ON q-DE RHAM COHOMOLOGY VIA Λ-RINGS

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Abstract. We show that Aomoto's q-deformation of de Rham cohomology arises as a natural cohomology theory for Λ-rings. Moreover, Scholze's $(q-1)$-adic completion of q-de Rham cohomology depends only on the Adams operations at each residue characteristic. This gives a fully functorial cohomology theory, including a lift of the Cartier isomorphism, for smooth formal schemes in mixed characteristic equipped with a suitable lift of Frobenius. If we attach p-power roots of q, the resulting theory is independent even of these lifts of Frobenius, refining a comparison by Bhatt, Morrow and Scholze.

Introduction

The q-de Rham cohomology of a polynomial ring is a $\mathbb{Z}[q]$-linear complex given by replacing the usual derivative with the Jackson q-derivative $\nabla_q(x^n) = [n]_q x^{n-1}dx$, where $[n]_q$ is Gauss' q-analogue $\frac{q^n-1}{q-1}$ of the integer n. In [Sch2], Scholze discussed the $(q-1)$-adic completion of this theory for smooth rings, explaining relations to p-adic Hodge theory and singular cohomology, and conjecturing that it is independent of co-ordinates, so functorial for smooth algebras over a fixed base [Sch2, Conjectures 1.1, 3.1 and 7.1].

We show that q-de Rham cohomology with q-connections naturally arises as a functorial invariant of Λ-rings (Theorems 1.17, 1.23 and Proposition 1.25), and that its $(q-1)$-adic completion depends only on a Λ_P-ring structure (Theorem 2.8), for P the set of residue characteristics; a Λ_P-ring has a lift of Frobenius for each $p \in P$. This recovers the known equivalence between de Rham cohomology and complete q-de Rham cohomology over the rationals, while giving no really new functoriality statements for smooth schemes over $\mathbb{Z}$. However, in mixed characteristic, it means that complete q-de Rham cohomology depends only on a lift Ψ^p of absolute Frobenius locally generated by co-ordinates with $\Psi^p(x_i) = x_i^p$. Given such data, we construct (Proposition 2.10) a quasi-isomorphism between Hodge cohomology and q-de Rham cohomology modulo $[p]_q$, extending the local lift of the Cartier isomorphism in [Sch2, Proposition 3.4].

Taking the Frobenius stabilisation of the complete q-de Rham complex of A yields a complex resembling the de Rham–Witt complex. We show (Theorem 3.11) that up to $(q^{1/p^\infty} - 1)$-torsion, the p-adic completion of this complex depends only on the p-adic completion of $A[\zeta_{p^\infty}]$ (where ζ_n denotes a primitive nth root of unity), with no requirement for a lift of Frobenius or a choice of co-ordinates. The main idea is to show that the stabilised q-de Rham complex is in a sense given by applying Fontaine's period ring construction A_{inf} to the best possible perfectoid approximation to $A[\zeta_{p^\infty}]$. As a consequence, this shows (Corollary 3.13) that after attaching all p-power roots of q, q-de Rham cohomology in mixed characteristic is independent of choices, which was already known after base change to a period ring, via the comparisons of [BMS] between q-de Rham cohomology and their theory $A\Omega$.

The cohomology theories we construct thus depend either on Adams operations at the residue characteristics (for de Rham) or on p-power roots of q (for variants of de Rham–Witt), establishing correspondingly weakened versions of the conjectures of [Sch2]; in Remark 3.15, we suggest a possible candidate for a theory without those restrictions. The essence of our construction of q-de Rham cohomology of A over R is to set q to be an element of rank 1 for the Λ-ring structure, and to look at flat Λ-rings B over $R[q]$ equipped with morphisms $A \to B/(q-1)$ of Λ-rings over R. If these seem unfamiliar, reassurance should be provided by the observation that $(q-1)B$ carries q-analogues of divided power operations (Remark 1.4). For the variants of de Rham–Witt cohomology in §3, the key to giving a characterisation independent of lifts of Frobenius is the factorisation of the tilting equivalence for perfectoid algebras via a category of Λ_p-rings, leading to constructions similar to [BMS].

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Contents

1. Comparisons for Λ-rings

We will follow standard notational conventions for Λ-rings. These are commutative rings equipped with operations λ^i resembling alternating powers, in particular satisfying $\lambda^k(a+b) = \sum_{i=0}^k \lambda^i(a)\lambda^{k-i}(b)$, with $\lambda^0(a) = 1$ and $\lambda^1(a) = a$. For background, see [Bor] and references therein. The Λ-rings we encounter are all torsion-free, in which case [Wil] shows the Λ-ring structure is equivalent to giving ring endomorphisms Ψ^n for $n \in \mathbb{Z}_{>0}$ with $\Psi^{mn} = \Psi^m \circ \Psi^n$ and $\Psi^p(x) \equiv x^p \mod p$ for all primes p. If we write $\lambda_t(f) := \sum_{i\geq 0} \lambda^i(f)t^i$ and $\Psi_t(f) := \sum_{n\geq 1} \Psi^n(f)t^n$, then the families of operations are related by the formula $\Psi_t = -t\frac{d \log \lambda_{-t}}{dt}$.

We refer to elements x with $\lambda^i(x) = 0$ for all $i > 1$ (or equivalently $\Psi^n(x) = x^n$ for all n) as elements of rank 1.

1.1. **The Λ-ring $\mathbb{Z}[q]$.**

Definition 1.1. Define $\mathbb{Z}[q]$ to be the Λ-ring with operations determined by setting q to be of rank 1.

We now consider the q-analogues $[n]_q := \frac{q^n-1}{q-1} \in \mathbb{Z}[q]$ of the integers, with $[n]_q! = [n]_q[n-1]_q \dots [1]_q$, and $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$.

Remark 1.2. To see the importance of regarding $\mathbb{Z}[q]$ as a Λ-ring observe that the binomial expressions

$$\lambda^k(n) = \tbinom{n}{k}, \quad \lambda^k(-n) = (-1)^k\tbinom{n+k-1}{k}$$

have as q-analogues the Gaussian binomial theorems

$$\lambda^k([n]_q) = q^{k(k-1)/2}\tbinom{n}{k}_q, \quad \lambda^k(-[n]_q) = (-1)^k\tbinom{n+k-1}{k}_q,$$

as well as Adams operations

$$\Psi^i([n]_q) = [n]_{q^i}.$$

For any torsion-free Λ-ring, localisation at a set of elements closed under the Adams operations always yields another Λ-ring, since $\Psi^p(a^{-1}) - a^{-p} = (\Psi^p(a)a^p)^{-1}(a^p - \Psi^p(a))$ is divisible by p.

Lemma 1.3. *For the Λ-ring structure on $\mathbb{Z}[x,y]$ with x,y of rank 1, the elements*

$$\lambda^n(\tfrac{y-x}{q-1}) \in \mathbb{Z}[q, \{(q^n-1)^{-1}\}_{n\geq 1}, x, y]$$

are given by

$$\lambda^k(\tfrac{y-x}{q-1}) = \frac{(y-x)(y-qx)\dots(y-q^{k-1}x)}{(q-1)^k[k]_q!},$$
$$= \sum_{j=0}^{k} \frac{q^{j(j-1)/2}(-x)^j y^{k-j}}{[j]_q![k-j]_q!}.$$

Proof. The second expression comes from multiplying out the Gaussian binomial expansions. The easiest way to prove the first is to observe that $\lambda^k(\frac{y-x}{q-1})$ must be a homogeneous polynomial of degree k in x, y, with coefficients in the integral domain $\mathbb{Z}[q, \{(q^n-1)^{-1}\}_{n\geq 1}]$, and to note that

$$\lambda^k(\tfrac{q^n x - x}{q-1}) = \lambda^k([n]_q x) = q^{k(k-1)/2}\tbinom{n}{k}_q x^k.$$

Thus $\lambda^k(\frac{y-x}{q-1})$ agrees with the homogeneous polynomial above for infinitely many values of $\frac{y}{x}$, so must be equal to it. ☐

Remark 1.4. Note that as $q \to 1$, Lemma 1.3 gives $(q-1)^k\lambda^k(\frac{y-x}{q-1}) \to \frac{(x-y)^k}{k!}$. Indeed, for any rank 1 element x in a Λ-ring we have

$$\lambda_{(q-1)t}(\frac{x}{q-1}) = \sum_{k\geq 0} \frac{(xt)^k}{[k]_q!},$$

which is just the q-exponential $e_q(xt)$. Multiplicativity and universality then imply that $\lambda_{(q-1)t}(\frac{a}{q-1})$ is a q-deformation of $\exp(at)$ for all a. Thus $(q-1)^k\lambda^k(\frac{a}{q-1})$ is a q-analogue of the kth divided power $(a^k/k!)$. An explicit expression comes recursively from the formula

$$[k]_q(q-1)\lambda^k(\tfrac{a}{q-1}) = \sum_{i>0} \lambda^i(a)\lambda^{k-i}(\tfrac{a}{q-1}),$$

obtained by subtracting $\lambda_t(\frac{a}{q-1})$ from each side of the expression $\lambda_{qt}(\frac{a}{q-1}) = \lambda_t(a)\lambda_t(\frac{a}{q-1})$, which arises because q is of rank 1 and $\frac{qa}{q-1} = a + \frac{a}{q-1}$.

Lemma 1.5. *For elements x, y of rank 1, the Λ-subring of $\mathbb{Z}[q, \{(q^n-1)^{-1}\}_{n\geq 1}, x, y]$ generated by $q, x, y, \frac{y-x}{q-1}$ has basis $\lambda^k(\frac{y-x}{q-1})$ as a $\mathbb{Z}[q,x]$-module.*

Proof. The Λ-subring clearly contains the $\mathbb{Z}[q,x]$-module M generated by the elements $\lambda^k(\frac{y-x}{q-1})$, which are also clearly $\mathbb{Z}[q,x]$-linearly independent. Since $\mathbb{Z}[q,x]$ is a Λ-ring, it suffices to show that M is closed under multiplication.

By Lemma 1.3, we know that

$$\lambda^i(\tfrac{y-x}{q-1})\lambda^j(\tfrac{y-q^ix}{q-1}) = \binom{i+j}{i}_q \lambda^{i+j}(\tfrac{y-x}{q-1}).$$

We can rewrite $\frac{y-q^ix}{q-1} = \frac{y-x}{q-1} - [i]_q x$, so $\lambda^j(\frac{y-q^ix}{q-1}) - \lambda^j(\frac{y-x}{q-1})$ lies in the $\mathbb{Z}[q,x]$-module spanned by $\lambda^m(\frac{y-x}{q-1})$ for $m < j$. By induction on j, it thus follows that

$$\lambda^i(\tfrac{y-x}{q-1})\lambda^j(\tfrac{y-q^ix}{q-1}) - \lambda^i(\tfrac{y-x}{q-1})\lambda^j(\tfrac{y-x}{q-1}) \in M,$$

so the binomial expression above implies $\lambda^i(\frac{y-x}{q-1})\lambda^j(\frac{y-x}{q-1}) \in M$. ☐

1.2. q**-cohomology of Λ-rings.**

Definition 1.6. Given a Λ-ring R, say that A is a Λ-ring over R if it is a Λ-ring equipped with a morphism $R \to A$ of Λ-rings. We say that A is a flat Λ-ring over R if A is flat as a module over the commutative ring underlying R.

Definition 1.7. Given a morphism $R \to A$ of Λ-rings, we define the category $\mathrm{Strat}^q_{A/R}$ to consist of flat Λ-rings B over $R[q]$ equipped with a compatible morphism $f\colon A \to B/(q-1)$, such that f admits a lift to B; a choice of lift is not taken to be part of the data, so need not be preserved by morphisms.

More concisely, $\mathrm{Strat}^q_{A/R}$ is the Grothendieck construction of the set-valued functor

$$(\mathrm{Spec}\, A)^q_{\mathrm{strat}}\colon\ B \mapsto \mathrm{Im}\,(\mathrm{Hom}_{\Lambda,R}(A,B) \to \mathrm{Hom}_{\Lambda,R}(A, B/(q-1)))$$

on the category $f\Lambda(R[q])$ of flat Λ-rings over $R[q]$.

Definition 1.8. Given a flat morphism $R \to A$ of Λ-rings, define $\mathrm{qDR}(A/R)$ to be the cochain complex of $R[q]$-modules given by taking the homotopy limit (in the sense of [BK]) of the functor

$$\begin{aligned} \mathrm{Strat}^q_{A/R} &\to \mathrm{Ch}(R[q]) \\ B &\mapsto B. \end{aligned}$$

The cochain complex $\mathrm{qDR}(A/R)$ naturally carries $(R[q], \Psi^n)$-semilinear operations Ψ^n coming from the morphisms $\Psi^n\colon\ B \otimes_{R[q],\Psi^n} R[q] \to B$ of $R[q]$-modules, for $B \in \mathrm{Strat}^q_{A/R}$.

