

Deformations via Simplicial Deformation Complexes

J.P.Pridham*

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Introduction

There has long been a philosophy that every deformation problem in characteristic 0 should give rise to a differential graded Lie algebra (DGLA). This DGLA should not only determine the deformation functor (and indeed the deformation groupoid up to equivalence), but also somehow encapsulate more information (corresponding to the higher cohomology) about the original problem. We have:

Deformation Problem \rightsquigarrow DGLA \rightsquigarrow Deformation Groupoid \rightsquigarrow Deformation Functor.

However, there are several respects in which DGLAs are not wholly satisfactory. Finding a DGLA to govern a given problem is not easy — Kontsevich describes it as an “art” ([Kon94]), so the arrow

Deformation Problem \rightsquigarrow DGLA

above is merely aspirational. Kontsevich also remarks that, where DGLAs are constructed for geometric problems, the methods used are predominantly analytic. This proves inadequate for the more algebraic problems, such as deformations of an algebraic variety. However, in [KS00], Kontsevich and Soibelman go some way towards constructing DGLAs for algebraic problems, by providing DGLAs to govern deformations over an operad. In non-zero characteristic, DGLAs do not, in general, even give rise to deformation functors.

The reason for these problems is that differential graded objects are usually natural things to consider only in characteristic zero. This statement is implied by omission in papers such as [Qui69], showing that DG objects suffice when considering homotopy theory over the rationals. In non-zero characteristic, it is necessary to use simplicial objects in their stead. This equivalence of DG and simplicial objects, together with the relative facility of DG methods, has resulted in a comparative neglect of simplicial methods. As I will demonstrate, there are many scenarios in which simplicial structures arise more naturally.

In this paper, I define a simplicial object to replace DGLAs, namely a Simplicial Deformation Complex (SDC). This gives rise to the picture:

Deformation Problem \rightsquigarrow SDC \rightsquigarrow Deformation Groupoid \rightsquigarrow Deformation Functor.

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Here we have the advantage not only that this works in non-zero characteristic, but also that the first arrow is reasonably canonical.

Whereas Kontsevich and Soibelman used operads for their notion of algebraicity, I use a weaker notion — that of monadic adjunctions. Given the existence of a suitable monadic adjunction, an SDC can be constructed to govern the relevant deformation problem. In particular, all operads give rise to monadic adjunctions.

The frequent existence of such adjunctions, and their rôle in cohomology and homology, are well documented (e.g.[Mac71] Ch.VII §6, and Notes at the end of Ch. VI). It is thus unsurprising that monadic adjunctions should be useful in deformation theory, and they immediately enable the construction of SDCs for many deformation problems.

One major way in which this approach diverges from previous approaches is that the dual construction is equally straightforward, providing SDCs whenever there is a suitable comonadic adjunction. As an example, this immediately yields an SDC governing deformations of a local system on an arbitrary topological space.

The real power of this approach, however, lies in the ability to construct SDCs from a combination of monadic and comonadic adjunctions. For example, to deform a scheme X is equivalent to deforming its structure sheaf \mathcal{O}_X . The algebra structure of \mathcal{O}_X can be thought of as monadic, while the sheaf structure is comonadic. This provides an SDC for this example, under more general conditions than those for which a DGLA was constructed in [Hin05]. By contrast, most previous examples for which DGLAs were constructed were either purely comonadic (e.g. smooth schemes, for which the ring structure does not deform), or purely monadic (e.g. affine schemes, for which the sheaf structure does not deform). Another example of this type is deformation of a Hopf algebra.

Furthermore, I adapt the results of [Qui69] to prove that DGLAs and SDCs are equivalent in characteristic zero. However, while the arrow $\text{DGLA} \rightsquigarrow \text{SDC}$ is very natural, the arrow $\text{SDC} \rightsquigarrow \text{DGLA}$ is generally not, which helps to explain why constructing DGLAs is an “art”.

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0 Pre-requisites

This section contains a summary of pre-requisite material used throughout the rest of the paper, included for ease of reference.

0.1 Functors on Artinian Rings

This section consists merely of a list of well-known results and definitions, included for the convenience of the reader.

Fix Λ a complete Noetherian local ring, μ its maximal ideal, k its residue field. Define $\widehat{\mathcal{C}}_\Lambda$ to be the category of complete local Noetherian Λ -algebras with residue field k , and \mathcal{C}_Λ to be the category of Artinian local Λ -algebras with residue field k .

We require that all functors on \mathcal{C}_Λ satisfy

(H0) $F(k) = \bullet$, the one-point set.

We take the following definitions and results from [Sch68]:

Definition 0.1. For $p : B \rightarrow A$ in \mathcal{C}_Λ surjective, p is a small extension if $\ker p = (t)$, a principal ideal, such that $\mathfrak{m}_B(t) = (0)$.

Note that any surjection can be factorised as a composition of small extensions.

For $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$, let $\hat{F} : \widehat{\mathcal{C}}_\Lambda \rightarrow \text{Set}$ by $\hat{F}(R) = \varprojlim F(R/\mathfrak{m}_R^n)$. Note that $\hat{F}(R) \xrightarrow{\sim} \text{Hom}(h_R, F)$, where $h_R : \mathcal{C}_\Lambda \rightarrow \text{Set}; A \mapsto \text{Hom}(R, A)$.

Definition 0.2. We will say a functor $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ is pro-representable if it is isomorphic to h_R , for some $R \in \widehat{\mathcal{C}}_\Lambda$. By the above remark, this isomorphism is determined by an element $\xi \in \hat{F}(R)$. We say the pro-couple (R, ξ) pro-represents F .

Definition 0.3. A natural transformation $\phi : F \rightarrow G$ in $[\mathcal{C}_\Lambda, \text{Set}]$ is called:

1. unramified if $\phi : t_F \rightarrow t_G$ is injective, where $t_F = F(k[\epsilon])$.
2. smooth if for every $B \rightarrow A$ in \mathcal{C}_Λ , we have $F(B) \rightarrow G(B) \times_{G(A)} F(A)$
3. étale if it is smooth and unramified.

Definition 0.4. $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ is smooth if and only if $F \rightarrow \bullet$ is smooth.

Definition 0.5. A pro-couple (R, ξ) is a hull for F if the induced map $h_R \rightarrow F$ is étale.

Theorem 0.6. Let $(R, \xi), (R', \xi')$ be hull of F . Then there exists an isomorphism $u : R \rightarrow R'$ such that $F(u)(\xi) = \xi'$.

Lemma 0.7. Suppose F is a functor such that

$$F(k[V] \times_k k[W]) \xrightarrow{\sim} F(k[V]) \times F(k[W])$$

for vector spaces V and W over k , where $k[V] := k \oplus V \in \mathcal{C}_\Lambda$ in which $V^2 = 0$. Then $F(k[V])$, and in particular t_F , has a canonical vector space structure, and $F(k[V]) \cong t_F \otimes V$.

Theorem 0.8. Let $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$. Let $A' \rightarrow A$ and $A'' \rightarrow A$ be morphisms in \mathcal{C}_Λ , and consider the map:

$$(\dagger) F(A' \times_A A'') \rightarrow F(A') \times_{F(A)} F(A'').$$

Then

1. F has a hull if and only if F has properties (H1), (H2) and (H3) below:

(H1) (\dagger) is a surjection whenever $A'' \rightarrow A$ is a small extension.

(H2) (\dagger) is a bijection when $A = k, A'' = k[\epsilon]$.

(H3) $\dim_k(t_F) < \infty$.

2. F is pro-representable if and only if F has the additional property (H4):

(H4) $F(A' \times_A A'') \xrightarrow{\sim} F(A') \times_{F(A)} F(A'')$.

for any small extension $A' \rightarrow A$.

Proof. [Sch68], Theorem 2.11. □

Definition 0.9. $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ is homogeneous if $\eta : F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C)$ is an isomorphism for every $B \twoheadrightarrow A$.

Note that a homogeneous functor satisfies conditions (H1), (H2) and (H4).

Definition 0.10. $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ is a deformation functor if:

1. η is surjective whenever $B \twoheadrightarrow A$.
2. η is an isomorphism whenever $A = k$.

Note that a deformation functor satisfies conditions (H1) and (H2).

The following results are proved by Manetti (in [Man99]):

Theorem 0.11. (*Standard Smoothness Criterion*) Given $\phi : F \rightarrow G$ a morphism of deformation functors, with $(V, v_e) \xrightarrow{\phi'} (W, w_e)$ a compatible morphism of obstruction theories, if (V, v_e) is complete, $V \xrightarrow{\phi'} W$ injective, and $t_F \rightarrow t_G$ surjective, then ϕ is smooth.

Proof. [Man99], Proposition 2.17. □

For functors $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ and $G : \mathcal{C}_\Lambda \rightarrow \text{Grp}$, we say that G acts on F if we have a functorial group action $G(A) \times F(A) \xrightarrow{*} F(A)$, for each A in \mathcal{C}_Λ . The quotient functor F/G is defined by $(F/G)(A) = F(A)/G(A)$.

Theorem 0.12. If $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$, a deformation functor, and $G : \mathcal{C}_\Lambda \rightarrow \text{Grp}$ a smooth deformation functor, with G acting on F , then $D := F/G$ is a deformation functor, and if $\nu : t_G \rightarrow t_F$ denotes $h \mapsto h * 0$, then $t_D = \text{coker } \nu$, and the universal obstruction theories of D and F are isomorphic.

Proof. [Man99], Lemma 2.20. □

Theorem 0.13. For $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ is homogeneous, and $G : \mathcal{C}_\Lambda \rightarrow \text{Grp}$ a deformation functor, given $a, b \in F(R)$, define $\text{Iso}(a, b) : \mathcal{C}_R \rightarrow \text{Set}$ by

$$\text{Iso}(a, b)(R \xrightarrow{f} A) = \{g \in G(A) \mid g * f(a) = f(b)\}.$$

Then $\text{Iso}(a, b)$ is a deformation functor, with tangent space $\ker \nu$ and, if G is moreover smooth, complete obstruction space $\text{coker } \nu = t_D$. Furthermore, if G is homogeneous, then so is $\text{Iso}(a, b)$.

Proof. [Man99], Proposition 2.21. □

Theorem 0.14. If G, G' smooth deformation functors, acting on F, F' respectively, with F, F' homogeneous, $\ker \nu \rightarrow \ker \nu'$ surjective, and $\text{coker } \nu \rightarrow \text{coker } \nu'$ injective, then $F/G \rightarrow F'/G'$ is injective.

Proof. [Man99], Corollary 2.22. □

This final result does not appear in [Man99], but proves extremely useful:

Corollary 0.15. If $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ and $G : \mathcal{C}_\Lambda \rightarrow \text{Grp}$ are deformation functors, with G acting on F , let $D := F/G$, then:

1. If G is smooth, then η_D is surjective for every $B \rightarrow A$ (i.e. D is a deformation functor).
2. If F is homogeneous and $\ker \nu = 0$, then η_D is injective for every $B \rightarrow A$.

Thus, in particular, F/G will be homogeneous if F is homogeneous, G is a smooth deformation functor and $\ker \nu = 0$.

Proof. The first part is just Theorem 0.12. For the second part, assume we are given

$$\alpha, \beta \in D(B \times_A C) \text{ such that } \eta(\alpha) = \eta(\beta),$$

so $\alpha|_B = \beta|_B$, $\alpha|_C = \beta|_C$. Lift α, β to $\tilde{\alpha}, \tilde{\beta} \in F(B \times_A C)$. We have $g_B \in G(B)$, $g_C \in G(C)$ such that

$$g_B * \tilde{\alpha}|_B = \tilde{\beta}|_B, \quad g_C * \tilde{\alpha}|_C = \tilde{\beta}|_C.$$

Thus $g_B|_A, g_C|_A \in \text{Iso}(\tilde{\alpha}|_A, \tilde{\beta}|_A)(A)$. By Theorem 0.13, $\text{Iso}(\tilde{\alpha}|_A, \tilde{\beta}|_A)$ has a hull, with tangent space $\ker \nu = 0$, so $\text{Iso}(\tilde{\alpha}|_A, \tilde{\beta}|_A)(A)$ contains at most one element, and $g_B|_A = g_C|_A$. Hence

$$(g_B, g_C) \in G(B) \times_{G(A)} G(C),$$

and we may use surjectivity of η_G to lift this to $g \in G(B \times_A C)$. Now $\eta_F(g * \tilde{\alpha}) = \eta_F(\tilde{\beta})$, so homogeneity of F implies that $g * \tilde{\alpha} = \tilde{\beta}$, so $\alpha = \beta$. \square

To summarise the results concerning the pro-representability of the quotient $D = F/G$, we have:

1. If F is a deformation functor and G a smooth deformation functor, with $\text{coker } \nu$ finite dimensional, then D has a hull.
2. If F is homogeneous and G a smooth deformation functor, with $\ker \nu = 0$ and $\text{coker } \nu$ finite dimensional, then D is pro-representable.

0.2 Differential Graded Lie Algebras

The results of this section can all be found in [Man99].

Definition 0.16. A DGLA over a field k of characteristic 0 is a graded vector space $L = \bigoplus_{i \in \mathbb{Z}} L^i$ over k , equipped with operators $[\cdot, \cdot] : L \times L \rightarrow L$ bilinear and $d : L \rightarrow L$ linear, satisfying:

1. $[L^i, L^j] \subset L^{i+j}$.
2. $[a, b] + (-1)^{\bar{a}\bar{b}}[b, a] = 0$.
3. $(-1)^{\bar{c}\bar{a}}[a, [b, c]] + (-1)^{\bar{a}\bar{b}}[b, [c, a]] + (-1)^{\bar{b}\bar{c}}[c, [a, b]] = 0$.
4. $d(L^i) \subset L^{i+1}$.
5. $d \circ d = 0$.

$$6. \quad d[a, b] = [da, b] + (-1)^{\bar{a}}[a, db]$$

Here \bar{a} denotes the degree of a , mod 2, for a homogeneous.

Remark 0.17. Note that this definition of a DGLA does allow for negative degrees; however, it is natural to require that $H^{<0}(L \otimes k) = 0$, and in Section 5 it will be shown that all the deformation problems we consider can be described by DGLAs with $L^{<0} = 0$.

Fix a DGLA L .

Definition 0.18. The Maurer-Cartan functor $\text{MC}_L : \mathcal{C}_k \rightarrow \text{Set}$ is defined by

$$\text{MC}_L(A) = \{x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x, x] = 0\}.$$

Observe that for $\omega \in L^1 \otimes \mathfrak{m}_A$,

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \Rightarrow (d + \text{ad}_\omega) \circ (d + \text{ad}_\omega) = 0,$$

so $(L \otimes A, [,], d + \text{ad}_\omega)$ is a DGLA over A .

Definition 0.19. Define the gauge functor $\mathfrak{gl} : \mathcal{C}_k \rightarrow \text{Grp}$ by $\mathfrak{gl}(A) = \exp(L^0 \otimes \mathfrak{m}_A)$.

We may define another DGLA, L_d , by

$$L_d^i = \begin{cases} L^1 \oplus kd & i = 1 \\ L^i & i \neq 1, \end{cases}$$

with

$$d_d(d) = 0, \quad [d, d] = 0, \quad [d, a]_d = da, \quad \forall a \in L.$$

Lemma 0.20. $\exp(L^0 \otimes \mathfrak{m}_A)$ commutes with $[,]$ when acting on $L_d \otimes A$ via the adjoint action.

Corollary 0.21. Since $\exp(L^0 \otimes \mathfrak{m}_A)$ preserves $L^1 \otimes \mathfrak{m}_A + d \subset L_d \otimes A$ under the adjoint action, and

$$x \in \text{MC}_L(A) \Leftrightarrow [x + d, x + d] = 0,$$

the adjoint action of $\exp(L^0 \otimes \mathfrak{m}_A)$ on $L^1 \otimes A + d$ induces an action of $\mathfrak{gl}(A)$ on $\text{MC}_L(A)$, which we will call the gauge action.

Definition 0.22. $\text{Def}_L = \text{MC}_L / \mathfrak{gl}$, the quotient being given by the gauge action $\alpha(x) = \text{ad}_\alpha(x + d) - d$. Observe that \mathfrak{gl} is homogeneous.

For $a \in \text{MC}_L(A)$, define $K_a : \mathcal{C}_A \rightarrow \text{Grp}$ by

$$K_a(B) = \exp((d + \text{ad}_a)L^{-1} \otimes \mathfrak{m}_B).$$

Note that this makes sense, since $(d + \text{ad}_a)^2 = 0$, so

$$(d + \text{ad}_a)L^{-1} \otimes \mathfrak{m}_B \leq L^0 \otimes \mathfrak{m}_B$$

is a Lie subalgebra. Note that K_a is then a subfunctor of $\text{Iso}(a, a) \leq \mathfrak{gl}(A)$, so acts on $\text{Iso}(a, b)$ by right multiplication.

Define the deformation groupoid \mathfrak{Def}_L to have objects MC_L , and morphisms given by $\text{Iso}(a, b)/K_a$.

Now, $t_{\mathfrak{gt}} = L^0\epsilon$, and $t_{\mathrm{MC}_L} = Z^1(L\epsilon)$, with action

$$\begin{aligned} t_{\mathfrak{gt}} \times t_{\mathrm{MC}_L} &\rightarrow t_{\mathrm{MC}_L}; \\ (b, x) &\mapsto x + db, \text{ so} \\ t_{\mathrm{Def}_L} &= H^1(L). \end{aligned}$$

Lemma 0.23. $H^2(L)$ is a complete obstruction space for MC_L .

Proof. Given a small extension

$$e : 0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0,$$

and $x \in \mathrm{MC}_L(B)$, lift x to $\tilde{x} \in L^1 \otimes \mathfrak{m}_A$, and let

$$h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes \mathfrak{m}_A.$$

In fact, $h \in L^2 \otimes J$, as $dx + \frac{1}{2}[x, x] = 0$.

Now,

$$dh = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h - \frac{1}{2}[\tilde{x}, \tilde{x}], \tilde{x}] = [h, \tilde{x}] = 0,$$

since $[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$ and $J\mathfrak{m}_A = 0$. Let

$$v_e(x) = [h] \in H^2(L \otimes J) = H^2(L) \otimes_k J.$$

This is well-defined: if $y = \tilde{x} + z$, for $z \in L^1 \otimes J$, then

$$dy + \frac{1}{2}[y, y] = d\tilde{x} + dz + \frac{1}{2}[\tilde{x}, \tilde{x}] + \frac{1}{2}[z, z] + [\tilde{x}, z] = h + dz,$$

as $J\mathfrak{m}_A = 0$.

This construction is clearly functorial, so it follows that $(H^2(L), v_e)$ is a complete obstruction theory for MC_L . \square

Now Theorem 0.12 implies the following:

Theorem 0.24. Def_L is a deformation functor, $t_{\mathrm{Def}_L} \cong H^1(L)$, and $H^2(L)$ is a complete obstruction theory for Def_L .

The other theorems can be used to prove:

Theorem 0.25. If $\phi : L \rightarrow M$ is a morphism of DGLAs, and

$$H^i(\phi) : H^i(L) \rightarrow H^i(M)$$

are the induced maps on cohomology, then:

1. If $H^1(\phi)$ is bijective, and $H^2(\phi)$ injective, then $\mathrm{Def}_L \rightarrow \mathrm{Def}_M$ is étale.
2. If also $H^0(\phi)$ is surjective, then $\mathrm{Def}_L \rightarrow \mathrm{Def}_M$ is an isomorphism.

3. Provided condition 1 holds, $\mathfrak{Def}_L \rightarrow \mathfrak{Def}_M$ is an equivalence of functors of groupoids if and only if $H^0(\phi)$ is an isomorphism.

Proof. [Man99], Theorem 3.1, mutatis mutandis. \square

Theorem 0.26. *If $H^0(L) = 0$, then Def_L is homogeneous.*

Proof. This is essentially Theorem 0.15, with some allowance made for K_a . \square

Thus, in particular, a quasi-isomorphism of DGLAs gives an isomorphism of deformation functors and deformation groupoids.

Example 0.27. Deformation of algebras (due to [Kon94]).

We wish to describe flat deformations R_A/A of any algebra R_0/k :

$$\begin{array}{ccc} X = \text{Spec } R_0 & \longrightarrow & \text{Spec } R_A = X_A \\ \text{flat} \downarrow & & \downarrow \text{flat} \\ \text{Spec } k & \longrightarrow & \text{Spec } A. \end{array}$$

Explicitly, the deformation groupoid has objects flat R_A/A , such that $R_A \otimes_A k = R_0$, with morphisms given by infinitesimal isomorphisms of R_A , i.e. A -algebra isomorphisms θ such that $\theta \equiv \text{id} \pmod{\mathfrak{m}_A}$.

Take any free resolution $R_\bullet \rightarrow R$ of R , i.e. a free graded algebra R_* , with a differential δ of degree 1, such that

$$H_i(R_\bullet) = \begin{cases} R & i = 0 \\ 0 & i > 0. \end{cases}$$

Now, let

$$L^* = \text{Der}_k(R_*, R_*),$$

with differential ad_δ .

Let $\mathfrak{Def}_{R_0/k}(A)$ denote the groupoid of deformations R_A/A of R_0 , with morphisms given by infinitesimal isomorphisms. Given $\omega \in \text{MC}_L(A)$, it follows that $(\delta + \omega)^2 = 0$. Define a map

$$\mathfrak{Def}_L(A) \xrightarrow{\Phi} \mathfrak{Def}_{R_0/k}(A),$$

sending $\omega \in \text{MC}_L(A)$ to the flat A -algebra

$$H_0(R_* \otimes A, \delta + \omega).$$

In fact, Φ gives an essential surjection.

Next, observe that $\mathfrak{gl}(A)$ consists precisely of infinitesimal A -algebra automorphisms of $R_* \otimes A$, and that the gauge action of θ sends $\delta + \omega$ to $\theta \circ (\delta + \omega) \circ \theta^{-1}$.

Now, $\theta \in \text{Hom}(\omega, \omega')$ maps under Φ to

$$H_0(\theta) : H_0(R_* \otimes A, \delta + \omega) \rightarrow H_0(R_* \otimes A, \delta + \omega'),$$

and it turns out that Φ is full.

To see that Φ is faithful, note that for $\theta \in \text{Iso}(\omega, \omega)(A)$,

$$H_0(\theta) : H_0(R_* \otimes A, \delta + \omega) \rightarrow H_0(R_* \otimes A, \delta + \omega)$$

is the identity if and only if $\theta \in K_\omega(A)$.

0.3 Monadic Adjunctions

This section just recalls some standard definitions concerning adjunctions, and fixes notation which will be used throughout the paper.

Definition 0.28. For categories \mathcal{D}, \mathcal{E} , and a pair of functors

$$\mathcal{D} \begin{array}{c} \xrightarrow{G} \\ \xleftarrow{F} \end{array} \mathcal{E},$$

recall that an adjunction $F \dashv G$ is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(FA, B) \cong \mathrm{Hom}_{\mathcal{E}}(A, GB).$$

We say that F is left adjoint to G , or G is right adjoint to F . Let $\perp = FG$, and $\top = GF$. To give an adjunction is equivalent to giving two natural transformations, the unit

$$\eta : \mathrm{id}_{\mathcal{E}} \rightarrow \top,$$

and the co-unit

$$\varepsilon : \perp \rightarrow \mathrm{id}_{\mathcal{D}},$$

satisfying the triangle identities:

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow \mathrm{id}_F & \downarrow \varepsilon F \\ & & F \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow \mathrm{id}_G & \downarrow G\varepsilon \\ & & G. \end{array}$$

In terms of the unit and co-unit, the adjunction is given as follows:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(FA, B) &\cong \mathrm{Hom}_{\mathcal{E}}(A, GB) \\ \theta &\mapsto G\theta \circ \eta_A \\ \varepsilon_B \circ F\phi &\leftarrow \phi. \end{aligned}$$

Note that the η_A and ε_B can be recovered from the adjunction, since they correspond under the adjunction to the identity maps on FA and GB , respectively.

