm{HC}^R(X) \otimes^{\mathbf{L}}_R M[-1])$ *of* Ξ *is given by the composition*

$$\mathbf{R}\underline{\mathrm{Hom}}_{O(X)}(\mathscr{E},\mathscr{E} \otimes_R M) \xrightarrow{\mathcal{L}_{O(X)/R}} \mathrm{HH}^R(X) \otimes^{\mathbf{L}}_R M \xrightarrow{I} \mathrm{HC}^R(X) \otimes^{\mathbf{L}}_R M.$$

Proof. Writing $C = A \oplus M$, the tangent map is the composition

$$\begin{aligned} K(R \otimes^{\mathbf{L}}_A C)_{\mathscr{E}} \xrightarrow{\mathrm{ch}^- - \mathrm{ch}^-(\mathscr{E})} \mathrm{HN}^A(R \otimes^{\mathbf{L}}_A C \to R) &\xrightarrow{B^{-1}} \mathrm{HC}^A(R \otimes^{\mathbf{L}}_A C \to R)[-1] \\ &\to \mathrm{HC}^A(R) \otimes^{\mathbf{L}}_A M[-1], \end{aligned}$$

which by Lemma 1.13 is the same as the composition

$$K(R \otimes^{\mathbf{L}}_A C)_{\mathscr{E}} \xrightarrow{\mathrm{ch} - \mathrm{ch}(\mathscr{E})} \mathrm{HH}^A(R) \otimes^{\mathbf{L}}_A \mathrm{HH}^A(C \to A) \xrightarrow{I\otimes\delta} \mathrm{HC}^A(R) \otimes^{\mathbf{L}}_A M[-1],$$

since the composition $K \xrightarrow{\mathrm{ch}^-} \mathrm{HN} \xrightarrow{B} \mathrm{HH}$ is just the Dennis trace ch.

Now, the Dennis trace is given by sending a perfect complex $\mathscr{F}$ to the image $\mathcal{L}(\mathrm{id}_{\mathscr{F}})$ of the identity under the Lefschetz map. We wish to describe the projection of

$$\tau_{\geq 0}\mathbf{R}\underline{\mathrm{Hom}}_{O(X)}(\mathscr{E},\mathscr{E} \otimes_R M)[-1] \simeq T_{\mathscr{E}}(\mathrm{Perf}, M) \xrightarrow{\mathrm{ch} - \mathrm{ch}(\mathscr{E})} \mathrm{HH}^A(R) \otimes^{\mathbf{L}}_A \mathrm{HH}^A(C \to A)$$

to $\mathrm{HH}^A(R) \otimes^{\mathbf{L}}_A \mathcal{W}_1\mathrm{HH}^A(C \to A) \simeq \mathrm{HH}^A(R) \otimes^{\mathbf{L}}_A (M \oplus M[-1])$, for the grading $\mathcal{W}$ from the proof of Lemma 1.13. Calculating the Lefschetz map for extensions, and replacing M with $M[n]$ to analyse higher degrees, we see that this map is given by $(0, \mathcal{L}_{O(X)/R})$. Thus $\delta \circ (\mathrm{ch} - \mathrm{ch}(\mathscr{E})) = \mathcal{L}_{O(X)/R}$, which completes the proof. □

2. The Abel–Jacobi map for schemes and stacks

From now on, all rings will be commutative.

2.1. Derived de Rham cohomology.

Definition 2.1. Given $A \in s\mathrm{CAlg}_{\mathbb{Q}}$ and $B \in s\mathrm{CAlg}_A$, define the de Rham complex to be the chain complex

$$\mathrm{DR}(B/A) := \prod_n \Omega^n(B/A)[n] = \prod_n (\Lambda^n_B \Omega^1(B/A))[n],$$

with differential given by combining the differentials on the chain complexes $\Omega^n(B/A)$ with the derivation induced by $\partial\colon B \to \Omega^1(B/A)$. This has a Hodge filtration given by $F^p\mathrm{DR}(B/A) := \prod_{n\geq p} \Omega^n(B/A)[n]$.

Beware that the de Rham complex is usually regarded as a cochain complex, so negative homology groups will correspond to positive cohomology groups.

