q-de Rham cohomology via Λ-rings

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q-analogues (Gauss)

$$[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

$$[n]_q! := [n]_q \ldots [2]_q [1]_q, \ \binom{n}{k}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!}$$

$$\prod_{i=0}^{n-1} (1 + q^i t) = \sum_k q^{k(k-1)/2} \binom{n}{k}_q t^k$$

$$\prod_{i=0}^{n-1} \frac{1}{(1 + q^i t)} = \sum_k (-1)^k \binom{n+k-1}{k}_q t^k.$$

Aomoto cohomology

Jackson differential

$$\begin{aligned} \partial_{x,q} \colon\ R[x,q] &\to R[x,q], \\ f(x) &\mapsto \frac{f(qx)-f(x)}{(q-1)x} \\ x^n &\mapsto [n]_q x^{n-1}. \end{aligned}$$

Aomoto's q-de Rham cohomology

$$\begin{aligned} q\text{-}\Omega^{\bullet}_{R[x_1,\dots,x_d]/R} &:= (\Omega^{*}_{R[x_1,\dots,x_d]/R}[q], \nabla_q) \\ \nabla_q \omega &:= \sum_i (\partial_{x_i,q}\omega) \wedge dx_i. \end{aligned}$$

Scholze's q-de Rham cohomology

For $\square\colon\ R[x_1,\ldots,x_d]\to A$ étale, unique automorphism $\gamma_i\colon\ A[\![q-1]\!]\to A[\![q-1]\!]$ with $\gamma_i x_j = q^{\delta_{ij}}x_j$.

$$\partial_{x_i,q}\colon\ A[\![q-1]\!]\to A[\![q-1]\!],$$
$$f\mapsto \frac{\gamma_i(f)-f}{(q-1)x_i}.$$

$$\widehat{q\text{-}\Omega}^{\bullet}_{A/R,\square} := (\Omega^*_{A/R}[\![q-1]\!],\nabla_q)$$
$$\nabla_q\omega := \sum_i(\partial_{x_i,q}\omega)\wedge dx_i.$$

Examples

- $\mathrm{H}^0(q\text{-}\Omega^\bullet_{R[x]/R}) = R,$

  $$\mathrm{H}^1(q\text{-}\Omega^\bullet_{R[x]/R}) = \bigoplus_{n\geqslant 2}(R[q]/[n]_q)x^{n-1}dx.$$

- $\mathrm{H}^0(q\text{-}\Omega^\bullet_{R[x,x^{-1}]/R}) = R,$

  $$\mathrm{H}^1(q\text{-}\Omega^\bullet_{R[x,x^{-1}]/R}) = \bigoplus_{n\in\mathbb{Z}}(R[q]/[n]_q)x^{n-1}dx.$$

- If $\mathbb{Q} \subset R$, then

  $$\widehat{q\text{-}\Omega}^\bullet_{A/R,\square} \simeq \Omega^\bullet_{A/R}[\![q-1]\!].$$

Conjecture (Scholze)

The complex $\widehat{q\text{-}\Omega}^\bullet_{A/R,\square}$ is essentially independent of the framing $\square$.

Evidence in mixed characteristic:

- Cartier isomorphism from Hodge cohomology lifts to q-de Rham cohomology modulo $[p]_q$.
- Bhatt–Morrow–Scholze: $(q-1)$-adic completion of $q\text{-}\Omega^\bullet_{A/R,\square}[q^{1/p^\infty}]$ is independent of $\square$.

Λ-rings

- Commutative rings A equipped with operations λ^i.
- Example: $K(X)$, with $\Lambda^i[\mathscr{E}] = [\Lambda^i\mathscr{E}]$.
- Relations
  - $\lambda^0(x) = 1$, $\lambda^1(x) = x$,
  - $\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$
  - $\lambda^i(xy) = \ldots$, $\lambda^i(\lambda^j(x)) = \ldots$.
- Write $\lambda_t(x) = \sum_{i \geqslant 0} \lambda^i(x)t^i$.
- Rank 1 elements: x with $\lambda_t(x) = 1 + x$.
- If x, y rank 1, then so is xy.

Λ_P-rings

- Λ-ring has commuting Adams ops $\{\Psi^p\}_p$: ring homomorphisms with

$$\Psi^p(a) \equiv a^p \mod p.$$

- For torsion-free rings, these determine λ-ops.
- P a set of primes; torsion-free Λ_P-ring defined to just have $\{\Psi^p\}_{p\in P}$ lifting Frobenius.