Examples 0.29. 1. Given a morphism $X \xrightarrow{f} Y$ of schemes, there is an adjunction

$$\mathrm{Shf}(X) \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{f^{-1}} \end{array} \mathrm{Shf}(Y),$$

between the category of sheaves on Y and the category of sheaves on X , i.e. a natural isomorphism

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}),$$

determined by the canonical maps

$$\varepsilon_{\mathcal{F}} : f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}, \quad \eta_{\mathcal{G}} : \mathcal{G} \rightarrow f_*f^{-1}\mathcal{G}.$$

2. There is also an adjunction

$$\mathcal{O}_X\text{-Mod} \begin{array}{c} \xrightarrow{f_*} \\ \dashv \\ \xleftarrow{f^*} \end{array} \mathcal{O}_Y\text{-Mod},$$

$$\begin{aligned} \text{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) &\cong \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}), \\ \varepsilon_{\mathcal{F}} : f^*f_*\mathcal{F} &\rightarrow \mathcal{F}, \quad \eta_{\mathcal{G}} : \mathcal{G} \rightarrow f_*f^*\mathcal{G}. \end{aligned}$$

If a functor has a left adjoint, then it preserves all (inverse) limits. Conversely, provided the categories involved satisfy various additional conditions, the Special Adjoint Functor Theorem ([Mac71] Ch. V.8) proves that any functor which preserves inverse limits has a left adjoint.

Given an adjunction

$$\mathcal{D} \begin{array}{c} \xrightarrow{U} \\ \dashv \\ \xleftarrow{F} \end{array} \mathcal{E}$$

with unit $\eta : \text{id} \rightarrow UF$ and co-unit $\varepsilon : FU \rightarrow \text{id}$, we let $\top = UF$, and define the category of \top -algebras, \mathcal{E}^\top , to have objects

$$\top E \xrightarrow{\theta} E,$$

such that the following diagrams commute:

$$\begin{array}{ccc} E & \xrightarrow{\eta_E} & \top E \\ & \searrow \text{id} & \downarrow \theta \\ & & E \end{array} \quad \text{and} \quad \begin{array}{ccc} \top^2 E & \xrightarrow{\top\theta} & \top E \\ U\varepsilon_{FE} \downarrow & & \downarrow \theta \\ \top E & \xrightarrow{\theta} & E. \end{array}$$

A morphism

$$g : \begin{pmatrix} \top E_1 \\ \downarrow \theta \\ E_1 \end{pmatrix} \rightarrow \begin{pmatrix} \top E_2 \\ \downarrow \phi \\ E_2 \end{pmatrix}$$

of \top -algebras is a morphism $g : E_1 \rightarrow E_2$ such that

$$\begin{array}{ccc} \top E_1 & \xrightarrow{\top g} & \top E_2 \\ \theta \downarrow & & \downarrow \phi \\ E_1 & \xrightarrow{g} & E_2 \end{array}$$

commutes.

We define the comparison functor $K : \mathcal{D} \rightarrow \mathcal{E}^\top$ by

$$B \mapsto \begin{pmatrix} UFUB \\ \downarrow U\varepsilon_B \\ UB \end{pmatrix}$$

on objects, and $K(g) = U(g)$ on morphisms.

Definition 0.30. The adjunction

$$\mathcal{D} \begin{array}{c} \xrightarrow{U} \\ \perp \\ \xleftarrow{F} \end{array} \mathcal{E},$$

is said to be *monadic* if $K : \mathcal{D} \rightarrow \mathcal{E}^\top$ is an equivalence.

Intuitively, monadic adjunctions correspond to algebraic theories, such as the adjunction

$$k\text{-Alg} \begin{array}{c} \xrightarrow{U} \\ \perp \\ \xleftarrow{\text{Symm}_k} \end{array} k\text{-Vect},$$

between k -algebras and vector spaces over k , U being the forgetful functor. Other examples are Lie algebras or non-commutative algebras, with their respective free functors, over vector spaces, or vector spaces over sets.

Example 0.31. Probably the clearest example of a monadic adjunction is that between sets and G -sets. Given a group G , The category $G\text{-Set}$ consists of sets S , with a G -action $G \times S \rightarrow S$. We have the adjunction:

$$G\text{-Set} \begin{array}{c} \xrightarrow{U} \\ \perp \\ \xleftarrow{G \times -} \end{array} \text{Set},$$

with $\eta_S = (e, \text{id}) : S \rightarrow G \times S$, and $\varepsilon_T = * : G \times T \rightarrow T$.

A \top -algebra consists of a set S , together with a map $\top S = G \times S \xrightarrow{f} S$, such that:

$$\begin{array}{ccc} S & \xrightarrow{(e, \text{id}_S)} & G \times S & \text{and} & G \times G \times S & \xrightarrow{(\text{id}_G, f)} & G \times S \\ & \searrow \text{id}_S & \downarrow f & & \downarrow (*, \text{id}_S) & & \downarrow f \\ & & S & & G \times S & \xrightarrow{f} & S \end{array}$$

commute, which is precisely the same as a G -set, hence the equivalence.

Dually, if we have an adjunction

$$\mathcal{D} \begin{array}{c} \xrightarrow{V} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{E},$$

with co-unit $\gamma : VG \rightarrow \text{id}$ and unit $\alpha : \text{id} \rightarrow GV$, we let $\perp = VG$, and can define the category of \perp -coalgebras, \mathcal{E}^\perp , with a functor $K : \mathcal{D} \rightarrow \mathcal{E}^\perp$.

Definition 0.32. The adjunction

$$\mathcal{D} \begin{array}{c} \xrightarrow{V} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{E},$$

is said to be *comonadic* if $K : \mathcal{D} \rightarrow \mathcal{E}^\perp$ is an equivalence.

Finally, the equivalence version of Beck's monadicity theorem gives a criterion for an adjunction to be monadic:

Definition 0.33. A split fork is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{e} & C, \\ & \xrightarrow{g} & & \xleftarrow{s} & \\ & \searrow & \curvearrowright & & \\ & & t & & \end{array}$$

such that $ef = eg$, $es = 1$, $ft = 1$, $gt = se$. A split coequaliser of f and g is the arrow e of some split fork.

Definition 0.34. A functor $U : \mathcal{D} \rightarrow \mathcal{E}$ is said to reflect coequalisers for a pair $f, g : A \rightarrow B$ if for every diagram $A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{e} C$, with $ef = eg$ and Ue a coequaliser of Uf and Ug , the morphism e is a coequaliser of f and g .

U is said to reflect isomorphisms if f is an isomorphism whenever Uf is. Observe that if U preserves a coequaliser of f, g in \mathcal{D} , and U reflects isomorphisms, then U reflects all coequalisers of f, g .

Theorem 0.35. *The following conditions are equivalent:*

1. *The adjunction*

$$\mathcal{D} \begin{array}{c} \xrightarrow{U} \\ \dashv \\ \xleftarrow{F} \end{array} \mathcal{E},$$

is monadic.

2. *If $f, g : D \rightarrow D'$ is a parallel pair in \mathcal{D} for which Uf, Ug has a split coequaliser, then \mathcal{D} has a coequaliser for f, g , and U preserves and reflects coequalisers for these pairs.*

Proof. [Mac71] Ch. VI.7 Ex. 6. □

Throughout this paper, the sign “=” will be used to denote canonical isomorphism.

1 Simplicial Deformation Complexes

1.1 Definitions

Definition 1.1. A simplicial deformation complex E^\bullet consists of smooth homogeneous functors $E^n : \mathcal{C}_\Lambda \rightarrow \text{Set}$ for each $n \geq 0$, together with maps

$$\begin{aligned} \partial^i : E^n &\rightarrow E^{n+1} & 1 \leq i \leq n \\ \sigma^i : E^n &\rightarrow E^{n-1} & 0 \leq i < n, \end{aligned}$$

an associative product $* : E^m \times E^n \rightarrow E^{m+n}$, with identity $1 : \bullet \rightarrow E^0$, where \bullet is the constant functor $\bullet(A) = \bullet$ on \mathcal{C}_Λ , such that:

1. $\partial^j \partial^i = \partial^i \partial^{j-1} \quad i < j.$
2. $\sigma^j \sigma^i = \sigma^i \sigma^{j+1} \quad i \leq j.$

$$3. \sigma^j \partial^i = \begin{cases} \partial^i \sigma^{j-1} & i < j \\ \text{id} & i = j, i = j + 1. \\ \partial^{i-1} \sigma^j & i > j + 1 \end{cases}$$

$$4. \partial^i(e) * f = \partial^i(e * f).$$

$$5. e * \partial^i(f) = \partial^{i+m}(e * f), \text{ for } e \in E^m.$$

$$6. \sigma^i(e) * f = \sigma^i(e * f).$$

$$7. e * \sigma^i(f) = \sigma^{i+m}(e * f), \text{ for } e \in E^m.$$

Remark 1.2. Observe that E^\bullet is not quite a cosimplicial complex — we have omitted the maps ∂^0 and ∂^{n+1} . In fact, if we define $\check{E}_{\bullet-1} = E^\bullet$, with $\sigma_i = \partial^{i+1}$ and $\partial_i = \sigma^i$, then \check{E}_\bullet is an augmented simplicial complex.

Definition 1.3. Let $C^\bullet(E)$ be the tangent space of E^\bullet , i.e. $C^n(E) = E^n(k[\epsilon])$.

Definition 1.4. Define the Maurer-Cartan functor MC_E by

$$\text{MC}_E(A) = \{\omega \in E^1(A) : \omega * \omega = \partial^1(\omega)\}.$$

Lemma 1.5. E^0 is a group under multiplication.

Proof. We need only show existence of inverses. Given $g \in E^0(A)$, we have a morphism $\phi_g : E^0 \rightarrow E^0$ of functors on \mathcal{C}_A , given by $h \mapsto g * h$. Since E^0 is smooth, its obstruction theory is trivial. Now, on tangent spaces, $\phi_g(a) = \bar{g} * a$, but the map $A \rightarrow k[\epsilon]$ factors through k , so $\bar{g} = 1$, and $\phi_g : C^0(E) \rightarrow C^0(E)$ is the identity. Thus the Standard Smoothness Criterion (Theorem 0.11) implies that ϕ_g is smooth, and in particular surjective, hence $\phi_g : E^0(A) \rightarrow E^0(A)$ is surjective, and there exists an h with $g * h = 1$. We construct a left inverse similarly, and then $h_l = h_l * g * h_r = h_r$. This also gives uniqueness of the inverse. \square

Now, if $\omega \in \text{MC}_E(A)$ and $g \in E^0(A)$, then $g * \omega * g^{-1} \in \text{MC}_E(A)$. We may therefore make the following definition:

Definition 1.6.

$$\text{Def}_E = \text{MC}_E / E^0,$$

the quotient being with respect to the adjoint action. The deformation groupoid

$$\mathfrak{Def}_E$$

has objects MC_E , and morphisms given by E^0 .

Lemma 1.7. The action $E^0 \times E^n \rightarrow E^n$ is faithful (i.e. $g * e = e$ for some e only if $g = 1$).

Proof. From Theorem 0.13, it will suffice to show that $\nu : C(E)^0 \rightarrow C(E)^n$ is injective, since a deformation functor F with tangent space 0 has at most one element in $F(A)$.

Let ω_0 be the unique element of $E^1(k)$; thus $\overbrace{\omega_0 * \dots * \omega_0}^n$ is the unique element of $E^n(k)$. Note that $\sigma^0(\omega_0) = 1$. Now, for $g \in C^0(E)$,

$$(\sigma^0)^n(\nu(g)) = (\sigma^0)^n(g * \omega_0^n) = (\sigma^0)^{n-1}(g * \sigma^0(\omega_0) * \omega_0^{n-1}) = (\sigma^0)^{n-1}(g * \omega_0^{n-1}) = g,$$

by induction. Therefore ν is injective. \square

Observe that, for $\omega \in MC_E(A)$,

$$\sigma^0(\omega) * \omega = \sigma^0(\omega * \omega) = \sigma^0 \partial^1(\omega) = \omega,$$

so the previous Lemma implies that $\sigma^0(\omega) = 1$. As an immediate consequence, we have:

Lemma 1.8. *For $\omega \in MC_E(A)$, if we let $\partial_\omega^0(e) = \omega * e$, and $\partial_\omega^{n+1}(e) = e * \omega$, for $e \in E^n(A)$, then $E^*(A)$ becomes a cosimplicial complex, ∂_ω^0 and ∂_ω^{n+1} also satisfying the following identities, for $e \in E^m, f \in E^n$:*

$$\begin{aligned} \partial_\omega^0(e) * f &= \partial_\omega^0(e * f), \\ e * \partial_\omega^{n+1}(f) &= \partial_\omega^{m+n+1}(e * f), \\ \partial_\omega^{m+1}(e) * f &= e * \partial_\omega^0(f). \end{aligned}$$

*Note, however, that not all the identities of Definition 1.1 hold for ∂_ω^0 and ∂_ω^{n+1} — in general $\partial_\omega^{m+1}(e) * f \neq \partial_\omega^{m+1}(e * f)$, and $e * \partial_\omega^0(f) \neq \partial_\omega^m(e * f)$.*

In particular, we obtain, on $C_k \subset C_\Lambda$, $\partial^0 := \partial_{\omega_0}^0$ and $\partial^{n+1} := \partial_{\omega_0}^{n+1}$. Thus $C^\bullet(E)$ has the natural structure of a cosimplicial complex.

Definition 1.9. Define the cohomology groups of E to be

$$H^i(E) := H^i(C^\bullet(E)),$$

the cohomotopy groups of the cosimplicial complex $C^\bullet(E)$.

Lemma 1.10. *The tangent space of MC_E is $Z^1(C^\bullet(E))$, with the action of E^0 giving $\nu(g) = \partial^1(g) - \partial^0(g)$.*

Proof. On tangent spaces all maps are linear. Thus the map $C^1(E) \times C^1(E) \rightarrow C^2(E)$ given by $(\alpha, \beta) \mapsto \alpha * \beta$ can be written $\alpha * \beta = f_1(\alpha) + f_2(\beta)$. Evaluating this at $\beta = \omega_0$ we obtain $f_1(\alpha) = \alpha * \omega_0 = \partial^2(\alpha)$, and similarly $f_2(\beta) = \partial^0(\beta)$. Thus

$$t_{MC_E} = \{\omega \in C^1(E) : \partial^2(\omega) + \partial^0(\omega) = \partial^1(\omega)\} = Z^1(C^\bullet(E)).$$

Now, $\nu(g) = g * \omega_0 * g^{-1}$, and, similarly to above,

$$f * \omega_0 * h = f * \omega_0 * 1 + 1 * \omega_0 * h = \partial^1(f) + \partial^0(h).$$

Hence $\nu(g) = \partial^1(g) - \partial^0(g)$. \square

Lemma 1.11. $H^2(E)$ is a complete obstruction space for MC_E .

Proof. Take a small extension

$$e : 0 \rightarrow k \xrightarrow{\epsilon} A \rightarrow B \rightarrow 0.$$

Note that ϵ provides an isomorphism $A \times_k k[\epsilon] \rightarrow A \times_B A$, $(\alpha, a\epsilon) \mapsto (\alpha, \alpha + a\epsilon)$, and thus, for any homogeneous functor F , $F(A) \times_{t_F \epsilon} \rightarrow F(A \times_B A)$. Given $f \in F(A)$ and $v \in t_F$, we will write the image under this isomorphism as $(f, f + v\epsilon)$. Note that all the properties which this terminology suggests do hold.

Now, given $\omega \in \text{MC}_E(B)$, use the smoothness of E^1 to take a lift $\tilde{\omega} \in E^1(A)$. Since $\omega * \omega = \partial^1(\omega)$,

$$(\tilde{\omega} * \tilde{\omega}, \partial^1(\tilde{\omega})) \in E^2(A) \times_{E^2(B)} E^2(A),$$

so we have $\tilde{\omega} * \tilde{\omega} = \partial^1(\tilde{\omega}) + c\epsilon$, for $c \in C^2(E)$.

Now,

$$\begin{aligned} (\tilde{\omega} * \tilde{\omega}) * \tilde{\omega} &= \partial^1(\tilde{\omega}) * \tilde{\omega} + \partial^3(c)\epsilon \\ \tilde{\omega} * (\tilde{\omega} * \tilde{\omega}) &= \tilde{\omega} * \partial^1(\tilde{\omega}) + \partial^0(c)\epsilon, \end{aligned}$$

so

$$\partial^1(\tilde{\omega} * \tilde{\omega}) - \partial^2(\tilde{\omega} * \tilde{\omega}) = (\partial^0(c) - \partial^3(c))\epsilon.$$

Also,

$$\begin{aligned} \partial^1(\tilde{\omega} * \tilde{\omega}) &= \partial^1 \partial^1(\tilde{\omega}) + \partial^1(c)\epsilon \\ \partial^2(\tilde{\omega} * \tilde{\omega}) &= \partial^2 \partial^1(\tilde{\omega}) + \partial^2(c)\epsilon, \end{aligned}$$

so

$$\partial^1(\tilde{\omega} * \tilde{\omega}) - \partial^2(\tilde{\omega} * \tilde{\omega}) = (\partial^1(c) - \partial^2(c))\epsilon,$$

since $\partial^1 \partial^1 = \partial^2 \partial^1$.

Thus $dc = (\partial^0 - \partial^1 + \partial^2 - \partial^3)(c) = 0$, so $c \in Z^2(C^\bullet(E))$.

If we take another lift $\tilde{\omega}_1 = \tilde{\omega} + b\epsilon$, then

$$\tilde{\omega}_1 * \tilde{\omega}_1 = \tilde{\omega} * \tilde{\omega} + \partial^0(b)\epsilon + \partial^2(b)\epsilon,$$

since $\tilde{\omega} * b = \omega_0 * b + \partial^0(b)$, &c., and

$$\partial^1(\tilde{\omega}_1) = \partial^1(\tilde{\omega}) + \partial^1(b)\epsilon,$$

so $c_1 = c + db$.

We therefore define $v_e(\omega) = [c]\epsilon \in H^2(E)$, and it follows from the above working that this is a complete obstruction theory. \square

Theorems 0.11 to 0.15 now imply:

Theorem 1.12. Def_E is a deformation functor, with tangent space $H^1(E)$ and complete obstruction space $H^2(E)$. For $\omega, \omega' \in \text{MC}_E(A)$, $\text{Iso}(\omega, \omega')$ is homogeneous, with tangent space $H^0(E)$ and complete obstruction space $H^1(E)$. Moreover, if $H^0(E) = 0$, then Def_E is homogeneous.

Proof. Theorem 0.12 and Corollary 0.15. □

Theorem 1.13. *If $\phi : E \rightarrow F$ is a morphism of SDCs, and*

$$H^i(\phi) : H^i(E) \rightarrow H^i(F)$$

are the induced maps on cohomology, then:

1. *If $H^1(\phi)$ is bijective, and $H^2(\phi)$ injective, then $\text{Def}_E \rightarrow \text{Def}_F$ is étale.*
2. *If also $H^0(\phi)$ is surjective, then $\text{Def}_E \rightarrow \text{Def}_F$ is an isomorphism.*
3. *Provided condition 1 holds, $\mathfrak{D}\text{ef}_E \rightarrow \mathfrak{D}\text{ef}_F$ is an equivalence of functors of groupoids if and only if $H^0(\phi)$ is an isomorphism.*

Proof.

1. Theorems 0.11 and 0.12.
2. Theorem 0.14.
3. It remains only to show that $\phi^0 : E^0 \rightarrow F^0$ is injective. E^0 acts on F^0 by multiplication, so $\ker \phi^0 = \text{Iso}(1, 1)$, which, by Theorem 0.13, is a deformation functor with tangent space $\ker H^0(\phi) = 0$. Hence $\ker \phi^0 = 0$.

□

Call a morphism $\phi : E \rightarrow F$ a quasi-isomorphism if the $H^i(\phi) : H^i(E) \rightarrow H^i(F)$ are all isomorphisms.

Definition 1.14. Given a morphism $\phi : E \rightarrow F$ of SDCs, and a point $x \in \text{MC}_F(\Lambda)$, define the groupoid

$$\mathfrak{D}\text{ef}_{\phi, x}$$

to be the fibre of the morphism

$$\mathfrak{D}\text{ef}_E \rightarrow \mathfrak{D}\text{ef}_F$$

over x .

Explicitly, $\mathfrak{D}\text{ef}_{\phi, x}(A)$ has objects

$$\{(\omega, h) \in \text{MC}_E(A) \times F^0(A) \quad : \quad h\phi(\omega)h^{-1} = x\},$$

and morphisms

$$E^0(A), \quad \text{where} \quad g(\omega, h) = (g\omega g^{-1}, h\phi(g)^{-1}).$$

Theorem 1.15. *Let $E_{\phi, x}^n(A)$ be the fibre of $E^n(A) \xrightarrow{\phi} F^n(A)$ over x^n . Then E_x^\bullet is an SDC, and the canonical map*

$$\mathfrak{D}\text{ef}_{E_{\phi, x}^\bullet} \rightarrow \mathfrak{D}\text{ef}_{\phi, x}$$

is an equivalence of groupoids.

Proof. $\text{Def}_{\phi,x}$ is a deformation functor, with tangent space $H^1(E, F)$ and complete obstruction space $H^2(E, F)$. For $\omega, \omega' \in \text{Ob } \mathfrak{Def}_{\phi,x}(A)$, $\text{Iso}(\omega, \omega')$ is homogeneous, with tangent space $H^0(E, F)$ and complete obstruction space $H^1(E, F)$. Here, $H^i(E, F)$ is the cohomology of the cone complex

$$\begin{array}{ccc}
\vdots & & \vdots \\
d \uparrow & \nearrow -\phi & d \uparrow \\
C^3(E) & \oplus & C^2(F) \\
d \uparrow & \nearrow \phi & d \uparrow \\
C^2(E) & \oplus & C^1(F) \\
d \uparrow & \nearrow -\phi & d \uparrow \\
C^1(E) & \oplus & C^0(F) \\
d \uparrow & \nearrow \phi & \\
C^0(E), & &
\end{array}$$

so we have the long exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(E, F) & \longrightarrow & H^0(E) & \xrightarrow{H^0(\phi)} & H^0(F) & \longrightarrow \\
& & H^1(E, F) & \longrightarrow & H^1(E) & \xrightarrow{H^1(\phi)} & H^1(F) & \longrightarrow \\
& & H^2(E, F) & \longrightarrow & H^2(E) & \xrightarrow{H^2(\phi)} & H^2(F) & \longrightarrow \dots
\end{array}$$

Now, the definition of E_x^\bullet provides a short exact sequence:

$$0 \rightarrow C^\bullet(E_{\phi,x}) \rightarrow C^\bullet(E) \xrightarrow{\phi} C^\bullet(F) \rightarrow 0,$$

giving the long exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(E_{\phi,x}) & \longrightarrow & H^0(E) & \xrightarrow{H^0(\phi)} & H^0(F) & \longrightarrow \\
& & H^1(E_{\phi,x}) & \longrightarrow & H^1(E) & \xrightarrow{H^1(\phi)} & H^1(F) & \longrightarrow \\
& & H^2(E_{\phi,x}) & \longrightarrow & H^2(E) & \xrightarrow{H^2(\phi)} & H^2(F) & \longrightarrow \dots
\end{array}$$

Thus the map $\mathfrak{Def}_{E_{\phi,x}} \rightarrow \mathfrak{Def}_{E,x}$ gives isomorphisms on all relevant tangent and obstruction spaces, so is an equivalence of groupoids. \square

1.2 Examples

1.2.1 Deformation of algebras

Let $\Lambda_n := \Lambda/\mu^{n+1}$. Take a flat μ -adic system of algebras B_n/Λ_n , i.e flat algebras B_n/Λ_n such that $B_{n+1} \otimes_{\Lambda_{n+1}} \Lambda_n = B_n$, and let $B := \varprojlim B_n$. The simplest (and most

common) case to consider is when $B_n = \Lambda_n$. We wish to create an SDC describing flat deformations $R_A/(B \otimes_\Lambda A)$ of any flat algebra R_0/B_0 :

$$\begin{array}{ccc} X_0 = \text{Spec } R_0 & \xrightarrow{i} & \text{Spec } R_A = X_A \\ \text{flat} \downarrow & & \downarrow \text{flat} \\ \text{Spec } B_0 & \longrightarrow & \text{Spf } B \times_{\text{Spf } \Lambda} \text{Spec } A. \end{array}$$

Explicitly, the deformation groupoid has objects flat $R_A/(B \otimes A)$, such that $R_A \otimes_A k = R_0$, with morphisms given by infinitesimal isomorphisms of R_A , i.e. $(B \otimes A)$ -algebra isomorphisms θ such that $\theta \equiv \text{id} \pmod{\mathfrak{m}_A}$.