Definition 2.2. Given $A \in s\mathrm{CAlg}_{\mathbb{Q}}$ and $B \in s\mathrm{CAlg}_A$, define the left-derived de Rham complex $\mathbf{L}\mathrm{DR}(B/A)$ by first taking a cofibrant replacement $\tilde{B} \to B$ over A, then setting

$$\mathbf{L}\Omega^p(B/A) := \Omega^p(\tilde{B}/A), \quad \mathbf{L}\mathrm{DR}(B/A) := \mathrm{DR}(\tilde{B}/A).$$

Note that this is well-defined up to quasi-isomorphism, that such replacements can be chosen functorially, and that $\mathbf{L}\Omega^1(B/A)$ is a model for the cotangent complex $\mathbb{L}(B/A)$.

The complex $\mathbf{L}\mathrm{DR}(B/A)$ has a Hodge filtration $F^p\mathbf{L}\mathrm{DR}(B/A) = F^p\mathrm{DR}(\tilde{B}/A)$, and we write $\mathbf{L}\mathrm{DR}(B/A)/F^p := \mathbf{L}\mathrm{DR}(B/A)/F^p\mathbf{L}\mathrm{DR}(B/A)$.

Remark 2.3. Following the ideas of [Gro] as developed in [GR], there is a more conceptual interpretation of the derived de Rham complex. For $A \to B$ as above, [TV] uses derived Hom in the model category $s\mathrm{CAlg}_A$ to give a $s\mathrm{Set}$-valued hypersheaf $\mathbf{R}\mathrm{Spec}\, B = \underline{\mathrm{Hom}}_A(\tilde{B}, -)$ on $s\mathrm{CAlg}_A$. For any such functor F, we may define $F_{\mathrm{inf}}(C) := F((\mathrm{H}^0 C)^{\mathrm{red}})$ and $F_{\mathrm{strat}}(C) = \mathrm{Im}\,(\pi_0 F(C) \to F_{\mathrm{inf}}(C))$. Note that these set-valued functors are equal if and only if B is smooth over A in the sense of [TV].

Now, $F_{\mathrm{strat}}(C)$ is equivalent to the Čech nerve of $F(C)$ over $F_{\mathrm{inf}}(C)$, which is represented in level n by formal completions of the diagonal map $\tilde{B}^{\otimes_A (n+1)} \to \tilde{B}$. Cohomology of symmetric powers then shows that $\mathbf{L}\mathrm{DR}(B/A) \simeq \mathbf{R}\Gamma((\mathbf{R}\mathrm{Spec}\, B)_{\mathrm{strat}}, \mathscr{O})$, where $\mathscr{O}$ is the hypersheaf given by $\mathscr{O}(C) = C$.

Proposition 2.4. *For a morphism $A \to B$ in $s\mathrm{CAlg}_{\mathbb{Q}}$ and all $p \in \mathbb{Z}$, there are canonical quasi-isomorphisms*

$$\mathrm{HP}^A(B)^{(p)} \simeq \mathbf{L}\mathrm{DR}(B/A)[-2p], \qquad \mathrm{HC}^A(B)^{(p)} \simeq (\mathbf{L}\mathrm{DR}(B/A)/F^{p+1})[-2p],$$
$$\mathrm{HN}^A(B)^{(p)} \simeq F^p\mathbf{L}\mathrm{DR}(B/A)[-2p], \qquad \mathrm{HH}^A(B)^{(p)} \simeq \mathbf{L}\Omega^p(B/A)[-p],$$

with the SBI sequence corresponding to the short exact sequences $0 \to F^p\mathbf{L}\mathrm{DR} \to \mathbf{L}\mathrm{DR} \to \mathbf{L}\mathrm{DR}/F^p \to 0$ *and* $0 \to \mathbf{L}\Omega^p[-p] \to \mathbf{L}\mathrm{DR}/F^{p+1} \to \mathbf{L}\mathrm{DR}/F^p \to 0$.

Proof. When A and B are concentrated in degree 0, with B smooth over A, this is a well-known consequence of the Hochschild–Kostant–Rosenberg theorem. As observed in [Maj, §5], the general case follows by taking a cofibrant replacement for B and passing to the diagonal of the resulting bisimplicial diagram. □

Remark 2.5. Note that [Emm, Theorem 2.2] (following [FT, Theorem 5]) shows that for finitely generated k-algebras B the complex $\mathrm{HP}^A(B)^{(p)}$ is quasi-isomorphic to the infinitesimal cohomology complex, or equivalently to Hartshorne's algebraic de Rham cohomology ([Har]) over A.