The Λ-ring $\mathbb{Z}[q]$

- Λ-ring structure on $\mathbb{Z}[q]$, determined by setting q to be of rank 1.
- Gaussian binomial theorems:

$$\lambda^k([n]_q) = q^{k(k-1)/2} \binom{n}{k}_q,$$

$$\lambda^k(-[n]_q) = (-1)^k \binom{n+k-1}{k}_q,$$

- Also $\Psi^r(q) = q^r$, $\Psi^r([n]_q) = [n]_{q^r}$.

$\Lambda_p \rightsquigarrow \widehat{q\text{-}\Omega}^{\bullet}$

- Take k of characteristic p, and $A/W(k)$ a formal deformation of smooth algebra.
- Étale framing $\square\colon\ W(k)[x_1, \ldots, x_d]^{\wedge p} \to A$ gives Adams op Ψ^p on A with $\Psi^p(x_i) = x_i^p$.

Theorem (P)

The Λ_p-ring (A, Ψ^p) determines $\widehat{q\text{-}\Omega}^{\bullet}_{A/W(k),\square}$.

Motivation

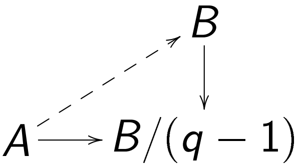
If $\nabla(f) := \frac{f\otimes 1 - 1\otimes f}{q-1}$, then for x of rank 1,

$$\begin{aligned}\nabla(x^n) &= \frac{x^n \otimes 1 - 1 \otimes x^n}{q-1} \\ &= (\frac{q^n-1}{q-1})\Psi^n(\frac{x\otimes 1 - 1\otimes x}{q-1}) \\ &= [n]_q \Psi^n(\nabla(x)),\end{aligned}$$

so $[n]_q \mid \nabla(x^n)$.

Site for q-$\Omega^\bullet_{A/R}$ from Λ-rings

- $\mathrm{Strat}^q_{A/R}$: flat Λ-rings B over $R[q]$, with



- Variant $\widehat{\mathrm{Strat}}^q_{A/R}$ for $\widehat{q\text{-}\Omega}^\bullet_{A/R}$: take B $(q-1)$-adically complete.
- Also $\widehat{\mathrm{Strat}}^{q,P}_{A/R}$ for Λ_P-rings.

Crystalline analogy

- On ideal $(q-1)B$, have operations $b \mapsto (q-1)^k \lambda^k(\frac{b}{q-1})$,
- q-analogues of divided powers.
- e.g. for x, y of rank 1, we have

$$\lambda^k(\frac{x}{q-1}) = \frac{x^k}{(q-1)^k [k]_q!}$$

$$\lambda^k(\frac{x-y}{q-1}) = \frac{(x-y)(x-qy)\dots(x-q^{k-1}y)}{(q-1)^k [k]_q!}.$$

qDR

- For Λ-rings $R \to A$, let $\mathrm{qDR}(A/R)$ be complex of $R[q]$-modules given by homotopy limit of

$$\begin{aligned} \mathrm{Strat}^q_{A/R} &\to \mathrm{Ch}(R[q]) \\ B &\mapsto B. \end{aligned}$$

- Regard as qu-coh cohomology of functor sending flat Λ-ring B over $R[q]$ to

$$\mathrm{Im}\left(\mathrm{Hom}_{\Lambda,R}(A,B) \to \mathrm{Hom}_{\Lambda,R}(A,B/(q-1))\right).$$

Key calculation

For R a Λ-ring, give $R[\underline{x}] := R[x_1, \ldots, x_d]$ the Λ-ring structure with x_i of rank 1,

Lemma

$\mathrm{qDR}(R[\underline{x}]/R) \simeq (\Omega^*_{R[\underline{x}]/R}[q], (q-1)\nabla_q)$.

Proof Uses modified Čech nerve & $\lambda^k(\frac{x-y}{q-1})$.

