Shifted symplectic and Poisson structures

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arXiv: 1504.01940

de Rham complexes

- $A = (A, \delta)$ a CDGA over R.
- Kähler differentials $\Omega^1_{A/R}$ (a complex).
- Exterior powers $\Omega^p_{A/R}$.
- de Rham differential $d\colon \Omega^p_{A/R} \to \Omega^{p+1}_{A/R}$.
- de Rham complex

$$\mathrm{DR}(A/R) = \big(\prod_p \Omega^p_{A/R}[-p], d \pm \delta\big).$$

- Hodge filtration $F^p\mathrm{DR}(A) = \prod \Omega^{\geqslant p}_A$.

Derived de Rham cohomology

- Derived de Rham $\mathbf{L}\mathrm{DR}(A/R) := \mathrm{DR}(\tilde{A}/R)$, for $\tilde{A} \to A$ a cofibrant ($\approx$ quasi-free) resolution.
- $\mathbf{L}F^p\mathrm{DR}(A/R) := F^p\mathrm{DR}(\tilde{A}/R)$.
- Also write $\mathbf{L}\Omega^p(A) := \Omega^p(\tilde{A})$.
- $\equiv$ Pieces of cyclic homology (Feigin–Tsygan, HKR).

n-shifted pre-symplectic structures

- $\omega \in \mathrm{Z}^{n+2}\mathbf{L}F^2\mathrm{DR}(A/R)$ [PTVV].
- Explicitly, $\omega = \sum_{p \geqslant 2} \omega_p$, with

$$\delta\omega_2 = 0, \quad d\omega_p = \delta\omega_{p+1}.$$

- Morphisms given by chain homotopies.
- Sheafify for global definition (e.g. via hypercovers).
- Symplectic if non-degenerate:

$$\omega_2^\sharp \colon (\mathbf{L}\Omega^1_X)^\vee \xrightarrow{\sim} \mathbf{L}\Omega^1_X[n].$$

n-shifted polyvectors

- $\widehat{\mathrm{Pol}}(A/R, n)$:

$$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathrm{CoSymm}_A(\mathbf{L}\Omega^1_{A/R}[n+1]), A),$$

$$(\approx \widehat{\mathrm{Symm}}_A(T_{A/R}[n+1])).$$

- Filtration $F^p\widehat{\mathrm{Pol}}(A/R, n)$:

$$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathrm{CoSymm}_A^{\geqslant p}(\mathbf{L}\Omega^1_{A/R}[n+1]), A).$$

- Commutative product $F^p \cdot F^q \subset F^{p+q}$.
- Schouten–Nijenhuis Lie bracket on $\widehat{\mathrm{Pol}}(A/R, n)[n+1]$, $[F^p, F^q] \subset F^{p+q-1}$.

n-Poisson structures (affine case)

- $\pi = \sum_{p \geqslant 2} \pi_p \in F^2 \widehat{\mathrm{Pol}}(A/R, n)^{n+2}$,

$$\delta\pi + \frac{1}{2}[\pi, \pi] = 0 \text{ (Maurer–Cartan)}.$$

- Morphisms from Thom–Sullivan homotopies.
- Non-degenerate if

$$\pi_2^\sharp \colon \mathbf{L}\Omega^1_X[n] \xrightarrow{\sim} (\mathbf{L}\Omega^1_X)^\vee.$$

n-Poisson algebras

- π gives L_∞-structure on $A[n]$.
- n-Poisson structures $\iff$ homotopy P_{n+1}-algebra structures on A [Melani].
- Global definition of Poisson structure:
  - $\check{X}_\bullet$ Čech nerve or analogue — simplicial derived affine.
  - $\mathcal{P}(X/R, n)$ the space of cosimplicial P_{n+1}-algebra structures on cosimplicial CDGA $O(\check{X}_\bullet)$.
  - (Well-defined.)

Symplectic versus Poisson

- Classical case: 2-form ω is symplectic iff inverse π is Poisson.
- Standard proof uses Darboux theorem (cotangent bundle).
- Shifted Darboux theorems [B-BBBJ], [BG] give local comparison for shifted structures.

A more direct approach

- ω_2 homotopy inverse to π_2.
- Higher components??
- Look to generalise

$$\pi^\sharp \circ \omega^\sharp \circ \pi^\sharp = \pi^\sharp.$$

- Then globalise (subtle).

The canonical tangent vector

- Tangent space $T_\pi\mathcal{P}$ of Poisson structures at π:

$$\alpha \in F^2\widehat{\mathrm{Pol}}(A/R, n)^{n+2},$$
$$\delta\alpha + [\pi, \alpha] = 0$$

(i.e. $\pi + \alpha\epsilon$ Poisson).