2.2. **Perfect complexes and derived de Rham complexes.** As in [Wei1], we now use naturality of the affine constructions to pass from local to global.

By [TV, §2.2] or the proof of [Pri1, Theorem 4.12], it follows that the functor Perf satisfies smooth hyperdescent on $s\mathrm{CAlg}_{\mathbb{Q}}$. Thus it extends to a hypersheaf from the smooth site of derived geometric stacks to the model category $s\mathrm{Set}$ of simplicial sets.

Definition 2.6. Denote the hypersheaf above by Perf, so given a strongly quasi-compact derived geometric Artin n-stack $\mathfrak{X}$, we have a simplicial set $\mathrm{Perf}(\mathfrak{X})$.

Remarks 2.7. Using the explicit hyperdescent formulae of [Pri3, Examples 1.15], the simplicial semiring $\mathrm{Perf}(\mathfrak{X})$ can be constructed as follows. First, [Pri3] provides the existence of a suitable resolution of $\mathfrak{X}$ by a derived Artin hypergroupoid $X_\bullet$, which is a simplicial derived affine scheme satisfying certain properties. We then define a cosimplicial simplicial semiring given by $\mathrm{C}^n(X_\bullet, \mathrm{Perf}(\mathscr{O}_{\mathfrak{X}})) := \mathrm{Perf}(X_n)$, and set

$$\mathrm{Perf}(\mathfrak{X}) = \mathbf{R}\mathrm{Tot}_{\,s\mathrm{Set}}\mathrm{C}^\bullet(X_\bullet, \mathrm{Perf}(\mathscr{O}_{\mathfrak{X}})),$$

where $\mathbf{R}\mathrm{Tot}_{\,s\mathrm{Set}}$ is the derived total functor from cosimplicial simplicial sets to simplicial sets, as in [GJ, §VIII.1].

Because Perf forms an étale hypersheaf, the definition of $\mathrm{Perf}(\mathfrak{X})$ above agrees with the standard definition for underived schemes, and indeed for stacks. In the case when $\mathfrak{X}$ is a quasi-compact semi-separated scheme, $X_\bullet$ can just be constructed by taking the Čech nerve of an affine cover, in which case $\mathrm{C}^\bullet$ is just a Čech complex.

In all of our applications in §2.5, $\mathfrak{X}$ will be of the form $Y \otimes_R A$, for Y a smooth quasi-compact semi-separated scheme over a Noetherian ring R, and $A \in s\mathrm{CAlg}_R$. Thus we can regard $\mathfrak{X}$ as being the derived scheme associated to a dg scheme (or even a dg manifold) in the sense of [CFK]. However, we need the greater flexibility provided by the theory of [TV, Pri3] in order to obtain a satisfactory construction of Perf invariant under quasi-isomorphism.

The functors $\mathbf{L}\Omega^p(-/A)$ satisfy $\mathbf{L}\Omega^p(B \otimes^{\mathbf{L}}_A A'/A) \simeq \mathbf{L}\Omega^p(B/A) \otimes_A A'$ for étale morphisms (in the sense of [TV, Theorem 2.2.2.6]) $A \to A'$ of simplicial $\mathbb{Q}$-algebras, so satisfy étale hyperdescent. The functors $F^p\mathbf{L}\mathrm{DR}(-/A)$ thus satisfy étale hyperdescent over A, so determine hypersheaves from the étale site of derived geometric Deligne–Mumford stacks over A to the model category of A-modules in unbounded complexes.

Definition 2.8. Denote the hypersheaves above by $F^p\mathbf{L}\mathrm{DR}(-/A)$, giving complexes $F^p\mathbf{L}\mathrm{DR}(\mathfrak{X}/A) := \mathbf{R}\Gamma(\mathfrak{X}, \mathbf{L}\mathrm{DR}(\mathscr{O}_{\mathfrak{X}}/A))$ for any strongly quasi-compact derived geometric Deligne–Mumford n-stack $\mathfrak{X}$ over $A \in s\mathrm{CAlg}_{\mathbb{Q}}$.