Corollary

For derived décalage $\mathbf{L}\eta_{(q-1)}$,

$$\mathbf{L}\eta_{(q-1)}\mathrm{qDR}(R[\underline{x}]/R) \simeq q\text{-}\Omega^\bullet_{R[\underline{x}]/R}.$$

Further comparisons

- $\mathrm{q}\hat{\mathrm{D}}\mathrm{R}(A/R)$ defined with $\hat{\mathrm{Strat}}^q_{A/R}$ instead of $\mathrm{Strat}^q_{A/R}$.
- $\mathrm{q}\hat{\mathrm{D}}\mathrm{R}(A/R)$ is $(q-1)$-adic completion of $\mathrm{qDR}(A/R)$.

Lemma

For étale Λ-homomorphism $\square\colon\ R[\underline{x}] \to A$,
$\mathrm{q}\hat{\mathrm{D}}\mathrm{R}(A/R) \simeq (\Omega^*_{R[\underline{x}]/R,\square}[\![q-1]\!], (q-1)\nabla_q)$.

Proof Much as before.

- $\mathrm{q\hat{D}R}_P(A/R)$ defined for Λ_P-rings with $\hat{\mathrm{Strat}}^{q,P}_{A/R}$ instead of $\hat{\mathrm{Strat}}^{q}_{A/R}$.

Proposition

If P contains all residue chars of A, then the natural map $\mathrm{q\hat{D}R}_P(A/R) \to \mathrm{q\hat{D}R}(A/R)$ is a quasi-isomorphism.

Corollary

$\widehat{q\text{-}\Omega}^{\bullet}_{A/R}$ *depends only on Λ_P-structure.*

Steps in proof of proposition

- Comparison uses right adjoint $W^{(\notin P)}$ to forgetful functor: Λ-rings $\to$ Λ_P-rings.
- Over $\mathbb{Z}_{(P)}$, $W^{(\notin P)}(B) \cong \prod_{(n,P)=1} B$, with $(qb)_n = q^n b_n$
- $(W^{(\notin P)}B)/(q-1) \to W^{(\notin P)}(B/(q-1))$ consists of $B/(q^n-1) \to B/(q-1)$.
- $[n]_q$ a unit in $\mathbb{Z}_{(P)}[\![q-1]\!]$. □

Consequences

- $\widehat{q\text{-}\Omega}^{\bullet}_{A/R}$ in mixed characteristic depends only on lift Ψ^p of Frobenius.
- $\rightsquigarrow$ Scholze's Cartier isomorphism.
- But how to explain [BMS] functoriality (Ψ^p not needed if we attach q^{1/p^∞})?
- Their constructions all involve $W\Omega^\bullet$, and $W^{(p)}$ turns commutative rings into Λ_p-rings.

q-de Rham–Witt cohomology?

- $W\Omega^\bullet_{\mathbb{F}_p[\underline{x}]/\mathbb{F}_p}$ is p-adic completion of a certain subspace of $\Omega^\bullet_{\mathbb{Q}_p[\underline{x}^{1/p^\infty}]/\mathbb{Q}_p}$.
- so look at $(q-1)\nabla_q$:

$$R[q^{1/p^\infty}, x^{1/p^\infty}] \to R[q^{1/p^\infty}, x^{1/p^\infty}]x^{1/p^\infty} d\log x$$
$$x^\alpha \mapsto (q^\alpha - 1)x^\alpha d\log x,$$

noting that $\frac{q^{m/n}-1}{q-1} = \frac{[m]_{q^{1/n}}}{[n]_{q^{1/n}}}$.

- Décalage $\rightsquigarrow$ q^{1/p^∞}-analogue of $W\Omega^\bullet$.

Towards functorial q-dR–W

- $\Psi^{1/p^\infty} B := \varinjlim(B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} \dots)$.
- $\Psi^{1/p^\infty}\mathrm{q\hat{D}R}_p(A/\mathbb{Z}_p)$ is invariant of Λ_p-ring $\mathrm{H}^0(\Psi^{1/p^\infty}\mathrm{q\hat{D}R}_p(A/\mathbb{Z}_p)/(q-1))$,
- which is dense Λ_p-subring of $W^{(p)}(A[q^{1/p^\infty}]/[p]_{q^{1/p}}) = W^{(p)}A[\zeta_{p^\infty}]$.
- Tweaking construction of $\Psi^{1/p^\infty}\mathrm{qDR}$ with ideal $\bigcup_n(\frac{q-1}{q^{1/n}-1})$ replacing $(q-1)$ gives functorial invariant of $A[\zeta_{p^\infty}]$,
- which resists calculation.