Equivalently, can we follow the approach of [Gro, Sim] towards the stratified site and de Rham stack by regarding $\mathrm{qDR}(A/R)$ as the quasi-coherent cohomology complex of $(\mathrm{Spec}\, A)^q_{\mathrm{strat}}$, as follows.

Definition 1.9. Given a category $\mathcal{C}$, write $[\mathcal{C}, \mathrm{Set}]$ and $[\mathcal{C}, \mathrm{Ab}]$ for the categories of functors on $\mathcal{C}$ taking values in sets and abelian groups, respectively. For any functor

$X\colon \mathcal{C} \to \mathrm{Set}$, we then denote by $\mathbf{R}\mathrm{Hom}_{[\mathcal{C},\mathrm{Set}]}(X,-)$ the functor from $[\mathcal{C},\mathrm{Ab}]$ to cochain complexes given by taking the right-derived functor of the functor

$$\mathrm{Hom}_{[\mathcal{C},\mathrm{Set}]}(X,-)\colon\ [\mathcal{C},\mathrm{Ab}] \to \mathrm{Ab}$$

of natural transformations with source X.

For the forgetful functor $\mathscr{O}\colon\ f\Lambda(R[q]) \to \mathrm{Mod}(R[q])$ to the category of $R[q]$-modules, we then have

$$\mathrm{qDR}(A/R) = \mathbf{R}\mathrm{Hom}_{[f\Lambda(R[q]),\mathrm{Set}]}((\mathrm{Spec}\,A)^q_{\mathrm{strat}}, \mathscr{O}),$$

with Adams operations $\Psi^n\colon\ \mathscr{O} \otimes_{R[q],\Psi^n} R[q] \to \mathscr{O}$ giving the $(R[q],\Psi^n)$-semilinear operations Ψ^n on $\mathrm{qDR}(A/R)$.

Remark 1.10. The cochain complex $\mathrm{qDR}(A/R)$ naturally carries much more structure than these Adams operations. Whenever we can factor the functor $\mathscr{O}$ through a model category $\mathcal{C}$ equipped with a forgetful functor to $\mathrm{Ch}(R[q])$ preserving weak equivalences and homotopy limits, we can regard $\mathrm{qDR}(A/R)$ as an object of the homotopy category of $\mathcal{C}$ by taking the defining homotopy limit in $\mathcal{C}$.

The universal such example for $\mathcal{C}$ is given by the model category of cosimplicial Λ-rings over $R[q]$, with weak equivalences being quasi-isomorphisms (i.e. cohomology isomorphisms) and fibrations being surjections; the underlying cochain complex has differential $\sum(-1)^j\partial^j$. That this determines a model structure follows from Kan's transfer theorem [Hir, Theorem 11.3.2] applied to the cosimplicial Dold–Kan normalisation functor taking values in unbounded chain complexes with the projective model structure; the conditions of that theorem are satisfied because the left adjoint functor sends acyclic cofibrant complexes to cosimplicial Λ-rings which automatically have a contracting homotopy in the form of an extra codegeneracy map.

In particular, $\mathrm{qDR}(A/R)$ naturally underlies a quasi-isomorphism class of cosimplicial Λ-rings over $R[q]$; forgetting the λ-operations gives a cosimplicial commutative $R[q]$-algebra, and stabilisation then gives an E_∞-algebra over $R[q]$, all with underlying cochain complex $\mathrm{qDR}(A/R)$.

Definition 1.11. Given a polynomial ring $R[x]$, recall from [Sch2] that the q-de Rham (or Aomoto–Jackson) cohomology $q\text{-}\Omega^\bullet_{R[x]/R}$ is given by the complex

$$R[x][q] \xrightarrow{\nabla_q} R[x][q]dx, \qquad \text{where} \qquad \nabla_q(f) = \frac{f(qx)-f(x)}{x(q-1)}dx,$$

so $\nabla_q(x^n) = [n]_q x^{n-1}dx$.

Given a polynomial ring $R[x_1,\ldots,x_d]$, the q-de Rham complex $q\text{-}\Omega^\bullet_{R[x_1,\ldots,x_d]/R}$ is then set to be

$$q\text{-}\Omega^\bullet_{R[x_1]/R} \otimes_{R[q]} q\text{-}\Omega^\bullet_{R[x_2]/R} \otimes_{R[q]} \cdots \otimes_{R[q]} q\text{-}\Omega^\bullet_{R[x_d]/R},$$

so takes the form

$$R[x_1,\ldots,x_d][q] \xrightarrow{\nabla_q} \Omega^1_{R[x_1,\ldots,x_d]/R}[q] \xrightarrow{\nabla_q} \ldots \xrightarrow{\nabla_q} \Omega^d_{R[x_1,\ldots,x_d]/R}[q].$$

Definition 1.12. Given a flat morphism $R \to A$ of Λ-rings with $X = \mathrm{Spec}\,A$, define the functor $\tilde{X}^q_{\mathrm{strat}}$ from flat Λ-rings over $R[q]$ to simplicial sets by taking the Čech nerve of $\mathrm{Hom}_{\Lambda,R}(A,B) \to \mathrm{Hom}_{\Lambda,R}(A,B/(q-1))$, so

$$(\tilde{X}^q_{\mathrm{strat}})_n(B) := \overbrace{\mathrm{Hom}_{\Lambda,R}(A,B) \times_{\mathrm{Hom}_{\Lambda,R}(A,B/(q-1))} \ldots \times_{\mathrm{Hom}_{\Lambda,R}(A,B/(q-1))} \mathrm{Hom}_{\Lambda,R}(A,B)}^{n+1}$$
$$= \mathrm{Hom}_{\Lambda,R}(A, \overbrace{B \times_{B/(q-1)} \ldots \times_{B/(q-1)} B}^{n+1}),$$

with simplicial operations

$$\partial_j(f_0, f_1, \ldots, f_n) := (f_0, f_1, \ldots, f_{j-1}, f_{j+1}, f_{j+2}, \ldots, f_n),$$
$$\sigma_j(f_0, f_1, \ldots, f_n) := (f_0, f_1, \ldots, f_j, f_j, f_{j+1}, \ldots, f_n).$$

Definition 1.13. Given a cosimplicial abelian group $V^\bullet$, we write NV for the Dold–Kan normalisation of V ([Wei, Lemma 8.3.7] applied the opposite category). This is a cochain complex with $N^rV = V^r \cap_{j<r} \ker \sigma^j$ and differential $d = \sum_{j=0}^{r+1}(-1)^j\partial^j\colon N^rV \to N^{r+1}V$.

Lemma 1.14. *If, for $X = \mathrm{Spec}\, A$, the functors $(\tilde{X}^q_{\mathrm{strat}})_n$ are represented by flat Λ-rings $\Gamma((\tilde{X}^q_{\mathrm{strat}})_n, \mathscr{O})$ over $R[q]$, then a model for $\mathrm{qDR}(A/R)$ is given by the Dold–Kan normalisation of the cosimplicial module $n \mapsto \Gamma((\tilde{X}^q_{\mathrm{strat}})_n, \mathscr{O})$.*

Proof. The set-valued functor $X^q_{\mathrm{strat}} = (\mathrm{Spec}\, A)^q_{\mathrm{strat}}$ of Definition 1.7 is resolved by the simplicial functor $\tilde{X}^q_{\mathrm{strat}}$ of Definition 1.12. In the notation of Definition 1.9, this implies that the functor $\mathrm{Hom}_{[f\Lambda(R[q]),\mathrm{Set}]}(X^q_{\mathrm{strat}}, -)$ on $[f\Lambda(R[q]), \mathrm{Ab}]$ is resolved by the cochain complex

$$N\mathrm{Hom}_{[f\Lambda(R[q]),\mathrm{Set}]}((\tilde{X}^q_{\mathrm{strat}})_\bullet, -).$$

Although X^q_{strat} is not representable on the category of flat Λ-rings over $R[q]$, our hypotheses ensure that each functor $(\tilde{X}^q_{\mathrm{strat}})_n$ is so. Thus the functors $\mathrm{Hom}_{[f\Lambda(R[q]),\mathrm{Set}]}((\tilde{X}^q_{\mathrm{strat}})_n, -)$ and their direct summands $N^n\mathrm{Hom}_{[f\Lambda(R[q]),\mathrm{Set}]}((\tilde{X}^q_{\mathrm{strat}})_\bullet, -)$ are exact, and are their own right-derived functors. This implies that the cochain complex of functors above models $\mathbf{R}\mathrm{Hom}_{[f\Lambda(R[q]),\mathrm{Set}]}(X^q_{\mathrm{strat}}, -)$, and the result follows by evaluation at $\mathscr{O}$. ☐

Proposition 1.15. *If R is a Λ-ring and x of rank 1, then $\mathrm{qDR}(R[x]/R)$ can be calculated by Dold–Kan normalisation of the cosimplicial $R[q]$-module $U^\bullet$ given by setting U^n to be the Λ-subring*

$$U^n \subset R[q, \{(q^m-1)^{-1}\}_{m\ge1}, x_0, \ldots, x_n]$$

generated by q and the elements x_i and $\frac{x_i - x_j}{q-1}$, with cosimplicial operations

$$\partial^j x_i := \begin{cases} x_i & j > i \\ x_{i+1} & j \le i, \end{cases} \qquad \sigma^j x_i := \begin{cases} x_i & j \ge i \\ x_{i-1} & j < i. \end{cases}$$

Proof. We verify the conditions of Lemma 1.14 by showing that each U^n is a flat Λ-ring over $R[q]$ representing $(\tilde{X}^q_{\mathrm{strat}})_n$. Taking $X = \mathrm{Spec}\, R[x]$, observe that any element of $(\tilde{X}^q_{\mathrm{strat}})_n(B)$ gives rise to a morphism $f\colon R[q, x_0, \ldots, x_n] \to B$ of Λ-rings over $R[q]$, with the image of $x_i - x_j$ divisible by $(q-1)$. Flatness of B then gives a unique element $f(x_i - x_j)/(q-1) \in B$, so we have a map f to B from the free Λ-ring L over $R[q, x_0, \ldots, x_n]$ generated by elements z_{ij} with $(q-1)z_{ij} = x_i - x_j$.

Since B is flat, it embeds in $B[\{(q^m-1)^{-1}\}_{m\ge1}]$ (the only hypothesis we really need) implying that the image of f factors through the image U^n of L in $R[q, \{(q^m -$

$1)^{-1}\}_{m\geq 1}, x_0, \dots, x_n]$. To see that $(\tilde{X}^q_{\mathrm{strat}})_n$ is represented by U^n, we only now need to check that U^n is itself flat over $R[q]$, which follows because the argument of Lemma 1.5 gives a basis

$$x_0^{r_0}\lambda^{r_1}(\tfrac{x_1-x_0}{q-1})\cdots\lambda^{r_n}(\tfrac{x_n-x_{n-1}}{q-1})$$

for U^n over $R[q]$. We therefore have $\mathrm{qDR}(R[x]/R) \simeq NU^\bullet$. □

In fact, the proofs of Lemma 1.14 and Proposition 1.15 show that the natural cosimplicial Λ-ring structure on $U^\bullet$ gives a model for the cosimplicial Λ-ring structure on $\mathrm{qDR}(R[x]/R)$ coming from Remark 1.10.

Definition 1.16. Following [Sch2, Proposition 5.4], we denote by $\mathbf{L}\eta_{(q-1)}$ the décalage functor with respect to the derived $(q-1)$-adic filtration. This is given on complexes $C^\bullet$ of $(q-1)$-torsion-free $R[q]$-modules by

$$(\eta_{(q-1)}C)^n := \{c \in (q-1)^n C^n \ : \ dc \in (q-1)^{n+1}C^{n+1}\},$$

and is extended to the derived category of $R[q]$-modules by taking torsion-free resolutions.

Theorem 1.17. *If R is a Λ-ring and if the polynomial ring $R[x_1, \dots, x_n]$ is given the Λ-ring structure for which the elements x_i are of rank 1, then there are $R[q]$-linear zigzags of quasi-isomorphisms*

$$\begin{aligned} \mathrm{qDR}(R[x_1,\dots,x_n]/R) &\simeq (\Omega^*_{R[x_1,\dots,x_n]/R}[q], (q-1)\nabla_q) \\ \mathbf{L}\eta_{(q-1)}\mathrm{qDR}(R[x_1,\dots,x_n]/R) &\simeq q\text{-}\Omega^\bullet_{R[x_1,\dots,x_n]/R}. \end{aligned}$$

Proof. It suffices to prove the first statement, the second following immediately by décalage. We have $(\mathrm{Spec}\, A\otimes_R A')^q_{\mathrm{strat}}(B) = (\mathrm{Spec}\, A)^q_{\mathrm{strat}}(B) \times (\mathrm{Spec}\, A')^q_{\mathrm{strat}}(B)$, and similarly for the simplicial functor $(\mathrm{Spec}\, A \otimes_R A')^q_{\mathrm{strat}}$ of Definition 1.12. Since coproduct of flat Λ-rings over $R[q]$ is given by $\otimes_{R[q]}$, it follows from Lemma 1.14 and Proposition 1.15 that $\mathrm{qDR}(R[x_1,\dots,x_n]/R)$ can be calculated as the Dold–Kan normalisation of $(U^\bullet)^{\otimes_{R[q]}n}$ (given by the n-fold tensor product $(U^m)^{\otimes_{R[q]}n}$ in cosimplicial level m), for the cosimplicial module $U^\bullet$ of Proposition 1.15.