Explicitly, for a resolution $X_\bullet$ as in Remarks 2.7, this is computed by the product total complex

$$F^p\mathbf{L}\mathrm{DR}(\mathfrak{X}/A) = \mathrm{Tot}^{\Pi}\mathrm{C}^\bullet(X_\bullet, F^p\mathbf{L}\mathrm{DR}(\mathscr{O}_{\mathfrak{X}}/A));$$

by hyperdescent, this agrees with the existing definition of $\mathbf{L}\mathrm{DR}(\mathfrak{X}/A)$ whenever $\mathfrak{X}$ is affine.

2.3. **Generalised Abel–Jacobi maps.** Fix a simplicial commutative k-algebra R, and a strongly quasi-compact derived geometric Deligne–Mumford n-stack $\mathfrak{X}$ over R. Given $A \in s\mathrm{CAlg}_R$, from now on we will write $\mathfrak{X}_A := \mathfrak{X} \otimes^{\mathbf{L}}_R A$.

Definition 2.9. For $A \in s\mathrm{CAlg}_R$, define the étale presheaf $\mathscr{J}^p_{\mathfrak{X}}(A,k)$ to be

$$J^p_{\mathscr{O}_{\mathfrak{X}}}(A,k) = \mathrm{cocone}(\mathbf{L}\mathrm{DR}(\mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_R A/k)[-2p] \to (\mathbf{L}\mathrm{DR}(\mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_R A/A)/F^p)[-2p]).$$

Since the $F^m\mathbf{L}\mathrm{DR}$ are étale hypersheaves, so is $\mathscr{J}^p_{\mathfrak{X}}(A,k)$, and we may set

$$J^p_{\mathfrak{X}}(A,k) := \mathbf{R}\Gamma(\mathfrak{X}, \mathscr{J}^p_{\mathfrak{X}}(A,k)).$$

Definition 2.10. Define the Abel–Jacobi map

$$\Xi\colon \mathrm{Perf}(\mathfrak{X}_A) \to \tau_{\geq 0}\prod_{p\geq 0} J^p_{\mathfrak{X}}(A,k)$$

by applying $\mathbf{R}\Gamma(\mathfrak{X}, -)$ to the Abel–Jacobi maps

$$\Xi_p\colon \mathrm{Perf}(\mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_R A) \to \tau_{\geq 0}\mathscr{J}^p_{\mathfrak{X}}(A,k)$$

of Definition 1.3, then composing with the natural map

$$\mathbf{R}\Gamma(\mathfrak{X}, \tau_{\geq 0}\mathscr{J}^p_{\mathfrak{X}}(A,k)) \to \tau_{\geq 0}\mathbf{R}\Gamma(\mathfrak{X}, \mathscr{J}^p_{\mathfrak{X}}(A,k)).$$

Definition 2.11. For $\mathscr{E} \in \mathrm{Perf}_{\mathfrak{X}}(A)$ and a simplicial A-module M, write

$$\xi^i\colon \mathbb{E}\mathrm{xt}^{i+1}_{\mathscr{O}_{\mathfrak{X}_A}}(\mathscr{E}, \mathscr{E} \otimes^{\mathbf{L}}_A M) \to \prod_{p\geq 0} \mathrm{H}^{2p+i-1}((\mathbf{L}\mathrm{DR}(\mathfrak{X}/R)/F^p) \otimes^{\mathbf{L}}_R M)$$

for the tangent map

$$\mathrm{D}^i_{\mathscr{E}}(\Xi, M)\colon \mathrm{D}^i_{\mathscr{E}}(\mathrm{Perf}_{\mathfrak{X}}, M) \to \prod_{p\geq 0} \mathrm{D}^i_{\Xi_p(\mathscr{E})}(\tau_{\geq 0}J^p_{\mathfrak{X}}(-,k), M).$$

Hypersheafifying Proposition 1.15 yields:

Proposition 2.12. *The map* ξ^i_p *is given by composing the Lefschetz map* $\mathcal{L}_{\mathfrak{X}/R}\colon\ \mathbb{E}\mathrm{xt}^{i+1}_{\mathscr{O}_{\mathfrak{X}}\otimes^{\mathbf{L}}_A}(\mathscr{E},\mathscr{E}\otimes^{\mathbf{L}}_A M) \to \mathbb{H}^{p+i}(\mathfrak{X},\mathbf{L}\Omega^{p-1}_{\mathfrak{X}/R}\otimes^{\mathbf{L}}_R M)$ *with the canonical map* $I\colon\ \mathbb{H}^{p+i}(\mathfrak{X},\mathbf{L}\Omega^{p-1}_{\mathfrak{X}/R}\otimes^{\mathbf{L}}_R M)\to\mathbb{H}^{2p+i-1}(\mathfrak{X},(\mathbf{L}\mathrm{DR}(\mathscr{O}_{\mathfrak{X}}/R)/F^p)\otimes^{\mathbf{L}}_R M)$.

