

SHIFTED POISSON AND SYMPLECTIC STRUCTURES ON DERIVED N -STACKS

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ABSTRACT. We show that on a derived Artin N -stack, there is a canonical equivalence between the spaces of n -shifted symplectic structures and non-degenerate n -shifted Poisson structures.

INTRODUCTION

In [PTVV], the notion of an n -shifted symplectic structure for a derived Artin stack was introduced. The definition of an n -Poisson structure was also sketched, and an approach suggested to prove equivalence of n -shifted symplectic structures and non-degenerate n -Poisson structures, by first establishing deformation quantisation for symplectic structures. The Darboux theorems of [BBBBJ, BG] imply the existence of shifted Poisson structures locally on derived Deligne–Mumford stacks, but not globally.

We prove existence of shifted Poisson structures by a direct approach, not involving deformation quantisation or Darboux theorems. The key new notion is that of compatibility between an n -shifted pre-symplectic (i.e. closed, possibly degenerate) structure ω and an n -Poisson structure π . In the case of unshifted structures on underived manifolds, compatibility simply says that the associated maps between tangent and cotangent bundles satisfy

$$\pi^\sharp \circ \omega^\sharp \circ \pi^\sharp = \pi^\sharp,$$

which ensures that $\omega^\sharp = (\pi^\sharp)^{-1}$ whenever π is non-degenerate. We demonstrate that this notion admits a natural generalisation incorporating shifts and higher coherence data. We expect that this notion of compatibility between Poisson and pre-symplectic structures is the same as in Definition 1.4.14 of the contemporaneous treatment [CPT⁺], whose proof also does not involve deformation quantisation. Our notion of compatibility is extended in [Pri6, Pri5, Pri4] from Poisson structures to quantisations.

Our first main observation is that there is a canonical global section σ of the tangent space $T\mathcal{P}(X, n)$ of the space $\mathcal{P}(X, n)$ of n -Poisson structures, given by differentiating the \mathbb{G}_m -action on the differential graded Lie algebra of polyvectors. For an unshifted Poisson structure on a smooth underived scheme, this just maps π to itself. In general, if we write $\pi = \sum_{r \geq 2} \pi_r$ with π_r an r -vector, then

$$\sigma(\pi) = \sum_{r \geq 2} (r-1)\pi_r.$$

For any n -shifted Poisson structure π , contraction gives a multiplicative map $\mu(-, \pi)$ from de Rham to Poisson cohomology, so any n -shifted pre-symplectic structure ω defines another global section $\mu(\omega, -)$ of the tangent space of the space of Poisson structures. In the unshifted case on a smooth scheme, the associated map from the cotangent

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space to the tangent space is simply given by

$$\mu(\omega, \pi)^\sharp = \pi^\sharp \circ \omega^\sharp \circ \pi^\sharp.$$

We then say that a pair (ω, π) is compatible if $\mu(\omega, \pi) \simeq \sigma(\pi)$. More formally, the space of compatible pairs is the homotopy limit of the diagram

$$\mathrm{PreSp}(X, n) \times \mathcal{P}(X, n) \begin{array}{c} \xrightarrow{\mu} \\ \xrightarrow{\sigma} \end{array} T\mathcal{P}(X, n) \longrightarrow \mathcal{P}(X, n).$$

In the unshifted case, this amounts to seeking compatible pairs in the sense above.

Poisson structures are functorial with respect to étale morphisms, so the notions above are readily defined for derived DM N -stacks. For derived Artin N -stacks, the formulation of Poisson structures is more subtle: we show that a derived Artin N -stack admits an étale cover by derived stacks coming from commutative bidifferential bigraded algebras, and put Poisson structures on them. These algebras perform the same role as the formal affine derived stacks of $[\mathrm{CPT}^+]$, and we refer to them as stacky CDGAs.

We then show that:

- (1) If π is non-degenerate, then every compatible pre-symplectic structure ω is also non-degenerate, hence symplectic, and the space of such structures is contractible.
- (2) If ω is symplectic, then the space of *non-degenerate* compatible Poisson structures is contractible.

Thus the spaces of n -shifted symplectic structures and non-degenerate n -Poisson structures are both weakly equivalent to the space of non-degenerate compatible pairs (ω, π) . These results have recently been obtained in $[\mathrm{CPT}^+]$, using a very different method to formulate compatibility.

The structure of the paper is as follows.

Section 1 addresses the case of a single, fixed, derived affine scheme. The compatibility operator μ is introduced in Definition 1.16, and the key technical result is Lemma 1.17, which shows how μ interpolates between the de Rham differential and the Schouten–Nijenhuis bracket. §1.4 shows how to realise the spaces of pre-symplectic structures, Poisson structures and compatible pairs as towers of homotopy fibres. This uses the obstruction theory associated to pro-nilpotent L_∞ -algebras, regarding the construction as a generalised deformation problem. The correspondence between symplectic and non-degenerate Poisson structures in the affine case is then given in Corollary 1.38.

Section 2 then applies these results to derived stacks. To date, the only proof that the cotangent complex governs deformations of a (derived) algebraic stack (rather than just morphisms to it from a fixed object) uses simplicial resolutions by affine schemes (cf. [Aok, Theorem 1.2], [Pri3, Theorem 8.8]). Since the construction of Poisson structures is a kind of generalised deformation problem, it is unsurprising that it can be studied using such resolutions. For derived Deligne–Mumford N -stacks, we look at spaces of structures on the diagrams given by suitable simplicial resolutions, and show that they are independent of the resolution; the key being étale functoriality. Theorem 2.19 then establishes an equivalence between the spaces of n -symplectic and non-degenerate n -Poisson structures.

In section 3, these results are extended to derived Artin N -stacks. Given a simplicial resolution by derived affines, we can form an associated stacky CDGA, and indeed a simplicial resolution by stacky CDGAs (Corollary 3.14). The morphisms in this resolution resemble étale maps, allowing a suitably functorial generalisation of the results

of §1 to stacky CDGAs. The main result is then Theorem 3.33, which establishes an equivalence between the spaces of n -symplectic and non-degenerate n -Poisson structures on derived Artin N -stacks.