There is, up to isomorphism, a unique flat μ -adic Λ -module M such that $M \otimes k = R_0$ (since flat modules over nilpotent rings are free). There is an adjunction

$$(B \otimes A)\text{FAlg} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow[\text{F}_{B \otimes A}]{\top} \end{array} \text{AFMod},$$

between the categories of flat $(B \otimes A)$ -algebras and of flat A -modules, U being the forgetful functor, and

$$F_{B \otimes A} = (B \otimes A) \otimes_A \text{Symm}_A = B \otimes_\Lambda \text{Symm}_A.$$

Let $\perp = FU$ and $\top = UF$. We have the unit and co-unit:

$$\eta_M : M = S^1(M) \hookrightarrow \text{Symm}_A M \rightarrow B \otimes_\Lambda \text{Symm}_A M = \top M,$$

and

$$\begin{array}{ccc} \varepsilon_R : \perp R & \rightarrow & R \\ b[r_1 \otimes \dots \otimes r_n] & \mapsto & b \cdot r_1 \dots r_n. \end{array}$$

As in [Wei94], we may form the canonical augmented cosimplicial complex

$$M \longrightarrow \top M \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \top^2 M \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \top^3 M \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \top^4 M \quad \dots \dots$$

with face operators

$$\top^i \eta \top^{n-i} : \top^n M \rightarrow \top^{n+1} M$$

and degeneracy operators

$$\top^i U \varepsilon F \top^{n-i} : \top^{n+1} M \rightarrow \top^n M.$$

Definition 1.16. Let

$$E^n = \text{Hom}_\Lambda(\top^n M, M)_{U(\varepsilon_{R_0} \circ \dots \circ \varepsilon_{\perp^{n-1} R_0})},$$

where, for flat μ -adic Λ -modules M and N , and a k -linear map $\alpha : M \otimes k \rightarrow N \otimes k$, we define

$$\text{Hom}_\Lambda(M, N)_\alpha(A)$$

to be the fibre of

$$\mathrm{Hom}_A(M \otimes A, N \otimes A) \rightarrow \mathrm{Hom}_k(M \otimes k, N \otimes k)$$

over α .

We make E^* into an SDC by giving it the product $g * h = g \circ \top^n h$, and \check{E}_* is the dual complex to the canonical augmented complex $M \rightarrow \top^* M$.

Since flat modules over Artinian rings do not deform (being free), to give a deformation $R/(B \otimes_\Lambda A)$ of R_0/B_0 is the same as to give the module $M \otimes_\Lambda A$ the structure of a $(B \otimes_\Lambda A)$ -algebra, compatible with the B_0 -algebra structure of R_0 , and the A -module structure of $M \otimes_\Lambda A$.

A $(B \otimes_\Lambda A)$ -algebra structure on module $M_A := M \otimes_\Lambda A$ is the same as a map

$$f \in \mathrm{Hom}_{\mathrm{AFMod}}(\top M_A, M_A); \quad b[m_1 \otimes \dots \otimes m_n] \mapsto b \cdot m_1 \dots m_n,$$

satisfying

$$\begin{aligned} f \circ \top f &= f \circ U\varepsilon_{FM_A}, \\ f \circ \eta_{M_A} &= \mathrm{id}_{M_A}. \end{aligned}$$

Hence elements of $\mathrm{MC}_E(A)$ correspond to $(B \otimes A)$ -algebra structures on $M \otimes A$, compatible with R_0 (recall that, for $f \in \mathrm{MC}_E$, we automatically have $\sigma^0(f) = 1$). Elements f_1, f_2 will give isomorphic ring structures precisely when we have $g \in E^0(A)$ such that

$$\begin{array}{ccc} \top M \otimes A & \xrightarrow{f_1} & M \otimes A \\ \top g \downarrow & & \downarrow g \\ \top M \otimes A & \xrightarrow{f_2} & M \otimes A \end{array}$$

commutes.

Therefore $\mathfrak{Def}_E \simeq \mathfrak{Def}_{R_0/B}$.

Remark 1.17. Observe that, if $\omega \in \mathrm{MC}_E(A)$ gives rise to a ring R , then we have an isomorphism of cosimplicial complexes

$$(E^n(A), \partial_\omega) \cong \mathrm{Hom}_{B \otimes A\text{-Alg}}(\perp_* R, R)_{\varepsilon_R \circ \varepsilon_{\perp R} \circ \dots \circ \varepsilon_{\perp^n R}},$$

where $\perp_n R = \perp^{n+1} R$. The natural product $(f, g) \mapsto f \circ \perp^{n+1} g$, for $f : \perp_n R \rightarrow R$, corresponds to the product $(\alpha, \beta) \mapsto \alpha * \omega * \beta$ in $E^*(A)$.

In particular, there is an isomorphism of cosimplicial complexes

$$\mathbf{C}^\bullet(E) \cong \mathrm{Der}_{B_0}(\perp_\bullet R_0, R_0) = \mathrm{Hom}_{R_0}(\mathbf{L}_\bullet^{R_0/B_0}, R_0),$$

where $\mathbf{L}_\bullet^{R_0/B_0}$ is one presentation of the cotangent complex. Illusie ([Ill71] and [Ill72]) and André ([And67]) define the cotangent complex using the resolution arising from the adjunction between algebras and sets, whereas we have used the adjunction between flat algebras and flat modules. However, Quillen ([Qui70]) proves that all simplicial resolutions give the same object in the derived category.

Therefore $H^*(E)$ is André-Quillen cohomology.

If R_0/B_0 is smooth, and lifts to some (μ -adic) smooth algebra R/B , then we have a morphism of SDCs

$$\mathrm{Hom}_{B\mathrm{FAlg}}(R, R)_{\mathrm{id}} \rightarrow E^n,$$

where the complex on the left has the same object at every level, the obvious multiplication map, and all face and degeneracy operators given by the identity. Since R/B is smooth, this is a quasi-isomorphism, so the first complex also describes deformations of R_0 .

1.2.2 Deformation of sheaves — Čech resolution

Take a sheaf of k -vector spaces \mathcal{M}_0 on a topological space X . The deformation functor will associate to A , flat sheaves \mathcal{M}_A of A -modules such that $\mathcal{M}_A \otimes k = \mathcal{M}_0$, modulo infinitesimal isomorphisms.

Take an open cover \mathfrak{U} of X such that $U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_n}$ is contractible for all i_1, \dots, i_n . Examples for which this is possible include X a manifold or a geometric simplicial complex. Set

$$X' = \coprod_i U_i.$$

We have inclusion maps $u_i : U_i \rightarrow X$, so may combine them to give $u : X' \rightarrow X$.

Now, since flat modules do not deform, there are, up to isomorphism, unique μ -adic Λ -modules \mathcal{N}_i lifting $\Gamma(U_i, \mathcal{M}_0)$. Combining these, we obtain a sheaf \mathcal{N} on X' with $\mathcal{N} \otimes k = u^{-1}\mathcal{M}_0$, and there is an adjunction

$$\mathrm{Shf}(X') \begin{array}{c} \xrightarrow{u_*} \\ \xleftarrow{u^{-1}} \end{array} \mathrm{Shf}(X),$$

with unit

$$\begin{aligned} \alpha : \mathcal{F} &\rightarrow u_* u^{-1} \mathcal{F}, \\ \Gamma(U, \mathcal{F}) &\rightarrow \Gamma(U, u_* u^{-1} \mathcal{F}) = \prod_i \Gamma(U \cap U_i, \mathcal{F}) \\ f &\mapsto \prod_i \rho_{U, U \cap U_i}(f), \end{aligned}$$

and co-unit

$$\begin{aligned} \gamma : u^{-1} u_* \mathcal{G} &\rightarrow \mathcal{G}, \\ \prod_j (\mathcal{G}_j)|_{U_{j_i}} = \Gamma(U_i, u^{-1} u_* \mathcal{G}) &\rightarrow \mathcal{G}_i \\ g &\mapsto g_i \in \mathcal{G}_i = \mathcal{G}_i. \end{aligned}$$

Definition 1.18. We define the SDC E^* by:

$$E^n = \mathrm{Hom}_{\Lambda}(\mathcal{N}, (u^{-1} u_*)^n(\mathcal{N}))_{\alpha^n},$$

where $\alpha^n = u^{-1}(\alpha_{(u_* u^{-1})^{n-1} \mathcal{M}_0} \circ \dots \circ \alpha_{\mathcal{M}_0})$, with the SDC structure arising exactly as in the previous section (modulo contravariance).

Now, to give a sheaf \mathcal{M} of A -modules on X , such that $u^{-1}\mathcal{M} = \mathcal{N} \otimes_{\Lambda} A$, is to give transition functions

$$\theta_{ji} : \mathcal{N}_i \otimes A \rightarrow (\mathcal{N}_j \otimes A)|_{U_{ij}},$$

satisfying

$$\theta_{kj} \circ \theta_{ji} = \theta_{ki}; \quad \theta_{ii} = \text{id}.$$

This is precisely the same as a map

$$\Theta : \mathcal{N} \otimes A \rightarrow u^{-1}u_*(\mathcal{N} \otimes A),$$

satisfying

$$(u^{-1}u_*\Theta) \circ \Theta = u^{-1}\alpha_{u_*\mathcal{N}A} \circ \Theta; \quad \gamma_{\mathcal{N}} \circ \Theta = \text{id}_{\mathcal{N}}.$$

If we include the additional condition that these transition functions must be compatible with those of \mathcal{M}_0 , mod \mathfrak{m}_A , then we have precisely an element of $\text{MC}_E(A)$. It is easy to see that equivalences of transition functions correspond to elements of $E^0(A)$, acting via the adjoint action.

Note that we may regard E^n as

$$\Gamma(X', \mathcal{H}om_{\Lambda}(\mathcal{N}, (u^{-1}u_*)^n(\mathcal{N}))_{\alpha^n}),$$

and that setting

$$\mathcal{E}^n = u_*\mathcal{H}om_{\Lambda}(\mathcal{N}, (u^{-1}u_*)^n(\mathcal{N}))_{\alpha^n},$$

with tangent space $C^n(\mathcal{E})$, we have that the chain complex $C^{\bullet}(\mathcal{E})$ is a flabby resolution of the sheaf $\mathcal{H}om_k(\mathcal{M}_0, \mathcal{M}_0)$ (it is, in fact, a version of the Čech resolution), so that

$$H^i(E) = H^i(C^{\bullet}(E)) = \mathbb{H}^i(X, C^{\bullet}(\mathcal{E})) = H^i(X, \mathcal{H}om_k(\mathcal{M}_0, \mathcal{M}_0)),$$

as expected. Equivalently,

$$H^i(C^{\bullet}(E)) = \text{Ext}_k^i(\mathcal{M}_0, \mathcal{M}_0) = H^i(X, \mathcal{H}om_k(\mathcal{M}_0, \mathcal{M}_0)),$$

since we are working with sheaves of vector spaces over k , so $\mathcal{H}om_k$ is exact.

2 A Procedure for Computing SDCs

In this section we make formal the approach which has been used so far to compute SDCs. The idea is that we throw away properties of the object which we wish to deform, until we obtain something whose deformations are trivial.

Throughout this section, we will encounter functors $\mathcal{D} : \mathcal{C}_{\Lambda} \rightarrow \text{Cat}$. We will *not* require that these functors satisfy (H0) (the condition that $F(k) = \bullet$).

Definition 2.1. Given a functor $\mathcal{D} : \mathcal{C}_{\Lambda} \rightarrow \text{Cat}$, and an object $D \in \text{Ob } \mathcal{D}(k)$, define $\mathfrak{D}\mathfrak{e}\mathfrak{f}_{\mathcal{D}, D} : \mathcal{C}_{\Lambda} \rightarrow \text{Grpd}$ by setting $\mathfrak{D}\mathfrak{e}\mathfrak{f}_{\mathcal{D}, D}(A)$ to be the fibre of $\mathcal{D}(A) \rightarrow \mathcal{D}(k)$ over (D, id) .

All the deformation problems we encounter are of this form.

Definition 2.2. We say a functor $\mathcal{B} : \mathcal{C}_{\Lambda} \rightarrow \text{Cat}$ has uniformly trivial deformation theory if

1. for all $A \in \mathcal{C}_\Lambda$ and all $B_1, B_2 \in \text{Ob } \mathcal{B}(A)$, the functor $\text{Mor}_{\mathcal{B}}(B_1, B_2) : \mathcal{C}_A \rightarrow \text{Set}$ of morphisms from B_1 to B_2 is smooth and homogeneous;
2. for $A' \twoheadrightarrow A$ in \mathcal{C}_Λ , $\mathcal{B}(A') \rightarrow \mathcal{B}(A)$ is essentially surjective.

A typical example of such a functor is that which sends A to the category of flat A -modules.

Observe that, if \mathcal{B} is uniformly trivial, then given $B \in \text{Ob } \mathcal{B}(k)$, we may lift it to $\tilde{B} \in \text{Ob } \mathcal{B}(\Lambda)$, and we have an equivalence of functors of groupoids

$$(\tilde{B}, \text{Mor}_{\mathcal{B}}(\tilde{B}, \tilde{B})) \xrightarrow{\sim} \mathfrak{Def}_{\mathcal{B}, B}.$$

2.1 The SDC associated to a monadic adjunction

Assume we have a monadic adjunction

$$\mathcal{D} \begin{array}{c} \xrightarrow{U} \\ \leftarrow \top \\ \xleftarrow{F} \end{array} \mathcal{B},$$

with unit $\eta : \text{id} \rightarrow UF = \top$ and co-unit $\varepsilon : FU \rightarrow \text{id}$, such that \mathcal{B} has uniformly trivial deformation theory. Then, given $D \in \mathcal{D}(k)$, let B be any lift of UD to $\mathcal{B}(\Lambda)$, and define the SDC E^* by

$$E^n = \text{Hom}_{\mathcal{B}}(\top^n B, B)_{U(\varepsilon_D \circ \dots \circ \varepsilon_{\perp^{n-1} D})},$$

the fibre of

$$\text{Hom}_{\mathcal{B}}(\top^n B, B) \rightarrow \text{Hom}_{\mathcal{B}(k)}(\top^n B(k), B(k))$$

over $U(\varepsilon_D \circ \dots \circ \varepsilon_{\perp^{n-1} D})$.

We make E^* into an SDC by giving it the product $g * h = g \circ \top^n h$, and \check{E}_* is the dual complex to the canonical augmented complex $B \rightarrow \top^* B$. Explicitly, for $g \in E^n$,

$$\begin{aligned} \partial^i(g) &= g \circ \top^{i-1} U \varepsilon_{F \top^{n-i} B} \\ \sigma^i(g) &= g \circ \top^i \eta_{\top^{n-i-1} B}. \end{aligned}$$

Theorem 2.3. *With the notation as above, we have an equivalence of functors of groupoids*

$$\mathfrak{Def}_{\mathcal{D}, D} \xrightarrow{\sim} \mathfrak{Def}_E.$$

Proof. Since the $F \dashv U$ is monadic, we have an equivalence $K : \mathcal{D} \rightarrow \mathcal{B}^\top$. Now observe that MC_E is precisely the fibre of

$$\text{Ob } \mathcal{B}^\top \rightarrow \text{Ob } \mathcal{B}^\top(k)$$

over $K(D)$ (note the comment after Lemma 1.7 gives $\sigma^0(\omega) = 1$), with morphisms in the fibre

$$\mathcal{B}^\top \rightarrow \mathcal{B}^\top(k)$$

over $(K(D), \text{id})$ corresponding to E^0 acting on MC_E via the adjoint action. Hence

$$\mathfrak{Def}_E = \mathfrak{Def}_{\mathcal{B}^\top, K(D)}.$$

But $K : \mathcal{D} \rightarrow \mathcal{B}^\top$ is an equivalence, giving an equivalence

$$K : \mathfrak{Def}_{\mathcal{D}, \mathcal{D}} \xrightarrow{\sim} \mathfrak{Def}_{\mathcal{B}^\top, K(\mathcal{D})} = \mathfrak{Def}_E.$$

□

Observe that this result is precisely the same as the description above of deformations of algebras, and that the dual result (for comonadic adjunctions) describes deformations of sheaves.

2.1.1 Deformation of sheaves — Godement resolution

Take a sheaf of k -vector spaces \mathcal{M}_0 on a site X with enough points. In examples, we will be considering the Zariski or étale sites of a scheme, or the site associated to a topological space. The deformation functor will associate to A , sheaves \mathcal{M}_A of flat A -modules such that $\mathcal{M}_A \otimes k = \mathcal{M}_0$, modulo infinitesimal isomorphisms.

Let X' be the set of points of X . Since X has enough points, the inverse image functor $\mathrm{Shf}(X) \rightarrow \prod_{x \in X'} \mathrm{Shf}(x)$ reflects isomorphisms. Explicitly, this says that a morphism $\theta : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves on X is an isomorphism whenever the morphisms $\theta_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ are for all $x \in X'$. In the reasoning which follows, it will suffice to replace X' by any subset with this property.

If we are working on the Zariski site, we may take $X' = \coprod_{x \in X} x$. On the étale site, we may take

$$X' = \coprod_{x \in X} \bar{x},$$

where for each $x \in X$, a geometric point $\bar{x} \rightarrow X$ has been chosen. For Jacobson schemes, we may consider only closed points $x \in X$. We define the category $\mathrm{Shf}(X')$ by $\mathrm{Shf}(X') := \prod_{x \in X'} \mathrm{Shf}(x)$.

There is an adjunction

$$\mathrm{Shf}(X') \begin{array}{c} \xrightarrow{u_*} \\ \xleftarrow{u^*} \end{array} \mathrm{Shf}(X),$$

the maps $u_x : x \rightarrow X$ combining to form maps $u^* : \mathrm{Shf}(X) \rightarrow \mathrm{Shf}(X')$, and $u_* : \mathrm{Shf}(X') \rightarrow \mathrm{Shf}(X)$ given by $u_* \mathcal{F} = \prod_{x \in X'} u_{x*} \mathcal{F}_x$. Observe that the category of flat A -modules on X' has uniformly trivial deformation theory.

It follows from Theorem 0.35 that this adjunction is comonadic, and from Theorem 2.5, the SDC governing this problem is

$$E^n = \mathrm{Hom}_\Lambda(\mathcal{N}, (u^* u_*)^n(\mathcal{N}))_{\alpha^n},$$

where \mathcal{N} is a flat μ -adic Λ -module on X' with $\mathcal{N} \otimes k = u^* \mathcal{M}_0$, and $\alpha^n = u^*(\alpha_{(u_* u^*)^{n-1}} \mathcal{M}_0 \circ \dots \circ \alpha_{\mathcal{M}_0})$.

To see this more clearly in the case of a topological space, observe, as in [God58], Remark II.4.3.2, that an element $n \in \Gamma(U, u^{-1} u_* \mathcal{N})$ can be represented by a function of the form

$$n(x_0, x_1) \in \mathcal{N}_{x_1},$$

defined on a set of the form

$$x_0 \in U; x_1 \in U(x_0),$$

where $U(x_0)$ is some open étale neighbourhood of x_0 in X . Two functions n_1, n_2 define the same section if they agree on some set of the above form.

Thus an element Θ of $\text{MC}_E(A)$ can be regarded as local transition functions; given $n \in \mathcal{N}_x$, we obtain, for some étale neighbourhood U of x , $\Theta(n)_{xy} \in \mathcal{N}_y$ for all $y \in U$, which can be regarded as lifting the germ and then projecting down to each stalk. The Maurer-Cartan equation is transitivity, and, as before, guarantees that $\Theta(n)_{xx} = n$.

Again, we get

$$\mathbb{H}^i(E) = \mathbb{H}^i(\mathbf{C}^\bullet(E)) = \mathbb{H}^i(X, \mathbf{C}^\bullet(\mathcal{E})) = \mathbb{H}^i(X, \mathcal{H}om_k(\mathcal{M}_0, \mathcal{M}_0)) = \text{Ext}_k^i(\mathcal{M}_0, \mathcal{M}_0),$$

as expected.

2.1.2 Deformation of co-algebras

Given a flat (co-associative) co-algebra (with co-unit) C_0/k , we wish to create an SDC describing flat deformations C_A/A such that $C_A \otimes_A k = C_0$, modulo infinitesimal isomorphisms.

There is, up to isomorphism, a unique flat μ -adic Λ -module M such that $M \otimes_\Lambda k = C_0$. There is an adjunction

$$A\text{-FCoAlg} \begin{array}{c} \xrightarrow{V} \\ \perp \\ \xleftarrow{G} \end{array} \text{AFMod},$$

between the category of flat co-algebras over A , and the category of flat modules over A , where V is the forgetful functor and the free functor G exists by the Special Adjoint Functor Theorem (since $A\text{-FCoAlg}$ has all colimits, and V preserves these). Note that in this case the free functor is hard to write down explicitly, but this is unnecessary for our purposes. See [Swe69] for such a description.

By the Theorem 0.35, this adjunction is comonadic, so that deformations of C_0 are given by the SDC

$$E^n = \text{Hom}_\Lambda(M, \perp^n M)_{U(\alpha_{\tau^{n-1}C_0} \circ \dots \circ \alpha_{C_0})}.$$

2.2 The general approach

In general, we will not be able to pass from the category \mathcal{D} to a category \mathcal{B} with uniformly trivial deformation theory via a single monadic or comonadic adjunction. However, we should be able to pass from \mathcal{D} to some \mathcal{B} via a chain of monadic and comonadic adjunctions.

Not only should we have monadic and comonadic adjunctions, but, informally, the forgetful functors should commute with one another. More precisely, assume that we have a diagram

$$\begin{array}{ccc} \mathcal{D} & \begin{array}{c} \xrightarrow{U} \\ \perp \\ \xleftarrow{F} \end{array} & \mathcal{E} \\ \begin{array}{c} \downarrow V \\ \uparrow G \end{array} & & \begin{array}{c} \downarrow V \\ \uparrow G \end{array} \\ \mathcal{A} & \begin{array}{c} \xrightarrow{U} \\ \perp \\ \xleftarrow{F} \end{array} & \mathcal{B}, \end{array}$$

where \mathcal{B} has uniformly trivial deformation theory, with $F \dashv U$ monadic and $G \vdash V$ comonadic. Let

$$\begin{aligned} \top_{\mathbf{h}} &= UF & \perp_{\mathbf{h}} &= FU \\ \perp_{\mathbf{v}} &= VG & \top_{\mathbf{v}} &= GV, \end{aligned}$$

with

$$\eta : 1 \rightarrow \top_{\mathbf{h}}, \quad \gamma : \perp_{\mathbf{v}} \rightarrow 1, \quad \varepsilon : \perp_{\mathbf{h}} \rightarrow 1 \text{ and } \alpha : 1 \rightarrow \top_{\mathbf{v}}.$$

The commutativity condition is that the following identities hold:

$$GU = UG \quad \text{or} \quad FV = VF \tag{1}$$

$$UV = VU \tag{2}$$

$$V\varepsilon = \varepsilon V \quad \text{or} \quad U\alpha = \alpha U \tag{3}$$

$$V\eta = \eta V \quad \text{or} \quad U\gamma = \gamma U. \tag{4}$$

Observe that the adjoint properties ensure that identities on the same line are equivalent.