The proof now proceeds in a similar fashion to the comparison between crystalline and de Rham cohomology in [Bd]. We consider the cochain complexes $\tilde{\Omega}^\bullet(U^m)$ given by

$$U^m \xrightarrow{(q-1)\nabla_q} \bigoplus_i U^m dx_i \xrightarrow{(q-1)\nabla_q} \bigoplus_{i<j} U^m dx_i\wedge dx_j \xrightarrow{(q-1)\nabla_q} \dots.$$

In order to see that this differential takes values in the codomains given, observe that

$$\begin{aligned} (q-1)\nabla_{q,y}\lambda^k(\tfrac{y-x}{q-1}) &= y^{-1}(\lambda^k(\tfrac{qy-x}{q-1}) - \lambda^k(\tfrac{y-x}{q-1}))dy \\ &= y^{-1}(\lambda^k(y+\tfrac{y-x}{q-1}) - \lambda^k(\tfrac{y-x}{q-1}))dy \\ &= \lambda^{k-1}(\tfrac{y-x}{q-1})dy, \end{aligned}$$

and similarly

$$(q-1)\nabla_{q,x}\lambda^k(\tfrac{y-x}{q-1}) = \sum_{i\geq 1}(-1)^i x^{i-1}\lambda^{k-i}(\tfrac{y-x}{q-1})dx.$$

The first calculation also shows that the inclusion $\tilde{\Omega}^\bullet(U^{m-1}) \hookrightarrow \tilde{\Omega}^\bullet(U^m)$ is a quasi-isomorphism, since for $\omega \in \tilde{\Omega}^\bullet(U^{m-1})$, we have

$$(q-1)\nabla_{q,x_m}\omega\lambda^k(\tfrac{x_m-x_{m-1}}{q-1}) = \omega\lambda^{k-1}(\tfrac{x_m-x_{m-1}}{q-1})dx_m$$

for $k \geq 1$, allowing us to define a contracting homotopy

$$h(\omega\lambda^{k-1}(\tfrac{x_m - x_{m-1}}{q-1})dx_m) := \omega\lambda^k(\tfrac{x_m - x_{m-1}}{q-1}),$$
$$h(\omega\lambda^{k-1}(\tfrac{x_m - x_{m-1}}{q-1})) := 0.$$

Since contracting homotopies interact well with tensor products, it also follows that the inclusion $\tilde{\Omega}^\bullet(U^{m-1})^{\otimes_{R[q]}n} \hookrightarrow \tilde{\Omega}^\bullet(U^m)^{\otimes_{R[q]}n}$ is a quasi-isomorphism. By induction on m we deduce that the inclusions $\tilde{\Omega}^\bullet(U^0)^{\otimes_{R[q]}n} \hookrightarrow \tilde{\Omega}^\bullet(U^m)^{\otimes_{R[q]}n}$, and hence their retractions given by diagonals $U^m \to U^0$, are quasi-isomorphisms. These combine to give a quasi-isomorphism

$$\mathrm{Tot}\, N(\tilde{\Omega}^\bullet(U^\bullet)^{\otimes_{R[q]}n}) \to \tilde{\Omega}^\bullet(U^0)^{\otimes_{R[q]}n} = \tilde{\Omega}^\bullet(R[x])^{\otimes_{R[q]}n}$$

on total complexes of normalisations.

Now, the cosimplicial module $\tilde{\Omega}^r(U^\bullet)$ is given by the cosimplicial (i.e. levelwise) tensor product of $U^\bullet$ with the cosimplicial $\mathbb{Z}$-module

$$j \mapsto \bigoplus_{0 \leq i_1 < i_2 < \ldots < i_r \leq j} \mathbb{Z}dx_{i_1} \wedge \ldots \wedge dx_{i_r},$$

with operations induced by those in Proposition 1.15. For $r > 0$, this cosimplicial $\mathbb{Z}$-module is contractible, via the extra codegeneracy map given by

$$\sigma^{-1}(dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = \begin{cases} dx_{i_1-1} \wedge \ldots \wedge dx_{i_r-1} & i_1 > 0, \\ 0 & i_1 = 0. \end{cases}$$

The Eilenberg–Zilber theorem ([Wei, §8.5] applied to the opposite category) ensures that the normalisation of a cosimplicial tensor product is quasi-isomorphic to the tensor product of the normalisations. Tensoring with a complex which has an extra codegeneracy map always produces an acyclic complex, so $\tilde{\Omega}^r(U^\bullet)$ and its tensor powers are all acyclic for $r > 0$.

The brutal truncation maps

$$\mathrm{Tot}\, N(\tilde{\Omega}^\bullet(U^\bullet)^{\otimes_{R[q]}n}) \to N(U^\bullet)^{\otimes_{R[q]}n} \simeq \mathrm{qDR}(R[x_1, \ldots, x_n]/R)$$

are therefore quasi-isomorphisms of flat cochain complexes over $R[q]$, so

$$\mathrm{qDR}(R[x_1, \ldots, x_n]/R) \simeq \tilde{\Omega}^\bullet(R[x])^{\otimes_{R[q]}n},$$

and we just observe that $\tilde{\Omega}^\bullet(R[x]) = (\Omega^*_{R[x]/R}[q], (q-1)\nabla_q)$. □

Remark 1.18. Note that Theorem 1.17 and Remark 1.10 together imply that $q\text{-}\Omega^\bullet_{R[x_1,\ldots,x_n]/R}$ naturally underlies the décalage of a cosimplicial Λ-ring over $R[q]$. Even the underlying cosimplicial commutative ring structure carries more information than an E_∞-structure when $\mathbb{Q} \not\subseteq R$.

1.3. **Completed q-cohomology.**

Definition 1.19. Given a morphism $R \to A$ of Λ-rings, we define the category $\hat{\mathrm{Strat}}^q_{A/R} \subset \mathrm{Strat}^q_{A/R}$ to consist of those objects which are $(q-1)$-adically complete.

Equivalently, $\hat{\mathrm{Strat}}^q_{A/R}$ is the Grothendieck construction of the functor

$$(\widehat{\mathrm{Spec}\, A})^q_{\mathrm{strat}} \colon\ B \mapsto \mathrm{Im}\,(\mathrm{Hom}_{\Lambda,R}(A, B) \to \mathrm{Hom}_{\Lambda,R}(A, B/(q-1)))$$

on the category of flat $(q-1)$-adically complete Λ-rings over $R[q]$.

Definition 1.20. Given a flat morphism $R \to A$ of Λ-rings, define $\widehat{\mathrm{qDR}}(A/R)$ to be the cochain complex of $R[\![q-1]\!]$-modules given by taking the homotopy limit of the functor

$$\begin{aligned} \hat{\mathrm{Strat}}^q_{A/R} &\to \mathrm{Ch}(R[\![q-1]\!]) \\ B &\mapsto B. \end{aligned}$$

The following is immediate:

Lemma 1.21. *Given a flat morphism $R \to A$ of Λ-rings, the complex $\widehat{\mathrm{qDR}}(A/R)$ is the derived $(q-1)$-adic completion of $\mathrm{qDR}(A/R)$.*

Definition 1.22. As in [Sch2, §3], given a formally étale map $\Box\colon R[x_1, \ldots, x_d] \to A$, define $\widehat{q\text{-}\Omega}^{\bullet}_{A/R,\Box}$ to be the complex

$$A[\![q-1]\!] \xrightarrow{\nabla_q} \Omega^1_{A/R}[\![q-1]\!] \xrightarrow{\nabla_q} \ldots \xrightarrow{\nabla_q} \Omega^d_{A/R}[\![q-1]\!],$$

where ∇_q is defined as follows. First note that the $R[\![q-1]\!]$-linear ring endomorphisms γ_i of $R[x_1, \ldots, x_d][\![q-1]\!]$ given by $\gamma_i(x_j) = q^{\delta_{ij}} x_j$ extend uniquely to endomorphisms of $A[\![q-1]\!]$ which are the identity modulo $(q-1)$, then set

$$\nabla_q(f) := \sum_i \frac{\gamma_i(f) - f}{(q-1)x_i} dx_i.$$

Note that $\widehat{q\text{-}\Omega}^{\bullet}_{R[x_1,\ldots,x_d]/R}$ is just the $(q-1)$-adic completion of $q\text{-}\Omega^{\bullet}_{R[x_1,\ldots,x_d]/R}$.

Theorem 1.23. *If R is a flat Λ-ring over $\mathbb{Z}$ and $\Box\colon R[x_1, \ldots, x_d] \to A$ is a formally étale map of Λ-rings, the elements x_i having rank 1, then there are zigzags of $R[\![q]\!]$-linear quasi-isomorphisms*

$$\widehat{\mathrm{qDR}}(A/R) \simeq (\Omega^*_{A/R}[\![q-1]\!], (q-1)\nabla_q), \qquad \mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}(A/R) \simeq \widehat{q\text{-}\Omega}^{\bullet}_{A/R,\Box}.$$

The induced quasi-isomorphisms

$$\widehat{\mathrm{qDR}}(A/R) \otimes^{\mathbf{L}}_{R[\![q-1]\!]} R \simeq (\Omega^*_{A/R}, 0), \qquad (\mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}(A/R)) \otimes^{\mathbf{L}}_{R[\![q-1]\!]} R \simeq \Omega^{\bullet}_{A/R}$$

are independent of the choice of framing.

Proof. Since the framing $\Box$ is formally étale, for any $(q-1)$-adically complete commutative $R[q]$-algebra B, any commutative square

$$\begin{array}{ccc} R[x_1, \ldots, x_d] & \longrightarrow & B \\ {\scriptstyle\Box}\downarrow & \nearrow & \downarrow \\ A & \longrightarrow & B/(q-1). \end{array}$$

of R-algebra homomorphisms admits a unique dashed arrow as shown.

For any $(q-1)$-adically complete flat Λ-ring B over R, we then have the same property for Λ-ring homomorphisms over R instead of R-algebra homomorphisms: the diagram above gives a unique dashed R-algebra homomorphism, and uniqueness of lifts ensures that it commutes with Adams operations, so is a Λ-ring homomorphism (R being flat over $\mathbb{Z}$). Similarly (taking $B = A[\![q-1]\!]$) uniqueness of lifts ensures that the operations γ_i are Λ-ring endomorphisms of $A[\![q-1]\!]$.

We can now proceed as in the proof of Theorem 1.17. The complex $\widehat{\mathrm{qDR}}(A/R)$ can be realised as the cochain complex underlying a cosimplicial Λ-ring $\hat{U}(A)$, representing the functor $\tilde{X}^q_{\mathrm{strat}}$ of Definition 1.12 for $X = \operatorname{Spec} A$, restricted to $(q-1)$-adically complete Λ-rings B. By the consequences of formal étaleness, we have

$$\begin{aligned}&\mathrm{Hom}_{\Lambda,R}(A,B)\times_{\mathrm{Hom}_{\Lambda,R}(A,B/(q-1))}\mathrm{Hom}_{\Lambda,R}(A,B)\\&\cong \mathrm{Hom}_{\Lambda,R}(A,B)\times_{\mathrm{Hom}_{\Lambda,R}(R[x_1,\ldots,x_d],B/(q-1))}\mathrm{Hom}_{\Lambda,R}(R[x_1,\ldots,x_d],B),\end{aligned}$$

giving $(\tilde{X}^q_{\mathrm{strat}})_n \cong \mathrm{Hom}_{\Lambda,R}(A,B)\times_{\mathrm{Hom}_{\Lambda,R}(R[x_1,\ldots,x_d],B)}(\tilde{Y}^q_{\mathrm{strat}})_n$ for each n, where $Y = \operatorname{Spec} R[x_1,\ldots,x_d]$ and the fibre product is given via the projection of $(\tilde{Y}^q_{\mathrm{strat}})_n$ onto the first factor.