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CONTENTS

Introduction	1
1. Compatible structures on derived affines	3
1.1. Shifted Poisson structures	4
1.2. Shifted pre-symplectic structures	6
1.3. Compatible pairs	7
1.4. The obstruction tower	10
1.5. The equivalence	14
2. Compatible structures on derived DM stacks	15
2.1. Compatible structures on diagrams	15
2.2. Derived N -hypergroupoids	18
3. Compatible structures on derived Artin stacks	20
3.1. Stacky CDGAs	21
3.2. Comparing DG Artin hypergroupoids and stacky CDGAs	25
3.3. Poisson and symplectic structures	28
3.4. Diagrams and functoriality	31
References	33

1. COMPATIBLE STRUCTURES ON DERIVED AFFINES

For the purposes of this section, we will fix a graded-commutative algebra $R = R_\bullet$ in chain complexes over \mathbb{Q} , and a cofibrant graded-commutative R_\bullet -algebra $A = A_\bullet$. We will denote the differential on A by δ .

In particular, the cofibrancy condition holds whenever the underlying morphism $R_\# \rightarrow A_\#$ of graded commutative algebras is freely generated in non-negative chain degrees. It ensures that the module $\Omega_{A/R}^1$ of Kähler differentials is a model for the cotangent complex of A over R . For the purposes of this section, this latter property is all we need, so we could relax the cofibrancy condition slightly to include morphisms which are ind-smooth rather than freely generated.

We define $\Omega_{A/R}^p := \Lambda_A^p \Omega_{A/R}^1$, and we denote its differential (inherited from A) by δ . There is also a de Rham cochain differential $\Omega_{A/R}^p \rightarrow \Omega_{A/R}^{p+1}$, which we denote by d .

For a chain (resp. cochain) complex M , we write $M_{[i]}$ (resp. $M^{[j]}$) for the complex $(M_{[i]})_m = M_{i+m}$ (resp. $(M^{[j]})^m = M^{j+m}$). Given A -modules M, N in chain complexes, we write $\underline{\mathrm{Hom}}_A(M, N)$ for the cochain complex given by

$$\underline{\mathrm{Hom}}_A(M, N)^i = \mathrm{Hom}_{A_\#}(M_\#, N_{\#[i]}),$$

with differential $\delta f = \delta_N \circ f \pm f \circ \delta_M$, where $V_\#$ denotes the graded vector space underlying a chain complex V .

1.1. Shifted Poisson structures.

1.1.1. Polyvector fields.

Definition 1.1. Define the cochain complex of n -shifted polyvector fields on A by

$$\widehat{\text{Pol}}(A, n) := \underline{\text{Hom}}_A(\text{CoS}_A((\Omega_{A/R}^1)_{[-n-1]}), A),$$

with graded-commutative multiplication following the usual conventions for symmetric powers. [Here, $\text{CoS}_A^p(M) = \text{CoSymm}_A^p(M) = (M^{\otimes_A p})^{\Sigma_p}$ and $\text{CoS}_A(M) = \bigoplus_{p \geq 0} \text{CoS}_A^p(M)$.]

The Lie bracket on $\text{Hom}_A(\Omega_{A/R}^1, A)$ then extends to give a bracket (the Schouten–Nijenhuis bracket)

$$[-, -]: \widehat{\text{Pol}}(A, n) \times \widehat{\text{Pol}}(A, n) \rightarrow \widehat{\text{Pol}}(A, n)^{[-1-n]},$$

determined by the property that it is a bi-derivation with respect to the multiplication operation.

Thus $\widehat{\text{Pol}}(A, n)$ has the natural structure of a P_{n+2} -algebra (i.e. an $(n+1)$ -shifted Poisson algebra), and in particular $\widehat{\text{Pol}}(A, n)^{[n+1]}$ is a differential graded Lie algebra (DGLA) over R .

Note that the cochain differential δ on $\widehat{\text{Pol}}(A, n)$ can be written as $[\delta, -]$, where $\delta \in \widehat{\text{Pol}}(A, n)^{n+2}$ is the element defined by the derivation δ on A .

Strictly speaking, $\widehat{\text{Pol}}$ is the complex of multiderivations, as polyvectors are usually defined as symmetric powers of the tangent complex. The two definitions agree (modulo completion) whenever the tangent complex is perfect, and Definition 1.1 is the more natural object when the definitions differ.

Definition 1.2. Define a decreasing filtration F on $\widehat{\text{Pol}}(A, n)$ by

$$F^i \widehat{\text{Pol}}(A, n) := \underline{\text{Hom}}_A\left(\bigoplus_{j \geq i} \text{CoS}_A^j((\Omega_{A/R}^1)_{[-n-1]}), A\right);$$

this has the properties that $\widehat{\text{Pol}}(A, n) = \varprojlim_i \widehat{\text{Pol}}(A, n)/F^i$, with $[F^i, F^j] \subset F^{i+j-1}$, $\delta F^i \subset F^i$, and $F^i F^j \subset F^{i+j}$.

Observe that this filtration makes $F^2 \widehat{\text{Pol}}(A, n)^{[n+1]}$ into a pro-nilpotent DGLA.

1.1.2. Poisson structures.

Definition 1.3. Given a DGLA L , define the the Maurer–Cartan set by

$$\text{MC}(L) := \{\omega \in L^1 \mid \delta\omega + \frac{1}{2}[\omega, \omega] = 0 \in \bigoplus_n L^2\}.$$

Following [Hin], define the Maurer–Cartan space $\underline{\text{MC}}(L)$ (a simplicial set) of a nilpotent DGLA L by

$$\underline{\text{MC}}(L)_n := \text{MC}(L \otimes_{\mathbb{Q}} \Omega^\bullet(\Delta^n)),$$

where

$$\Omega^\bullet(\Delta^n) = \mathbb{Q}[t_0, t_1, \dots, t_n, \delta t_0, \delta t_1, \dots, \delta t_n] / (\sum t_i - 1, \sum \delta t_i)$$

is the commutative dg algebra of de Rham polynomial forms on the n -simplex, with the t_i of degree 0.

Definition 1.4. Given an inverse system $L = \{L_\alpha\}_\alpha$ of nilpotent DGLAs, define

$$\mathrm{MC}(L) := \varprojlim_\alpha \mathrm{MC}(L_\alpha) \quad \underline{\mathrm{MC}}(L) := \varprojlim_\alpha \underline{\mathrm{MC}}(L_\alpha).$$

Note that $\mathrm{MC}(L) = \mathrm{MC}(\varprojlim_\alpha L_\alpha)$, but $\underline{\mathrm{MC}}(L) \neq \underline{\mathrm{MC}}(\varprojlim_\alpha L_\alpha)$.