Remark 2.13. Note that the construction of the Atiyah–Hochschild character $\mathrm{AH}(\mathscr{E})$ of [BF2, §5] just makes it the dual of the Lefschetz map, in the sense that $\mathcal{L}(\alpha)=\mathrm{tr}(\mathrm{AH}(\mathscr{E})\circ\alpha)$. Thus [BF2, Theorem 5.1.3 and Proposition 6.2.1] ensure that $\mathcal{L}$ is the same as the semiregularity map σ of [BF1], given by applying the exponential of the Atiyah class then taking the trace.

Lemma 2.14. *Take a square-zero extension* $e: I\to A\to B$ *in* $s\mathrm{CAlg}_R$*, admitting a section in the derived category of* R*-modules (when* $R=k$*, this just says that* $\pi_*A\twoheadrightarrow\pi_*B$*). Then the obstruction map*

$$o_e\colon\ \mathrm{H}_0(J^p_{\mathfrak{X}}(B,k))\to\mathrm{H}_0((\mathbf{L}\mathrm{DR}(\mathfrak{X}/R)/F^p)\otimes_R I)$$

for the functor $\tau_{\geq 0}J^p_{\mathfrak{X}}(-,k)$ *is identically* 0.

Proof. The R-module splitting of $A\to B$ splits the long exact sequence of homology, in particular ensuring that

$$\mathrm{H}_0(J^p_{\mathfrak{X}}(A,k))\to\mathrm{H}_0(J^p_{\mathfrak{X}}(B,k))$$

is surjective, and hence that $o_e=0$. □

Since Lemma 1.7 is functorial, we may apply it to the Abel–Jacobi map, giving:

Corollary 2.15. *Under the hypotheses of Lemma 2.14, for any* $\mathscr{E}\in\mathrm{Perf}_{\mathfrak{X}}(B)$ *the obstruction*

$$o_e(\mathscr{E})\in\mathrm{D}^1_{\mathscr{E}}(\mathrm{Perf}_{\mathfrak{X}},I)=\mathbb{E}\mathrm{xt}^2_{\mathfrak{X}_B}(\mathscr{E},\mathscr{E}\otimes_R I)$$

lies in the kernel of

$$\xi^1_{\mathscr{E},I}\colon\ \mathbb{E}\mathrm{xt}^2_{\mathfrak{X}}(\mathscr{E},\mathscr{E}\otimes I)\to\prod_{p\geq 1}\mathrm{H}^{2p}((\mathbf{L}\mathrm{DR}(\mathfrak{X}/R)/F^p)\otimes_R I).$$

2.4. **Horizontal sections.** Retain R and $\mathfrak{X}$ as in the previous section. Take $A\in s\mathrm{CAlg}_R$ local Artinian with residue field k; write $\mathfrak{X}_0:=\mathfrak{X}\otimes^{\mathbf{L}}_A k$.

Proposition 2.16. *In the derived category of simplicial* A*-modules, there is a canonical (unfiltered) quasi-isomorphism*

$$\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)\simeq\mathbf{L}\mathrm{DR}(\mathfrak{X}_0/k)\otimes_k A.$$

Proof. The diagonal map $A\otimes_k A\to A$ is a nilpotent extension, so pulling back $\mathfrak{X}_A$ along it gives us a nilpotent embedding $\iota\colon\ \mathfrak{X}_A\hookrightarrow\mathfrak{X}_A\otimes_k A$. Proposition 1.9 then implies that $\iota^*\colon\ \mathbf{L}\mathrm{DR}(\mathfrak{X}_A\otimes_k A/A)\to\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)$ is a quasi-isomorphism, where the map $\mathfrak{X}_A\otimes_k A\to\mathrm{Spec}\,A$ is taken to be projection on the second factor.