Lemma 2.4. *For $E \in \mathcal{E}, A \in \mathcal{A}$, consider the diagram of isomorphisms given by the adjunctions:*

$$\begin{array}{ccccc} & & \text{Hom}_{\mathcal{D}}(FE, GA) & & \\ & \swarrow & & \searrow & \\ \text{Hom}_{\mathcal{A}}(VFE, A) & & & & \text{Hom}_{\mathcal{E}}(E, UGA) \\ \parallel & & & & \parallel \\ \text{Hom}_{\mathcal{A}}(FVE, A) & & & & \text{Hom}_{\mathcal{E}}(E, GUA) \\ & \swarrow & \text{Hom}_{\mathcal{B}}(VE, UA) & \searrow & \end{array}$$

The identities (1)–(4) ensure that all squares in this diagram commute. In particular, for any $B \in \mathcal{B}$, the map

$$\eta_B \circ \gamma_B : VGB \rightarrow UFB$$

corresponds to a map

$$\rho_B = \varepsilon_{GFB} \circ FG\eta_B = GF\gamma_B \circ \alpha_{GFB} : FGB \rightarrow GFB,$$

with

$$\begin{aligned} \gamma_{\top_{\mathbf{h}}B} \circ UV(\rho_B) \circ \eta_{\perp_{\mathbf{v}}B} &= \eta_B \circ \gamma_B, \\ U(\rho_B) \circ \eta_{GB} &= G(\eta_B), \\ \gamma_{FB} \circ V(\rho_B) &= F(\gamma_B). \end{aligned}$$

By the naturality of $\varepsilon : \perp_{\mathbf{h}} \rightarrow \text{id}$, we have $\varepsilon \circ (\perp_{\mathbf{h}}\varepsilon) = \varepsilon \circ (\varepsilon\perp_{\mathbf{h}})$, so we obtain a canonical map $\varepsilon^n : \perp_{\mathbf{h}}^n \rightarrow \text{id}$, given by any such composition of ε 's. [For instance

$$\varepsilon^n = \varepsilon \circ (\perp_{\mathbf{h}}\varepsilon) \circ \dots \circ (\perp_{\mathbf{h}}^{n-1}\varepsilon)$$

is one such composition.] We have similar maps for each of the units and co-units, giving:

$$\begin{aligned}\varepsilon^n : \perp_{\mathfrak{h}}^n &\rightarrow \text{id}, & \eta^n : \text{id} &\rightarrow \top_{\mathfrak{h}}^n \\ \alpha^n : \text{id} &\rightarrow \top_{\mathfrak{v}}^n, & \gamma^n : \perp_{\mathfrak{v}}^n &\rightarrow \text{id}\end{aligned}$$

Let

$$\delta = UV(\rho) : \top_{\mathfrak{h}}\perp_{\mathfrak{v}} \rightarrow \perp_{\mathfrak{v}}\top_{\mathfrak{h}}.$$

Since δ is natural,

$$(\delta\perp_{\mathfrak{v}}\top_{\mathfrak{h}}) \circ (\top_{\mathfrak{h}}\perp_{\mathfrak{v}}\delta) = (\perp_{\mathfrak{v}}\top_{\mathfrak{h}}\delta) \circ (\delta\top_{\mathfrak{h}}\perp_{\mathfrak{v}}).$$

Therefore any composition of δ 's gives us the same canonical map

$$\delta^{m,n} : \top_{\mathfrak{h}}^m\perp_{\mathfrak{v}}^n \rightarrow \perp_{\mathfrak{v}}^n\top_{\mathfrak{h}}^m.$$

Theorem 2.5. *Suppose we have a diagram as above. Then, for $D \in \text{Ob } \mathcal{D}(k)$, let B be any lift of $UV D \in \text{Ob } \mathcal{B}(k)$ to $\text{Ob } \mathcal{B}(\Lambda)$ (by the hypothesis on \mathcal{B} , such a lift must exist and be unique up to isomorphism). Set*

$$E^n = \text{Hom}_{\mathcal{B}}(\top_{\mathfrak{h}}^n B, \perp_{\mathfrak{v}}^n B)_{UV(\alpha_D^n \circ \varepsilon_D^n)}.$$

We give E^* the multiplication

$$g * h = \perp_{\mathfrak{v}}^n(g) \circ \delta^{m,n} \circ \top_{\mathfrak{h}}^m(h),$$

and \check{E}_* gets its simplicial structure from the diagonal of the canonical double augmented complex

$$\text{Hom}_{\mathcal{B}}(B \rightarrow B^*, B_* \rightarrow B).$$

Explicitly, for $g \in E^n$,

$$\begin{aligned}\partial^i(g) &= \perp_{\mathfrak{v}}^{n-i} V \alpha_{G\perp_{\mathfrak{v}}^{i-1} B} \circ g \circ \top_{\mathfrak{h}}^{i-1} U \varepsilon_{F\top_{\mathfrak{h}}^{n-i} B} \\ \sigma^i(g) &= \perp_{\mathfrak{v}}^{n-i-1} \gamma_{\perp_{\mathfrak{v}}^i B} \circ g \circ \top_{\mathfrak{h}}^i \eta_{\top_{\mathfrak{h}}^{n-i-1} B}.\end{aligned}$$

Then

$$\mathfrak{D}\text{ef}_{\mathcal{D}, D} \simeq \mathfrak{D}\text{ef}_E.$$

Proof. First observe that this is, indeed, an SDC. We need to check that the product is associative. For $f \in S^l, g \in S^m, h \in S^n$, we have

$$\begin{aligned}f * (g * h) &= \perp_{\mathfrak{v}}^{m+n} f \circ \delta^{l, m+n} \circ \top_{\mathfrak{h}}^l (\perp_{\mathfrak{v}}^n g \circ \delta^{m,n} \circ \top_{\mathfrak{h}}^m h) \\ &= \perp_{\mathfrak{v}}^{m+n} f \circ (\perp_{\mathfrak{v}}^n \delta^{l,m} \circ \delta^{l,n} \perp_{\mathfrak{v}}^m) \circ \top_{\mathfrak{h}}^l \perp_{\mathfrak{v}}^n g \circ \top_{\mathfrak{h}}^l \delta^{m,n} \circ \top_{\mathfrak{h}}^{l+m} h \\ &= \perp_{\mathfrak{v}}^{m+n} f \circ \perp_{\mathfrak{v}}^n \delta^{l,m} \circ \perp_{\mathfrak{v}}^n \top_{\mathfrak{h}}^l g \circ \delta^{l,n} \top_{\mathfrak{h}}^m \circ \top_{\mathfrak{h}}^l \delta^{m,n} \circ \top_{\mathfrak{h}}^{l+m} h \\ &= \perp_{\mathfrak{v}}^n (\perp_{\mathfrak{v}}^m f \circ \delta^{l,m} \circ \top_{\mathfrak{h}}^l g) \circ \delta^{l+m,n} \circ \top_{\mathfrak{h}}^{l+m} h \\ &= (f * g) * h.\end{aligned}$$

We now have a lemma:

Lemma 2.6. *To give a map $f : \mathbb{T}_h B \rightarrow \perp_v B$ satisfying the Maurer-Cartan equation*

$$\perp_v f \circ \delta \circ \mathbb{T}_h f = V\alpha_{GB} \circ f \circ U\varepsilon_{FB}$$

is the same as giving maps $\theta : \mathbb{T}_h B \rightarrow B$ and $\phi : B \rightarrow \perp_v B$ satisfying the Maurer-Cartan equations:

$$\begin{aligned}\theta \circ \mathbb{T}_h \theta &= \theta \circ U\varepsilon_{FB} \\ \perp_v \phi \circ \phi &= V\alpha_{GB} \circ \phi,\end{aligned}$$

and the compatibility condition

$$\perp_v \theta \circ \delta_B \circ \mathbb{T}_h \phi = \phi \circ \theta.$$

Proof. (Of Lemma.)

Given such an f , we have

$$\begin{aligned}f &= \gamma_{\perp_v B} \circ V\alpha_{GB} \circ f \circ U\varepsilon_{FB} \circ \eta_{\mathbb{T}_h B} \\ &= \gamma_{\perp_v B} \circ \perp_v f \circ \delta_B \circ \mathbb{T}_h f \circ \eta_{\mathbb{T}_h B} \\ &= f \circ \gamma_{\mathbb{T}_h B} \circ \delta_B \circ \eta_{\perp_v B} \circ f \\ &= f \circ \eta_B \circ \gamma_B \circ f.\end{aligned}$$

Let $\phi = f \circ \eta_B$, and $\theta = \gamma_B \circ f$. Now,

$$\begin{aligned}\gamma_B \circ \gamma_{\perp_v B} \circ \perp_v f \circ \delta_B \circ \mathbb{T}_h f &= \gamma_B \circ \gamma_{\perp_v B} \circ V\alpha_{GB} \circ f \circ U\varepsilon_{FB} \\ \gamma_B \circ f \circ \gamma_{\mathbb{T}_h B} \circ \delta_B \circ \mathbb{T}_h f &= \gamma_B \circ f \circ U\varepsilon_{FB} \\ \gamma_B \circ f \circ U(\gamma_{FB} \circ V(\rho_B)) \circ \mathbb{T}_h f &= \gamma_B \circ f \circ U\varepsilon_{FB} \\ \gamma_B \circ f \circ \mathbb{T}_h \gamma_B \circ \mathbb{T}_h f &= \gamma_B \circ f \circ U\varepsilon_{FB} \\ \theta \circ \mathbb{T}_h \theta &= \theta \circ U\varepsilon_{FB},\end{aligned}$$

using the Maurer-Cartan equations and Lemma 2.4. Similarly we obtain the Maurer-Cartan equation for ϕ , and finally

$$\begin{aligned}\perp_v \theta \circ \delta_B \circ \mathbb{T}_h \phi &= \perp_v \gamma_B \circ \perp_v f \circ \delta_B \circ \mathbb{T}_h f \circ \mathbb{T}_h \eta_B \\ &= \perp_v \gamma_B \circ V\alpha_{GB} \circ f \circ U\varepsilon_{FB} \circ \mathbb{T}_h \eta_B \\ &= f \\ &= \phi \circ \theta.\end{aligned}$$

Conversely, given such θ and ϕ , set $f = \phi \circ \theta$. We obtain:

$$\begin{aligned}\perp_v f \circ \delta_B \circ \mathbb{T}_h f &= \perp_v \phi \circ \perp_v \theta \circ \delta_B \circ \mathbb{T}_h \phi \circ \mathbb{T}_h \theta \\ &= \perp_v \phi \circ \phi \circ \theta \circ \mathbb{T}_h \theta \\ &= V\alpha_{GB} \circ \phi \circ \theta \circ U\varepsilon_{FB} \\ &= V\alpha_{GB} \circ f \circ U\varepsilon_{FB}.\end{aligned}$$

□

Recall, from the comment after Lemma 1.7, that every element f of $\text{MC}_E(A)$ must satisfy $\sigma^0(f) = 1$, which in this case is $\gamma_B \circ f \circ \eta_B = \text{id}$. This means that the pair (θ, ϕ) above satisfy $\theta \circ \eta_B = \text{id}$, and $\gamma_B \circ \phi = \text{id}$.

We obtain an adjunction

$$\mathcal{E}^{\top_h} \begin{array}{c} \xrightarrow{V} \\ \xleftarrow[G]{\perp} \end{array} \mathcal{B}^{\top_h},$$

$$\begin{array}{ccc} \left(\begin{array}{c} \top_h E \\ f \downarrow \\ E \end{array} \right) & \mapsto & \left(\begin{array}{c} \top_h VE \\ Vf \downarrow \\ VE \end{array} \right) \\ \left(\begin{array}{c} \top_h GB \\ G(g) \circ U(\rho_B) \downarrow \\ GB \end{array} \right) & \leftarrow & \left(\begin{array}{c} \top_h B \\ g \downarrow \\ B \end{array} \right), \end{array}$$

and such a pair (θ, ϕ) is precisely the same as an element of $(\mathcal{B}^{\top_h})^{\perp_v}$. We also have a correspondence between the action of E^0 on MC_E and morphisms in $(\mathcal{B}^{\top_h})^{\perp_v}$, so that \mathfrak{Def}_E corresponds to the fibre of

$$(\mathcal{B}^{\top_h})^{\perp_v} \rightarrow (\mathcal{B}^{\top_h})^{\perp_v}(k)$$

over the image of (D, id) .

Now, the equivalence

$$K : \mathcal{A} \rightarrow \mathcal{B}^{\top_h}$$

$$A \mapsto \left(\begin{array}{c} \top_h UA \\ U\varepsilon_A \downarrow \\ UA \end{array} \right)$$

arising from the monadic adjunction $F \dashv U$ satisfies

$$\begin{aligned} \perp_v KA &= VG \left(\begin{array}{c} \top_h UA \\ U\varepsilon_A \downarrow \\ UA \end{array} \right) \\ &= \left(\begin{array}{c} \top_h \perp_v UA \\ \perp_v U\varepsilon_A \circ VU(\rho_{UA}) \downarrow \\ \perp_v UA \end{array} \right). \end{aligned}$$

Moreover,

$$\begin{aligned} \perp_v U\varepsilon_A \circ VU(\rho_{UA}) &= UV(G\varepsilon_A \circ (\varepsilon_{GFUA} \circ FG\eta_{UA})) \\ &= UV((\varepsilon_{GA} \circ FUG\varepsilon_A) \circ FG\eta_{UA}) \\ &= UV(\varepsilon_{GA} \circ FG(U\varepsilon_A \circ \eta_{UA})) \\ &= UV(\varepsilon_{GA}), \end{aligned}$$

so

$$\perp_v KA = \left(\begin{array}{c} \top_h \perp_v UA \\ UV(\varepsilon_{GA}) \downarrow \\ \perp_v UA \end{array} \right) = K \perp_v A.$$

This induces an equivalence of categories

$$\mathcal{A}^{\perp_v} \xrightarrow{K_h} (\mathcal{B}^{\top_h})^{\perp_v}.$$

But $G \vdash V$ is comonadic, so

$$\mathcal{D} \xrightarrow{K_v} \mathcal{A}^{\perp_v}$$

is an equivalence. Thus we have an equivalence of groupoids

$$\mathfrak{Def}_{\mathcal{D}, \mathcal{D}} \xrightarrow{\sim} \mathfrak{Def}_{\mathcal{A}^{\perp_v}, K_v \mathcal{D}} \xrightarrow{\sim} \mathfrak{Def}_{(\mathcal{B}^{\top_h})^{\perp_v}, K_h K_v \mathcal{D}} = \mathfrak{Def}_E.$$

□

Given a category \mathcal{D} , we now need a procedure for finding a category \mathcal{B} with uniformly trivial deformation theory. We do this by thinking of the functors U and V as being forgetful functors, and looking for structure in \mathcal{D} to discard.

Definition 2.7. Given a category $\mathcal{B} : \mathcal{C}_\Lambda \rightarrow \text{Cat}$, define a set of structures Σ over \mathcal{B} to consist of the following data:

1. A finite set $\Sigma = \Sigma^+ \sqcup \Sigma^-$,
2. For each subset $S \subset \Sigma$, a category \mathcal{B}^S , with $\mathcal{B}^\emptyset = \mathcal{B}$,
3. (a) for each $s \in \Sigma^+$ and each $S \subset \Sigma$ not containing s , a monadic adjunction

$$\mathcal{B}^{S \cup \{s\}} \begin{array}{c} \xrightarrow{U_s} \\ \leftarrow \top \\ \xleftarrow{F_s} \end{array} \mathcal{B}^S.$$

- (b) for each $s \in \Sigma^-$ and each $S \subset \Sigma$ not containing s , a comonadic adjunction

$$\mathcal{B}^{S \cup \{s\}} \begin{array}{c} \xrightarrow{V_s} \\ \leftarrow \perp \\ \xleftarrow{G_s} \end{array} \mathcal{B}^S,$$

satisfying the commutativity conditions given below.

For $s \in \Sigma^+$ we write

$$\top_s = U_s F_s \quad \text{and} \quad \perp_s = F_s U_s,$$

with

$$\eta_s : 1 \rightarrow \top_s \quad \text{and} \quad \varepsilon_s : \perp_s \rightarrow 1.$$

For $s \in \Sigma^-$ we write

$$\perp_s = V_s G_s \quad \text{and} \quad \top_s = G_s V_s,$$

with

$$\gamma_s : \perp_s \rightarrow 1, \quad \text{and} \quad \alpha_s : 1 \rightarrow \top_s.$$

The commutativity conditions are that the following hold:

1. For each distinct pair $s_1, s_2 \in \Sigma^+$, and each $S \subset \Sigma$ not containing s_1, s_2 , the morphisms in the diagram

$$\begin{array}{ccc}
\mathcal{B}^{S \cup \{s_1, s_2\}} & \begin{array}{c} \xrightarrow{U_1} \\ \dashv \\ \xleftarrow{F_1} \end{array} & \mathcal{B}^{S \cup \{s_2\}} \\
U_2 \downarrow \dashv \uparrow F_2 & & U_2 \downarrow \dashv \uparrow F_2 \\
\mathcal{B}^{S \cup \{s_1\}} & \begin{array}{c} \xrightarrow{U_1} \\ \dashv \\ \xleftarrow{F_1} \end{array} & \mathcal{B}^S
\end{array}$$

satisfy

$$F_1 F_2 = F_2 F_1 \quad \text{or} \quad U_1 U_2 = U_2 U_1$$

$$U_1 \varepsilon_2 = \varepsilon_2 U_1$$

$$U_2 \varepsilon_1 = \varepsilon_1 U_2$$

$$U_1 \eta_2 = \eta_2 U_1$$

$$U_2 \eta_1 = \eta_1 U_2.$$

2. For each distinct pair $s_1, s_2 \in \Sigma^-$, and each $S \subset \Sigma$ not containing s_1, s_2 , the morphisms in the diagram

$$\begin{array}{ccc}
\mathcal{B}^{S \cup \{s_1, s_2\}} & \begin{array}{c} \xrightarrow{V_1} \\ \perp \\ \xleftarrow{G_1} \end{array} & \mathcal{B}^{S \cup \{s_2\}} \\
V_2 \downarrow \dashv \uparrow G_2 & & V_2 \downarrow \dashv \uparrow G_2 \\
\mathcal{B}^{S \cup \{s_1\}} & \begin{array}{c} \xrightarrow{V_1} \\ \perp \\ \xleftarrow{G_1} \end{array} & \mathcal{B}^S
\end{array}$$

satisfy

$$G_1 G_2 = G_2 G_1 \quad \text{or} \quad V_1 V_2 = V_2 V_1$$

$$V_1 \alpha_2 = \alpha_2 V_1$$

$$V_2 \alpha_1 = \alpha_1 V_2$$

$$V_1 \gamma_2 = \gamma_2 V_1$$

$$V_2 \gamma_1 = \gamma_1 V_2.$$

3. For each $s_1 \in \Sigma^+$ and $s_2 \in \Sigma^-$, and each $S \subset \Sigma$ not containing s_1, s_2 , the morphisms in the diagram

$$\begin{array}{ccc}
\mathcal{B}^{S \cup \{s_1, s_2\}} & \begin{array}{c} \xrightarrow{U} \\ \dashv \\ \xleftarrow{F} \end{array} & \mathcal{B}^{S \cup \{s_2\}} \\
V \downarrow \dashv \uparrow G & & V \downarrow \dashv \uparrow G \\
\mathcal{B}^{S \cup \{s_1\}} & \begin{array}{c} \xrightarrow{U} \\ \dashv \\ \xleftarrow{F} \end{array} & \mathcal{B}^S,
\end{array}$$

satisfy

$$GU = UG \quad \text{or} \quad FV = VF$$

$$UV = VU$$

$$V\varepsilon = \varepsilon V \quad \text{or} \quad U\alpha = \alpha U$$

$$V\eta = \eta V \quad \text{or} \quad U\gamma = \gamma U.$$

Lemma 2.8. *Given a set of structures Σ over a category \mathcal{B} , we have a diagram*

$$\begin{array}{ccc}
 \mathcal{B}^\Sigma & \xrightleftharpoons[U]{F} & \mathcal{B}^{\Sigma^-} \\
 \downarrow \dashv G & & \downarrow \dashv G \\
 \mathcal{B}^{\Sigma^+} & \xrightleftharpoons[U]{F} & \mathcal{B},
 \end{array}$$

satisfying the equations (1)–(4) on p. 26, with $F \dashv U$ monadic and $G \vdash V$ comonadic.

Proof. We define F (resp. U) to be the composition of the F_s (resp U_s) for all $s \in \Sigma^+$, noting that the order of composition does not matter, since these functors commute with one another. We define G and V analogously. It is immediate that $F \dashv U$ and $G \vdash V$ are adjoint pairs, and that the commutativity conditions (1)–(4) are satisfied.

It thus remains only to show that $F \dashv U$ is monadic, and $G \vdash V$ comonadic. This can be done by using a similar approach to the proof of Theorem 2.5. The statement is that $\mathcal{B}^{\Sigma^+} \simeq \mathcal{B}^{\top_+}$, where $\top_+ = FU$, which is proved by induction on the cardinality of Σ^+ . \square

Remarks 2.9. 1. This lemma shows that every set of structures can be replaced by a set of at most two elements, so it might seem that introducing the notion of a set of structures was not helpful. However, it is frequently easier to find the individual adjunctions than the composites, as will be seen in later examples (notably the deformation of a group scheme in Section 3.6).

2. When considering those deformation problems for which DGLAs have been successfully constructed, it seems that there is a set of structures which is either wholly monadic or wholly comonadic.

2.2.1 Deformation of Hopf algebras

Given a flat (associative, commutative, co-associative) Hopf algebra (with unit and co-unit) R_0/k , we wish to create an SDC describing flat deformations R_A/A of R_0 such that $R_A \otimes_A k = R_0$, modulo infinitesimal isomorphisms.

The structures are:

$$\Sigma^+ = \{\text{Algebra}\}, \quad \Sigma^- = \{\text{Co-Algebra}\},$$

over the category of flat A -modules.

This gives the following commutative diagram of monadic and comonadic adjunctions:

$$\begin{array}{ccc}
 A\text{-FHopfAlg} & \xrightleftharpoons[\text{Symm}_A]{\top} & A\text{-FCoAlg} \\
 \downarrow \dashv G & & \downarrow \dashv G \\
 A\text{FAlg} & \xrightleftharpoons[\text{Symm}_A]{\top} & A\text{FMod},
 \end{array}$$

where $A\text{-FHopfAlg}$ is the category of flat Hopf algebras over A , $A\text{-FCoAlg}$ is the category of flat co-associative co-algebras with co-unit and co-inverse over A , and G is the free co-algebra functor of Section 2.1.2. Since we cannot describe G explicitly, we use the alternative form of the axioms involving Symm_A instead, when verifying the conditions of Theorem 2.5.