Definition 1.5. Define an R -linear n -shifted Poisson structure on A to be an element of

$$\mathrm{MC}(F^2\widehat{\mathrm{Pol}}(A, n)^{[n+1]}),$$

and the space $\mathcal{P}(A, n)$ of R -linear n -shifted Poisson structures on A to be given by the simplicial set

$$\mathcal{P}(A, n) := \underline{\mathrm{MC}}(\{F^2\widehat{\mathrm{Pol}}(A, n)^{[n+1]}/F^{i+2}\}_i).$$

Also write $\mathcal{P}(A, n)/F^{i+2} := \underline{\mathrm{MC}}(F^2\widehat{\mathrm{Pol}}(A, n)^{[n+1]}/F^{i+2})$, so $\mathcal{P}(A, n) = \varprojlim_i \mathcal{P}(A, n)/F^{i+2}$.

Remark 1.6. Observe that elements of $\mathcal{P}_0(A, n) = \mathrm{MC}(F^2\widehat{\mathrm{Pol}}(A, n)^{[n+1]})$ consist of infinite sums $\pi = \sum_{i \geq 2} \pi_i$ with

$$\pi_i: \mathrm{CoS}_A^i((\Omega_{A/R}^1)_{[-n-1]}) \rightarrow A_{[-n-2]}$$

satisfying $\delta(\pi_i) + \frac{1}{2} \sum_{j+k=i+1} [\pi_j, \pi_k] = 0$. This is precisely the condition which ensures that π defines an L_∞ -algebra structure on $A_{[-n]}$. This then makes A into a \widehat{P}_{n+1} -algebra in the sense of [Mel, Definition 2.9]. Equivalently, this is an algebra for the operad $\mathrm{Com} \circ (L_\infty[-n])$ via a distributive law $(L_\infty[-n]) \circ \mathrm{Com} \rightarrow \mathrm{Com} \circ (L_\infty[-n])$.

It is important to remember that $\mathcal{P}(A, n)$ is just one explicit presentation of the right-derived functor of $\mathcal{P}_0(A, n)$ for varying R .

When we need to compare chain and cochain complexes, we silently make use of the equivalence u from chain complexes to cochain complexes given by $(uV)^i := V_{-i}$. On suspensions, this has the effect that $u(V_{[n]}) = (uV)^{[-n]}$.

Definition 1.7. We say that an n -shifted Poisson structure $\pi = \sum_{i \geq 2} \pi_i$ is non-degenerate if $\pi_2: \mathrm{CoS}_A^2((\Omega_{A/R}^1)_{[-n-1]}) \rightarrow A_{[-n-2]}$ induces a quasi-isomorphism

$$\pi_2^\sharp: (\Omega_{A/R}^1)_{[-n]} \rightarrow \underline{\mathrm{Hom}}_A(\Omega_{A/R}^1, A)$$

and $\Omega_{A/R}^1$ is perfect.

Define $\mathcal{P}(A, n)^{\mathrm{nondeg}} \subset \mathcal{P}(A, n)$ to consist of non-degenerate elements — this is a union of path-components.

1.1.3. *The canonical tangent vector σ .* The space $\mathcal{P}(A, n)$ admits an action of $\mathbb{G}_m(R_0)$, which is inherited from the scalar multiplication on $\widehat{\mathrm{Pol}}(A, n)$ in which $\mathrm{Hom}_A(\mathrm{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), A)$ is given weight $p - 1$. Differentiating this action gives us a global tangent vector on $\mathcal{P}(A, n)$, as follows. Take ϵ to be a variable of degree 0, with $\epsilon^2 = 0$.

Definition 1.8. Define the tangent spaces

$$\begin{aligned} T\mathcal{P}(A, n) &:= \underline{\mathrm{MC}}(\{F^2\widehat{\mathrm{Pol}}(A, n)^{[n+1]} \otimes_{\mathbb{Q}} \mathbb{Q}[\epsilon]/F^{i+2}\}_i) \\ T\mathcal{P}(A, n)/F^{i+2} &:= \underline{\mathrm{MC}}(F^2\widehat{\mathrm{Pol}}(A, n)^{[n+1]} \otimes_{\mathbb{Q}} \mathbb{Q}[\epsilon]/F^{i+2}). \end{aligned}$$

These are simplicial sets over $\mathcal{P}(A, n)$ (resp. $\mathcal{P}(A, n)/F^{i+2}$), fibred in simplicial abelian groups.

Definition 1.9. Given $\pi \in \mathcal{P}_0(A, n)$, observe that $\delta + [\pi, -]$ defines a square-zero derivation on $\widehat{\text{Pol}}(A, n)$, and denote the resulting complex by

$$\widehat{\text{Pol}}_\pi(A, n).$$

The product and bracket on polyvectors make this a P_{n+2} -algebra, and it inherits the filtration F .

Given $\pi \in \mathcal{P}_0(A, n)/F^p$, we define $\widehat{\text{Pol}}_\pi(A, n)/F^p$ similarly. This is a CDGA, and $F^1\widehat{\text{Pol}}_\pi(A, n)/F^p$ is a P_{n+2} -algebra, because $F^i \cdot F^j \subset F^{i+j}$ and $[F^i, F^j] \subset F^{i+j-1}$.

Note that $\widehat{\text{Pol}}_\pi(A, n)$ is just the natural n -shifted analogue of the complex computing Poisson cohomology.

The following is an instance of a standard result on square-zero extensions of DGLAs:

Lemma 1.10. *The fibre $T_\pi\mathcal{P}(A, n)$ of $T\mathcal{P}(A, n)$ over $\pi \in \mathcal{P}(A, n)$ is canonically homotopy equivalent to the Dold–Kan denormalisation of the good truncation $\tau^{\leq 0}(F^2\widehat{\text{Pol}}_\pi(A, n)^{[n+2]})$. In particular, its homotopy groups are given by*

$$\pi_i T_\pi\mathcal{P}(A, n) = H^{n+2-i}(F^2\widehat{\text{Pol}}_\pi(A, n), \delta + [\pi, -]).$$

The corresponding statements for $T_\pi\mathcal{P}(A, n)/F^{i+2}$ also hold.