In particular, this means that $\hat{U}(A)^n$ is the $(q-1)$-adic completion of

$$A\otimes_{R[x_1,\ldots,x_d]}(U(R[x_1])^n\otimes_{R[q]}\ldots\otimes_{R[q]}U(R[x_d])^n),$$

where each $U(R[x_i])$ is a copy of the cosimplicial ring U from Proposition 1.15. This isomorphism respects the cosimplicial operations; note that ∂^0 is not linear for the left multiplication by A, but is still determined via formal étaleness of the framing.

We now define a cosimplicial cochain complex $\tilde{\Omega}^\bullet(\hat{U}(A))$ by setting $\tilde{\Omega}^\bullet(\hat{U}(A)^n)$ to be the $(q-1)$-adic completion of

$$\begin{aligned}&(A\otimes_{R[x_1,\ldots,x_d]}(\tilde{\Omega}^*(U(R[x_1])^n)\otimes_{R[q]}\ldots\otimes_{R[q]}\tilde{\Omega}^*(U(R[x_d])^n)),(q-1)\nabla_q)\\&\cong(\hat{U}(A)^n\otimes_{A^{\otimes(n+1)}}(\Omega^*_{A/R})^{\otimes(n+1)},(q-1)\nabla_q)).\end{aligned}$$

where each $\tilde{\Omega}^\bullet(U(R[x_i]))$ is a copy of the complex $\tilde{\Omega}^\bullet(U^n)$ from the proof of Theorem 1.17. Compatibility of this construction with the cosimplicial operations follows because the γ_i are Λ-ring homomorphisms.

The calculations contributing to the proof of Theorem 1.17 are still valid after base change, with contracting homotopies giving quasi-isomorphisms

$$(\Omega^*_{A/R}[\![q]\!],(q-1)\nabla_q)\leftarrow \operatorname{Tot} N\tilde{\Omega}^\bullet(\hat{U}(A)^\bullet)\to N\hat{U}(A)^\bullet.$$

Reduction of this modulo $(q-1)^2$, or of its décalage modulo $(q-1)$ (cf. [BMS, Proposition 6.12]), replaces ∇_q with d throughout, removing any dependence on co-ordinates. □

As in [Sch2, Definition 7.3], there is a notion of q-connection $\nabla_q = (\nabla_{1,q},\ldots,\nabla_{d,q})$ on a finite projective $A[\![q-1]\!]$-module M, in the form of commuting $R[\![q-1]\!]$-linear operators $\nabla_{i,q}$ on M, with each $\nabla_{i,q}$ satisfying $\nabla_{i,q}(av) = \nabla_{q,x_i}(a)v + \gamma_i(a)\nabla_{i,q}(v)$ for $a\in A, v\in M$.

Definition 1.24. Given a flat morphism $R\to A$ of Λ-rings with $X := \operatorname{Spec} A$, denote the forgetful functor $(B,f)\mapsto B$ from $\hat{\mathrm{Strat}}^q_{A/R}$ to rings by $\mathscr{O}_{\hat{X}^q,\mathrm{strat}}$.

There is then a notion of $\mathscr{O}_{\hat{X}^q,\mathrm{strat}}$-modules in the category of functors from $\hat{\mathrm{Strat}}^q_{A/R}$ to abelian groups; we will simply refer to these as $\mathscr{O}_{\hat{X}^q,\mathrm{strat}}$-modules. Given a property P of modules, we will say that an $\mathscr{O}_{\hat{X}^q,\mathrm{strat}}$-module $\mathscr{F}$ has the property P if for each $(B,f)\in\hat{\mathrm{Strat}}^q_{A/R}$, the B-module $\mathscr{F}(B,f)$ has property P.

We say that an $\mathscr{O}_{\hat{X}^q,\mathrm{strat}}$-module $\mathscr{F}$ is Cartesian if for each morphism $(B,f)\to(B',f')$ in $\hat{\mathrm{Strat}}^q_{A/R}$, the map $\mathscr{F}(B,f)\otimes_B B'\to\mathscr{F}(B',f')$ is an isomorphism.

Given an $\mathscr{O}_{\hat{X}^q,\mathrm{strat}}$-module $\mathscr{F}$, we define $\Gamma(\hat{X}^q_{\mathrm{strat}},\mathscr{F}) := \varprojlim_{\hat{\mathrm{Strat}}^q_{A/R}}\mathscr{F}$.

In [Sch2, Conjecture 7.5], Scholze predicted that the category of q-connections on finite projective $A[\![q-1]\!]$-module is independent of co-ordinates on A. The following proposition gives the weaker statement that the category depends only on the Λ-ring structure on A.

Proposition 1.25. *Under the conditions of Theorem 1.23, with* $X := \operatorname{Spec} A$*, the category of finite projective* $A[\![q-1]\!]$*-modules* (M, ∇) *with* q*-connection is equivalent to the category of those finite projective* $\mathscr{O}_{\hat{X}^q,\mathrm{strat}}$*-modules* $\mathscr{N}$ *for which the map*

$$\Gamma(\hat{X}^q_{\mathrm{strat}}, \mathscr{N}/(q-1)) \otimes_A (\mathscr{O}_{\hat{X}^q,\mathrm{strat}}/(q-1)) \to \mathscr{N}/(q-1)$$

is an isomorphism.

Proof. The restriction on $\mathscr{N}/(q-1)$ ensures that it is Cartesian; this also implies that $\mathscr{N}$ is Cartesian, because finite projective modules are flat and $(q-1)$-adically complete.

Now, the cosimplicial Λ-ring $\hat{U}(A)$ realising $\widehat{\mathrm{qDR}}(A/R)$ in the proof of Theorem 1.23 admits a natural map $A \to \hat{U}(A)/(q-1)$ from the constant cosimplicial diagram. Thus $\hat{U}(A)$ defines a cosimplicial diagram in $\hat{\mathrm{Strat}}^q_{A/R}$. Since the functor $\tilde{X}^q_{\mathrm{strat}}$ of Definition 1.12 resolves X^q_{strat}, it follows that the functor $\hat{U}(A)\colon \Delta \to \hat{\mathrm{Strat}}^q_{A/R}$ from the simplex category is initial in the sense of [Mac, §IX.3].

In particular, this means that the category of Cartesian $\mathscr{O}_{\hat{X}^q,\mathrm{strat}}$-modules $\mathscr{N}$ is equivalent to the category of Cartesian cosimplicial $\hat{U}(A)$-modules N, where the Cartesian condition amounts to saying that the maps $N^m \otimes_{\hat{U}(A)^m,\partial^i} \hat{U}(A)^{m+1} \to N^{m+1}$ are all isomorphisms. Setting $M = N^0$, Cartesian $\hat{U}(A)$-modules are equivalent to $\hat{U}(A)^0 = A[\![q-1]\!]$-modules M with isomorphisms $\Delta\colon (\partial^1)^*M \cong (\partial^0)^*M$ satisfying the cocycle condition $\partial^1\Delta = (\partial^0\Delta) \circ (\partial^2\Delta)\colon (\partial^2\partial^0)^*M \to (\partial^0\partial^0)^*M$.

The map Δ is determined by its restriction to M, so using the basis for U^1 from Lemma 1.5, and taking $v \in M$, we have

$$\Delta(v) = \sum_{\underline{k} \in \mathbb{N}_0^d} \partial^0(\Delta_{\underline{k}}(v))\lambda^{k_1}(\tfrac{\partial^1 x_1 - \partial^0 x_1}{q-1}) \cdots \lambda^{k_d}(\tfrac{\partial^1 x_d - \partial^0 x_d}{q-1})$$

for $R[\![q-1]\!]$-linear endomorphisms $\Delta_{\underline{k}}$ of M. Since $\lambda_t(a+b) = \lambda_t(a)\lambda_t(b)$, the cocycle condition becomes $\Delta_{\underline{j}+\underline{k}} = \Delta_{\underline{j}} \circ \Delta_{\underline{k}}$, meaning Δ is determined by the operators Δ_{e_i} at the basis vectors, which must moreover commute.

Linearity of Δ with respect to $\hat{U}(A)^1$ then reduces to the condition that $\Delta(av) = \partial^1(a)\Delta(v)$ for $a \in A$, $v \in M$. Writing A for $\partial^0 A$ and $h_i^{[k]} := \lambda^k(\frac{\partial^1 x_i - \partial^0 x_i}{q-1})$, the ideal $J := (h_i^{[\geq 2]}, h_i h_j)_{i \neq j}$ satisfies $U^1 = A \oplus \bigoplus_i A h_i \oplus J$. The proof of Theorem 1.23 gives $\partial^1(a) \equiv a + (q-1)\sum_i \nabla_{q,x_i}(a) h_i \mod J$, and in U^1/J we have $[h_i]^2 \equiv x_i[h_i]$. Comparing coefficients of h_i in the equation $\Delta(av) \equiv \partial^1(a)\Delta(v) \mod J$ then gives

$$\begin{aligned}\Delta_{e_i}(av) &= (q-1)\nabla_{q,x_i}(a)v + a\Delta_{e_i}(v) + (q-1)x_i\nabla_{q,x_i}(a)\Delta_{e_i}(v) \\ &= (q-1)\nabla_{q,x_i}(a)v + \gamma_i(a)\Delta_{e_i}(v).\end{aligned}$$

Finally, note that the condition that $\mathscr{N}/(q-1)$ be the pullback of an A-module (necessarily $\Gamma(\hat{X}^q_{\mathrm{strat}}, \mathscr{N}/(q-1))$) is equivalent to saying that $\partial^0_N \equiv \partial^1_N \mod (q-1)$, or that $(q-1)$ divides $\Delta_{\underline{k}}$ whenever $\underline{k} \neq 0$. In particular, $(q-1)$ divides Δ_{e_i}, and setting $\nabla_{i,q} := (q-1)^{-1}\Delta_{e_i}$ gives a q-connection $(\nabla_{i,q})_{1 \leq i \leq d}$ on $M = N^0$ uniquely determining Δ.

The inverse construction is given by $\Delta_{\underline{k}} = (q-1)^{\sum k_i}\nabla_{1,q}^{k_1} \circ \cdots \circ \nabla_{d,q}^{k_d}$. □

2. Comparisons for Λ_P-rings

Since very few étale maps $R[x_1, \ldots, x_d] \to A$ give rise to Λ-ring structures on A, Theorem 1.23 is fairly limited in its scope for applications. We now show how the construction of $\widehat{\mathrm{qDR}}$ and the comparison quasi-isomorphism survive when we weaken the Λ-ring structure by discarding Adams operations at invertible primes.

2.1. **q-cohomology for Λ_P-rings.** Our earlier constructions for Λ-rings all carry over to Λ_P-rings, as follows.

Definition 2.1. Given a set P of primes, we define a Λ_P-ring A to be a $\Lambda_{\mathbb{Z},P}$-ring in the sense of [Bor]. This means that it is a coalgebra in commutative rings for the comonad given by the functor $W^{(P)}$ of P-typical Witt vectors. When a commutative ring A is flat over $\mathbb{Z}$, giving a Λ_P-ring structure on A is equivalent to giving commuting Adams operations Ψ^p for all $p \in P$, with $\Psi^p(a) \equiv a^p \mod p$ for all a.

Thus when P is the set of all primes, a Λ_P-ring is just a Λ-ring; a $\Lambda_\emptyset$-ring is just a commutative ring; for a single prime p, we write $\Lambda_p := \Lambda_{\{p\}}$, and note that a Λ_p-ring is a δ-ring in the sense of [Joy].

Definition 2.2. Given a Λ_P-ring R, say that A is a Λ_P-ring over R if it is a Λ_P-ring equipped with a morphism $R \to A$ of Λ_P-rings. We say that A is a flat Λ_P-ring over R if A is flat as a module over the commutative ring underlying R.

Definition 2.3. Given a morphism $R \to A$ of Λ_P-rings, we define the category $\mathrm{Strat}^{q,P}_{A/R}$ to consist of flat Λ_P-rings B over $R[q]$ equipped with a compatible morphism $A \to B/(q-1)$, such that the map $A \to B/(q-1)$ admits a lift to B. We define the category $\hat{\mathrm{Strat}}^{q,P}_{A/R} \subset \mathrm{Strat}^q_{A/R}$ to consist of those objects which are $(q-1)$-adically complete.

More concisely, $\mathrm{Strat}^{q,P}_{A/R}$ (resp. $\hat{\mathrm{Strat}}^{q,P}_{A/R}$) is the Grothendieck construction of the functor $(\mathrm{Spec}\, A)^{q,P}_{\mathrm{strat}}$ (resp. $(\widehat{\mathrm{Spec}\, A})^{q,P}_{\mathrm{strat}}$) given by

$$B \mapsto \mathrm{Im}\left(\mathrm{Hom}_{\Lambda_P,R}(A, B) \to \mathrm{Hom}_{\Lambda_P,R}(A, B/(q-1))\right)$$

on the category of flat Λ_P-rings (resp. $(q-1)$-adically complete flat Λ_P-rings) over $R[q]$.