We thus obtain the SDC

$$E^n = \text{Hom}_\Lambda((\text{Symm}_\Lambda)^n M, G^n M)_{\alpha^n \circ \varepsilon^n},$$

where M is the flat μ -adic Λ -module (unique up to isomorphism) lifting the k -vector space R_0 . with α^n, ε^n the canonical maps

$$\varepsilon^n : (\text{Symm}_k)^n R_0 \rightarrow R_0, \quad \alpha^n : R_0 \rightarrow G^n R_0$$

associated to R_0 .

3 Deformation of Schemes

Given a flat μ -adic system of schemes $S_n/\text{Spec } \Lambda_n$, and a flat scheme X_0/S_0 , our deformation functor consists of flat schemes X_A/S_A such that $X_A \times_{\text{Spec } A} \text{Spec } k = X_0$, modulo infinitesimal isomorphisms (isomorphisms which pull back to the identity on X_0), where $\mathfrak{S} = \varinjlim S_n$ and $S_A = \mathfrak{S} \times_{\text{Spf } \Lambda} \text{Spec } A$:

$$\begin{array}{ccc} X_0 & \longrightarrow & X_A \\ f \text{ flat} \downarrow & & \downarrow \text{flat} \\ S_0 & \longrightarrow & S_A. \end{array}$$

Since the topological space $|X_0|$ underlying X_0 does not deform, this is just a question of deforming the sheaf \mathcal{O}_{X_0} of algebras. Observe that, since X_0 is flat over S_0 , any scheme X_A/S_A deforming X , and flat over A , will necessarily be flat over S_A . It therefore suffices to deform \mathcal{O}_{X_0} as an \mathcal{O}_{S_A} -algebra, flat over A .

In the notation of Section 2.2, we have the set of structures

$$\Sigma^+ = \{\text{Algebra}, \mathcal{O}_{S_A}\text{-Module}\}, \quad \Sigma^- = \{X_0\text{-Sheaf}\},$$

over the category of sheaves of flat A -modules on X'_0 , where X'_0 is defined as in Section 2.1.1. To these structures correspond the monadic and comonadic adjunctions

$$\{\text{Symm}_A \dashv U, \mathcal{O}_{\mathfrak{S} \otimes \Lambda} \dashv U\}, \quad \{u_* \vdash u^{-1}\},$$

where U denotes the relevant forgetful functor.

This yields the following diagram of Cat-valued functors:

$$\begin{array}{ccc} \mathcal{O}_{S_A} \backslash (\text{AFAlg}(X_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{S} \otimes \Lambda} \text{Symm}_A]{\top} & \text{AFMod}(X_0) \\ \begin{array}{c} \downarrow u^{-1} \\ \uparrow u_* \end{array} & & \begin{array}{c} \downarrow u^{-1} \\ \uparrow u_* \end{array} \\ \mathcal{O}_{S_A} \backslash (\text{AFAlg}(X'_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{S} \otimes \Lambda} \text{Symm}_A]{\top} & \text{AFMod}(X'_0). \end{array}$$

where we write $\mathcal{O}_{\mathfrak{S}}$ for the sheaf $f^{-1}\mathcal{O}_{\mathfrak{S}}$ (resp. $u^{-1}f^{-1}\mathcal{O}_{\mathfrak{S}}$) on X_0 (resp. X'_0), and $\mathcal{O}_{S_A} \setminus (\text{AFAlg})$ consists of those \mathcal{O}_{Y_A} -algebras which are flat over A . The only non-trivial commutativity condition is the observation that pull-backs commute with tensor operations.

Hence, by Theorem 2.5, deformations are described by the SDC

$$E^n = \text{Hom}_{\Lambda}((\mathcal{O}_{\mathfrak{S}} \otimes_{\Lambda} \text{Symm}_{\Lambda})^n \mathcal{M}, (u^{-1}u_*)^n \mathcal{M})_{u^{-1}(\alpha^n \circ \varepsilon^n)},$$

where \mathcal{M} is a lift of the sheaf $u^{-1}\mathcal{O}_{X_0}$ of vector spaces on X'_0 to a sheaf of flat μ -adic Λ -modules, and α^n, ε^n are the canonical maps $\alpha^n_{\mathcal{O}_{X_0}}, \varepsilon^n_{\mathcal{O}_{X_0}}$ in Theorem 2.5, given by the adjunctions.

Remark 3.1. Observe that the description above allows us to construct an SDC governing deformations of a sheaf of algebras on any site with enough points. This is more general than the problem in [Hin05] for which a DGLA was constructed.

Define a sheaf of SDCs on X by

$$\mathcal{E}^n = u_* \mathcal{H}om_{\Lambda}((\mathcal{O}_{\mathfrak{S}} \otimes_{\Lambda} \text{Symm}_{\Lambda})^n \mathcal{M}, (u^{-1}u_*)^n \mathcal{M})_{u^{-1}(\alpha^n \circ \varepsilon^n)}.$$

Combining the observations in Sections 1.2.1, 2.1.1, we see that this has tangent space

$$\begin{aligned} C^n(\mathcal{E}) &= u_* \mathcal{D}er_{\mathcal{O}_{S_0}}((\mathcal{O}_{S_0} \otimes_k \text{Symm}_k)^{n+1} u^{-1}\mathcal{O}_{X_0}, (u^{-1}u_*)^n u^{-1}\mathcal{O}_{X_0}) \\ &\cong \mathcal{D}er_{\mathcal{O}_{S_0}}((\mathcal{O}_{S_0} \otimes_k \text{Symm}_k)^{n+1} \mathcal{O}_{X_0}, (u_* u^{-1})^{n+1} \mathcal{O}_{X_0}) \\ &\cong \mathcal{H}om_{\mathcal{O}_{X_0}}(\mathbf{L}_n^{X_0/S_0}, \mathcal{C}^n(\mathcal{O}_{X_0})), \end{aligned}$$

where $\mathbf{L}_{\bullet}^{X_0/S_0}$ is the standard form of the cotangent complex, as described in [Ill71], and \mathcal{C}^n denotes the Godement resolution.

Since $C^n(\mathcal{E})$ has the structure of a diagonal complex of a bicosimplicial complex, it follows from the Eilenberg-Zilber Theorem that it is quasi-isomorphic to the total complex of the double complex

$$\mathcal{H}om_{\mathcal{O}_{X_0}}(\mathbf{L}_{\bullet}^{X_0/S_0}, \mathcal{C}^{\bullet}(\mathcal{O}_{X_0}))$$

Therefore the cohomology of our SDC is

$$H^i(C^{\bullet}(E)) = H^i(\Gamma(X_0, C^{\bullet}(\mathcal{E}))) \cong \text{Ext}_{\mathcal{O}_{X_0}}^i(\mathbf{L}_{\bullet}^{X_0/S_0}, \mathcal{O}_{X_0}),$$

which is André-Quillen hypercohomology, the second isomorphism following because $\mathbf{L}_{\bullet}^{X_0/S_0}$ is locally projective as an \mathcal{O}_{X_0} -module, and $\mathcal{C}^{\bullet}(\mathcal{O}_{X_0})$ is flabby.

3.1 Separated Noetherian schemes

If X_0 is separated and Noetherian, then we may replace Godement resolutions by Čech resolutions. Take an open affine cover $(X_{\alpha})_{\alpha \in I}$ of X_0 , and set $\check{X} := \coprod_{\alpha \in I} X_{\alpha}$. We then have a diagram

$$\begin{array}{ccc} \check{X}' & \xrightarrow{\check{u}} & \check{X} \\ v' \downarrow & \searrow w & \downarrow v \\ X'_0 & \xrightarrow{u} & X_0. \end{array}$$

Since $v^{-1}\mathcal{O}_{X_0} = \mathcal{O}_{\check{X}}$ is a quasi-coherent sheaf on \check{X} , $\text{Ext}^i(\mathcal{O}_{\check{X}}, \mathcal{O}_{\check{X}}) = 0$ for all $i > 0$, so deformations of the sheaf $\mathcal{O}_{\check{X}}$ of k -vector spaces on \check{X} are unobstructed and we may lift $v^{-1}\mathcal{O}_{X_0}$ to some sheaf \mathcal{N} of flat μ -adic Λ -modules on \check{X} , unique up to non-unique isomorphism. There must also be (non-canonical) isomorphisms $\check{u}^{-1}\mathcal{N} \cong v'^{-1}\mathcal{M}$. If we define \check{E}^\bullet by

$$\check{E}^n = \text{Hom}_\Lambda((\mathcal{O}_{\mathfrak{S}} \otimes_\Lambda \text{Symm}_\Lambda)^n \mathcal{N}, (v^{-1}v_*)^n \mathcal{N})_{v^{-1}(\alpha^n \circ \varepsilon^n)},$$

and F^\bullet similarly, replacing v by w , then we have morphisms of SDCs

$$E^\bullet \xrightarrow{\gamma_{v'}} F^\bullet \xleftarrow{\gamma_{\check{u}}} \check{E}^\bullet.$$

To see that these are quasi-isomorphisms, let $\mathcal{T}^\bullet := \mathcal{H}om_{\mathcal{O}_{X_0}}(\mathbf{L}_{\bullet}^{X_0/S_0}, \mathcal{O}_{X_0})$, which is a complex of quasi-coherent sheaves on X_0 , and observe that the induced maps on cohomology are

$$\mathbb{H}^i(X_0, \mathcal{T}^\bullet) \rightarrow \mathbb{H}^i(X_0, \mathcal{T}^\bullet) \leftarrow \check{\mathbb{H}}^i(X_0, \mathcal{T}^\bullet).$$

The final map is an isomorphism since X_0 is separated and Noetherian, so Čech cohomology agrees with sheaf cohomology.

3.2 Smooth schemes

If X_0/S_0 is smooth, then the cohomology groups are just $\mathbb{H}^i(X_0, \mathcal{T}_{X_0/S_0})$, the cohomology of the tangent sheaf. If X_0 is also separated and Noetherian (for instance if S_0 is so), then we may lift \check{X} to some smooth μ -adic formal scheme $\check{\mathfrak{X}}$ over \mathfrak{S} .

Consider the diagram

$$\mathcal{O}_{S_A} \setminus (\text{AFAlg}(\check{X})) \begin{array}{c} \xrightarrow{v_*} \\ \leftarrow \text{T} \\ \xleftarrow{v^*} \end{array} \mathcal{O}_{S_A} \setminus (\text{AFAlg}(X_0));$$

although the former category does not have uniformly trivial deformation theory, all the morphisms in it which we encounter do, which gives us an SDC and canonical maps

$$\text{Hom}_{\mathcal{O}_{\mathfrak{S}}\text{-Alg}}(\mathcal{O}_{\check{\mathfrak{X}}}, (v^{-1}v_*)^n \mathcal{O}_{\check{\mathfrak{X}}})_{v^{-1}(\alpha^n)} \xrightarrow{(\varepsilon^n)^*} \check{E}^n,$$

which give a quasi-isomorphism of SDCs.

Explicitly, we may write our SDC as

$$E^n = \prod_{\alpha_0, \dots, \alpha_n \in I} \Gamma(X_{\alpha_0, \dots, \alpha_n}, \mathcal{H}om_{\mathcal{O}_{\mathfrak{S}}\text{-Alg}}(\mathcal{O}_{\check{\mathfrak{X}}_{\alpha_n}}, \mathcal{O}_{\check{\mathfrak{X}}_{\alpha_0}})_{\text{id}}),$$

where $X_{\alpha_0, \dots, \alpha_n} = \bigcap_{i=1}^n X_{\alpha_i}$, and $\check{\mathfrak{X}} = \coprod \check{\mathfrak{X}}_{\alpha}$. The product is given by

$$(\phi * \psi)_{\alpha_0, \dots, \alpha_{m+n}} = \phi_{\alpha_0, \dots, \alpha_m} \circ \psi_{\alpha_m, \dots, \alpha_{m+n}},$$

and operations are

$$\begin{aligned} (\partial^i \phi)_{\alpha_0, \dots, \alpha_{n+1}} &= \phi_{\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n+1}} \\ (\sigma^i \phi)_{\alpha_0, \dots, \alpha_{n-1}} &= \phi_{\alpha_0, \dots, \alpha_i, \alpha_i, \dots, \alpha_{n-1}} \end{aligned}$$

3.3 Deformation of a subscheme

The standard application of the previous section would be to set $S_n = \text{Spec } \Lambda_n$. However, exactly the same proof applies to a far wider class of problems. Rather than requiring both X_A/S_A and $\mathfrak{S}/\text{Spf } \Lambda$ to be flat, we only used the fact that $X_A/\text{Spec } A$ was flat (flatness of X_A/S_A then being automatic, since X_0/S_0 flat). This gives us the following immediate generalisation:

Given

$$\begin{array}{ccc} X_0 & \longrightarrow & Y_0 \\ & \searrow \text{flat} & \downarrow \text{flat} \\ & & S_0 \\ & & \downarrow \\ & & \text{Spec } k, \end{array}$$

for flat μ -adic families

$$\mathfrak{Y} \xrightarrow{\text{flat}} \mathfrak{S} \xrightarrow{\text{flat}} \text{Spf } \Lambda,$$

let $Y_A := \mathfrak{Y} \otimes_{\Lambda} A$, $S_A := \mathfrak{S} \otimes_{\Lambda} A$, and consider deformations

$$\begin{array}{ccccc} X_0 & \longrightarrow & Y_0 & & \\ & \searrow & \downarrow & \searrow & \\ & & S_0 & \longrightarrow & X_A & \longrightarrow & Y_A \\ & & & \searrow \text{flat} & \downarrow \text{flat} \\ & & & & S_A, \end{array}$$

such that $X_A \otimes_A k = X_0$.

Thus we need to consider \mathcal{O}_{Y_A} -algebras deforming \mathcal{O}_{X_0} , flat over \mathcal{O}_{S_A} . These are the same as \mathcal{O}_{Y_A} -algebras deforming \mathcal{O}_{X_0} , flat over A .

We have the set of structures

$$\Sigma^+ = \{\text{Algebra, } \mathcal{O}_{Y_A}\text{-Module}\}, \quad \Sigma^- = \{X_0\text{-Sheaf}\},$$

over the category $\text{AFMod}(X'_0)$, giving the diagram:

$$\begin{array}{ccc} \mathcal{O}_{Y_A} \backslash (\text{AFAlg}(X_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{Y} \otimes_{\Lambda} \text{Symm}_A}]{\top} & \text{AFMod}(X_0) \\ \downarrow u^{-1} \dashv u_* & & \downarrow u^{-1} \dashv u_* \\ \mathcal{O}_{Y_A} \backslash (\text{AFAlg}(X'_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{Y} \otimes_{\Lambda} \text{Symm}_A}]{\top} & \text{AFMod}(X'_0) \end{array}$$

and we obtain the same SDC as before, with \mathfrak{S} replaced by \mathfrak{Y} . Hence deformations are described by the SDC

$$E^n = \text{Hom}_{\Lambda}((\mathcal{O}_{\mathfrak{Y}} \otimes_{\Lambda} \text{Symm}_{\Lambda})^n \mathcal{M}, (u^{-1}u_*)^n \mathcal{M})_{u^{-1}(\alpha^n \circ \varepsilon^n)},$$

for a flat μ -adic Λ -module \mathcal{M} lifting $u^{-1}\mathcal{O}_{X_0}$.

In particular this applies to deformation of a subscheme $X_0 \hookrightarrow Y_0$, since a flat deformation of a subscheme as a Y_A -scheme will be a subscheme. If $X_0 \hookrightarrow Y_0$ is a regular embedding (e.g. if X_0 and Y_0 are both smooth over S_0), then the cohomology groups will be

$$H^i(\mathbf{C}^\bullet(E)) = H^{i+1}(X_0, \mathcal{N}_{X_0/Y_0}),$$

where \mathcal{N}_{X_0/Y_0} is the normal sheaf.

If X_0 and Y_0 are both smooth over S_0 , we can simplify the SDC of Section 3.1 still further. Consider the diagram

$$\begin{array}{ccc} \mathcal{O}_{Y_A} \backslash (\text{AFAlg}(X_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{y}} \otimes \sigma_{\mathfrak{S}}]{\top} & \mathcal{O}_{S_A} \backslash (\text{AFAlg}(X_0)) \\ \downarrow v^{-1} \uparrow v_* & & \downarrow v^{-1} \uparrow v_* \\ \mathcal{O}_{Y_A} \backslash (\text{AFAlg}(\check{X})) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{y}} \otimes \sigma_{\mathfrak{S}}]{\top} & \mathcal{O}_{S_A} \backslash (\text{AFAlg}(\check{X})). \end{array}$$

Although the last category does not have uniformly trivial deformation theory, all the morphisms in it which we encounter do. If we let $(f^\#)^{\otimes n}$ denote the canonical ring homomorphism $\mathcal{O}_{\mathfrak{y}}^{\otimes \sigma_{\mathfrak{S}} n} \otimes_{\mathcal{O}_{\mathfrak{S}}} \mathcal{O}_{\hat{\mathfrak{x}}} \rightarrow \mathcal{O}_{\hat{\mathfrak{x}}}$, then there is an SDC and a quasi-isomorphism

$$\text{Hom}_{\mathcal{O}_{\mathfrak{S}}\text{-Alg}}(\mathcal{O}_{\mathfrak{y}}^{\otimes \sigma_{\mathfrak{S}} n} \otimes_{\mathcal{O}_{\mathfrak{S}}} \mathcal{O}_{\hat{\mathfrak{x}}}, (v^{-1}v_*)^n \mathcal{O}_{\hat{\mathfrak{x}}})_{v^{-1}\alpha^n \circ (f^\#)^{\otimes n}} \rightarrow \check{E}^n.$$

The map is defined by composing the maps $\text{Symm}_\Lambda(\mathcal{O}_{\mathfrak{y}}^{\otimes n} \otimes \mathcal{O}_{\hat{\mathfrak{x}}}) \rightarrow \mathcal{O}_{\mathfrak{y}}^{\otimes n} \otimes \mathcal{O}_{\hat{\mathfrak{x}}}$, arising from the natural ring structure on the tensor product.

Explicitly, we may write our SDC as

$$E^n = \prod_{\alpha_0, \dots, \alpha_n \in I} \Gamma(X_{\alpha_0, \dots, \alpha_n}, \mathcal{H}om_{\mathcal{O}_{\mathfrak{S}}\text{-Alg}}(\mathcal{O}_{\mathfrak{y}}^{\otimes \sigma_{\mathfrak{S}} n} \otimes_{\mathcal{O}_{\mathfrak{S}}} \mathcal{O}_{\hat{\mathfrak{x}}_{\alpha_n}}, \mathcal{O}_{\hat{\mathfrak{x}}_{\alpha_0}})_{(f^\#)^{\otimes n}}),$$

The product is given by

$$\begin{aligned} & (\phi * \psi)_{\alpha_0, \dots, \alpha_{m+n}}(r_1 \otimes \dots \otimes r_{m+n} \otimes a) = \\ & \phi_{\alpha_0, \dots, \alpha_m}(r_1 \otimes \dots \otimes r_m \otimes \psi_{\alpha_m, \dots, \alpha_{m+n}}(r_{m+1} \otimes \dots \otimes r_{m+n} \otimes a)) \quad , \end{aligned}$$

and operations are

$$\begin{aligned} & (\partial^i \phi)_{\alpha_0, \dots, \alpha_{n+1}}(r_1 \otimes \dots \otimes r_{n+1} \otimes a) = \\ & \phi_{\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n+1}}(r_1 \otimes \dots \otimes r_{i-1} \otimes (r_i r_{i+1}) \otimes r_{i+2} \dots \otimes r_{n+1} \otimes a) \\ & (\sigma^i \phi)_{\alpha_0, \dots, \alpha_{n-1}}(r_1 \otimes \dots \otimes r_{n-1} \otimes a) = \\ & \phi_{\alpha_0, \dots, \alpha_i, \alpha_i, \dots, \alpha_{n-1}}(r_1 \otimes \dots \otimes r_i \otimes 1 \otimes r_{i+1} \dots \otimes r_{n-1} \otimes a). \end{aligned}$$

To see that the map is a quasi-isomorphism, observe that the Eilenberg-Zilber Theorem allows us to regard the cohomology of E as hypercohomology of the complex $\mathcal{T}_{X_0/S_0} \rightarrow f^* \mathcal{T}_{Y_0/S_0}[-1]$, which is quasi-isomorphic to the tangent complex of the morphism $X_0 \rightarrow Y_0$. Note that this SDC works for any morphism of smooth schemes, not only for embeddings.

3.4 Constrained deformation of schemes

Given a morphism of flat μ -adic schemes $\mathfrak{Z} \xrightarrow{h} \mathfrak{Y}$ over \mathfrak{S} , and a diagram

$$Z_0 \xrightarrow{g} X_0 \xrightarrow{f} Y_0$$

over S_0 , with $fg = h$, our deformation functor consists of diagrams

$$Z_A \xrightarrow{g_A} X_A \xrightarrow{f_A} Y_A,$$

over S_A , with $X_A \rightarrow S_A$ flat and $f_A \circ g_A = h$, which pull back along $S_0 \rightarrow S_A$ to our original diagram. Here, Z_A is the scheme $\mathfrak{Z} \times_{\mathrm{Spf} \Lambda} \mathrm{Spec} A$, and $Z_0 = Z_k$. An example of such a deformation problem would be deformations of a subscheme, constrained to pass through a fixed set of points.

Since the topological space $|X_0|$ underlying X_0 does not deform, it suffices to deform \mathcal{O}_{X_0} as an \mathcal{O}_{Y_A} -algebra with an \mathcal{O}_{Z_A} -augmentation, flat over A .

In the notation of Section 2.2, we have the set of structures

$$\Sigma^+ = \{\text{algebra, } \mathcal{O}_{Y_A}\text{-module}\}, \quad \Sigma^- = \{X_0\text{-sheaf, } \mathcal{O}_{Z_A}\text{-augmented}\},$$

over the category of sheaves of flat A -modules on X'_0 , where X'_0 is defined as in Section 2.1.1.

To these structures correspond the monadic and comonadic adjunctions

$$\{\mathrm{Symm}_A \dashv U, \mathcal{O}_{\mathfrak{Y}} \otimes_{\Lambda} \dashv U\}, \quad \{u_* \vdash u^{-1}, \mathcal{O}_{\mathfrak{Z}} \times \vdash V\},$$

where U, V denote the relevant forgetful functors.

This yields the following diagram of Cat-valued functors:

$$\begin{array}{ccc} \mathcal{O}_{Y_A} \setminus (\mathrm{AFAlg}(X_0)) / \mathcal{O}_{Z_A} & \xrightleftharpoons[\mathcal{O}_{\mathfrak{Y}} \otimes_{\Lambda} \mathrm{Symm}_A]{\top} & \mathrm{AFMod}(X_0) / \mathcal{O}_{Z_A} \\ \downarrow u^{-1} \dashv \uparrow \mathcal{O}_{Z_A} \times u_* & & \downarrow u^{-1} \dashv \uparrow \mathcal{O}_{Z_A} \times u_* \\ \mathcal{O}_{Y_A} \setminus (\mathrm{AFAlg}(X'_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{Y}} \otimes_{\Lambda} \mathrm{Symm}_A]{\top} & \mathrm{AFMod}(X'_0). \end{array}$$

where we write $\mathcal{O}_{\mathfrak{Y}}$ for the sheaf $f^{-1}\mathcal{O}_{\mathfrak{Y}}$ (resp. $u^{-1}f^{-1}\mathcal{O}_{\mathfrak{Y}}$), and $\mathcal{O}_{\mathfrak{Z}}$ for the sheaf $g_*\mathcal{O}_{\mathfrak{Z}}$ (resp. $u^{-1}g_*\mathcal{O}_{\mathfrak{Z}}$) on X_0 (resp. X'_0). The category $\mathcal{O}_{Y_A} \setminus (\mathrm{AFAlg}) / \mathcal{O}_{Z_A}$ consists of those \mathcal{O}_{Y_A} -algebras which are flat over A , and equipped with an augmentation to \mathcal{O}_{Z_A} , compatible with h^\sharp . The only non-trivial commutativity condition is the observation that pull-backs commute with tensor operations.