Now observe that the map

$$\begin{aligned} \sigma: F^2\widehat{\text{Pol}}(A, n)^{[n+1]} &\rightarrow F^2\widehat{\text{Pol}}(A, n)^{[n+1]} \otimes_{\mathbb{Q}} \mathbb{Q}[\epsilon] \\ &\sum_{i \geq 2} \alpha_i \mapsto \sum_{i \geq 2} (\alpha_i + (i-1)\alpha_i\epsilon) \end{aligned}$$

is a morphism of filtered DGLAs, for $\alpha_i: \text{CoS}_A^i(\Omega_{A/R}^1[n+1]) \rightarrow A$. This can be seen either by direct calculation or by observing that σ is the differential of the \mathbb{G}_m -action on Pol .

Definition 1.11. Define the canonical tangent vector $\sigma: \mathcal{P}(A, n) \rightarrow T\mathcal{P}(A, n)$ on the space of n -shifted Poisson structures by applying $\underline{\text{MC}}$ to the morphism σ of DGLAs.

Explicitly, this sends $\pi = \sum \pi_i$ to $\sigma(\pi) = \sum_{i \geq 2} (i-1)\pi_i \in T_\pi\mathcal{P}(A, n)$, which thus has the property that $\delta\sigma(\pi) + [\pi, \sigma(\pi)] = 0$.

The map σ preserves the cofiltration in the sense that it comes from a system of maps $\sigma: \mathcal{P}(A, n)/F^{i+2} \rightarrow T\mathcal{P}(A, n)/F^{i+2}$.

1.2. Shifted pre-symplectic structures.

Definition 1.12. Define the de Rham complex $\text{DR}(A/R)$ to be the product total cochain complex of the double complex

$$A \xrightarrow{d} \Omega_{A/R}^1 \xrightarrow{d} \Omega_{A/R}^2 \xrightarrow{d} \dots,$$

so the total differential is $d \pm \delta$.

We define the Hodge filtration F on $\text{DR}(A/R)$ by setting $F^p\text{DR}(A/R) \subset \text{DR}(A/R)$ to consist of terms $\Omega_{A/R}^i$ with $i \geq p$.

Properties of the product total complex ensure that a map $f: A \rightarrow B$ induces a quasi-isomorphism $\mathrm{DR}(A/R) \rightarrow \mathrm{DR}(B/R)$ whenever the maps $\Omega_{A/R}^p \rightarrow \Omega_{B/R}^p$ are quasi-isomorphisms, which will happen whenever f is a weak equivalence between cofibrant R -algebras.

The complex $\mathrm{DR}(A/R)$ has the natural structure of a commutative DG algebra over R , filtered in the sense that $F^i F^j \subset F^{i+j}$.

Definition 1.13. Define an n -shifted pre-symplectic structure ω on A/R to be an element

$$\omega \in Z^{n+2} F^2 \mathrm{DR}(A/R).$$

Explicitly, this means that ω is given by an infinite sum $\omega = \sum_{i \geq 2} \omega_i$, with $\omega_i \in (\Omega_{A/R}^i)_{i-n-2}$ and $d\omega_i = \delta\omega_{i+1}$.

Definition 1.14. Define an n -shifted symplectic structure ω on A/R to be an n -shifted pre-symplectic structure ω for which the component $\omega_2 \in Z^n \Omega_{A/R}^2$ induces a quasi-isomorphism

$$\omega_2^\sharp: \mathrm{Hom}_A(\Omega_{A/R}^1, A) \rightarrow (\Omega_{A/R}^1)_{[-n]}.$$

and $\Omega_{A/R}^1$ is perfect as an A -module.

Now, we can regard $F^2 \mathrm{DR}(A/R)^{[n+1]}$ as a filtered DGLA with trivial bracket. This has the property that $\mathrm{MC}(F^2 \mathrm{DR}(A/R)^{[n+1]}) = Z^{n+2} F^2 \mathrm{DR}(A/R)$. We therefore make the following definition:

Definition 1.15. Define the space of n -shifted pre-symplectic structures on A/R to be the simplicial set

$$\mathrm{PreSp}(A, n) := \underline{\mathrm{MC}}(\{F^2 \mathrm{DR}(A/R)^{[n+1]}/F^{i+2}\}_i)$$

Also write $\mathrm{PreSp}(A, n)/F^{i+2} := \underline{\mathrm{MC}}(F^2 \mathrm{DR}(A/R)^{[n+1]}/F^{i+2})$, so $\mathrm{PreSp}(A, n) = \varprojlim_i \mathrm{PreSp}(A, n)/F^{i+2}$.

Set $\mathrm{Sp}(A, n) \subset \mathrm{PreSp}(A, n)$ to consist of the symplectic structures — this is a union of path-components.

Note that $\mathrm{PreSp}(A, n)/F^{i+2}$ is canonically weakly equivalent to the Dold–Kan de-normalisation of the complex $\tau^{\leq 0}(F^2 \mathrm{DR}(A)^{[n+2]}/F^{i+2})$ (and similarly for the limit $\mathrm{PreSp}(A, n)$), but the description in terms of $\underline{\mathrm{MC}}$ will simplify comparisons.

1.3. Compatible pairs. We will now develop the notion of compatibility between a pre-symplectic structure and a Poisson structure. Analogous results for the unshifted, undervived \mathcal{C}^∞ context can be found in [KSM, Proposition 6.4], and for the $\mathbb{Z}/2$ -graded \mathcal{C}^∞ context in [KV].

Definition 1.16. Given $\pi \in (F^2 \widehat{\mathrm{Pol}}(A, n)/F^p)^{n+2}$, define

$$\mu(-, \pi): \mathrm{DR}(A/R)/F^p \rightarrow \widehat{\mathrm{Pol}}(A, n)/F^p$$

to be the morphism of graded A -algebras given on generators $\Omega_{A/R}^1$ by

$$\mu(\mathrm{adf}, \pi) := \pi \lrcorner(\mathrm{adf}) = a[\pi, f].$$

Given $b \in (F^2 \widehat{\mathrm{Pol}}(A, n)/F^p)$, we then define

$$\nu(-, \pi, b): \mathrm{DR}(A/R)/F^p \rightarrow \widehat{\mathrm{Pol}}(A, n)/F^p$$

by setting $\mu(\omega, \pi + b\epsilon) = \mu(\omega, \pi) + \nu(-, \pi, b)\epsilon$ for $\epsilon^2 = 0$. More explicitly, $\nu(-, \pi, b)$ is the A -linear derivation with respect to the ring homomorphism $\mu(-, \pi)$ given on generators $\Omega_{A/R}^1$ by

$$\nu(\text{adf}, \pi, b) := b \lrcorner(\text{adf}) = a[b, f].$$

To see that these are well-defined in the sense that they descend to the quotients by F^p , observe that because $\pi \in F^2$, contraction has the property that

$$\mu(\Omega^1, \pi) \subset F^1, \quad \nu(\Omega^1, \pi, b) \subset F^1,$$

it follows that $\mu(F^p, \pi) \subset F^p$, $\nu(F^p, \pi, b) \subset F^p$ by multiplicativity.