Since A is finite-dimensional over k, base change then gives an isomorphism $\mathbf{L}\mathrm{DR}(\mathfrak{X}_A\otimes_k A/A)\cong\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/k)\otimes_k A$. Applying Proposition 1.9 to the nilpotent embedding $\mathfrak{X}_0\hookrightarrow\mathfrak{X}_A$ gives a quasi-isomorphism $\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/k)\to\mathbf{L}\mathrm{DR}(\mathfrak{X}_0/k)$, which completes the proof. □

Definition 2.17. Define the *horizontal sections* $\mathrm{H}_*\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)^{\mathrm{hor}}\subset\mathrm{H}_*\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)$ to be the image of $\mathrm{H}_*\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/k)\hookrightarrow\mathrm{H}_*\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)$, so $\mathrm{H}_*\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)\cong\mathrm{H}_*\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)^{\mathrm{hor}}\otimes_k\pi_*(A)$.

Remark 2.18. When $k=\mathbb{C}$ and $\mathfrak{X}$ is a smooth proper scheme over R, the horizontal sections above agree with those constructed in [Blo, Proposition 3.8]. To see this, note that for any finitely generated simplicial $\mathbb{C}$-algebra B, we can form an analytic simplicial algebra B^{an} by completing B over $\pi_0 B$ then taking the associated analytic algebra levelwise. This gives us an analytic sheaf $\mathbf{L}\mathrm{DR}(\mathscr{O}^{\mathrm{an}}_{\mathfrak{X}_A}/A)$ on the analytic site of $h^0\mathfrak{X}_A=\mathbf{Spec}_{\mathfrak{X}_A}\mathscr{H}_0\mathscr{O}_{\mathfrak{X}_A}$, and Proposition 1.9 shows that $A \to \mathbf{L}\mathrm{DR}(\mathscr{O}_{\mathfrak{X}_A,\mathrm{an}}/A)$ is a quasi-isomorphism of analytic sheaves. GAGA then reduces the proof of Proposition 2.16 to the canonical equivalence $\mathbf{R}\Gamma((h^0\mathfrak{X}_A)_{\mathrm{an}}, A) \simeq \mathbf{R}\Gamma((h^0\mathfrak{X}_A)_{\mathrm{an}}, \mathbb{C}) \otimes_{\mathbb{C}} A$.

We now describe the obstruction map in cases not covered by Lemma 2.14, in particular when R is not a field.

Proposition 2.19. *Take a square-zero extension $e : I \to A \to B$ (for A as above). Then the composition*

$$\mathrm{H}_n J^p_{\mathfrak{X}}(B,k) \xrightarrow{o_e} \mathrm{H}_{n-2p}((\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)/F^p) \otimes_A I) \to \mathrm{H}_{n-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)/F^p)$$

is equal to the composition

$$\begin{aligned}\mathrm{H}_n J^p_{\mathfrak{X}}(B,k) \to &\mathrm{H}_{n-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_B/B))^{\mathrm{hor}} \\ \cong &\mathrm{H}_{n-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A))^{\mathrm{hor}} \to \mathrm{H}_{n-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)/F^p),\end{aligned}$$

for the horizontal sections of Definition 2.17.

Proof. By Proposition 2.16, we have a canonical isomorphism $\mathrm{H}_*\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A) \simeq \mathrm{H}_*(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A))^{\mathrm{hor}} \otimes_k \pi_*A$, with $\mathrm{H}_*(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A))^{\mathrm{hor}}$ the isomorphic image of $\mathrm{H}_*\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/k)$, and similarly for $\mathfrak{X}_B$ over B. The second composition above then becomes

$$\mathrm{H}_n J^p_{\mathfrak{X}}(B,k) \to \mathrm{H}_{n-2p}\mathbf{L}\mathrm{DR}(\mathfrak{X}_B/k) \cong \mathrm{H}_{n-2p}\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/k) \to \mathrm{H}_{2n-p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)/F^p)$$

which is precisely the first composition. □

The following now follows immediately from functoriality of obstructions in §1.2.

Corollary 2.20. *Under the hypotheses of Proposition 2.19, for any $\mathscr{E} \in \mathrm{Perf}(\mathfrak{X}_B)$ the image of the obstruction $o_e(\mathscr{E})$ under the maps*

$$\mathbb{E}\mathrm{xt}^2_{\mathfrak{X}_B}(\mathscr{E}, \mathscr{E} \otimes_B I) \xrightarrow{\xi^1_{\mathscr{E},I,p}} \mathrm{H}_{-2p}((\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)/F^p) \otimes_A I) \to \mathrm{H}_{-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)/F^p)$$

is the obstruction to lifting $\mathrm{ch}_p(\mathscr{E})$ from $\mathrm{H}_{-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_B/B))^{\mathrm{hor}} \cap F^p\mathrm{H}_{-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_B/B))$ to $\mathrm{H}_{-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A))^{\mathrm{hor}} \cap F^p\mathrm{H}_{-2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A))$.