- Differentiating $\mathbb{G}_m$-action gives

$$\sigma(\pi) := \sum_{p\geqslant 2}(p-1)\pi_p \in T_\pi\mathcal{P}.$$

Compatibility (the key)

- μ defined by contraction and multiplication.
- "Derivative" ν.
- Relates de Rham / Schouten–Nijenhuis:

$$[\mu(\omega, \pi), \pi] = \mu(d\omega, \pi) \pm \nu(\omega, \pi; \frac{1}{2}[\pi, \pi]).$$

- $\mu(\omega, \pi) \in T_\pi \mathcal{P}$ for ω pre-symplectic, π Poisson.

In detail

When $\phi = adf_1 \wedge \ldots \wedge df_p$,

$$\mu(\phi, \pi) = a[f_1, \pi] \ldots [f_p, \pi],$$
$$\nu(\phi, \pi; b) = \sum_i \pm a[f_1, \pi] \ldots [f_i, b] \ldots [f_p, \pi].$$

- Thus

$$\mu(\omega_2, \pi_2)^\sharp = \pi_2^\sharp \circ \omega_2^\sharp \circ \pi_2^\sharp.$$

The affine equivalence

Weak equivalences of ∞-groupoids:

$$\{\text{Symplectic structures } \omega\}$$

$$\uparrow$$

$$\{\text{Compatible pairs } \mu(\omega,\pi) \simeq \sigma(\pi)\}$$

$$\downarrow$$

$$\{\text{Non-degenerate Poisson structures } \pi\}$$

Governing DGLAs

- Symplectic: $F^2\mathbf{L}\mathrm{DR}(A/R)[n+1]$ (abelian).
- Poisson: $F^2\widehat{\mathrm{Pol}}(A/R,n)[n+1]$.
- $T\mathcal{P}$: $F^2\widehat{\mathrm{Pol}}(A/R,n)[n+1][\epsilon]$.
- Compatible pairs a homotopy limit.
- Equivalence via obstruction theory.

Obstructions

$$\mathrm{gr}^p_F \to F^2\widehat{\mathrm{Pol}}(A)/F^{p+1} \to F^2\widehat{\mathrm{Pol}}(A)/F^p$$

central extension of DGLAs.

- Maurer–Cartan obstruction map $\mathcal{P}(A/R,n)/F^p \to \underline{\mathrm{MC}}(\mathrm{gr}^p_F[n+2])$, fibre $\mathcal{P}(A/R,n)/F^{p+1}$.
- Similar obstructions for symplectic structures, compatible pairs.
- Graded pieces equivalent via powers of $(\mathbf{L}\Omega^1_X)^\vee \simeq \mathbf{L}\Omega^1_X[n]$.

Global approach

- Filtration on polyvectors $\leftrightarrow$ truncated L_∞-structures. (**Not** L_n-structures — no higher relations.)
- Hence truncated Poisson structures.
- Forgetful functor to CDGAs not Cartesian — no Grothendieck construction.
- Also categories of symplectic structures, compatible pairs, truncated versions.

Diagrams

- I a category, $A\colon I \to CDGA$ a diagram.
- $\mathcal{P}(A, n)$: the space of diagrams $I \to P_{n+1}$-algebras, lifting A.
- Also diagrams of symplectic structures, compatible pairs.
- Obstruction theory for diagrams.
- Comparison stepwise: $(\omega_2, \ldots, \omega_p)$ compatible with $(\pi_2, \ldots, \pi_p)$ (includes L_∞-morphisms).

Derived DM N-stacks

- Nice simplicial affine resolution $X_\bullet \to \mathfrak{X}$ always exists.
- Look at structures on diagram $O(X)\colon \Delta \to CDGA$.
- Obstructions in terms of

$$\mathrm{H}^*(\mathfrak{X}, \mathbf{L}\Omega^p_{\mathfrak{X}})$$
$$\mathrm{Ext}^*_{\mathscr{O}_{\mathfrak{X}}}(\mathbf{L}\mathrm{CoSymm}^p_{\mathscr{O}_{\mathfrak{X}}}(\mathbf{L}\Omega^1_{\mathfrak{X}}[n+1]), \mathscr{O}_{\mathfrak{X}}).$$

- Hence equivalence symplectic $\leftrightarrow$ non-degenerate Poisson.

Why well-defined?

- Simplicial morphism $X_\bullet \to Y_\bullet$ gives a diagram

$$\Delta \times (0 \to 1) \to CDGA.$$

- $\mathcal{P}(X) \leftarrow \mathcal{P}(X \to Y) \to \mathcal{P}(Y)$ etc.
- Equivalences when X, Y resolve the same stack.