Explicitly, when $\phi = \text{adf}_1 \wedge \dots \wedge \text{df}_p$, the operations are given by

$$\begin{aligned} \mu(\phi, \pi) &= a[\pi, f_1] \dots [\pi, f_p], \\ \nu(\phi, \pi, b) &= \sum_i \pm a[\pi, f_1] \dots [b, f_i] \dots [\pi, f_p]. \end{aligned}$$

Lemma 1.17. *For $\omega \in \text{DR}(A)/F^p$ and $\pi \in F^2\widehat{\text{Pol}}(A, n)^{n+2}/F^p$, we have*

$$\begin{aligned} [\pi, \mu(\omega, \pi)] &= \mu(d\omega, \pi) + \frac{1}{2}\nu(\omega, \pi, [\pi, \pi]), \\ \delta_\pi \mu(\omega, \pi) &= \mu(D\omega, \pi) + \nu(\omega, \pi, \kappa(\pi)), \end{aligned}$$

where $\delta_\pi = [\delta + \pi, -]$ is the differential on $T_\pi\mathcal{P}(A, n)/F^p$, with $D = d \pm \delta$ the total differential on $(F^2\text{DR}(A)/F^p)$ and $\kappa(\pi) = [\delta, \pi] + \frac{1}{2}[\pi, \pi]$.

Proof. For fixed π and varying ω , all expressions are derivations with respect to $\mu(-, \pi)$, so it suffices to check this expression in the cases $p = 0$ and $p = 1$, $\omega = \text{df}$. In these cases, we have

$$\begin{aligned} [\pi, \mu(a, \pi)] &= [\pi, a] = \mu(da, \pi), \\ \delta_\pi \mu(a, \pi) &= \mu(Da, \pi), \\ [\pi, \mu(\text{df}, \pi)] &= [\pi, [\pi, f]] = \frac{1}{2}\nu(\text{df}, \pi, [\pi, \pi]), \\ \delta_\pi \mu(\text{df}, \pi) &= \delta_\pi[\pi, f] = \nu(\text{df}, \pi, [\delta, \pi] + \frac{1}{2}[\pi, \pi]). \end{aligned}$$

Because $\nu(a, \pi, [\pi, \pi]) = 0$ (ν being A -linear) and $d\text{df} = 0$, this gives the required results. \square

In particular, this implies that when π is Poisson, $\mu(-, \pi)$ defines a map from de Rham cohomology to Poisson cohomology.

Lemma 1.18. *There are maps*

$$(\text{pr}_2 + \mu\epsilon): \text{PreSp}(A, n)/F^p \times \mathcal{P}(A, n)/F^p \rightarrow T\mathcal{P}(A, n)/F^p$$

over $\mathcal{P}(A, n)/F^p$ for all p , compatible with each other. In particular, we have

$$(\text{pr}_2 + \mu\epsilon): \text{PreSp}(A, n) \times \mathcal{P}(A, n) \rightarrow T\mathcal{P}(A, n).$$

Proof. For $\omega \in \text{PreSp}(A, n)_0$, $\pi \in \mathcal{P}(A, n)_0$, Lemma 1.17 shows that $\mu(\omega, \pi) \in T_\pi\mathcal{P}(A, n)/F^p$. Replacing A, R with $A \otimes \Omega^\bullet(\Delta^m), R \otimes \Omega^\bullet(\Delta^m)$ then shows that the statement also holds on the m th level of the simplicial set. \square

An alternative approach to proving Lemma 1.18 is to observe that μ defines a filtered L_∞ -morphism

$$F^2\mathrm{DR}(A/R)^{[n+1]} \times F^2\widehat{\mathrm{Pol}}(A, n)^{[n+1]} \xrightarrow{\mathrm{pr}_2 + \mu\epsilon} F^2\widehat{\mathrm{Pol}}(A, n)^{[n+1]}[\epsilon]$$

(with respect to the filtration F), and then to apply the functor $\underline{\mathrm{MC}}$.

Example 1.19. If ω and π are pre-symplectic and Poisson, with $\omega_i = 0$ for $i > 2$ and $\pi_i = 0$ for $i > 2$, then observe that $\mu(\omega, \pi)$ induces the morphism

$$\mu(\omega, \pi)^\sharp: (\Omega_{A/R}^1)_{[-n]} \rightarrow \mathrm{Hom}_A(\Omega_{A/R}^1, A)$$

given by

$$\mu(\omega, \pi)^\sharp = \pi^\sharp \circ \omega^\sharp \circ \pi^\sharp.$$

Definition 1.20. We say that an n -shifted pre-symplectic structure ω and an n -Poisson structure π are compatible (or a compatible pair) if

$$[\mu(\omega, \pi)] = [\sigma(\pi)] \in \mathrm{H}^{n+2}(F^2\widehat{\mathrm{Pol}}_\pi(A, n)) = \pi_0 T_\pi \mathcal{P}(A, n),$$

where σ is the canonical tangent vector of Definition 1.11.

Example 1.21. Following Example 1.19, if $\omega_i = 0$ for $i > 2$ and $\pi_i = 0$ for $i > 2$, then (ω, π) are a compatible pair if and only if the map

$$\pi^\sharp \circ \omega^\sharp \circ \pi^\sharp: (\Omega_{A/R}^1)_{[-n]} \rightarrow \mathrm{Hom}_A(\Omega_{A/R}^1, A)$$

is homotopic to π^\sharp , because $\sigma(\pi) = \pi$ in this case.

In particular, if π is non-degenerate, this means that ω and π determine each other up to homotopy.