Definition 2.4. Given a flat morphism $R \to A$ of Λ_P-rings, define $\mathrm{qDR}_P(A/R)$ to be the cochain complex of $R[q]$-modules given by taking the homotopy limit of the functor

$$\begin{aligned} \mathrm{Strat}^{q,P}_{A/R} &\to \mathrm{Ch}(R[q]) \\ B &\mapsto B. \end{aligned}$$

Define $\widehat{\mathrm{qDR}}_P(A/R)$ to be the cochain complex of $R[\![q-1]\!]$-modules given by the corresponding homotopy limit over $\hat{\mathrm{Strat}}^{q,P}_{A/R}$.

For $p \in P$, the cochain complex $\mathrm{qDR}_P(A/R)$ naturally carries $(R[q], \Psi^p)$-semilinear operations Ψ^p coming from the morphisms $\Psi^p \colon B \otimes_{R[q],\Psi^p} R[q] \to B$ of $R[q]$-modules, for $B \in \mathrm{Strat}^{q,P}_{A/R}$.

Thus when P is the set of all primes, we have $\mathrm{qDR}_P(A/R) = \mathrm{qDR}(A/R)$. At the other extreme, for A smooth, $\widehat{\mathrm{qDR}}_\emptyset(A/R)$ is the Rees construction of the Hodge filtration on the infinitesimal cohomology complex [Gro] of A over R , with formal variable $(q-1)$. In more detail, there is a decreasing filtration F of $\mathscr{O}_{\mathrm{inf}}$ given by powers of the augmentation ideal of $\mathscr{O}_{\mathrm{inf}} \to \mathscr{O}_{\mathrm{Zar}}$ (with $F^\nu \mathscr{O}_{\mathrm{inf}} = \mathscr{O}_{\mathrm{inf}}$ for $\nu \le 0$), and then

$$\widehat{\mathrm{qDR}}_\emptyset(A/R) \simeq \prod_{\nu\in\mathbb{Z}} (q-1)^{-\nu}\mathbf{R}\Gamma(\operatorname{Spec} A, F^\nu\mathscr{O}_{\mathrm{inf}}).$$

Lemma 2.5. *For a set P of primes, the forgetful functor from Λ-rings to Λ_P-rings has a right adjoint $W^{(\notin P)}$. There is a canonical ghost component morphism*

$$W^{(\notin P)}(B) \to \prod_{\substack{n\in\mathbb{N}:\\ (n,p)=1\ \forall p\in P}} B,$$

which is an isomorphism when P contains all the residue characteristics of B.

Proof. Existence of a right adjoint follows from the comonadic definitions of Λ-rings and Λ_P-rings. The ghost component morphism is given by taking the Adams operations Ψ^n coming from the Λ-ring structure on $W^{(\notin P)}(B)$, followed by projection to B. When P contains all the residue characteristics of B, a Λ-ring structure is the same as a Λ_P-ring structure with compatible commuting Adams operations for all primes not in P, leading to the description above. ☐

Note that the big Witt vector functor W on commutative rings thus factorises as $W = W^{(\notin P)} \circ W^{(P)}$, for $W^{(P)}$ the P-typical Witt vectors.

Proposition 2.6. *Given a morphism $R \to A$ of Λ-rings, and a set P of primes, there are natural maps*

$$\mathrm{qDR}_P(A/R) \to \mathrm{qDR}(A/R), \qquad \widehat{\mathrm{qDR}}_P(A/R) \to \widehat{\mathrm{qDR}}(A/R),$$

and the latter map is a quasi-isomorphism when P contains all the residue characteristics of A.

Proof. We have functors

$$(\operatorname{Spec} A)^q_{\mathrm{strat}} \circ W^{(\notin P)}\colon\ B \mapsto \operatorname{Im}(\operatorname{Hom}_{\Lambda,R}(A, W^{(\notin P)}B) \to \operatorname{Hom}_{\Lambda,R}(A, (W^{(\notin P)}B)/(q-1)))$$
$$(\operatorname{Spec} A)^{q,P}_{\mathrm{strat}}\colon\ B \mapsto \operatorname{Im}(\operatorname{Hom}_{\Lambda_P,R}(A, B) \to \operatorname{Hom}_{\Lambda_P,R}(A, B/(q-1)))$$

on the category of flat Λ_P-rings over $R[q]$. There is an obvious map

$$(W^{(\notin P)}B)/(q-1) \to W^{(\notin P)}(B/(q-1)),$$

and hence a natural transformation $(\operatorname{Spec} A)^q_{\mathrm{strat}} \circ W^{(\notin P)} \to (\operatorname{Spec} A)^{q,P}_{\mathrm{strat}}$, which induces the morphism $\mathrm{qDR}_P(A/R) \to \mathrm{qDR}(A/R)$ on cohomology.

When P contains all the residue characteristics of A, the map $(W^{(\notin P)}B)/(q-1) \to W^{(\notin P)}(B/(q-1))$ is just

$$\prod_{\substack{n\in\mathbb{N}:\\ (n,p)=1\ \forall p\in P}} B/(q^n-1) \to \prod_{\substack{n\in\mathbb{N}:\\ (n,p)=1\ \forall p\in P}} B/(q-1),$$

since the morphism $R[q] \to W^{(\notin P)}B$ is given by Adams operations, with $\Psi^n(q-1) = q^n - 1$.

We have $(q^n-1) = (q-1)[n]_q$, and $[n]_q$ is a unit in $\mathbb{Z}[\frac{1}{n}][\![q-1]\!]$, hence a unit in B when n is coprime to the residue characteristics. Thus the map $(W^{(\notin P)}B)/(q-1) \to W^{(\notin P)}(B/(q-1))$ gives an isomorphism whenever B is $(q-1)$-adically complete and admits a map from A, so the transformation $(\operatorname{Spec} A)^q_{\mathrm{strat}} \circ W^{(\notin P)} \to (\operatorname{Spec} A)^{q,P}_{\mathrm{strat}}$ is a natural isomorphism on the category of flat $(q-1)$-adically complete Λ_P-rings over $R[q]$, and hence $\widehat{\mathrm{qDR}}_P(A/R) \xrightarrow{\simeq} \widehat{\mathrm{qDR}}(A/R)$. □

Remark 2.7. Remark 1.10 shows that $\mathrm{qDR}(A/R)$ can naturally be promoted to a cosimplicial Λ-ring, and the same reasoning promotes $\mathrm{qDR}_P(A/R)$ to a cosimplicial Λ_P-ring. The proof of Proposition 2.6 then ensures that the map $\mathrm{qDR}_P(A/R) \to \mathrm{qDR}(A/R)$ is naturally a morphism of cosimplicial Λ_P-rings,

Over $\mathbb{Z}[\{\frac{1}{p}\colon p \in P\}]$, every Λ_P-ring can be canonically made into a Λ-ring, by setting all the additional Adams operations to be the identity. However, this observation is of limited use in establishing functoriality of q-de Rham cohomology, because the resulting Λ-ring structure will not satisfy the conditions of Theorem 1.23. We now give a more general result which does allow for meaningful comparisons.

Theorem 2.8. *If R is a flat Λ_P-ring over $\mathbb{Z}$ and $\Box\colon R[x_1,\ldots,x_d] \to A$ is a formally étale map of Λ_P-rings, the elements x_i having rank 1, then there are zigzags of $R[\![q-1]\!]$-linear quasi-isomorphisms*

$$\widehat{\mathrm{qDR}_P}(A/R) \simeq (\Omega^*_{A/R}[\![q-1]\!], (q-1)\nabla_q), \qquad \mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}_P(A/R) \simeq \widehat{q\text{-}\Omega}^{\bullet}_{A/R,\Box}.$$

whenever P contains all the residue characteristics of A.

Proof. The key observation to make is that formally étale maps have a unique lifting property with respect to nilpotent extensions of flat Λ_P-rings, because the Adams operations must also lift uniquely. In particular, this means that the operations γ_i featuring in the definition of q-de Rham cohomology are necessarily endomorphisms of A as a Λ_P-ring.

Similarly to Theorem 1.23, $\widehat{\mathrm{qDR}}_P(A/R)$ is calculated using a cosimplicial Λ_P-ring given in level n by the $(q-1)$-adic completion $\hat{U}^{\bullet}_{P,A}$ of the Λ_P-ring over $R[q]$ generated by $A^{\otimes_R(n+1)}[q]$ and $(q-1)^{-1}\ker(A^{\otimes_R(n+1)} \to A)[q]$. The observation above shows that $\hat{U}^n_{P,A} \cong \hat{U}^n_{P,R[x_1,\ldots,x_d]}\hat{\otimes}_{R[x_1,\ldots,x_d]}A$, changing base along $\Box$ applied to the first factor.

As in Proposition 2.6, $\hat{U}^{\bullet}_{P,R[x_1,\ldots,x_d]}$ is just the $(q-1)$-adic completion of the complex $U^{\bullet}$ from Proposition 1.15. Further application of the key observation above then allows us to adapt the constructions of Theorem 1.17, giving the desired quasi-isomorphisms. □

2.2. Cartier isomorphisms in mixed characteristic.

In [Sch2, Conjecture 7.1], Scholze predicted that $\widehat{q\text{-}\Omega}^{\bullet}_{A/R,\Box}$ is a functorial invariant of the R-algebra A, independent of the choice of framing, so extends to all smooth schemes. Theorem 2.8 shows that $\widehat{q\text{-}\Omega}^{\bullet}_{A/R,\Box}$ is functorial invariant of the Λ_P-ring A over R.

The only setting in which Theorem 2.8 leads to results close to Scholze's conjecture is when $R = W^{(p)}(k)$, the p-typical Witt vectors of a perfect field of characteristic p, and $A = \varprojlim_n A_n$ is a formal deformation of a smooth k-algebra A_0. Then any formally étale morphism $W^{(p)}(k)[x_1,\ldots,x_d] \to A$ of topological rings gives rise to a unique compatible lift Ψ^p of absolute Frobenius on A with $\Psi^p(x_i) = x_i^p$, so gives A the structure of a topological Λ_p-ring. The framing still affects the choice of Λ_p-ring structure, but at least such a structure is guaranteed to exist, giving rise to a complex $\mathrm{qDR}_P(A/R)^{\wedge_p} := \mathbf{R}\varprojlim_n \mathrm{qDR}_p(A/R)\otimes^{\mathbf{L}}_R R_n$ depending only on the choice of Ψ^p, where $R_n = W^{(p)}_n(k)$.

Our constructions now allow us to globalise the quasi-isomorphism

$$(\widehat{q\text{-}\Omega^{\bullet}_{A/R,\square}})^{\wedge_p}/[p]_q \simeq (\Omega^*_{A/R})^{\wedge_p}[\![q-1]\!]/[p]_q$$

of [Sch2, Proposition 3.4], where $\Omega^*_{A/R}$ denotes the complex $A \xrightarrow{0} \Omega^1_{A/R} \xrightarrow{0} \Omega^2_{A/R} \xrightarrow{0} \ldots$.

Lemma 2.9. *Under the quasi-isomorphism $\widehat{\mathrm{qDR}}_p(A/R) \simeq (\Omega^*_{A/R}[\![q-1]\!], (q-1)\nabla_q)$ from Theorem 2.8, the semilinear Adams operation Ψ^p on $\widehat{\mathrm{qDR}}_p(A/R)$ described in Definition 1.8 corresponds to the operation on $\Omega^*_{A/R}[\![q-1]\!]$ given by setting*

$$\Psi^p(adx_{i_1} \wedge \ldots \wedge dx_{i_m}) := \Psi^p(a)x_{i_1}^{p-1}\ldots x_{i_m}^{p-1}dx_i \wedge \ldots \wedge dx_{i_m}.$$

for $a \in A[\![q-1]\!]$.

Proof. Just observe that this expression defines a chain map on $(\Omega^*_{A/R}[\![q-1]\!], (q-1)\nabla_q)$ (for instance $\Psi^p((q-1)\nabla_q x_i) = (q^p-1)\Psi^p(dx) = (q-1)\nabla_q x_i^p$), and that the quasi-isomorphisms in the proof of Theorem 1.23 commute with these operations. □

As in [Sch2, §4], we refer to formal schemes over $W^{(p)}(k)$ as smooth if they are flat deformations of smooth schemes over k. We refer to morphisms of such schemes as étale if they are flat deformations of étale morphisms over k.

Proposition 2.10. *Take a smooth formal scheme $\mathfrak{X}$ over $R = W^{(p)}(k)$ equipped with a lift Ψ^p of Frobenius which étale locally admits co-ordinates $\{x_i\}_i$ as above with $\Psi^p(x_i) = x_i^p$. Then there is a global quasi-isomorphism*

$$C_q^{-1}\colon (\Omega^*_{\mathfrak{X}/R})^{\wedge_p}[\![q-1]\!]/[p]_q \to (\mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}_p(\mathscr{O}_{\mathfrak{X}}/R))^{\wedge_p}/[p]_q$$

in the derived category of étale sheaves on $\mathfrak{X}$.