By Theorem 2.5, this deformation problem is governed by the SDC

$$E^n = \mathrm{Hom}_{\Lambda}((\mathcal{O}_{\mathfrak{Y}} \otimes_{\Lambda} \mathrm{Symm}_{\Lambda})^n \mathcal{M}, (\mathcal{O}_{\mathfrak{Z}} \times u^{-1}u_*)^n \mathcal{M})_{u^{-1}(\alpha^n \circ \varepsilon^n)},$$

where \mathcal{M} is a lift of the sheaf $u^{-1}\mathcal{O}_{X_0}$ of vector spaces on X'_0 to a sheaf of flat μ -adic Λ -modules, and α^n, ε^n are the canonical maps $\alpha^n_{\mathcal{O}_{X_0}}, \varepsilon^n_{\mathcal{O}_{X_0}}$ in Theorem 2.5, given by the adjunctions. The cohomology is

$$\mathrm{Ext}_{\mathcal{O}_{X_0}}^i(\mathbf{L}_{\bullet}^{X_0/Y_0}, \mathcal{O}_{X_0} \xrightarrow{g^\sharp} g_*\mathcal{O}_{Z_0}[-1]),$$

If X_0 and Y_0 are both smooth over S_0 , we may consider the diagram

$$\begin{array}{ccc}
\mathcal{O}_{Y_A} \backslash (\text{AFAlg}(X_0)) / \mathcal{O}_{Z_A} & \xrightleftharpoons[\mathcal{O}_{\mathfrak{y}} \otimes \mathcal{O}_{\mathfrak{S}}]{\top} & \mathcal{O}_{S_A} \backslash (\text{AFAlg}(X_0)) / \mathcal{O}_{Z_A} \\
\downarrow v^{-1} \dashv \uparrow \mathcal{O}_{Z_A} \times v_* & & \downarrow v^{-1} \dashv \uparrow \mathcal{O}_{Z_A} \times v_* \\
\mathcal{O}_{Y_A} \backslash (\text{AFAlg}(\check{X})) / \mathcal{O}_{Z_A} & \xrightleftharpoons[\mathcal{O}_{\mathfrak{y}} \otimes \mathcal{O}_{S_A}]{\top} & \mathcal{O}_{S_A} \backslash (\text{AFAlg}(\check{X})).
\end{array}$$

All the morphisms which we encounter in the last category have uniformly trivial deformation theory, allowing us to replace this SDC by

$$E^n = \prod_{\alpha_0, \dots, \alpha_n \in I} \Gamma(X_{\alpha_0, \dots, \alpha_n}, \mathcal{H}om_{\mathcal{O}_{\mathfrak{S}}\text{-Alg}}(\mathcal{O}_{\mathfrak{y}}^{\otimes \mathcal{O}_{\mathfrak{S}}} \otimes \mathcal{O}_{\mathfrak{S}} \mathcal{O}_{\mathfrak{x}_{\alpha_n}}, \mathcal{O}_{\mathfrak{x}_{\alpha_0}} \times \mathcal{O}_{\mathfrak{S}}^n)_{(g^\#)^{n_0} \circ (f^\#)^{\otimes n}}),$$

since cohomology in this case is hypercohomology of the complex

$$\mathcal{I}_{X_0/S_0} \rightarrow (f^* \mathcal{I}_{Y_0/S_0} \oplus \mathcal{I}_{X_0/S_0} \otimes_{\mathcal{O}_{X_0}} g_* \mathcal{O}_{Z_0})[-1] \rightarrow f^* \mathcal{I}_{Y_0/S_0} \otimes_{\mathcal{O}_{X_0}} g_* \mathcal{O}_{Z_0}[-2].$$

3.5 Deformation of a pair

Given a flat μ -adic formal scheme \mathfrak{S} , and flat schemes

$$\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & Y_0 \\
& \searrow & \swarrow \\
& & S_0
\end{array}$$

let $S_A := \mathfrak{S} \times_{\text{Spf } \Lambda} \text{Spec } A$. Our deformation functor consists of (infinitesimal) isomorphism classes of:

$$\begin{array}{ccccc}
X_0 & \xrightarrow{f_0} & Y_0 & & \\
& \searrow & \swarrow & \searrow & \\
& & S_0 & \xrightarrow{f_A} & Y_A \\
& & & \swarrow \text{flat} & \swarrow \text{flat} \\
& & & & S_A
\end{array}$$

such that $X_A \times_{\text{Spec } A} \text{Spec } k = X_0$, $Y_A \times_{\text{Spec } A} \text{Spec } k = Y_0$, and $f_A \otimes_A k = f_0$.

In this case, we obtain the SDC from the square of adjunctions

$$\begin{array}{ccc}
\left(\begin{array}{c} \mathcal{O}_{S_A} \setminus (\text{AFAlg}(Y_0)) \\ \downarrow f_0^\# \\ \mathcal{O}_{S_A} \setminus (\text{AFAlg}(X_0)) \end{array} \right) & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow[\mathcal{O}_{\mathfrak{S}} \otimes_{\Lambda} \text{Symm}_A]{\top} \end{array} & \left(\begin{array}{c} \text{AFMod}(Y_0) \\ \downarrow f_0^\# \\ \text{AFMod}(X_0) \end{array} \right) \\
u_* \uparrow \downarrow u^{-1} & & u_* \uparrow \downarrow u^{-1} \\
\left(\begin{array}{c} \mathcal{O}_{S_A} \setminus (\text{AFAlg}(Y'_0)) \\ \downarrow (f'_0)^\# \\ \mathcal{O}_{S_A} \setminus (\text{AFAlg}(X'_0)) \end{array} \right) & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow[\mathcal{O}_{\mathfrak{S}} \otimes_{\Lambda} \text{Symm}_A]{\top} \end{array} & \left(\begin{array}{c} \text{AFMod}(Y'_0) \\ \downarrow (f'_0)^\# \\ \text{AFMod}(X'_0) \end{array} \right),
\end{array}$$

where for categories \mathcal{A}, \mathcal{B} , with a functor $F : \mathcal{A} \rightarrow \mathcal{B}$, $(\mathcal{A} \xrightarrow{F} \mathcal{B})$ is the category whose objects are triples

$$(A, B, FA \xrightarrow{h} B),$$

and whose morphisms are pairs $f : A \rightarrow A', g : B \rightarrow B'$ such that the following square commutes:

$$\begin{array}{ccc}
FA & \xrightarrow{h} & B \\
Ff \downarrow & & \downarrow g \\
FA' & \xrightarrow{h'} & B'.
\end{array}$$

3.6 Deformation of a group scheme

Given a group scheme G_0/k , we consider deformations

$$\begin{array}{ccc}
G_0 & \longrightarrow & G_A \\
\downarrow & & \downarrow \text{flat} \\
\text{Spec } k & \longrightarrow & \text{Spec } A,
\end{array}$$

where G_A is a group scheme over A with $G_A \otimes_A k = G_0$.

Since the topological space $|G_0|$ does not deform, we need only consider deformations of \mathcal{O}_{G_0} as a sheaf of Hopf algebras. The structures are:

$$\Sigma^+ = \{\text{Algebra}\}, \quad \Sigma^- = \{G_0\text{-Sheaf, Co-Algebra}\},$$

over the category of sheaves of flat A -modules on G'_0 .

This gives the following commutative diagram of monadic and comonadic adjunctions:

$$\begin{array}{ccc}
A\text{-FHopfAlg}(G_0) & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow[\text{Symm}_A]{\top} \end{array} & A\text{-FCoAlg}(G_0) \\
\begin{array}{c} \downarrow \\ u^{-1} \downarrow \uparrow \\ \downarrow \uparrow \end{array} \begin{array}{c} \uparrow \\ Q \circ u_* \\ \uparrow \end{array} & & \begin{array}{c} \downarrow \\ u^{-1} \downarrow \uparrow \\ \downarrow \uparrow \end{array} \begin{array}{c} \uparrow \\ Q \circ u_* \\ \uparrow \end{array} \\
A\text{FAlg}(G'_0) & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow[\text{Symm}_A]{\top} \end{array} & A\text{FMod}(G'_0),
\end{array}$$

where $A\text{-FHopfAlg}$ is the category of flat Hopf algebras over A , $A\text{-FCoAlg}$ is the category of flat co-associative co-algebras with co-unit and co-inverse over A , and Q is the free co-algebra functor of Section 2.1.2. We thus obtain the SDC

$$E^n = \text{Hom}_\Lambda((\text{Symm}_\Lambda)^n \mathcal{M}, (u_* \circ Q \circ u^{-1})^n \mathcal{M})_{u^{-1}(\alpha^n \circ \varepsilon^n)},$$

where \mathcal{M} is the lift (unique up to isomorphism) of the sheaf $u^{-1}\mathcal{O}_{G_0}$ of k -vector spaces to a flat μ -adic sheaf on G' , with α^n, ε^n the canonical maps

$$\varepsilon^n : (\text{Symm}_k)^n \mathcal{O}_{G_0} \rightarrow \mathcal{O}_{G_0}, \quad \alpha^n : \mathcal{O}_{G_0} \rightarrow (u_* \circ Q \circ u^{-1})^n \mathcal{O}_{G_0}$$

associated to \mathcal{O}_{G_0} .

3.6.1 Deformation of a sub-(group scheme)

The problem is now, given a flat μ -adic formal group scheme \mathfrak{H} over Λ , to deform a sub-(group scheme) $G_0 \rightarrow H_0$ as a sub-(group scheme) of $H_A := \mathfrak{H} \otimes_\Lambda A$, flat over A . In fact, we will not need to assume that $G_0 \rightarrow H_0$ is a subscheme — any morphism will do. The picture is:

$$\begin{array}{ccccc} G_0 & \longrightarrow & H_0 & & \\ & \searrow & \downarrow & \searrow & \\ & & \text{Spec } k & \longrightarrow & G_A & \longrightarrow & H_A \\ & & & \searrow & \downarrow \text{flat} & & \downarrow \text{flat} \\ & & & & \text{Spec } A, & & \end{array}$$

all morphisms as group schemes over Λ .

The structures are:

$$\Sigma^+ = \{\text{Algebra}, \mathcal{O}_{H_A}\text{-Module}\}, \quad \Sigma^- = \{G_0\text{-Sheaf}, \text{Coalgebra}\},$$

so the diagram

$$\begin{array}{ccc} \mathcal{O}_{H_A} \backslash (A\text{-FHopfAlg}(G_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{H} \otimes_\Lambda \text{Symm}_A}]{\top} & A\text{-FCoAlg}(G_0) \\ \downarrow u^{-1} \uparrow Q \circ u_* & & \downarrow u^{-1} \uparrow Q \circ u_* \\ \mathcal{O}_{H_A} \backslash (A\text{FAlg}(G'_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{H} \otimes_\Lambda \text{Symm}_A}]{\top} & A\text{FMod}(G'_0) \end{array}$$

gives us our SDC,

$$E^n = \text{Hom}_\Lambda((\mathcal{O}_{\mathfrak{H}} \otimes_\Lambda \text{Symm}_\Lambda)^n \mathcal{M}, (u_* \circ Q \circ u^{-1})^n \mathcal{M})_{u^{-1}(\alpha^n \circ \varepsilon^n)}.$$

3.7 G -invariant deformations

Given a flat μ -adic formal group scheme \mathfrak{G} over $\text{Spf } \Lambda$, a flat μ -adic formal scheme \mathfrak{T}/Λ , and a flat scheme X_0/k , with a morphism $X_0 \rightarrow T_0$, and compatible group actions

$$\mathfrak{T} \hat{\times}_\Lambda \mathfrak{G} \rightarrow \mathfrak{T}, \quad r : X_0 \times_k G_0 \rightarrow X_0,$$

where $G_0 = \mathfrak{G} \otimes_{\Lambda} k$, we seek deformations

$$\begin{array}{ccccc}
X_0 & \longrightarrow & T_0 & & \\
& \searrow & \downarrow & \searrow & \\
& & \text{Spec } k & \longrightarrow & X_A & \longrightarrow & T_A \\
& & & \searrow & \text{flat} & & \downarrow \text{flat} \\
& & & & & & \text{Spec } A,
\end{array}$$

where $T_A = \mathfrak{T} \otimes_{\Lambda} A$ and $G_A = \mathfrak{G} \otimes_{\Lambda} A$, together with a compatible group action $X_A \times_A G_A \rightarrow X_A$. Since the underlying topological space does not deform, we need only consider G_A -linearised sheaves \mathcal{F} of algebras, i.e sheaves with co-associative maps $\mathcal{F} \rightarrow r_*(\mathcal{F} \otimes_A \mathcal{O}_{G_A})$, deforming \mathcal{O}_{X_0} .

The structures are:

$$\Sigma^+ = \{\text{Algebra, } \mathcal{O}_{T_A}\text{-Module}\}, \quad \Sigma^- = \{X_0\text{-Sheaf, } G_A\text{-linear}\},$$

where a G_A -linearised module is a module M equipped with a co-associative morphism

$$M \rightarrow M \otimes \mathcal{O}_{G_A}.$$

The relevant diagram is then:

$$\begin{array}{ccc}
\mathcal{O}_{T_A} \setminus ((\text{AFAlg}(X_0))^{G_A}) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{T} \otimes_{\Lambda} \text{Symm}_A}]{\top} & (\text{AFMod}(X_0))^{G_A} \\
\downarrow u^{-1} \dashv \uparrow R \circ u_* & & \downarrow u^{-1} \dashv \uparrow R \circ u_* \\
\mathcal{O}_{T_A} \setminus (\text{AFAlg}(X'_0)) & \xrightleftharpoons[\mathcal{O}_{\mathfrak{T} \otimes_{\Lambda} \text{Symm}_A}]{\top} & \text{AFMod}(X'_0),
\end{array}$$

where $(\text{AFMod})^{G_A}$ denotes G_A -linearised module, and R is the co-free functor

$$R(\mathcal{F}) = r_*(\mathcal{F} \otimes_{\Lambda} \mathcal{O}_{\mathfrak{G}}),$$

whose \mathfrak{G} -action is given by:

$$r_*(\mathcal{F} \otimes_{\Lambda} \mathcal{O}_{\mathfrak{G}}) \xrightarrow{r_*(\text{id} \otimes m^{\sharp})} r_*(\mathcal{F} \otimes m_*(\mathcal{O}_{\mathfrak{G}} \otimes \mathcal{O}_{\mathfrak{G}})) = r_*(r_*(\mathcal{F} \otimes_{\Lambda} \mathcal{O}_{\mathfrak{G}}) \otimes \mathcal{O}_{\mathfrak{G}}),$$

as required. The unit of the adjunction $V \dashv R$, for forgetful functor V , is

$$\rho : \mathcal{F} \rightarrow RV(\mathcal{F}) = r_*(\mathcal{F} \otimes_{\Lambda} \mathcal{O}_{\mathfrak{G}}),$$

where ρ is the structure map of \mathcal{F} as a \mathfrak{G} -linearised sheaf. The co-unit is

$$r_*(\text{id} \otimes e^{\sharp}) : VR(\mathcal{F}) = r_*(\mathcal{F} \otimes_{\Lambda} \mathcal{O}_{\mathfrak{G}}) \rightarrow \mathcal{F}.$$

This gives the SDC

$$E^n = \text{Hom}_{\Lambda}((\mathcal{O}_{\mathfrak{T}} \otimes_{\Lambda} \text{Symm}_{\Lambda})^n \mathcal{M}, (u_* \circ R \circ u^{-1})^n \mathcal{M})_{u^{-1}(\alpha^n \circ \varepsilon^n)},$$

where \mathcal{M} is the lift (unique up to isomorphism) of the sheaf $u^{-1}\mathcal{O}_{X_0}$ of k -vector spaces to a flat μ -adic sheaf on X' .

4 Representations of the Algebraic Fundamental Group

Fix a connected scheme X , a geometric point $\bar{x} \rightarrow X$, and a ring Λ as in Section 0.1, with finite residue field k . Throughout, A will denote a ring in \mathcal{C}_Λ . Denote $\pi_1(X, \bar{x})$ by Γ .

4.1 Representations to GL_n

It is well known (e.g. [Mil80] Theorem 5.3), that there is an equivalence of categories between finite $\pi_1(X, \bar{x})$ -sets, and locally constant étale sheaves on X , with finite stalks. There is thus an equivalence of categories between representations

$$\rho : \pi_1(X, \bar{x}) \rightarrow \mathrm{GL}_n(A),$$

and locally constant étale sheaves \mathbb{V} on X with stalks A^n . Given ρ , the corresponding sheaf \mathbb{V}_ρ is given by

$$\mathbb{V}_\rho = A^n \times_{\Gamma, \rho} \mathcal{F}_Y = (\pi_* A^n)^{\Gamma, \rho},$$

where \mathcal{F}_Y is the sheaf on X whose espace étalé is Y , and $\rho|_{\pi_1(Y, \bar{y})} = 1$, which must happen for some Galois cover $Y \xrightarrow{\pi} X$, A being finite.

If we now fix a representation $\rho_0 : \Gamma \rightarrow \mathrm{GL}_n(k)$, then we have an equivalence of categories (or, rather, groupoids)

$$\mathfrak{R}_A(\rho_0) \leftrightarrow \mathfrak{V}_A(\mathbb{V}_0),$$

where $\mathfrak{R}_A(\rho_0)$ has objects

$$\rho \equiv \rho_0 \pmod{\mathfrak{m}_A},$$

while $\mathfrak{V}_A(\mathbb{V}_0)$ has objects

$$\mathbb{V} \equiv \mathbb{V}_0 \pmod{\mathfrak{m}_A}.$$

In both cases morphisms must be the identity mod \mathfrak{m}_A . Explicitly, a morphism in $\mathfrak{R}_A(\rho_0)$ is an element $g \in \ker(\mathrm{GL}_n(A) \rightarrow \mathrm{GL}_n(k))$, acting by conjugation.

Hence this deformation problem is just the same as deforming sheaves of modules, as in Section 2.1.1, giving the SDC

$$E^n = \mathrm{Hom}_\Lambda(\mathcal{V}, (u^* u_*)^n(\mathcal{V}))_{u^* \alpha^n},$$

where \mathcal{V} is the lift (unique up to isomorphism) of the sheaf $u^{-1}\mathbb{V}_0$ of k -vector spaces to a flat μ -adic sheaf on X' . The cohomotopy groups are, of course, isomorphic to

$$\mathrm{H}^*(X, \mathcal{H}om_k(\mathbb{V}_0, \mathbb{V}_0)).$$

Note that

$$\mathcal{H}om_k(\mathbb{V}_0, \mathbb{V}_0) = (\pi_* \mathfrak{gl}_n(k))^{\Gamma, \rho_0},$$

so the cohomotopy groups are

$$\mathrm{H}^i(X, (\pi_* \mathfrak{gl}_n(k))^{\Gamma, \rho_0}),$$

where Γ acts via the adjoint action.

4.2 Representations to a smooth Noetherian group scheme

Fix a system $\{G_n/\mathrm{Spec} \Lambda_n\}_{n \in \mathbb{N}}$ of smooth Noetherian group schemes, and let G be the μ -adic formal group scheme $\varinjlim G_n$. Fix a representation $\rho_0 : \Gamma \rightarrow G(k)$.

Definition 4.1. A principal $G(A)$ -sheaf on X is an étale sheaf \mathbb{B} on X , together with an action $\mathcal{G} \times \mathbb{B} \rightarrow \mathbb{B}$, where \mathcal{G} denotes the constant sheaf $G(A)$, such that

$$\mathcal{G} \times \mathbb{B} \cong \mathbb{B} \times \mathbb{B}, \quad (g, b) \mapsto (g \cdot b, b),$$

and the stalks $\mathbb{B}_{\bar{x}}$ are non-empty.

Given a map $A \rightarrow B$ in \mathcal{C}_Λ , there is a functorial map from $G(A)$ -sheaves to $G(B)$ -sheaves, given by

$$\mathbb{B} \mapsto G(B) \times^{G(A)} \mathbb{B},$$

i.e. $(gh, b) \sim (h, gb)$, for $g \in G(A)$, $h \in G(B)$ and $b \in B$.

Definition 4.2. Given a representation $\rho : \Gamma \rightarrow G(A)$, let

$$\mathbb{B}_\rho := \mathcal{F}_Y \times^{\Gamma, \rho} G(A) = (\pi_* G(A))^{\Gamma, \rho},$$

where \mathcal{F}_Y is the sheaf on X whose espace étalé is Y , and $\rho|_{\pi_1(Y, \bar{y})} = 1$, which must happen for some Galois cover $Y \xrightarrow{\pi} X$, $G(A)$ being finite (since G is Noetherian and A finite).

More precisely,

$$\mathcal{F}_Y(U \xrightarrow{u} X) = \left\{ \begin{array}{ccc} U & \xrightarrow{\tilde{u}} & Y \\ & \searrow u & \downarrow \\ & & X \end{array} \quad \text{a lift} \right\},$$

and the equivalence relation forming \mathbb{B}_ρ is $(\gamma \circ u, g) \sim (u, g\rho(\gamma))$, or equivalently $\eta \sim \gamma^* \eta \rho(\gamma)$. $\mathbb{B}_0 := \{\mathbb{B}_{\rho_0|_{\Lambda_n}}\}_{n \in \mathbb{N}}$.

Lemma 4.3. *There is an equivalence of categories*

$$\mathfrak{R}_A(\rho_0) \leftrightarrow \mathfrak{B}_A(\mathbb{B}_0),$$

where $\mathfrak{R}_A(\rho_0)$ has as objects representations

$$\rho \equiv \rho_0 \pmod{\mathfrak{m}_A},$$

while $\mathfrak{B}_A(\mathbb{B}_0)$ has as objects principal $G(A)$ -sheaves

$$\mathbb{B} \equiv \mathbb{B}_0 \pmod{\mathfrak{m}_A}$$

on X . In both cases morphisms must be the identity mod \mathfrak{m}_A .

Proof. We need to show that the functor $\Phi : \rho \mapsto \mathbb{B}_\rho$ is full, faithful and essentially surjective.

1. Φ is full and faithful:

Given $\theta : \mathbb{B}_\sigma \xrightarrow{\sim} \mathbb{B}_\tau$, take some finite Galois cover $Y \xrightarrow{\pi} X$ such that $\pi^{-1}\mathbb{B}_\sigma$ and $\pi^{-1}\mathbb{B}_\tau$ are both isomorphic to the trivial sheaf $G(A)$. Thus $\pi^{-1}(\theta)$ acts as a unique element α of $G(A)$ with $\alpha \equiv 1 \pmod{\mathfrak{m}_A}$, so $\theta = \Phi(\alpha)$.

2. Φ is essentially surjective:

Given $\mathbb{B} \in \mathfrak{B}_A(\mathbb{B}_0)$, \mathbb{B} is locally constant with finite stalks (since $G(A)$ is finite), so there is a Galois cover $Y \xrightarrow{\pi} X$ such that $\pi^{-1}\mathbb{B}$ is constant. Choose a geometric point $\bar{y} \rightarrow Y$, above \bar{x} . Γ acts on $(\pi^{-1}\mathbb{B})_{\bar{y}}$ via:

$$(\pi^{-1}\mathbb{B})_{\bar{y}} \cong (\pi^{-1}\mathbb{B})_{\gamma \circ \bar{y}} \cong \mathbb{B}_{\bar{x}} \cong (\pi^{-1}\mathbb{B})_{\bar{y}},$$

the first isomorphism being given by the constancy of $\pi^{-1}\mathbb{B}$, and the other two canonical. This is an isomorphism of $G(A)$ -sets, so gives an element $\rho \in \mathfrak{R}_A(\rho_0)$. It follows that $\Phi(\rho) \cong \mathbb{B}$.