Lemma 1.22. *If (ω, π) is a compatible pair and π is non-degenerate, then ω is symplectic.*

Proof. Even when the vanishing conditions of Example 1.19 are not satisfied, we still have

$$\pi_2^\sharp \circ \omega_2^\sharp \circ \pi_2^\sharp \simeq \pi_2^\sharp,$$

so if π_2^\sharp is a quasi-isomorphism, then ω_2^\sharp must be its homotopy inverse. \square

Definition 1.23. Given a simplicial set Z , an abelian group object A in simplicial sets over Z , and a section $s: Z \rightarrow A$, define the homotopy vanishing locus of s to be the homotopy limit of the diagram

$$Z \begin{array}{c} \xrightarrow{s} \\ \rightrightarrows \\ \xrightarrow{0} \end{array} A \longrightarrow Z.$$

We can write this as a homotopy fibre product $Z \times_{(s,0), A \times_{\frac{1}{2}Z} A}^h A$, for the diagonal map $A \rightarrow A \times_{\frac{1}{2}Z} A$. When A is a trivial bundle $A = Z \times V$, for V a simplicial abelian group, note that the homotopy vanishing locus is just the homotopy fibre of $s: Z \rightarrow V$ over 0.

Definition 1.24. Define the space $\mathrm{Comp}(A, n)$ of compatible n -shifted pairs to be the homotopy vanishing locus of

$$\mu - \sigma: \mathrm{PreSp}(A, n) \times \mathcal{P}(A, n) \rightarrow \mathrm{pr}_2^* T\mathcal{P}(A, n) = \mathrm{PreSp}(A, n) \times T\mathcal{P}(A, n).$$

We define a cofiltration on this space by setting $\mathrm{Comp}(A, n)/F^p$ to be the homotopy vanishing locus of

$$\mu - \sigma: \mathrm{PreSp}(A, n)/F^p \times \mathcal{P}(A, n)/F^p \rightarrow \mathrm{pr}_2^* T\mathcal{P}(A, n)/F^p.$$

We can rewrite $\text{Comp}(A, n)$ as the homotopy limit of the diagram

$$\text{PreSp}(A, n) \times \mathcal{P}(A, n) \begin{array}{c} \xrightarrow{(\text{pr}_2 + \mu\epsilon)} \\ \xrightarrow{\text{pr}_2 + \sigma\text{pr}_2\epsilon} \end{array} T\mathcal{P}(A, n) \longrightarrow \mathcal{P}(A, n)$$

of simplicial sets.

In particular, an object of this space is given by a pre-symplectic structure ω , a Poisson structure π , and a homotopy h between $\mu(\omega, \pi)$ and $\sigma(\pi)$ in $T_\pi\mathcal{P}(A, n)$.

Definition 1.25. Define $\text{Comp}(A, n)^{\text{nondeg}} \subset \text{Comp}(A, n)$ to consist of compatible pairs with π non-degenerate. This is a union of path-components, and by Lemma 1.22 has a natural projection

$$\text{Comp}(A, n)^{\text{nondeg}} \rightarrow \text{Sp}(A, n)$$

as well as the canonical map

$$\text{Comp}(A, n)^{\text{nondeg}} \rightarrow \mathcal{P}(A, n)^{\text{nondeg}}.$$

Proposition 1.26. *The canonical map*

$$\text{Comp}(A, n)^{\text{nondeg}} \rightarrow \mathcal{P}(A, n)^{\text{nondeg}}$$

is a weak equivalence.

Proof. For any $\pi \in \mathcal{P}(A, n)$, the homotopy fibre of $\text{Comp}(A, n)^{\text{nondeg}}$ over π is just the homotopy fibre of

$$\mu(-, \pi): \text{PreSp}(A, n) \rightarrow T_\pi\mathcal{P}(A, n)$$

over $\sigma(\pi)$.

The map $\mu(-, \pi): \text{DR}(A/R) \rightarrow (\widehat{\text{Pol}}(A, n), \delta_\pi)$ is a morphism of complete filtered CDGAs by Lemma 1.17, and non-degeneracy of π_2 implies that we have a quasi-isomorphism on the associated graded gr_F . We therefore have a quasi-isomorphism of filtered complexes, so we have isomorphisms on homotopy groups:

$$\begin{aligned} \pi_j \text{PreSp}(A, n) &\rightarrow \pi_j T_\pi\mathcal{P}(A, n) \\ \mathbb{H}^{n+2-j}(F^2 \text{DR}(A/R)) &\rightarrow \mathbb{H}^{n+2-j}(F^2 \widehat{\text{Pol}}(A, n), \delta_\pi). \end{aligned}$$

□

For an earlier analogue of this result, see [KV], which takes $\delta = 0$ and works in the $\mathbb{Z}/2$ -graded (rather than \mathbb{Z} -graded) context, allowing $\pi = \sum_{i \geq 0} \pi_i$ to have constant and linear terms. In that setting, they show that for π non-degenerate, there is a unique solution ω of the equation $\mu(\omega, \pi) = \sigma(\pi)$, given by Legendre transformations.

1.4. The obstruction tower. We will now show that the tower $\text{Comp}(A, n) \rightarrow \dots \rightarrow \text{Comp}(A, n)/F^{i+2} \rightarrow \dots \rightarrow \text{Comp}(A, n)/F^2$ does not contain nearly as much information as first appears.

1.4.1. Small extensions and obstructions.

Definition 1.27. Say that a surjection $L \rightarrow M$ of DGLAs is small if the kernel I satisfies $[I, L] = 0$.

Lemma 1.28. *Given a small extension $e: L \rightarrow M$ of DGLAs with kernel I , there is a sequence*

$$\pi_0 \underline{\text{MC}}(L) \xrightarrow{e} \pi_0 \underline{\text{MC}}(M) \xrightarrow{o_e} \mathbb{H}^2(I)$$

of sets, exact in the sense that $o_e^{-1}(0)$ is the image of e .

Proof. This is well-known. The obstruction map o_e is given by

$$o_e(\omega) := d_L \tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}]$$

for any lift $\tilde{\omega} \in L^1$ of $\omega \in \underline{\text{MC}}(M)$. \square

Proposition 1.29. *For any small extension $e: L \rightarrow M$ of DGLAs with kernel I , there is an obstruction map $\text{ob}_e: \underline{\text{MC}}(M) \rightarrow \underline{\text{MC}}(I^{[1]})$ in the homotopy category of simplicial sets, with homotopy fibre $\underline{\text{MC}}(L)$.*

Proof. This is essentially the same as [Pri1, Theorem 1.45]. Set M' to be the mapping cone of $I \rightarrow L$, with DGLA structure given by setting $[M', I^{[1]}] = 0$. Then we have a surjection $M' \rightarrow M \oplus I^{[1]}$ of DGLAs with kernel I . Since the map $M' \rightarrow M$ is a small extension with acyclic kernel, it follows that $\underline{\text{MC}}(M') \rightarrow \underline{\text{MC}}(M)$ is a trivial fibration, hence a weak equivalence.