Proof. The unique lifting property of formally étale morphisms ensures that each affine formal scheme $\mathfrak{U}$ étale over $\mathfrak{X}$ has a unique lift $\Psi^p|_{\mathfrak{U}}$ of Frobenius compatible with the given operation Ψ^p on $\mathfrak{X}$. Functoriality of the construction qDR_p for rings with Frobenius lifts thus gives us an étale presheaf $\widehat{\mathrm{qDR}}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}$ of complexes on $\mathfrak{X}$. As in Definition 1.8, the Adams operation Ψ^p on $\mathscr{O}_{\mathfrak{X}}$ then extends to $(R[\![q-1]\!], \Psi^p)$-semilinear maps

$$\Psi^p\colon \mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p} \to \mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}$$
$$\mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}/(q-1) \to \mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}/(q^p-1),$$

and thus, denoting good truncation by τ,

$$(q-1)^i\Psi^p\colon \tau^{\leq i}(\mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}/(q-1)) \to (\mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p})/[p]_q;$$

the left-hand side is quasi-isomorphic to $\bigoplus_{j\leq i}(\Omega^j_{\mathscr{O}_{\mathfrak{X}}/R})^{\wedge_p}[-j]$ by Theorem 1.23.

Extending the construction $R[q]$-linearly and restricting to top summands therefore gives us the global map C_q^{-1}. For a local choice of framing, Lemma 2.9 gives equivalences

$$(q-1)^i\Psi^p \simeq \sum_{j\leq i}(q-1)^{i-j}(\tilde{C}^{-1})^j$$

for Scholze's locally defined lifts $(\tilde{C}^{-1})^j\colon (\Omega^j_{A/R})^{\wedge_p}[-j] \to (\widehat{q\text{-}\Omega^{\bullet}_{A/R,\square}})^{\wedge_p}/[p]_q$ of the Cartier quasi-isomorphism. The local calculation of [Sch2, Proposition 3.4] then ensures that C_q^{-1} is a quasi-isomorphism. □

3. Functoriality via analogues of de Rham–Witt cohomology

In order to obtain a cohomology theory for smooth commutative rings rather than for Λ_P-rings, we now consider q-analogues of de Rham–Witt cohomology. Our starting point is to observe that if we allow roots of q, we can extend the Jackson differential to fractional powers of x by the formula

$$\nabla_q(x^{m/n}) = \frac{q^{m/n}-1}{q-1}x^{m/n}d\log x,$$

where $d\log x = x^{-1}dx$, so terms such as $[n]_{q^{1/n}}x^{m/n}$ have integral derivative, where $[n]_{q^{1/n}} = \frac{q-1}{q^{1/n}-1}$.

3.1. Motivation.

Definition 3.1. Given a Λ_P-ring B, define $\Psi^{1/P^\infty}B$ to be the smallest Λ_P-ring which is equipped with a morphism from B and for which the Adams operations are automorphisms.

In the case $P = \{p\}$, the Λ_p-ring $\Psi^{1/p^\infty}B$ is thus the colimit of the diagram

$$B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} \ldots.$$

By Remark 2.7, $\widehat{\mathrm{qDR}}_p(A/R)$ naturally underlies a cosimplicial Λ_p-ring, so applying Ψ^{1/p^∞} levelwise gives another cosimplicial Λ_p-ring. For the Adams operation Ψ^p of Definition 2.3, the underlying cochain complex is just $\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R) := \varinjlim_{\Psi^p}\widehat{\mathrm{qDR}}_p(A/R)$. As an immediate consequence of Lemma 2.9, we have:

Lemma 3.2. *If R is a flat Λ_p-ring over $\mathbb{Z}_{(p)}$ with Ψ^p an isomorphism, then $\Psi^{1/p^\infty}\widehat{qDR}_p(R[x]/R)$ is quasi-isomorphic to the complex*

$$(R[x^{1/p^\infty}, q^{1/p^\infty}] \xrightarrow{(q-1)\nabla_q} (x^{1/p^\infty})R[x^{1/p^\infty}, q^{1/p^\infty}]\mathrm{dlog}\, x)^{\wedge_{(q-1)}},$$

so the décalage $\mathbf{L}\eta_{(q-1)}\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(R[x]/R)$ and the complex

$$\{a \in R[x^{1/p^\infty}, q^{1/p^\infty}] \;:\; \nabla_q a \in R[x^{1/p^\infty}, q^{1/p^\infty}]d\log x\}$$
$$\xrightarrow{\nabla_q} (x^{1/p^\infty})R[x^{1/p^\infty}, q^{1/p^\infty}]\mathrm{dlog}\, x.$$

are quasi-isomorphic after $(q-1)$-adic completion.

Thus in level 0 (resp. level 1), $\mathbf{L}\eta_{(q-1)}\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}(R[x]/R)$ is spanned by elements of the form $[p^n]_{q^{1/p^n}}x^{m/p^n}$ (resp. $x^{m/p^n}d\log x$), so setting $q^{1/p^\infty} = 1$ gives a complex whose p-adic completion is the p-typical de Rham–Witt complex.

Lemma 3.3. *Let R and A be flat p-adically complete Λ_p-algebras over $\mathbb{Z}_p$, with Ψ^p an isomorphism on R. For elements x_i of rank 1, take a map $\Box\colon R[x_1, \ldots, x_d]^{\wedge_p} \to A$ of Λ_p-rings which is a flat p-adic deformation of an étale map. Then the map*

$$(R[q^{1/p^\infty}] \otimes_{R[q]} \mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p} \to \mathbf{L}\eta_{(q-1)}(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p}$$

is a quasi-isomorphism.

Proof. The map $\Psi^p\colon A \otimes_{R[x_1,\ldots,x_d]} R[x_1^{1/p},\ldots,x_d^{1/p}] \to A$ becomes an isomorphism on p-adic completion, because $\Box$ is flat and we have an isomorphism modulo p. Thus

$$\Psi^{1/p^\infty} A \cong A[x_1^{1/p^\infty},\ldots,x_d^{1/p^\infty}]^{\wedge_p} := (A \otimes_{R[x_1,\ldots,x_d]} R[x_1^{1/p^\infty},\ldots,x_d^{1/p^\infty}])^{\wedge_p}$$

Combined with the calculation of Lemma 2.9, this gives us a quasi-isomorphism between $(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p}$ and the $(p,q-1)$-adic completion of

$$(\bigoplus_I \bigoplus_\alpha A[\![q-1]\!] x_1^{\alpha_1}\ldots x_d^{\alpha_d} dx^I[-|I|], (q-1)\nabla_q),$$

where I ranges over finite subsets of $\{1,\ldots,d\}$ and α ranges over elements of $p^{-\infty}\mathbb{Z}^d$ with $0 \le \alpha_i < 1$ if $i \notin I$ and $-1 < \alpha_i \le 0$ if $i \in I$.

We then observe that the contributions to the décalage $\eta_{(q-1)}$ from terms with $\alpha \neq 0$ must be acyclic, via a contracting homotopy defined by the restriction to $\eta_{(q-1)}$ of the q-integration map

$$f x_1^{\alpha_1}\ldots x_d^{\alpha_d} dx^I \mapsto f x_1^{\alpha_1}\ldots x_d^{\alpha_d} \sum_{i\in I} \pm x_i [\alpha_i]_q^{-1} dx^{(I\setminus i)},$$

where $[\frac{m}{p^n}]_q^{-1} = [m]_{q^{1/p^n}}^{-1}[p^n]_{q^{1/p^n}}$ for m coprime to p, noting that $[m]_{q^{1/p^n}}$ is a unit in $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$. $\Box$

Remark 3.4. The endomorphism given on $\Psi^{1/P^\infty}\widehat{\mathrm{qDR}}_P(A/R)$ by

$$a \mapsto \Psi^{1/n}([n]_q a) = [n]_{q^{1/n}}\Psi^{1/n} a$$

descends to an endomorphism of $\mathrm{H}^0(\Psi^{1/P^\infty}\widehat{\mathrm{qDR}}_P(A/R)/(q-1))$, which we may denote by V_n because it mimics Verschiebung in the sense that $\Psi^n V_n = n\cdot \mathrm{id}$ (since $[n]_q \equiv n \mod (q-1)$). For A smooth over $\mathbb{Z}$, we then have

$$\begin{aligned}\mathrm{H}^0(\Psi^{1/P^\infty}\widehat{\mathrm{qDR}}_P(A/\mathbb{Z})/(q-1))/(V_p \ :\ p\in P) &\cong A[q^{1/P^\infty}]/([p]_{q^{1/p}} \ :\ p \in P)\\ &\cong A[\zeta_{P^\infty}],\end{aligned}$$

for ζ_n a primitive nth root of unity.

By adjunction, this gives an injective map

$$\mathrm{H}^0(\Psi^{1/P^\infty}\widehat{\mathrm{qDR}}_P(A/\mathbb{Z})/(q-1)) \hookrightarrow W^{(P)}A[\zeta_{P^\infty}]$$

of Λ_P-rings, which becomes an isomorphism on completing $\Psi^{1/P^\infty}\widehat{\mathrm{qDR}}(A/\mathbb{Z})$ with respect to the system $\{([n]_{q^{1/n}})\}_{n\in P^\infty}$ of ideals, where we write P^∞ for the set of integers whose prime factors are all in P. This implies that the cokernel is annihilated by all elements of $(q^{1/P^\infty}-1)$, so leads us to consider almost mathematics as in [GR].

3.2. **Almost isomorphisms.** From now on, we consider only the case $P=\{p\}$. Combined with Lemma 3.3, Remark 3.4 allows us to regard $\mathbf{L}\eta_{(q-1)}\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/\mathbb{Z}_p)^{\wedge_p}$ as being almost a q^{1/p^∞}-analogue of p-typical de Rham–Witt cohomology.

The ideal $(q^{1/p^\infty}-1)^{\wedge_{(p,q-1)}} = \ker(\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}} \to \mathbb{Z}_p)$ is equal to the p-adic completion of its square, since we may write it as the kernel $W^{(p)}(\mathfrak{m})$ of $W^{(p)}(\mathbb{F}_p[q^{1/p^\infty}]^{\wedge_{(q-1)}}) \to W^{(p)}(\mathbb{F}_p)$, for the idempotent maximal ideal $\mathfrak{m} = ((q-1)^{1/p^\infty})^{\wedge_{(q-1)}}$ in $\mathbb{F}_p[q^{1/p^\infty}]^{\wedge_{(q-1)}}$. If we set h^{1/p^n} to be the Teichmüller element

$$[q^{1/p^n}-1] = \lim_{r\to\infty}(q^{1/p^{nr}}-1)^{p^r} \in \mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}},$$

then $W^{(p)}(\mathfrak{m}) = (h^{1/p^\infty})^{\wedge(p,h)}$. Although $W^{(p)}(\mathfrak{m})/p^n$ is not maximal in $\mathbb{Z}[h^{1/p^\infty}]^{\wedge(h)}/p^n$, it is idempotent and flat, so gives a basic setup in the sense of [GR, 2.1.1]. We thus regard the pair $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)}, W(p)(\mathfrak{m}))$ as an inverse system of basic setups for almost ring theory.

We then follow the terminology and notation of [GR], studying p-adically complete $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)})^a$-modules (almost $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)}$-modules) given by localising at almost isomorphisms, the maps whose kernel and cokernel are $W^{(p)}(\mathfrak{m})$-torsion.

Definition 3.5. The obvious functor $(-)^a$ from modules to almost modules has a right adjoint $(-)_*$, given by $N_* := \mathrm{Hom}_{\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)}}(W^{(p)}(\mathfrak{m}), N)$, the module of almost elements.

Since the counit $(M_*)^a \to M$ of the adjunction is an (almost) isomorphism, we may also regard almost modules as a full subcategory of the category of modules, consisting of those M for which the natural map $M \to (M^a)_*$ is an isomorphism. We can define p-adically complete $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)})^a$-algebras similarly, forming a full subcategory of $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)}$-algebras.

3.3. **Perfectoid algebras.** We now relate Scholze's perfectoid algebras to a class of Λ_p-rings, by factorising the tilting equivalence. For simplicity, we work over $\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p}$, although Lemma 3.8 has natural analogues over the ring $K^o \subset K$ of power-bounded elements of any perfectoid field K in the sense of [Sch1].

Definition 3.6. Define Fontaine's period ring functor $\mathscr{A}_{\mathrm{inf}}$ from commutative rings to Λ_p-rings by $\mathscr{A}_{\mathrm{inf}}(C) := \varprojlim_{\Psi^p} W^{(p)}(C)$.