□

Observe that deformations of \mathbb{B}_0 as a principal G -sheaf are the same as its deformations as a faithful G -sheaf. We have the functorial comonadic adjunction:

$$\begin{array}{c} \text{Faithful } G(A)\text{-Sheaves}(X) \\ \begin{array}{c} \downarrow u^{-1} \\ \dashv \\ \uparrow u_* \end{array} \\ \text{Faithful } G(A)\text{-Sheaves}(X') \end{array}$$

using the canonical injection $G(A) \rightarrow u_*G(A)$ to make $u_*\mathcal{B}$ a faithful $G(A)$ -sheaf. Observe that the second category has uniformly trivial deformation theory, G being smooth, so that deformations are described by the SDC

$$E^n = \text{Hom}(\mathcal{B}, (u^*u_*)^n \mathcal{B})_{u^*\alpha^n}^{G(\Lambda)}$$

where \mathcal{B} is the lift (unique up to isomorphism) of the sheaf $u^{-1}\mathbb{B}_0$ of $G(k)$ -sheaves to a sheaf of continuous (in the μ -adic topology) $G(\Lambda)$ -sheaves on X' . The cohomotopy groups are isomorphic to

$$\mathrm{H}^*(X, \mathrm{ad}\mathbb{B}_0),$$

where $\mathrm{ad}\mathbb{B}_0$ is the tangent space of $\mathrm{aut}(\mathbb{B}_0)$ at the identity. Equivalently,

$$\mathrm{ad}\mathbb{B}_0 = \mathfrak{g} \times^{G(k)} \mathbb{B}_0 = (\pi_*\mathfrak{g})^{\Gamma, \rho_0},$$

for \mathfrak{g} the tangent space of $G(k)$ at the identity, and $G(k)$ acting on \mathfrak{g} via the adjoint action, so the cohomotopy groups are

$$\mathrm{H}^*(X, (\pi_*\mathfrak{g})^{\Gamma, \rho_0}).$$

5 Equivalence between SDCs and DGLAs in Characteristic Zero

Throughout this section, we will assume that $\Lambda = k$, a field of characteristic zero. We will also assume that all DGLAs are in non-negative degrees, i.e. $L^{<0} = 0$.

Remark 5.1. Throughout this section, we will encounter both chain complexes and simplicial complexes. In order to avoid confusion, chain complexes will be denoted by a bullet (C_\bullet), while simplicial complexes will be denoted by an asterisk (C_*). Asterisks will also be used to denote complexes without differentials, so that, for a chain complex C_\bullet , we may write $C_\bullet = (C_*, d)$. Homology groups of a chain complex C_\bullet will be denoted $H_i(C)$, while homotopy groups of a simplicial complex C_* will be denoted $\pi_i(C)$. The dual conventions will be used for cochain and cosimplicial complexes.

1. Consider, for simplicity, an SDC E^* for which each tangent space $\mathfrak{e}^n := C^n(E)$ is finite dimensional. Thus each E^n is pro-representable, by a smooth complete local Noetherian k -algebra Q_n , say. Since $\partial_{\omega_0}^0$ and $\partial_{\omega_0}^{n+1}$ are defined on C_k , E^* is a cosimplicial complex, so Q_* has the structure of a simplicial complex of algebras. The product on E^* gives us a comultiplication

$$*_{m,n}^\# : Q_{m+n} \rightarrow Q_m \hat{\otimes} Q_n,$$

satisfying various axioms. $\mathfrak{e}^* = \text{Der}_k(Q_*, k)$, so $\pi^i(\mathfrak{e}^*) = \pi^i(\text{Der}_k(Q_*, k))$.

2. To give a graded Lie algebra L^* (in non-negative degrees) over k is the same as giving the smooth homogeneous functor

$$\begin{aligned} \exp(L) : \mathcal{C}_k^{\mathbb{N}_0} &\rightarrow \text{Grp} \\ A_* &\mapsto \exp\left(\bigoplus_n L^n \otimes (\mathfrak{m}_A)_n\right), \end{aligned}$$

where $\mathcal{C}_k^{\mathbb{N}_0}$ is the category of nilpotent local Noetherian \mathbb{N}_0 -graded (supercommutative) k -algebras with residue field k (concentrated in degree 0). Observe that $\bigoplus_n L^n \otimes (\mathfrak{m}_A)_n$ is a Lie algebra, so the formula makes sense.

The category $\mathcal{C}_k^{\mathbb{N}_0}$ behaves similarly to \mathcal{C}_k , in that the analogue of Theorem 0.8 holds; in particular, if L^* is finite dimensional, then $\exp(L)$ can be pro-represented by a smooth complete local Noetherian graded k -algebra R_* , the group structure making R_* into a graded Hopf algebra:

$$*_{m,n}^\# : R_{m+n} \rightarrow R_m \hat{\otimes} R_n.$$

If we take a finite dimensional DGLA L^\bullet , the differential gives rise to a differential on R_* , giving it the structure of a DG Hopf algebra R_\bullet . Since $L^\bullet = \text{Der}_k(R_\bullet, k)$, we have $H^i(L^\bullet) = H^i(\text{Der}_k(R_\bullet, k))$.

Hence, given the equivalence, described in [Qui69], between DG and simplicial algebras in characteristic zero, it should be possible to obtain DGLAs from SDCs and *vice versa*. Moreover, given the similarity between our cohomology groups and André-Quillen cohomology, and the result in [Qui70] that, in characteristic 0, André-Quillen cohomology can be computed by either simplicial or DG resolutions, it seems that an SDC should have the same cohomology as the corresponding DGLA. It is then plausible that they should have the same deformation groupoid. The purpose of this section is to make this reasoning rigorous.

5.1 SDCs

Fix an SDC E^* . Although each E^n will only be pro-representable if $\dim \mathfrak{e}^n < \infty$, in general we may make use of the following lemma:

Lemma 5.2. *Given a smooth homogeneous functor $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$, let $\{t_i : i \in I\}$ be a basis for t_F . Then F is isomorphic to $\text{Hom}_{\text{cts.}}(\mathcal{R}, -)$, where \mathcal{R} is the pro-Artinian completion of $\Lambda[T_i : i \in I]$, localised at $(T_i : i \in I)$, for formal symbols T_i . Note that, if t_F is finite dimensional, this is just the expected ring $\Lambda[[T_1, \dots, T_m]]$.*

Proof. This is very similar to the proof of Theorem 0.8. A full proof in the analogous case of nilpotent Lie algebras (rather than Artinian rings) is given in [Pri07] Theorem ??.

Thus an SDC E^* is equivalent to a system of smooth local complete pro-Artinian rings \mathcal{Q}_* , with the dual structures to those described in Section 1.1. In particular, since we are only considering the case $\Lambda = k$, we have canonical $\partial_{\omega_0}^0, \partial_{\omega_0}^{n+1}$. Explicitly:

Lemma 5.3. *\mathcal{Q}_* is a simplicial complex of smooth local complete pro-Artinian k -algebras (with residue field k), together with a co-associative comultiplication*

$$\rho_{m,n} := *_{m,n}^\# : \mathcal{Q}_{m+n} \rightarrow \mathcal{Q}_m \hat{\otimes} \mathcal{Q}_n,$$

such that $\mathcal{Q}_0 \rightarrow k$ is the co-unit, satisfying:

1. $(\partial_i \otimes \text{id}) \circ \rho_{m+1,n} = \rho_{m,n} \circ \partial_i$ for $i \leq m$.
2. $(\text{id} \otimes \partial_i) \circ \rho_{m,n+1} = \rho_{m,n} \circ \partial_{m+i}$ for $i \geq 1$.
3. $(\partial_{m+1} \otimes \text{id}) \circ \rho_{m+1,n} = (\text{id} \otimes \partial_0) \circ \rho_{m,n+1}$.
4. $(\sigma_i \otimes \text{id}) \circ \rho_{m-1,n} = \rho_{m,n} \circ \sigma_i$.
5. $(\text{id} \otimes \sigma_i) \circ \rho_{m,n-1} = \rho_{m,n} \circ \sigma_{m+i}$.

Of course, given a system \mathcal{Q}_* satisfying these conditions, we can clearly recover an SDC as $\text{Hom}_{\text{cts.}}(\mathcal{Q}_*, -)$, so that we may regard this as an equivalent definition of an SDC.

Remark 5.4. We say that a simplicial complex A_* of local Λ -algebras is Artinian if it satisfies DCC on simplicial ideals. This is equivalent to saying that the algebras A_n are Artinian, and the normalised complex $N(A_*)$ is bounded. It is easy to see that $\mathcal{Q}_* = \varprojlim Q_*$, the limit being over quotient complexes $\mathcal{Q}_* \rightarrow Q_*$ Artinian. Moreover, \mathcal{Q}_* is really just a shorthand for this inverse system. Thus, for any functor F , by $F(\mathcal{Q}_*)$ we mean $\varprojlim F(Q_*)$. In particular, $\pi_*(\mathcal{Q}_*) := \varprojlim \pi_*(Q_*)$. We will apply the same convention to the limits of DG complexes in the next section.

5.2 DGLAs

Define $\mathcal{C}_k^{\mathbb{N}_0}$ to be the category of nilpotent local Noetherian graded (super-commutative) k -algebras (in non-negative degrees) with residue field k (concentrated in degree 0).

A GLA L^* over k (in non-negative degrees) gives rise to a smooth homogeneous functor

$$\begin{aligned} \exp(L) : \mathcal{C}_k^{\mathbb{N}_0} &\rightarrow \text{Grp} \\ A_* &\mapsto \exp\left(\bigoplus_n L^n \otimes (\mathfrak{m}_A)_n\right). \end{aligned}$$

Given a functor $F : \mathcal{C}_k^{\mathbb{N}_0} \rightarrow \text{Set}$, define $t_F^i := F(k[\epsilon_i])$, where ϵ_i is a variable in degree i , with $\epsilon_i^2 = 0$.

Lemma 5.5. *Given a smooth homogeneous functor $F : \mathcal{C}_k^{\mathbb{N}_0} \rightarrow \text{Set}$, let $\{t_i : i \in I\}$ be a homogeneous basis for t_F . Then F is isomorphic to $\text{Hom}_{\text{cts.}}(\mathcal{R}, -)$, where \mathcal{R} is the pro-Artinian completion of the ring $\Lambda[T_i : i \in I]_{\text{gr}}$, localised at $(T_i : i \in I)$, for formal symbols $T_i \in \deg(t_i)$. Note that, if t_F is finite dimensional, this is just the expected ring $\Lambda[[T_1, \dots, T_m]]_{\text{gr}}$.*

Proof. The analogue of Theorem 0.8 holds for graded rings, and this is just a generalisation of Lemma 5.2. \square

Let $\exp(L)$ be “represented” by the DG algebra \mathcal{T}_* . The group structure of $\exp(L)$ corresponds to a graded Hopf algebra map

$$\rho : \mathcal{T}_* \rightarrow \text{Tot}(\mathcal{T}_* \hat{\otimes} \mathcal{T}_*),$$

where

$$\text{Tot}(\mathcal{T}_* \hat{\otimes} \mathcal{T}_*)_n = \bigoplus_{r+s=n} \mathcal{T}_r \hat{\otimes} \mathcal{T}_s,$$

and the multiplication follows the usual (graded) convention:

$$(u \otimes v)(x \otimes y) = (-1)^{\bar{v}\bar{x}} ux \otimes vy,$$

for homogeneous elements. We can, of course, recover L^* from \mathcal{T}_* by

$$L^* = \text{Der}_{\text{cts.}}(\mathcal{T}_*, k).$$

Lemma 5.6. *Conversely, given a “representable” functor $G : \mathcal{C}_k^{\mathbb{N}_0} \rightarrow \text{Grp}$, with tangent space L^* , there is a canonical isomorphism*

$$\exp(L) \cong G.$$

In particular, this implies that G must be smooth.

Proof. Let G be “represented” by \mathcal{T}_* . Then

$$G(A_*) = \text{Hom}_{k\text{-Alg}}(\mathcal{T}_*, A_*), \quad \bigoplus_n L^n \otimes (\mathfrak{m}_A)_n = \text{Der}_k(\mathcal{T}_*, A_*).$$

We can embed both of these into $\text{Hom}_k(\mathcal{T}_*, A_*)$, on which we define the associative product $f * g = (f \otimes g) \circ \rho$. Now the maps

$$\begin{aligned} G(A_*) &\cong \bigoplus_n L^n \otimes (\mathfrak{m}_A)_n \\ g &\mapsto \sum_{n \geq 1} (-1)^{n-1} \frac{(g-e)^{*n}}{n} \\ \sum_{n \geq 0} \frac{l^{*n}}{n!} &\leftrightarrow l, \end{aligned}$$

where e is the identity map $\mathcal{T}_* \rightarrow k$, give us our isomorphism. \square

Given a DGLA L^\bullet , it remains only to understand what the differential d on L^\bullet gives rise to on \mathcal{T}_* .

To give d is the same as giving the map

$$\begin{array}{ccc} L^* & \xrightarrow{\text{id}+d\epsilon_1} & L^*[\epsilon_1] \\ & \searrow \text{id} & \downarrow \\ & & L^*, \end{array}$$

and $\exp(L^*[\epsilon_1])$ is “represented” by $\mathcal{T}_*[\epsilon_1]$. Thus d corresponds to a derivation $d^\sharp : \mathcal{T}_{n+1} \rightarrow \mathcal{T}_n$. Since $d^2 = 0$, we get $(d^\sharp)^2 = 0$. The condition that d be a Lie algebra derivation corresponds to the statement

$$\begin{aligned} \rho \circ d^\sharp &= (\text{id} \otimes d^\sharp \pm d^\sharp \otimes \text{id}) \circ \rho, \text{ where} \\ (\pm d^\sharp \otimes \text{id})(u \otimes v) &= (-1)^{\bar{v}} d^\sharp(u) \cdot v. \end{aligned}$$

In summary, a DGLA L^\bullet is equivalent to a smooth local complete pro-Artinian DG k -algebra \mathcal{T}_\bullet , with a co-associative chain map

$$\rho : \mathcal{T}_\bullet \rightarrow \text{Tot}(\mathcal{T}_\bullet \hat{\otimes} \mathcal{T}_\bullet),$$

for which $\mathcal{T}_\bullet \rightarrow k$ is the co-unit.

5.3 SDC \rightsquigarrow DGLA

Given \mathcal{Q}_* , as in Section 5.1, we form a DG Hopf algebra $C(\mathcal{Q})_\bullet$, with $C(\mathcal{Q})_n = \mathcal{Q}_n$ as a graded module. We let

$$d = \sum_{i=0}^{n+1} (-1)^i \partial_i : C(\mathcal{Q})_{n+1} \rightarrow C(\mathcal{Q})_n,$$

and give it an algebra structure via the Eilenberg-Zilber shuffle product: for $x \in C(\mathcal{Q})_m$ and $y \in C(\mathcal{Q})_n$, let

$$x \nabla y = \sum_{(\mu, \nu) \in \text{Sh}(m, n)} (-1)^{(\mu, \nu)} (\sigma_{\nu_n} \dots \sigma_{\nu_1} x) \cdot (\sigma_{\mu_m} \dots \sigma_{\mu_1} y),$$

where $\text{Sh}(m, n)$ is the set of (m, n) shuffle permutations, i.e. permutations

$$(\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n) \quad \text{of} \quad \{0, \dots, m+n-1\}$$

such that

$$\mu_1 < \dots < \mu_m \text{ and } \nu_1 < \dots < \nu_n,$$

and $(-1)^{(\mu, \nu)}$ denotes the sign of the permutation (μ, ν) . As explained in [Wei94] or [Qui69], the resulting product is supercommutative

$$x \nabla y = (-1)^{mn} y \nabla x$$

and associative, and d is a graded derivation with respect to ∇ , i.e.

$$d(x \nabla y) = (dx) \nabla y + (-1)^m x \nabla (dy).$$

Lemma 5.7. $C(\mathcal{Q})_*$ is a Hopf algebra under the map ρ .

Proof. Given $x \in C(\mathcal{Q})_m$, $y \in C(\mathcal{Q})_n$, and any $p+q = m+n$, we have:

$$\begin{aligned} \rho_{p,q}(x \nabla y) &= \rho_{p,q} \left(\sum_{(\mu, \nu)} (-1)^{(\mu, \nu)} \sigma_{\nu_n} \dots \sigma_{\nu_1} x \cdot \sigma_{\mu_m} \dots \sigma_{\mu_1} y \right) \\ &= \sum_{(\mu, \nu)} (-1)^{(\mu, \nu)} \rho_{p,q}(\sigma_{\nu_n} \dots \sigma_{\nu_1} x) \cdot \rho_{p,q}(\sigma_{\mu_m} \dots \sigma_{\mu_1} y) \\ &= \sum \pm (\sigma_{\nu'_{n'}} \dots \sigma_{\nu'_1} \otimes \sigma_{\nu''_{n''}} \dots \sigma_{\nu''_1}) \circ \rho_{m', m''}(x) \cdot (\sigma_{\mu'_{m'}} \dots \sigma_{\mu'_1} \otimes \sigma_{\mu''_{m''}} \dots \sigma_{\mu''_1}) \circ \rho_{n', n''}(y), \end{aligned}$$

the final sum being over

$$\begin{aligned} m' + m'' = m, \quad n' + n'' = n, \quad m' + n' = p, \quad m'' + n'' = q, \\ (\mu', \nu') \in \text{Sh}(m', n'), \quad (\mu'', \nu'') \in \text{Sh}(m'', n''), \end{aligned}$$

with

$$\pm = (-1)^{(\mu', \mu''+p, \nu', \nu''+p)}.$$

For this map to be a Hopf algebra map, we would need this expression to equal

$$\sum_{\substack{m'+m''=m, n'+n''=n, \\ m'+n'=p, m''+n''=q}} \rho_{m', m''}(x) \nabla \rho_{n', n''}(y),$$

The (p, q) component of $\rho(x) \nabla \rho(y)$. Now,

$$\begin{aligned} \rho_{m', m''}(x) \nabla \rho_{n', n''}(y) &= \\ \sum_{\substack{(\mu', \nu') \\ (\mu'', \nu'')}} \pm (\sigma_{\nu'_{n'}} \dots \sigma_{\nu'_1} \otimes \sigma_{\nu''_{n''}} \dots \sigma_{\nu''_1}) \circ \rho_{m', m''}(x) \cdot (\sigma_{\mu'_{m'}} \dots \sigma_{\mu'_1} \otimes \sigma_{\mu''_{m''}} \dots \sigma_{\mu''_1}) \circ \rho_{n', n''}(y), \end{aligned}$$

where

$$\pm = (-1)^{m''n'} (-1)^{(\mu', \nu')} (-1)^{(\mu'', \nu'')},$$

the $(-1)^{m''n'}$ arising from the (required) supercommutativity of $C(\mathcal{Q})_* \otimes C(\mathcal{Q})_*$. The result now follows from the observation that

$$(-1)^{(\mu', \mu''+p, \nu', \nu''+p)} = (-1)^{m''n'} (-1)^{(\mu', \nu')} (-1)^{(\mu'', \nu'')}.$$

□

Lemma 5.8. ρ is a chain map.

Proof. For $x \in C(\mathcal{Q})_{m+1}$,

$$\begin{aligned}
\rho(dx) &= \sum_{p+q=m} \rho_{p,q} \left(\sum_{i=0}^{m+1} (-1)^i \partial_i x \right) \\
&= \sum_{p+q=m} \left(\sum_{i=0}^p (-1)^i (\partial_i \otimes \text{id}) \circ \rho_{p+1,q}(x) + (-1)^p \sum_{j=1}^{q+1} (-1)^j (\text{id} \otimes \partial_j) \circ \rho_{p,q+1}(x) \right) \\
&= \sum_{p+q=m} \left(((d - (-1)^{p+1} \partial_{p+1}) \otimes \text{id}) \circ \rho_{p+1,q}(x) + (-1)^p (\text{id} \otimes (d - \partial_0)) \circ \rho_{p,q+1}(x) \right) \\
&= \sum_{p+q=m} (d \otimes \text{id}) \circ \rho_{p+1,q}(x) - (-1)^{p+1} (\partial_{p+1} \otimes \text{id}) \circ \rho_{p+1,q}(x) \\
&\quad + \sum_{p+q=m} (-1)^p (\text{id} \otimes d) \circ \rho_{p,q+1}(x) - (-1)^p (\text{id} \otimes \partial_0) \circ \rho_{p,q+1}(x) \\
&= \sum_{p+q=m} (d \otimes \text{id}) \circ \rho_{p+1,q}(x) + \sum_{p+q=m} (-1)^p (\text{id} \otimes d) \circ \rho_{p,q+1}(x) \\
&= d\rho(x).
\end{aligned}$$

□

There is now a problem with $C(\mathcal{Q}_*)$ as a candidate to define a DGLA L — it is not local. In [Qui69], rather than taking the unnormalised chain complex $C(\mathcal{Q}_*)$, Quillen takes the normalised chain complex $N(\mathcal{Q}_*)$. Recall that

$$C(\mathcal{Q})_n = N(\mathcal{Q})_n \oplus D(\mathcal{Q})_n, \text{ where}$$

$$N(\mathcal{Q})_n = \bigcap_{i=1}^n \ker(\partial_i : \mathcal{Q}_n \rightarrow \mathcal{Q}_{n-1}), \quad D(\mathcal{Q})_n = \sum_{i=1}^{n-1} \sigma_i(\mathcal{Q}_{n-1}).$$

Observe that $N(\mathcal{Q})_\bullet$ is local, as $\ker(\partial_1) \subset \mathfrak{m}_{\mathcal{Q}}$. It is pro-Artinian following Remark 5.4, since the normalisation of an ideal is a ∇ -ideal, and $N(I)\nabla N(J) \subset N(IJ)$. However, we cannot give $N(\mathcal{Q})$ the Hopf algebra structure ρ : in general, for $x \in N(\mathcal{Q})_{m+n}$, $(\partial_m \otimes \text{id}) \circ \rho_{m,n}(x) \neq 0$. Instead, we take the complex

$$\bar{N}(\mathcal{Q}) := C(\mathcal{Q})/D(\mathcal{Q}).$$

It follows immediately from the identities in Lemma 5.3 that

$$\rho_{m,n} : \bar{N}(\mathcal{Q})_{m+n} \rightarrow \bar{N}(\mathcal{Q})_m \hat{\otimes} \bar{N}(\mathcal{Q})_n$$

is well defined. We need only check that ∇ is well defined on $\bar{N}(\mathcal{Q})$, since then we will have $\bar{N}(\mathcal{Q})$ isomorphic to $N(\mathcal{Q})$ as an algebra.

Lemma 5.9. For $x \in D(\mathcal{Q})_m$ and $y \in C(\mathcal{Q})_n$, $x\nabla y \in D(\mathcal{Q})_{m+n}$.