2.5. **Reduced obstructions.**

Corollary 2.21. *Take a square-zero extension $e : I \to A \xrightarrow{f} B$ in $s\mathrm{CAlg}_R$, admitting a section in the derived category of R-modules (when $R = k$, this says $\pi_*A \twoheadrightarrow \pi_*B$). Then for any $\mathscr{E} \in \mathrm{Perf}_{\mathfrak{X}}(B)$ the obstruction*

$$o_e(\mathscr{E}) \in \mathrm{D}^1_{\mathscr{E}}(\mathrm{Perf}_{\mathfrak{X}}, I) = \mathbb{E}\mathrm{xt}^2_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{E}, \mathscr{E} \otimes I)$$

lies in the kernel of the composition

$$\begin{aligned}\mathbb{E}\mathrm{xt}^2_{\mathfrak{X}}(\mathscr{E}, \mathscr{E} \otimes_B I) &\xrightarrow{\mathcal{L}} \prod_{p \geq 1} \mathbb{H}^{p+1}(\mathfrak{X}, \mathbf{L}\Omega^{p-1}_{\mathfrak{X}/R} \otimes_R I) \\ &\to \prod_{p \geq 1} \mathbb{H}^{2p}(\mathfrak{X}, (\mathbf{L}\mathrm{DR}(\mathscr{O}_{\mathfrak{X}}/R)/F^p) \otimes_R I),\end{aligned}$$

where $\mathcal{L}$ is the Lefschetz map of Definition 1.14.

Proof. Combine Corollary 2.15 with Proposition 2.12, the composite map being $\xi^1_{\mathscr{E},I}$. □

Remark 2.22. Whenever the Hodge–de Rham spectral sequence for $\mathfrak{X}$ over R degenerates, the map

$$\mathbb{H}^{p+1}(\mathfrak{X}, \mathbf{L}\Omega^{p-1}_{\mathfrak{X}/R} \otimes_R I) \to \mathbb{H}^{2p}(\mathfrak{X}, (\mathbf{L}\mathrm{DR}(\mathscr{O}_{\mathfrak{X}}/R)/F^p) \otimes_R I)$$

is injective, so obstructions then lie in the kernel of $\mathcal{L}$. This applies whenever $\mathfrak{X}$ is a smooth proper scheme over a Noetherian $\mathbb{Q}$-algebra R ([Del]), or a smooth proper Deligne–Mumford stack over a field.

Corollary 2.23. *Take an Artinian local simplicial R-algebra A with residue field k, and a square-zero extension $e : I \to A \to B$. Then for any $\mathscr{E} \in \mathrm{Perf}_{\mathfrak{X}}(B)$ the image of the obstruction $o_e(\mathscr{E})$ under the maps*

$$\begin{aligned}\mathbb{E}\mathrm{xt}^2_{\mathfrak{X}}(\mathscr{E}, \mathscr{E} \otimes_B I) &\xrightarrow{\mathcal{L}_{p-1}} \mathbb{H}^{p+1}(\mathfrak{X}, \mathbf{L}\Omega^{p-1}_{\mathfrak{X}/R} \otimes_R I) \\ &\to \mathrm{H}^{2p}((\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)/F^p) \otimes_A I) \\ &\to \mathrm{H}^{2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A)/F^p)\end{aligned}$$

is the obstruction to lifting $\mathrm{ch}_p(\mathscr{E})$ from $\mathrm{H}^{2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_B/B))^{\mathrm{hor}} \cap F^p\mathrm{H}^{2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_B/B))$ to $\mathrm{H}^{2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A))^{\mathrm{hor}} \cap F^p\mathrm{H}^{2p}(\mathbf{L}\mathrm{DR}(\mathfrak{X}_A/A))$.