Definition 3.7. Define a perfectoid Λ_p-ring to be a flat p-adically complete Λ_p-algebra over $\mathbb{Z}_p$, on which the Adams operation Ψ^p is an isomorphism.

By analogy with [Bha, Notation 1.4], we say that a perfectoid Λ_p-ring over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)}$ is integral if the morphism $B \to B_*$ of Definition 3.5 is an isomorphism.

Lemma 3.8. *We have equivalences of categories*

$$\begin{array}{c}
\textit{perfectoid almost } \mathbb{Z}[\zeta_{p^\infty}]^{\wedge p}\textit{-algebras} \\
{\scriptstyle -/[p]_{q^{1/p}}}\Big\uparrow \Big\downarrow {\scriptstyle \mathscr{A}_{\mathrm{inf}}(-)_*} \\
\textit{integral perfectoid } \Lambda_p\textit{-rings over } \mathbb{Z}[q^{1/p^\infty}]^{\wedge(p,q-1)} \\
{\scriptstyle -/p}\Big\downarrow \Big\uparrow {\scriptstyle W^{(p)}(-)_*} \\
\textit{perfectoid almost } \mathbb{F}_p[q^{1/p^\infty}]^{\wedge(q-1)}\textit{-algebras.}
\end{array}$$

Proof. A perfectoid Λ_p-ring B is a deformation of the perfect $\mathbb{F}_p$-algebra B/p. As in [Sch1, Proposition 5.13], a perfect $\mathbb{F}_p$-algebra C has a unique deformation $W^{(p)}(C)$ over $\mathbb{Z}_p$, to which Frobenius must lift uniquely; this shows that $W^{(p)}$ gives an equivalence between perfect $\mathbb{F}_p$-algebras and perfectoid Λ_p-rings. To obtain the bottom equivalence of the diagram, we will show that the functor $W^{(p)}$ commutes with the respective functors $C \mapsto C_*$ of almost elements, then appeal to the tilting equivalence.

Because the idempotent ideals of the basic setups in each of our three categories are generated by the rank 1 elements $h^{p^{-n}}$ constructed before Definition 3.5, we can write $C_* = \bigcap_n h^{-p^{-n}} C$ in each setting. For a Teichmüller element $[c] \in W^{(p)}(C)$, the standard isomorphism $W^{(p)}(C) \cong C^{\mathbb{N}_0}$ of sets gives an isomorphism $[c]W^{(p)}(C) \cong \prod_{m\geq 0} c^{p^m} C$. Thus the natural map $W^{(p)}(C)_* \to W^{(p)}(C_*)$ of Λ_p-rings is an isomorphism, since

$$W^{(p)}(C)_* \cong \bigcap_{n\geq 0} \prod_{m\geq 0} h^{-p^{m-n}} C \cong \prod_{m\geq 0} C_* \cong W^{(p)}(C_*),$$

and taking inverse limits with respect to Ψ^p gives $\mathscr{A}_{\mathrm{inf}}(C)_* \cong \mathscr{A}_{\mathrm{inf}}(C_*)$ as well.

Next, we observe that since $B := \mathscr{A}_{\mathrm{inf}}(C)$ is a perfectoid Λ_p-ring for any flat p-adically complete $\mathbb{Z}_p$-algebra C, we must have $B \cong W^{(p)}(B/p)$. Comparing rank 1 elements then gives a monoid isomorphism $(B/p) \cong \varprojlim_{x\mapsto x^p} C$, from which it follows that

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathscr{A}_{\mathrm{inf}}(C) \cong \varprojlim_{\Phi}(C/p) = C^\flat$$

whenever C is perfectoid. Since tilting gives an equivalence of almost algebras by [Sch1, Theorem 5.2], this completes the proof. ☐

3.4. **Functoriality of q-de Rham cohomology.** Since $(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/\mathbb{Z}_p))^{\wedge_p}$ is represented by a cosimplicial perfectoid Λ_p-ring over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$ for any flat Λ_p-ring A over $\mathbb{Z}_p$, it corresponds under Lemma 3.8 to a cosimplicial perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebra, representing the following functor:

Lemma 3.9. *For a perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebra C, and a Λ_p-ring A over $\mathbb{Z}_p$ with $X = \operatorname{Spec} A$, there is a canonical isomorphism*

$$X^{q,p}_{\mathrm{strat}}(\mathscr{A}_{\mathrm{inf}}(C)_*) \cong \mathrm{Im}\,(\varprojlim_{\Psi^p} X(C_*) \to X(C_*)),$$

for the ring C_* of almost elements.

Proof. By definition, $X^{q,p}_{\mathrm{strat}}(\mathscr{A}_{\mathrm{inf}}(C)_*)$ is the image of

$$\mathrm{Hom}_{\Lambda_p}(A, \mathscr{A}_{\mathrm{inf}}(C)_*) \to \mathrm{Hom}_{\Lambda_p}(A, (\mathscr{A}_{\mathrm{inf}}(C)_*)/(q-1)).$$

Since right adjoints commute with limits and $\mathscr{A}_{\mathrm{inf}} = \varprojlim_{\Psi^p} W^{(p)}$, we may rewrite the first term as $\varprojlim_{\Psi^p} \mathrm{Hom}_{\Lambda_p}(A, W^{(p)}(C_*)) = \varprojlim_{\Psi^p} X(C_*)$.

Setting $B := \varprojlim_{\Psi^p} W^{(p)}(C)_*$, observe that because $[p^n]_{q^{1/p^n}}(q^{1/p^n} - 1) = (q-1)$, we have $\bigcap_n [p^n]_{q^{1/p^n}} B = (q-1)B$, any element on the left defining an almost element of $(q-1)B$, hence a genuine element since $B = B_*$ is flat. Then note that since the projection map $\theta\colon B \to C_*$ has kernel $([p]_{q^{1/p}})$, the map $\theta \circ \Psi^{p^{n-1}}$ has kernel $([p]_{q^{1/p^n}})$, and so $B \to W^{(p)}(C)_*$ has kernel $\bigcap_n [p^n]_{q^{1/p^n}} B$. Thus

$$\mathrm{Hom}_{\Lambda_p}(A, (\varprojlim_{\Psi^p} W^{(p)}(C)_*)/(q-1)) \hookrightarrow \mathrm{Hom}_{\Lambda_p}(A, W^{(p)}(C)_*) = X(C_*). \qquad ☐$$

In fact, the tilting equivalence gives $\varprojlim_{\Psi^p} X(C_*) \cong X(C_*^\flat)$, so the only dependence of $X^{q,p}_{\mathrm{strat}}(\mathscr{A}_{\mathrm{inf}}(C)_*)$, and hence $((\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/\mathbb{Z}_p))^{\wedge_p})^a$, on the Frobenius lift Ψ^p is in determining the image of $X(C_*^\flat) \to X(C_*)$ as C varies.

Although the map $X(C_*^\flat) \to X(C_*)$ is not surjective, it is almost so in a precise sense, which we now use to establish independence of Ψ^p, showing that, up to faithfully flat descent, $\widehat{\mathrm{qDR}}_p(A/\mathbb{Z}_p)^{\wedge_p}/[p]_{q^{1/p}}$ is the best possible perfectoid approximation to $A[\zeta_{p^\infty}]^{\wedge_p}$.

Definition 3.10. Given a functor X from $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebras to sets and a functor $\mathscr{A}$ from perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebras to abelian groups, we write

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}(X,\mathscr{A}) := \mathbf{R}\mathrm{Hom}_{[\mathrm{Pfd}((\mathbb{Z}_p[\zeta_{p^\infty}]^{\wedge_p})^a),\mathrm{Set}]}(X,\mathscr{A}),$$

where $\mathrm{Pfd}(S^a)$ denotes the category of perfectoid almost S-algebras, and $\mathbf{R}\mathrm{Hom}_{[\mathcal{C},\mathrm{Set}]}(-,-)$ is as in Definition 1.9.

When X is representable by a $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebra C, we simply denote $\mathbf{R}\Gamma_{\mathrm{Pfd}}(X,\mathscr{A})$ by $\mathbf{R}\Gamma_{\mathrm{Pfd}}(C,\mathscr{A})$ — when C is perfectoid, this will just be $\mathscr{A}(C)$.

Thus $\mathbf{R}\Gamma_{\mathrm{Pfd}}(C,\mathscr{A})$ is the homotopy limit of the functor $\mathscr{A}$ (regarded as taking values in cochain complexes) on the category of perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebras equipped with a map from C. This is closely related to the pushforward from the pro-étale site of the generic fibre, whose décalage for $\mathscr{A}=\mathscr{A}_{\mathrm{inf}}$ is the complex $A\Omega$ of [BMS, Definition 9.1].

Theorem 3.11. *If R is a p-adically complete Λ_p-ring over $\mathbb{Z}_p$, and A a formal R-deformation of a smooth ring over (R/p), then the complex*

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}((A[\zeta_{p^\infty}]\otimes_R \Psi^{1/p^\infty}R)^{\wedge_p},\mathscr{A}_{\mathrm{inf}})$$

of $(\Psi^{1/p^\infty}R[q])^{\wedge_{(p,q-1)}}$-modules is almost quasi-isomorphic to $(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p}$ for any Λ_p-ring structure on A coming from a framing over R as in Theorem 2.8.

Proof. Since passage to almost modules is an exact functor, it follows from the definition of qDR_p that the cochain complex $((\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p})^a$ is given by $\mathbf{R}\mathrm{Hom}_{[f\hat{\Lambda}_p(R[\![q-1]\!]),\mathrm{Set}]}(X^{q,p}_{\mathrm{strat}},((\Psi^{1/p^\infty}\mathscr{O})^{\wedge_p})^a)$ in the notation of Definition 1.9, where $f\hat{\Lambda}_p(R[\![q-1]\!])$ denotes the category of flat $(p,q-1)$-adically complete Λ_p-algebras over $R[\![q-1]\!]$.

Now note that $C\mapsto((\Psi^{1/p^\infty}C)^{\wedge_p})_*$ is left adjoint to the inclusion functor $i\colon \mathrm{Pfd}\Lambda_p(R[\![q-1]\!])\to f\hat{\Lambda}_p(R[\![q-1]\!])$ from the category of integral perfectoid Λ_p-rings over $\Psi^{1/p^\infty}(R[\![q-1]\!])^{\wedge_p}$. Thus $i^*\colon \mathrm{Ch}([f\hat{\Lambda}_p(R[\![q-1]\!]),\mathrm{Ab}])\to\mathrm{Ch}([\mathrm{Pfd}\Lambda_p(R[\![q-1]\!]),\mathrm{Ab}])$ has exact right adjoint $\mathscr{F}\mapsto(\mathscr{F}\circ(\Psi^{1/p^\infty})^{\wedge_p})_*)$. We therefore have

$$\mathbf{R}\mathrm{Hom}_{[f\hat{\Lambda}_p(R[\![q-1]\!]),\mathrm{Set}]}(X^{q,p}_{\mathrm{strat}},((\Psi^{1/p^\infty}\mathscr{O})^{\wedge_p})^a)\simeq\mathbf{R}\mathrm{Hom}_{[\mathrm{Pfd}\Lambda_p(R[\![q-1]\!]),\mathrm{Set}]}(i^*X^{q,p}_{\mathrm{strat}},\mathscr{O}^a).$$

It thus follows that the cochain complex $((\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p})^a$ is the homotopy limit of the functor $(B,x,y)\mapsto B^a$ on the category of triples (B,x,y) for integral perfectoid Λ_p-rings B over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$ and

$$(x,y)\in X^{q,p}_{\mathrm{strat}}(B)\times_{Y^{q,p}_{\mathrm{strat}}(B)}Y(B),$$

where $X=\operatorname{Spec}A$ and $Y=\operatorname{Spec}R$.