Proof. Without loss of generality, $x = \sigma_r u$, for some $0 \leq r < m$. We will show that each term in the sum of the shuffle product is in $D(\mathcal{Q})_{m+n}$. Consider

$$\sigma_{\nu_n} \dots \sigma_{\nu_1} x \cdot \sigma_{\mu_m} \dots \sigma_{\mu_1} y,$$

and let $a := \max\{j : \nu_j - j < r\}$, which makes sense, since $\nu_j - j$ is a non-decreasing function. With the obvious convention if this set is empty, we have $0 \leq a \leq n$. Now, the simplicial identities give

$$\sigma_{\nu_n} \dots \sigma_{\nu_1} \sigma_r u = \sigma_{\nu_n} \dots \sigma_{\nu_{a+1}} \sigma_{r+a} \sigma_{\nu_a} \dots \sigma_{\nu_1} u,$$

with

$$\nu_1 < \dots < \nu_a < r + a < \nu_{a+1} < \dots < \nu_n.$$

Now $r + a \leq p - 1 + q$, and $r + a$ does not equal any of the ν_j , so $r + a = \mu_i$, for some i . This gives

$$\begin{aligned} (\sigma_{\nu_n} \dots \sigma_{\nu_1} x) \cdot (\sigma_{\mu_m} \dots \sigma_{\mu_1} y) &= (\sigma_{\nu_n} \dots \sigma_{\nu_{a+1}} \sigma_{\mu_i} \sigma_{\nu_a} \dots \sigma_{\nu_1} u) \cdot (\sigma_{\mu_m} \dots \sigma_{\mu_1} y) \\ &= (\sigma_{\mu_i} \sigma_{\nu_{n-1}} \dots \sigma_{\nu_{a+1}-1} \sigma_{\nu_a} \dots \sigma_{\nu_1} u) \cdot (\sigma_{\mu_i} \sigma_{\mu_{m-1}} \dots \sigma_{\mu_{i+1}-1} \sigma_{\mu_{i-1}} \dots \sigma_{\mu_1} y) \\ &= \sigma_{\mu_i} ((\sigma_{\nu_{n-1}} \dots \sigma_{\nu_{a+1}-1} \sigma_{\nu_a} \dots \sigma_{\nu_1} u) \cdot (\sigma_{\mu_{m-1}} \dots \sigma_{\mu_{i+1}-1} \sigma_{\mu_{i-1}} \dots \sigma_{\mu_1} y)), \end{aligned}$$

which is in $D(\mathcal{Q})_{m+n}$, as required. \square

Definition 5.10. Given an SDC E^* , we define the DGLA $\mathcal{L}(E)^\bullet$ by

$$\mathcal{L}(E)^\bullet = \text{Der}_{k, \text{cts.}}(\bar{N}(\mathcal{Q})_\bullet, k),$$

where \mathcal{Q}_* “represents” E^* .

Lemma 5.11. *There are natural maps $\pi^i(\mathfrak{e}^*) \rightarrow \text{H}^i(\mathcal{L}(E)^\bullet)$, where \mathfrak{e}^* is the tangent space of E^* .*

Proof. We may compute π^i using the conormalised cocomplex

$$N^n(K^*) = \bigcap_{i=0}^{n-1} \ker(\sigma^i : K^n \rightarrow K^{n-1}), \quad d = \sum_{i=0}^n (-1)^i \partial^i.$$

Thus an element $\alpha \in N^i(\mathfrak{e}^*)$ is a derivation (with respect to the product on \mathcal{Q}) from $\bar{N}_i(\mathcal{Q})$ to k (by the definition of \bar{N}). A derivation to k is just something which annihilates both k and $\mathfrak{m}_{\mathcal{Q}}^2$, but $\mathfrak{m}_{\mathcal{Q}} \nabla \mathfrak{m}_{\mathcal{Q}} \subset \mathfrak{m}_{\mathcal{Q}}^2$, so α gives a derivation (with respect to ∇) from $\bar{N}_\bullet(\mathcal{Q})$ to k , of degree i , i.e.

$$\alpha \in \mathcal{L}(E)^i.$$

This correspondence preserves closed and exact forms, so we get

$$\pi^i(\mathfrak{e}^*) \rightarrow \text{H}^i(\mathcal{L}(E)^\bullet).$$

\square

Lemma 5.12. *There is a natural map $\mathfrak{D}\text{ef}_E \rightarrow \mathfrak{D}\text{ef}_{\mathcal{L}(E)}$, with the maps on cohomology compatible with the corresponding maps on tangent and obstruction spaces.*

Proof. 1. Given $\omega \in \text{MC}_E$, we may regard ω as a continuous ring homomorphism $\omega : \mathcal{Q}_1 \rightarrow A$, satisfying $\omega * \omega = \partial^1(\omega)$. We also know that $\sigma^0(\omega) = e$. Set $\alpha = \omega - \omega_0$. This gives $\sigma^0(\alpha) = 0$, therefore $\alpha : \bar{N}_1(\mathcal{Q}) \rightarrow A$. In fact, α is a ∇ -derivation: for $x \in \mathcal{Q}_0, y \in \mathcal{Q}_1$,

$$\begin{aligned} \alpha(x\nabla y) &= \alpha(\sigma_0(x)y) \\ &= (\omega - \omega_0)(\sigma_0(x)y) \\ &= \sigma^0(\omega)(x)\bar{y} - \sigma^0(\omega_0)(x)\bar{y} + \overline{\sigma_0(x)\omega(y)} - \overline{\sigma_0(x)\omega_0(y)} \\ &= \bar{x}\alpha(y) \\ &= \bar{x}\alpha(y) + \alpha(x)\bar{y}. \end{aligned}$$

Moreover, the Maurer-Cartan equation for ω becomes

$$\begin{aligned} (\omega_0 + \alpha) * (\omega_0 + \alpha) &= \partial^1(\omega_0 + \alpha) \\ \omega_0^2 + \partial^0(\alpha) + \partial^2(\alpha) + \alpha * \alpha &= \omega_0^2 + \partial^1(\alpha) \\ \partial^0(\alpha) - \partial^1(\alpha) + \partial^2(\alpha) + \alpha * \alpha &= 0 \\ d\alpha + \frac{1}{2}[\alpha, \alpha] &= 0. \end{aligned}$$

Therefore $\alpha \in \text{MC}_{\mathcal{L}(E)}$.

2. Given $g \in E^0(A)$, g is an algebra homomorphism $g : \mathcal{Q}_0 \rightarrow A$. Since ∇ agrees with the usual product on \mathcal{Q}_0 , and $\bar{N}_0(\mathcal{Q}) = \mathcal{Q}_0$, g is a ∇ -algebra homomorphism $g : \bar{N}_0(\mathcal{Q}) \rightarrow A$. Therefore

$$g \in \exp(\mathcal{L}(E) \otimes \mathfrak{m}_A).$$

It is easy to see that the adjoint action on E^* corresponds to the gauge action on $\mathcal{L}(E)^\bullet$.

We therefore have the map $\mathfrak{Def}_E \rightarrow \mathfrak{Def}_{\mathcal{L}(E)}$. The results on tangent and obstruction spaces follow immediately. \square

We now need only prove that the maps $\pi^i(\mathfrak{e}^*) \rightarrow \text{H}^i(\mathcal{L}(E)^\bullet)$ are isomorphisms, from which it will follow that the morphism of groupoids is an equivalence. This will be proved in Theorem 5.22.

5.4 DGLA \rightsquigarrow SDC

Definition 5.13. Given a chain complex V_\bullet of vector spaces, the left adjoint N^{-1} of the normalisation functor N is defined by

$$N^{-1}(V)_n = \bigoplus_{\substack{m+s=n \\ 0 \leq \nu_1 < \nu_2 < \dots < \nu_s < n}} \sigma_{\nu_s} \dots \sigma_{\nu_2} \sigma_{\nu_1} V_m,$$

where $\sigma_{\nu_s} \dots \sigma_{\nu_2} \sigma_{\nu_1} V_m$ is just a copy of V_m , and we define σ and ∂ on $N^{-1}(V)$ using the simplicial identities, subject to

$$\partial_i|_V = \begin{cases} d & i = 0 \\ 0 & i > 0. \end{cases}$$

Note that $N(N^{-1}(V)) \cong V$, and, given $v \in V_n$, we write $N^{-1}v$ for

$$v \in (\text{id})V_n \subset N^{-1}(V)_n.$$

Definition 5.14. Given a local complete pro-Artinian DG k -algebra \mathcal{T}_\bullet , we can follow Quillen ([Qui69] Proposition 4.4) in defining the left adjoint to N , on complexes of algebras (adapted slightly for local algebras), by

$$N^*(\mathcal{T})_n = F(N^{-1}(\mathfrak{m}_{\mathcal{T}}))_n / I_n,$$

recalling that F is our free functor for local pro-Artinian algebras, where I_n is the ideal generated by

$$\{N^{-1}(x)\nabla N^{-1}(y) - N^{-1}(x \wedge y) : x \in (\mathfrak{m}_{\mathcal{T}})_r, y \in (\mathfrak{m}_{\mathcal{T}})_s, r + s = n\},$$

where \wedge is the product on \mathcal{T}_\bullet , and ∇ is the shuffle product on $F(N^{-1}(\mathfrak{m}_{\mathcal{T}}))$, defined in terms of the (free) product on $F(N^{-1}(\mathfrak{m}_{\mathcal{T}}))$. We regard $(k \subset N^*(\mathcal{T})_n)$ as $\sigma_0^n(k \subset \mathcal{T}_0)$ so that there is a copy of $N^{-1}(\mathcal{T})$ (and not just $N^{-1}(\mathfrak{m}_{\mathcal{T}})$) in $N^*(\mathcal{T})$.

Theorem 5.15. *If \mathcal{T}_\bullet is smooth, then so is $N^*(\mathcal{T})_*$.*

Proof. This is essentially [Qui69] Proposition 4.4. \square

Theorem 5.16. *If \mathcal{T}_\bullet is smooth, then the unit $\eta : \mathcal{T}_\bullet \rightarrow N(N^*(\mathcal{T}))_\bullet$ is such that the $H_i(\eta)$ are isomorphisms for all i . Thus $H_i(\mathcal{T}_\bullet) \cong \pi_i(N^*(\mathcal{T})_*)$.*

Proof. Essentially [Qui69] (Theorem 4.6 and the remark following). Note that the assumption of characteristic zero is vital here. \square

It therefore seems that, if \mathcal{T}_\bullet is the smooth DG Hopf algebra corresponding to the DGLA L^\bullet , then a good candidate for the corresponding SDC $\mathcal{E}(L)^*$ should be that represented by $N^*(\mathcal{T})$. However, we need first to define Hopf maps on $N^*(\mathcal{T})$. Since the Hopf maps did not behave under N , it is unsurprising that they do not behave under N^* . We get round this by making use of the isomorphism $\bar{N} \cong N$.

Lemma 5.17. *If \mathcal{Q}_* “represents” an SDC E^* , then, under the isomorphism*

$$N(\mathcal{Q})_\bullet \rightarrow \bar{N}(\mathcal{Q})_\bullet,$$

the map ρ on \bar{N} corresponds to Δ on N , where

$$\Delta_{m,n} = ((\text{id} + \sum_{i=0}^{m-1} (-1)^{m-i} \sigma_i \partial_m) \otimes \text{id}) \circ \rho_{m,n} :$$

$$\begin{array}{ccc} N(\mathcal{Q})_\bullet & \xrightarrow{\Delta} & N(\mathcal{Q})_\bullet \hat{\otimes} N(\mathcal{Q})_\bullet \\ \cong \downarrow & & \downarrow \cong \\ \bar{N}(\mathcal{Q})_\bullet & \xrightarrow{\rho} & \bar{N}(\mathcal{Q})_\bullet \hat{\otimes} \bar{N}(\mathcal{Q})_\bullet \end{array}$$

On N , we may also recover ρ from Δ .

Proof. Observe that

$$\Delta(x) - \rho(x) \in D(\mathcal{Q}) \hat{\otimes} \mathcal{Q},$$

and verify that $\Delta(x) \in N(\mathcal{Q}) \hat{\otimes} N(\mathcal{Q})$.

Finally, note that we can apply the identities from Lemma 5.3 to get

$$\Delta_{m,n} = \rho_{m,n} + \sum_{i=0}^{m-1} (-1)^{m-i} (\sigma_i \otimes \partial_0) \circ \rho_{m-1,n+1}.$$

Hence

$$\begin{aligned} & \Delta_{m,n} - \sum_{i=0}^{m-1} (-1)^{m-i} (\sigma_i \otimes \partial_0) \circ \Delta_{m-1,n+1} \\ = & \rho_{m,n} + \sum_{i=0}^{m-1} (-1)^{m-i} (\sigma_i \otimes \partial_0) \circ \rho_{m-1,n+1} - \sum_{i=0}^{m-1} (-1)^{m-i} (\sigma_i \otimes \partial_0) \circ \rho_{m-1,n+1} \\ & - \sum_{i=0}^{m-1} \sum_{j=0}^{m-2} (-1)^{m-i} (-1)^{m-1-j} (\sigma_i \otimes \partial_0) \circ (\sigma_j \otimes \partial_0) \circ \rho_{m-2,n+2} \\ = & \rho_{m,n} - \sum_{i=0}^{m-1} \sum_{j=0}^{m-2} (-1)^{m-i} (-1)^{m-1-j} (\sigma_i \sigma_j \otimes \partial_0 \partial_0) \circ \rho_{m-2,n+2} \\ = & \rho_{m,n} \end{aligned}$$

on N , since $\partial_0 \partial_0 = \partial_0 \partial_1$, and $\text{id} \otimes \partial_1$ is zero on $\rho(N)$. \square

Definition 5.18. Therefore, given a DGLA L , “represented” by the smooth DG Hopf algebra \mathcal{T}_\bullet with comultiplication Δ , we define $\bar{N}^*(\mathcal{T})$ to be the algebra $N^*(\mathcal{T})$ (noting that this is smooth by Theorem 5.15), with Hopf maps defined on

$$\{N^{-1}(t) : t \in \mathfrak{m}_{\mathcal{T}}\} \subset N^{-1}(\mathfrak{m}_{\mathcal{T}})$$

by

$$\rho_{m,n} := \Delta_{m,n} - \sum_{i=0}^{m-1} (-1)^{m-i} (\sigma_i \otimes \partial_0) \circ \Delta_{m-1,n+1},$$

extending to the rest of $N^{-1}(\mathfrak{m}_{\mathcal{T}})$ by the rules in Lemma 5.3, and extending to $N^*(\mathcal{T})$ by multiplicativity. Note that Lemma 5.7 ensures that this is well-defined. Call the resulting SDC $\mathcal{E}(L)^*$.

Theorem 5.19. *There are canonical isomorphisms $H^i(L) \rightarrow \pi^i(e(L)^*)$, where $e(L)^*$ is the tangent space of $\mathcal{E}(L)^*$, and an equivalence $\mathfrak{D}\mathfrak{e}\mathfrak{f}_L \rightarrow \mathfrak{D}\mathfrak{e}\mathfrak{f}_{\mathcal{E}(L)^*}$.*

Proof.

1. $e(L)^* = \text{Der}(\bar{N}^*(\mathcal{T})_*, k)$. From the definition of $\bar{N}^*(\mathcal{T})$, it follows that

$$\text{Der}(\bar{N}^*(\mathcal{T})_*, k) \cong \{\alpha \in \text{Hom}(N^{-1}(\mathcal{T})^*, k) : \alpha(1) = 0, \alpha(N^{-1}(x \wedge y)) = 0 \forall x, y \in \mathfrak{m}_{\mathcal{T}}\}.$$

Now, one possible definition of cohomotopy groups is as the cohomology of the complex

$$\bar{N}^n(e) = e^n / \left(\sum_{i=1}^n \partial^i e^{n-1} \right).$$

Therefore

$$\begin{aligned} \pi^i(e(L)^*) &= \mathbf{H}^i(\bar{N}(e(L))^\bullet) \\ &= \mathbf{H}^i(\bar{N}(\mathrm{Der}(\bar{N}^*(\mathcal{T})_*, k))) \\ &= \mathbf{H}^i(\bar{N}(\{\alpha \in \mathrm{Hom}(N^{-1}(\mathfrak{m}_{\mathcal{T}})_*, k) : \alpha(N^{-1}(x \wedge y)) = 0 \quad \forall x, y \in \mathfrak{m}_{\mathcal{T}}\})) \\ &= \mathbf{H}^i(\{\alpha \in \mathrm{Hom}(NN^{-1}(\mathfrak{m}_{\mathcal{T}})_\bullet, k) : \alpha(N^{-1}(x \wedge y)) = 0 \quad \forall x, y \in \mathfrak{m}_{\mathcal{T}}\}) \\ &= \mathbf{H}^i(\mathrm{Der}(\mathcal{T}_\bullet, k)) \\ &= \mathbf{H}^i(L), \end{aligned}$$

since $NN^{-1}(\mathfrak{m}_{\mathcal{T}})_\bullet = (\mathfrak{m}_{\mathcal{T}})_\bullet$.

2. First observe that $\bar{N}^*(\mathcal{T})_0 = \mathcal{T}_0$, with the same product. Thus $\mathfrak{gl} = \mathcal{E}(L)^0$. Call the identity of both these groups e . Given $\alpha \in \mathrm{MC}_L(A)$, let

$$\omega : \bar{N}^*(\mathcal{T})_1 \rightarrow A$$

be the multiplicative map defined on the generators

$$N^{-1}(\mathcal{T})_1 = \mathcal{T}_1 \oplus \sigma_0 \mathcal{T}_0 \quad \text{by} \quad \omega = (\alpha, 0) + \omega_0,$$

where $\omega_0 = e \circ \partial_0$ [effectively this gives us $\omega = (\alpha, e \circ (\sigma_0)^{-1})$]. The proof of Lemma 5.12, and the discussion comparing N and \bar{N} , show that this is well-defined and satisfies the Maurer-Cartan equation. The gauge action of \mathfrak{gl} corresponds to the adjoint action of E^0 , combining to give us a morphism

$$\mathfrak{Def}_L \rightarrow \mathfrak{Def}_{\mathcal{E}(L)}.$$

That this is an equivalence follows from the first part, and Theorems 0.11 to 0.14. □

Lemma 5.20. *There is a quasi-isomorphism $E^* \rightarrow \mathcal{E}(\mathcal{L}(E))^*$ of SDCs, for any SDC E^* .*

Proof. To see that there is a morphism $E^* \rightarrow \mathcal{E}(\mathcal{L}(E))^*$ is immediate. It is the map

$$\bar{N}^* \bar{N}(\mathcal{Q})_* \rightarrow \mathcal{Q}_*$$

corresponding to the canonical map $N^* N(\mathcal{Q}_*) \rightarrow \mathcal{Q}_*$. Moreover

$$\pi_*(\bar{N}^* \bar{N}(\mathcal{Q})_*) \cong \mathbf{H}_*(\bar{N}(\mathcal{Q})_\bullet) \cong \pi_*(\mathcal{Q}_*),$$

the first isomorphism being a consequence of Theorem 5.16. Thus the map $\bar{N}^* \bar{N}(\mathcal{Q})_* \rightarrow \mathcal{Q}_*$ induces an isomorphism on homotopy groups. Now, we may use the completeness of these simplicial algebras to show that they are free in the sense of

[Qui69] Proposition I 4.4. Since free simplicial complexes are the analogue of projective resolutions, and we have a pair of homotopic free complexes, it follows that

$$\mathrm{Der}(\mathcal{Q}_*, k) \rightarrow \mathrm{Der}(\bar{N}^* \bar{N}(\mathcal{Q})_*, k)$$

gives an isomorphism on cohomotopy groups, i.e.

$$\pi^i(\mathfrak{e}^*) \cong \pi^i(e(\mathcal{L}(E))^*),$$

the proof being the same as the result in [Qui70] that the quasi-isomorphism class of the tangent complex is independent of the choice of cofibrant resolution. Hence $E^* \rightarrow \mathcal{E}(\mathcal{L}(E))^*$ is a quasi-isomorphism of SDCs. \square

Corollary 5.21. *For any DGLA L^\bullet , there is a quasi-isomorphism $L^\bullet \rightarrow \mathcal{L}(\mathcal{E}(L))^\bullet$.*

Proof. By Lemma 5.20, there is a quasi-isomorphism of SDCs

$$\mathcal{E}(L)^* \rightarrow \mathcal{E}(\mathcal{L}(\mathcal{E}(L)))^*.$$

Now, Theorem 5.19 gives a canonical isomorphism $\bar{N}(e(L))^\bullet \cong L^\bullet$, so we may regard $\bar{N}(e(L))^\bullet$ as a DGLA, with bracket

$$[\alpha, \beta] = (\alpha \otimes \beta) \circ \Delta \mp (\beta \otimes \alpha) \circ \Delta.$$

(Note that $\bar{N}(\mathfrak{e}^*)$, for the tangent space \mathfrak{e}^* of an arbitrary SDC E^* will *not* be closed under this bracket.)

The quasi-isomorphism of SDCs then gives us a quasi-isomorphism of DGLAs

$$L^\bullet \cong \bar{N}(e(L))^\bullet \xrightarrow{\sim} \bar{N}(e(\mathcal{L}(\mathcal{E}(L))))^\bullet \cong \mathcal{L}(\mathcal{E}(L))^\bullet.$$

\square

Theorem 5.22. *The maps $\pi^i(\mathfrak{e}^*) \rightarrow \mathrm{H}^i(\mathcal{L}(E)^\bullet)$ in Lemma 5.11 are isomorphisms, and hence the morphism $\mathfrak{Def}_E \rightarrow \mathfrak{Def}_{\mathcal{L}(E)}$ in Lemma 5.12 is an equivalence.*

Proof. We can factor the isomorphism

$$\pi^i(\mathfrak{e}^*) \xrightarrow{\cong} \pi^i(e(\mathcal{L}(E))^*)$$

from Lemma 5.20 as

$$\pi^i(\mathfrak{e}^*) \rightarrow \mathrm{H}^i(\mathcal{L}(E)^\bullet) \xrightarrow{\cong} \pi^i(e(\mathcal{L}(E))^*),$$

the second isomorphism coming from Theorem 5.19. Thus

$$\pi^i(\mathfrak{e}^*) \rightarrow \mathrm{H}^i(\mathcal{L}(E)^\bullet)$$

must be an isomorphism, and by Theorems 0.11 to 0.14 and Lemma 5.12,

$$\mathfrak{Def}_E \rightarrow \mathfrak{Def}_{\mathcal{L}(E)}$$

must be an equivalence. \square

5.4.1 An explicit description

Given a DGLA L , first form the denormalised cosimplicial complex

$$D^n L := \left(\bigoplus_{\substack{m+s=n \\ 0 \leq j_1 < \dots < j_s \leq n}} \partial^{j_s} \dots \partial^{j_1} L^m \right) / (dv \sim \sum_{i=0}^{n+1} (-1)^i \partial^i v \quad \forall v \in L^n),$$

where we define the ∂^j and σ^i using the simplicial identities, subject to the condition that $\sigma^i L = 0$.

We now have to define the Lie bracket $[[,]]$ from $D^n L \otimes D^n L$ to $D^n L$. On the conormalised complex $N^n(DL \otimes DL) \leq D^n L \otimes D^n L$, we define this as the composition

$$N^n(DL \otimes DL) \xrightarrow{\Sigma^{\nabla^{pq}}} \bigoplus_{p+q=n} L^p \otimes L^q \xrightarrow{[[,]]} L^n,$$

where

$$\nabla^{pq}(v \otimes w) = \sum_{(\mu, \nu) \in \text{Sh}(p, q)} (-1)^{(\mu, \nu)} (\sigma^{\nu_1} \dots \sigma^{\nu_q} v) \otimes (\sigma^{\mu_1} \dots \sigma^{\mu_p} w).$$

We then extend this bracket to the whole of $D^n L \otimes D^n L$ by setting

$$[[\partial^i a, \partial^i b]] := \partial^i [[a, b]].$$

Now set

$$\mathcal{E}(L)^n(A) = \exp(D^n(L) \otimes \mathfrak{m}_A),$$

making $\mathcal{E}(L)$ into a cosimplicial complex of group-valued functors. To make it an SDC, we must define a $*$ product. We do this as the Alexander-Whitney cup product

$$g * h = (\partial^{m+n} \dots \partial^{m+2} \partial^{m+1} g) \cdot (\partial^0)^m h,$$

for $g \in \mathcal{E}(L)^m$, $h \in \mathcal{E}(L)^n$.

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