Proof. This just combines Corollary 2.20 with Proposition 2.12. □

Remark 2.24. Assume that R is a Noetherian $\mathbb{Q}$-algebra, with X a smooth proper scheme over $\mathrm{Spec}\, R$. Then [Del] gives an isomorphism

$$\mathbb{H}^*(X, (\mathrm{DR}(\mathscr{O}_X/R)/F^p) \otimes_R M) \cong \mathbb{H}^*(X, (\mathrm{DR}(\mathscr{O}_X/R)/F^p)) \otimes_R \pi_*(M)$$

for all simplicial R-modules M, with $\mathbb{H}^*(X, (\mathrm{DR}(\mathscr{O}_X/R)/F^p))$ a projective R-module.

Thus whenever $\pi_* A \twoheadrightarrow \pi_* B$, the map

$$\mathrm{H}^{p+1}(X, \Omega^{p-1}_{X/R} \otimes_R I) \to \mathbb{H}^{2p}(X, (\mathrm{DR}(\mathscr{O}_X/R)/F^p) \otimes_R A)$$

is injective, so Corollary 2.23 shows that $\mathcal{L}_{p-1}(o_e(\mathscr{E})) \in \mathrm{H}^{p+1}(X, \Omega^{p-1}_{X/R} \otimes_R I)$ will vanish provided $\mathrm{ch}_p(\mathscr{E})$ stays in F^p when lifted as a horizontal section.

Taking an open substack $\mathfrak{M} \subset \mathrm{Perf}_X$ for which this holds (for instance by restricting to the Hodge locus of [Voi]), we thus obtain an obstruction theory for $\mathfrak{M}$ by

$$\mathscr{E} \mapsto \ker(\mathcal{L}_{p-1} \colon \mathbb{E}\mathrm{xt}^2_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathrm{H}^{p+1}(X, \Omega^{p-1}_{X/R})).$$

This gives a reduced obstruction theory for stable pairs, as required in the study of Pandharipande–Thomas invariants.

Remark 2.25. There is a morphism from the derived moduli stack of proper schemes over X ([Pri1, Theorem 3.32]) to the derived stack Perf_X, given by sending $f \colon Z \to X_B$ to $\mathbf{R}f_*\mathscr{O}_Z$. Since the obstruction maps of Lemma 1.7 are functorial, this gives rise to morphisms

$$\psi \colon \mathrm{Ext}^2_Z(\mathbb{L}^{Z/X_B}, \mathscr{O}_Z) \to \mathrm{Ext}^2_{X_B}(\mathbf{R}f_*\mathscr{O}_Z, \mathbf{R}f_*\mathscr{O}_Z)$$

of obstruction theories, so Corollaries 2.21 and 2.23 give conditions for $\xi^1_{\mathscr{E},I} \circ \psi$ to annihilate obstructions to deforming Z over X_B.

For X smooth and proper over R, Remark 2.24 then implies that $\mathcal{L}_{p-1} \circ \psi$ annihilates such obstructions provided $\mathrm{ch}_p(\mathbf{R}f_*Z)$ stays of Hodge type on deforming Z. For suitable moduli of proper schemes Z over X, this gives rise to a reduced global obstruction theory

$$Z \mapsto \ker(\mathcal{L}_{p-1} \circ \psi \colon \mathrm{Ext}^2_Z(\mathbb{L}^{Z/X}, \mathscr{O}_Z) \to \mathrm{H}^{p+1}(X, \Omega^{p-1}_{X/R}));$$

in particular this applies to stable curves Z over X, as required in the study of Gromov–Witten invariants (see for instance [KT, §2.2]).

Remark 2.26. The proof of Proposition 1.15 characterises $\mathcal{L}$ as a deformation of the Dennis trace. This means that for any proper LCI morphism $f \colon Z \to X_B$, the Riemann–Roch theorem allows us to interpret the semiregularity map $\mathcal{L} \circ \psi$ of Remark 2.25 as the deformation of $f_*(\mathrm{Td}(T_f))$ as f varies.

When $Z \subset X$ is a codimension p LCI subscheme of a smooth proper scheme over k, [BF1, Proposition 8.2] combines with Remark 2.13 to show that

$$\mathcal{L}_{p-1} \circ \psi \colon \mathrm{H}^1(Z, \mathscr{N}_{Z/X}) \to \mathrm{H}^{p+1}(X, \Omega^{p-1}_{X/k})$$

is just Bloch's semiregularity map from [Blo], which admits a relatively simple description in terms of Verdier duality.

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