By Lemma 3.8, such Λ_p-rings B are uniquely of the form $\mathscr{A}_{\mathrm{inf}}(C_*)$ for $C\in\mathrm{Pfd}((\mathbb{Z}_p[\zeta_{p^\infty}]^{\wedge_p})^a)$, so this homotopy limit becomes

$$((\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p})^a\simeq\mathbf{R}\Gamma_{\mathrm{Pfd}}((X^{q,p}_{\mathrm{strat}}\times_{Y^{q,p}_{\mathrm{strat}}}Y)\circ(\mathscr{A}_{\mathrm{inf}})_*,(\mathscr{A}_{\mathrm{inf}}))^a.$$

Writing $X^\infty(C):=\mathrm{Im}\,(\varprojlim_{\Psi^p}X(C_*)\to X(C_*))$, Lemma 3.9 then combines with the description above to give

$$\begin{aligned}(\widehat{\mathrm{qDR}}_p(A/R)^{\wedge_p})^a &\simeq\mathbf{R}\Gamma_{\mathrm{Pfd}}(X^\infty\times_{Y^\infty}\varprojlim_{\Psi^p}Y,(\mathscr{A}_{\mathrm{inf}}))^a,\\ &\simeq\mathbf{R}\Gamma_{\mathrm{Pfd}}(X^\infty\times_Y\varprojlim_{\Psi^p}Y,(\mathscr{A}_{\mathrm{inf}}))^a.\end{aligned}$$

We now introduce a Grothendieck topology on the category $[\mathrm{Pfd}_{(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge p})^a}, \mathrm{Set}]$ by taking covering morphisms to be those maps $C \to C'$ of perfectoid algebras which are almost faithfully flat modulo p. Since $C^\flat = \varprojlim_\Phi (C/p)$, the functor $\mathscr{A}_{\mathrm{inf}}$ satisfies descent with respect to these coverings, so the map

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}((X^\infty \times_Y \varprojlim_{\Psi^p} Y)^\sharp, \mathscr{A}_{\mathrm{inf}})^a \to \mathbf{R}\Gamma_{\mathrm{Pfd}}(X^\infty \times_Y \varprojlim_{\Psi^p} Y, \mathscr{A}_{\mathrm{inf}})^a$$

is a quasi-isomorphism, where $(-)^\sharp$ denotes sheafification.

In other words, the calculation of $(\widehat{\mathrm{qDR}}_p(A/R)^{\wedge p})^a$ is not affected if we tweak the definition of X^∞ by taking the image sheaf instead of the image presheaf. We then have

$$(X^\infty)^\sharp(C) = \bigcup_{C \to C'} \mathrm{Im}\,(X(C_*) \times_{X(C'_*)} \varprojlim_{\Psi^p} X(C'_*) \to X(C_*)),$$

where $C \to C'$ runs over all covering morphisms.

Now, $\varprojlim_{\Psi^p} X$ is represented by the perfectoid algebra $(\Psi^{1/p^\infty} A)^{\wedge p}$, which is isomorphic to $A[x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\wedge p}$ as in the proof of Lemma 3.3. This allows us to appeal to André's results [And, §2.5] as generalised in [Bha, Theorem 2.3]. For any morphism $f\colon A \to C$, there exists a covering morphism $C \to C_i$ such that $f(x_i)$ has arbitrary p-power roots in C_i. Setting $C' := C_1 \otimes_C \ldots \otimes_C C_d$, this means that the composite $A \xrightarrow{f} C \to C'$ extends to a map $(\Psi^{1/p^\infty} A)^{\wedge p} \to C'$, so $f \in (X^\infty)^\sharp(C)$. We have thus shown that $(X^\infty)^\sharp = X$, giving the required equivalence

$$((\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge p})^a \simeq \mathbf{R}\Gamma_{\mathrm{Pfd}}(X \times_Y \varprojlim_{\Psi^p} Y, (\mathscr{A}_{\mathrm{inf}})_*)^a.$$

Finally, compatibility of these equivalences with the $(\Psi^{1/p^\infty} R[q])^{\wedge (p,q-1)}$-module structures is given by functoriality, multiplicativity and the identification $(\Psi^{1/p^\infty} R[q])^{\wedge (p,q-1)} \simeq (\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(R/R))^{\wedge p}$. □

Remark 3.12. Corresponding to the cohomology theory $((\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge p})^a$, it is natural to consider q-connections on finite projective modules M over

$$\begin{aligned}
&\eta^0_{(q-1)}((\Psi^{1/p^\infty}(\Omega^*_{A/R}[\![q-1]\!], (q-1)\nabla_q))^{\wedge p, a}) \\
&= \{a \in (\Psi^{1/p^\infty}(A[\![q-1]\!]))^{\wedge (p,q-1), a} \ :\ \nabla_q a \in (\Psi^{1/p^\infty}(\Omega^1_A[\![q-1]\!]))^{\wedge (p,q-1), a}\} \\
&= ((\sum_n [p^n]_{q^{1/p^n}} \Psi^{1/p^n} A[q^{1/p^\infty}])^{\wedge (p,q-1)})^a.
\end{aligned}$$

It follows from the proof of Proposition 1.25 that these are equivalent, for $X = \mathrm{Spec}\, A$, to finite projective almost $(\Psi^{1/p^\infty}\mathscr{O}_{\hat{X}^q,\mathrm{strat}})^{\wedge p}$-modules $\mathscr{N}$ for which $\mathscr{N}/(q-1)$ is the pullback of the almost $\mathrm{H}^0((\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge p}/(q-1))$-module $\Gamma(\hat{X}^q_{\mathrm{strat}}, \mathscr{N}/(q-1)) =: M_0$.

Up to almost isomorphism, these correspond via the proof of Theorem 3.11 to those finite projective $\mathscr{A}_{\mathrm{inf}}$-modules N on the site of integral perfectoid algebras C over $A[\zeta_{p^\infty}]^{\wedge p} \otimes_R \Psi^{1/p^\infty} R$ for which there exists a $W^{(p)}(A[\zeta_{p^\infty}]^{\wedge p})$-module M_0 with $W^{(p)}(C)$-linear isomorphisms

$$N(C) \otimes_{\mathscr{A}_{\mathrm{inf}}(C)} W^{(p)}(C) \cong M_0 \otimes_{W^{(p)}(A[\zeta_{p^\infty}]^{\wedge p})} W^{(p)}(C),$$

functorial in C.

This establishes a weakened form of [Sch2, Conjecture 7.5] on co-ordinate independence of the category of q-connections, giving the statement for almost $(\sum_n [p^n]_{q^{1/p^n}} \Psi^{1/p^n} A[q^{1/p^\infty} - 1])^{\wedge_p}$)-modules rather than $A[\![q-1]\!]$-modules.

The following gives a slight partial refinement of [BMS, Theorem 1.17]:

Corollary 3.13. *If R is a p-adically complete Λ_p-ring over $\mathbb{Z}_p$, and A a formal R-deformation of a smooth ring over (R/p), then the q-de Rham cohomology complex $(q\text{-}\Omega^\bullet_{A/R,\square} \otimes_{R[q]} (\Psi^{1/p^\infty} R)[q^{1/p^\infty}])^{\wedge_p}$ is, up to almost quasi-isomorphism, independent of a choice of co-ordinates $\square$. As such, it is naturally an invariant of the commutative p-adically complete $(\Psi^{1/p^\infty} R)[\zeta_{p^\infty}]^{\wedge_p}$-algebra $(A[\zeta_{p^\infty}] \otimes_R \Psi^{1/p^\infty} R)^{\wedge_p}$.*

Proof. Since

$$\Psi^{1/p^\infty} \mathrm{qDR}_p(A/R) = \Psi^{1/p^\infty} \mathrm{qDR}_p((A \otimes_R \Psi^{1/p^\infty} R)/\Psi^{1/p^\infty} R),$$

Theorem 2.8 combines with Lemma 3.3 to give

$$(q\text{-}\Omega^\bullet_{A/R,\square} \otimes_{R[q]} (\Psi^{1/p^\infty} R)[q^{1/p^\infty}])^{\wedge_p} \simeq \mathbf{L}\eta_{(q-1)}((\Psi^{1/p^\infty} \widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p}),$$

and by Theorem 3.11, we know that this depends only on $(A[\zeta_{p^\infty}] \otimes_R \Psi^{1/p^\infty} R)^{\wedge_p}$ up to almost quasi-isomorphism. □

Remark 3.14. The almost quasi-isomorphism in Corollary 3.13 should be a genuine quasi-isomorphism when we impose some conditions on the base ring R. By [BMS, Lemma 8.11], it would suffice to verify that $\mathrm{H}^*((\Psi^{1/p^\infty}(\Omega^*_{A/R}[\![q-1]\!], (q-1)\nabla_q))^{\wedge_p})$ and its quotient by $(q-1)$ have no $(q^{1/p^\infty} - 1)$-torsion, which should follow for R smooth by an argument similar to [BMS, Proposition 8.9].

Remark 3.15 (Eliminating roots of q). The key feature of the comparison results in this section is that, up to faithfully flat descent, the functor $X^{q,p}_{\mathrm{strat}}$ does not depend on Adams operations when restricted to the category of integral perfectoid Λ_p-rings B over $\mathbb{Z}[q]$, since the proof of Theorem 3.11 gives $(X^{q,p}_{\mathrm{strat}})^\sharp(B) \cong X(B/[p]_{q^{1/p}})$. We can extend the latter functor to more general Λ_p-rings over $\mathbb{Z}[q]$ by setting

$$X^{q,p}(B) := X(B/(\Psi^p)^{-1}([p]_q B)),$$

which does not depend on any Adams operations on X.

When $\mathscr{O}_X$ has a Λ_p-ring structure, there is then a natural map $\alpha\colon X^{q,p}_{\mathrm{strat}} \to X^{q,p}$ because $\Psi^p((q-1)B) \subset [p]_q B$. This induces a transformation

$$\alpha^*\colon \mathbf{R}\mathrm{Hom}_{[f\hat{\Lambda}_p(R[\![q-1]\!]),\mathrm{Set}]}(X^{q,p}, \mathscr{O}) \to \widehat{\mathrm{qDR}}_p(A/R)^{\wedge_p}$$

for $X = \mathrm{Spec}\, A$. But for integral perfectoid Λ_p-rings B, we know that $X^{q,p}(B) = (X^{q,p}_{\mathrm{strat}})^\sharp(B)$, so by adjunction, as in the proof of Theorem 3.11, α^* becomes an almost quasi-isomorphism on applying a form of completed stabilisation $\Psi^{1/p^\infty}(-)^{\wedge_p}$. Thus $\mathrm{H}^*(X^{q,p}, \mathscr{O})$ might be a candidate for the co-ordinate independent q-de Rham cohomology theory proposed in [Sch2]. It naturally carries an Adams operation Ψ^p, which would correspond to the operation ϕ_p of [Sch2, Conjecture 6.1].

Any $a \in A$ defines an element of $\mathrm{H}^0(X^{q,p}, \mathscr{O}/(\Psi^p)^{-1}([p]_q\mathscr{O}))$ so $\Psi^p(a) \in \mathrm{H}^0(X^{q,p}, \mathscr{O}/[p]_q)$ and applying the connecting homomorphism associated to $[p]_q\colon \mathscr{O} \to \mathscr{O}$ gives an element $\beta_{[p]_q}\Psi^p(a) \in \mathrm{H}^1(X^{q,p}, \mathscr{O})$ whose image under $\mathrm{H}^1(\alpha^*)$ is

$$[p]_q^{-1}\Psi^p((q-1)\nabla_q a) = (q-1)\Psi^p(\nabla_q a).$$

Moreover, to $a \in A$ we may associate elements $a_n \in \mathrm{H}^0(X^{q,p}, \mathscr{O}/[p^n]_q)$ for $n \geq 1$, determined by the property that $a_n \equiv \Psi^{p^i} a^{p^{n-i}} \mod [p]_{q^{p^{i-1}}}$ for $1 \leq i \leq n$, and these give rise to elements $\beta_{[p^n]_q} a_n \in \mathrm{H}^1(X^{q,p}, \mathscr{O})$. Explicitly, if we define operations ε_i on $\mathscr{O}$ by $\varepsilon_0 = \mathrm{id}$ and $\varepsilon_{i+1}(a) := (a^{p^{i+1}} - \Psi^p(a^{p^i}))/p^{i+1}$, then for a local lift $\tilde{a} \in \mathscr{O}$ of $a \in \mathscr{O}/(\Psi^p)^{-1}([p]_q)$, we have

$$a_n = \sum_{i=0}^{n-1} [p^i]_{q^{p^{n-i}}} \Psi^{p^{n-i}}(\varepsilon_i \tilde{a}) + [p^n]_q \mathscr{O},$$

so

$$\mathrm{H}^1(\alpha^*)(\beta_{[p^n]_q} a_n) = (q-1) \sum_{i=0}^{n-1} \Psi^{p^{n-i}}(\nabla_q \varepsilon_i \tilde{a}).$$

In particular, for $A = R[x]$ these include all the elements $(q-1)[m]_{q^{p^n}} x^{p^n m - 1} dx$, since $\varepsilon_i(x^m) = 0$ for all $i > 0$, x^m having rank 1. This suggests that in general the image of $\mathrm{H}^1(\alpha^*)$ might be $(q-1)\mathrm{H}^1\widehat{\mathrm{qDR}}_p(A/R)^{\wedge_p}$, tying in well with $(q-1)$-adic décalage. Explicit descriptions for much of the functoriality from Corollary 3.13 can also be inferred from this analysis, since it implies that the transformations $\sum_{i=0}^{n-1} \Psi^{p^{n-i}} \circ \nabla_q \circ \varepsilon_i \colon A \to \mathrm{H}^1(q\text{-}\Omega^\bullet_{A/R,\square})$ are all natural in A.

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