The obstruction map is then given by $\underline{\text{MC}}(M') \rightarrow \underline{\text{MC}}(I^{[1]})$, the homotopy fibre being $\underline{\text{MC}}(L)$, as required. \square

Proposition 1.30. *Let A be an abelian group object over a simplicial set Z , with homotopy fibre A_z over $z \in Z$. If $s: Z \rightarrow A$ is a section with homotopy vanishing locus Y , there is a long exact sequence*

$$\begin{aligned} \cdots &\longrightarrow \pi_i(Y, z) \longrightarrow \pi_i(Z, z) \xrightarrow{s} \pi_i(A_z, 0) \longrightarrow \pi_{i-1}(Y, z) \longrightarrow \cdots \\ \cdots &\longrightarrow \pi_0 Y \longrightarrow \pi_0 Z \xrightarrow{s} \pi_0(A_z), \end{aligned}$$

of groups and sets, where the final map sends z to $s(z) \in \pi_0(A_s)$

Proof. Since Y is the homotopy fibre product

$$Z \times_{(s,0), (A \times_Z^h A)} A,$$

the long exact sequence of homotopy gives

$$\cdots \rightarrow \pi_i(Y, z) \rightarrow \pi_i(Z, z) \times \pi_i(A, z) \rightarrow \pi_i(A \times_Z^h A, z) \rightarrow \cdots$$

Since $(A \times_Z^h A) \times_A^h \{z\} = A_z$, this simplifies to

$$\cdots \rightarrow \pi_i(Y, z) \rightarrow \pi_i(Z, z) \rightarrow \pi_i(A_z, z).$$

\square

Combining Propositions 1.29 and 1.30 gives:

Corollary 1.31. *Given a small extension $e: L \rightarrow M$ of DGLAs with kernel I , there is a canonical long exact sequence*

$$\begin{aligned} \cdots &\xrightarrow{e_*} \pi_i \underline{\text{MC}}(M) \xrightarrow{o_e} \text{H}^{2-i}(I) \longrightarrow \pi_{i-1} \underline{\text{MC}}(L) \xrightarrow{e_*} \cdots \\ \cdots &\xrightarrow{e_*} \pi_1 \underline{\text{MC}}(M) \xrightarrow{o_e} \text{H}^1(I) \longrightarrow \pi_0 \underline{\text{MC}}(L) \xrightarrow{e_*} \pi_0 \underline{\text{MC}}(M) \xrightarrow{o_e} \text{H}^2(I). \end{aligned}$$

1.4.2. *Towers of obstructions.* The following is the long exact sequence of cohomology:

Proposition 1.32. *For each p , there is a canonical long exact sequence*

$$\cdots \rightarrow \text{H}_{p+i-n-2}(\Omega_{A/R}^p) \rightarrow \pi_i(\text{PreSp}/F^{p+1}) \rightarrow \pi_i(\text{PreSp}/F^p) \rightarrow \text{H}_{p+i-n-3}(\Omega_{A/R}^p) \rightarrow \cdots$$

of homotopy groups, where $\text{PreSp} = \text{PreSp}(A, n)$.

Definition 1.33. Given a compatible pair $(\omega, \pi) \in \text{Comp}(A, n)/F^3$ and $p \geq 0$, define the cochain complex $M(\omega, \pi, p)$ to be the cocone of the map

$$\begin{aligned} & (\Omega_{A/R}^p)^{[n-p+1]} \oplus \underline{\text{Hom}}_A(\text{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), A)^{[n+1]} \\ & \rightarrow \underline{\text{Hom}}_A(\text{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), A)^{[n+1]} \end{aligned}$$

given by combining

$$\text{Symm}^p(\pi^\sharp): (\Omega_{A/R}^p)^{[-p]} \rightarrow \underline{\text{Hom}}_A(\text{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), A)$$

with the map

$$\nu(\omega, \pi) - (p-1): \underline{\text{Hom}}_A(\text{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), A) \rightarrow \underline{\text{Hom}}_A(\text{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), A),$$

where

$$\nu(\omega, \pi)(b) := \nu(\omega, \pi, b).$$

Because ω and π lie in gr_F^2 , the description of Definition 1.16 simplifies to give

$$\nu(\text{adf}_1 \wedge \text{df}_2, \pi)(b) = \pm a[\pi, f_1][b, f_2] \pm a[b, f_1][\pi, f_2].$$

Lemma 1.34. *If π is non-degenerate, then the projection*

$$M(\omega, \pi, p) \rightarrow (\Omega_{A/R}^p)^{[n-p+1]}$$

is a quasi-isomorphism.

Proof. On $\underline{\text{Hom}}_A((\Omega_{A/R}^1)_{[-n-1]}, A)$ (the case $p = 1$), observe that the map $\nu(\omega, \pi)$ is just given by $\pi^\sharp \circ \omega^\sharp$. Moreover, contraction with a 1-form defines a derivation with respect to the commutative multiplication on polyvectors, so $\nu(\omega, \pi)$ is the derivation

$$\nu(\omega, \pi): \widehat{\text{Pol}}(A, n) \rightarrow \widehat{\text{Pol}}(A, n)$$

given on generators by $\pi^\sharp \circ \omega^\sharp$.

If π is non-degenerate, then (by compatibility) $\pi^\sharp \circ \omega^\sharp$ is homotopic to the identity map, and thus $\text{gr}_F^p \nu(\omega, \pi)$ is homotopic to p . In particular, $\text{gr}_F^p \nu(\omega, \pi) - (p-1)$ is a quasi-isomorphism in this case, so the projection $M(\omega, \pi, p) \rightarrow (\Omega_{A/R}^p)^{[n-p+1]}$ is a quasi-isomorphism. \square

If we apply Corollary 1.31 to the small extensions $\text{gr}_F^p L \rightarrow L/F^{p+1} \rightarrow L/F^p$ for the DGLA of polyvectors, and taking the long exact sequence of homotopy groups, we get:

Proposition 1.35. *For each $p \geq 3$, there is a canonical long exact sequence*

$$\begin{array}{ccccc}
 \vdots & & \vdots & & \vdots \\
 \downarrow e_* & & \downarrow e_* & & \downarrow e_* \\
 \pi_i(\text{Comp}(A, n)/F^p) & & \pi_1(\text{Comp}(A, n)/F^p) & & \pi_0(\text{Comp}(A, n)/F^p) \\
 \downarrow o_e & & \downarrow o_e & & \downarrow o_e \\
 H^{2-i}M(\omega_2, \pi_2, p) & & H^1M(\omega_2, \pi_2, p) & & H^2M(\omega_2, \pi_2, p) \\
 \downarrow & & \downarrow & & \downarrow \\
 \pi_{i-1}(\text{Comp}(A, n)/F^{p+1}) & & \pi_0(\text{Comp}(A, n)/F^{p+1}) & & \\
 \downarrow e_* & & & & \\
 \vdots & & & & \\
 \vdots & & \vdots & & \vdots \\
 & & \downarrow e_* & & \\
 & & \pi_0(\text{Comp}(A, n)/F^{p+1}) & & \\
 & & \downarrow e_* & & \\
 & & \vdots & &
 \end{array}$$

of homotopy groups and sets, where π_i indicates the homotopy group at basepoint (ω, π) , and the target of the final map is understood to mean

$$o_e(\omega, \pi) \in H^2M(\omega_2, \pi_2, p).$$

Proof. Proposition 1.29 gives fibration sequences

$$\begin{aligned}
 \mathcal{P}(A, n)/F^{p+1} &\rightarrow \mathcal{P}(A, n)/F^p \rightarrow \underline{\text{MC}}(\text{gr}_F^p \widehat{\text{Pol}}(A, n)^{[n+2]}) \\
 \mathcal{TP}(A, n)/F^{p+1} &\rightarrow \mathcal{TP}(A, n)/F^p \rightarrow \underline{\text{MC}}(\text{gr}_F^p \widehat{\text{Pol}}(A, n)^{[n+2]}[\epsilon]) \\
 \text{PreSp}(A, n)/F^{p+1} &\rightarrow \text{PreSp}(A, n)/F^p \rightarrow \underline{\text{MC}}((\Omega_{A/R}^p)^{[n+2-p]}),
 \end{aligned}$$

for $\epsilon^2 = 0$.

We can regard these as homotopy vanishing loci for sections of trivial bundles. Combined with the description of $\text{Comp}(A, n)$ as a homotopy vanishing locus, this gives $\text{Comp}(A, n)/F^{p+1}$ as a homotopy vanishing locus on $\text{Comp}(A, n)/F^p$.

In more detail, we pull the sequences above back along

$$(i_1, i_2): \text{Comp}(A, n)/F^p \rightarrow \text{PreSp}(A, n)/F^p \times \mathcal{P}(A, n)/F^p,$$

giving a morphism

$$\begin{aligned}
 &\text{Comp}(A, n)/F^p \times \underline{\text{MC}}(\text{gr}_F^p \widehat{\text{Pol}}(A, n)^{[n+2]}) \oplus (\Omega_{A/R}^p)^{[n+2-p]} \\
 &\xrightarrow{\text{gr}_F^p(\mu-\sigma)} \text{Comp}(A, n)/F^p \times \underline{\text{MC}}(\text{gr}_F^p \widehat{\text{Pol}}(A, n)^{[n+2]})
 \end{aligned}$$

of trivial bundles on $\text{Comp}(A, n)/F^p$. If we write N for the homotopy kernel of this map, we obtain a bundle over $\text{Comp}(A, n)/F^p$ equipped with a section s whose homotopy vanishing locus is $\text{Comp}(A, n)/F^{p+1}$.

It therefore remains to describe the tangents of the maps μ, σ . Since σ is linear on gr_F^p , it is its own tangent, equal to $(p-1)$. To calculate the tangent of μ , take e with $e^2 = 0$, $a \in F^p \text{DR}(A/R)$ and $b \in F^p \widehat{\text{Pol}}(A, n)$. Then

$$\begin{aligned}
 \mu(\omega + ae, \pi + be) - \mu(\omega, \pi) &= \mu(ae, \pi + be) + \mu(\omega, \pi + be) - \mu(\omega, \pi) \\
 &= \mu(a, \pi)e + \nu(\omega, \pi, b)e,
 \end{aligned}$$

because $e^2 = 0$.

Since terms in F^{p+1} vanish, and μ preserves the filtration F^{*+2} , the only contributions left are

$$\mu(a, \pi_2)e + \nu(\omega_2, \pi_2, b)e.$$

At a point $(\omega, \pi) \in \text{Comp}(A, n)/F^p$, the homotopy fibre $N_{\omega, \pi}$ of the bundle N above is therefore just $\underline{\text{MC}}(M(\omega_2, \pi_2, p)^{[1]})$. Then $\text{Comp}(A, n)/F^{p+1}$ is the homotopy vanishing locus of the obstruction map (which takes values in N), giving the long exact sequence of homotopy groups from Proposition 1.30. \square

1.5. The equivalence.

Corollary 1.36. *The canonical map*

$$\text{Comp}(A, n)^{\text{nondeg}} \rightarrow (\text{Comp}(A, n)/F^3)^{\text{nondeg}} \times_{(\text{PreSp}(A, n)/F^3)}^h \text{PreSp}(A, n)$$

is a weak equivalence.

Proof. Lemma 1.34 shows that for non-degenerate ω_2 , the map

$$M(\omega_2, \pi_2, p) \rightarrow (\Omega_{A/R}^p)^{[n-p+1]}$$

is a quasi-isomorphism. Propositions 1.32 and 1.35 thus combine to show that the maps $\text{Comp}(A, n)^{\text{nondeg}}/F^{p+1} \rightarrow (\text{Comp}(A, n)^{\text{nondeg}}/F^p) \times_{(\text{PreSp}(A, n)/F^p)}^h (\text{PreSp}(A, n)/F^{p+1})$ are weak equivalences for all $p \geq 3$. We then just take the limit over all p . \square

Proposition 1.37. *The canonical map*

$$\text{Comp}(A, n)^{\text{nondeg}}/F^3 \rightarrow \text{Sp}(A, n)/F^3$$

is a weak equivalence.

Proof. We may apply Proposition 1.30 to the definition of Comp as a homotopy vanishing locus, giving a long exact sequence

$$\begin{array}{ccc}
 \vdots & & \\
 \downarrow & & \\
 \pi_i(\text{Comp}(A, n)/F^3, (\omega, \pi)) & & \pi_0(\text{Comp}(A, n)/F^3) \\
 \downarrow & & \downarrow \\
 \pi_i(\text{PreSp}(A, n)/F^3, \omega) \times \pi_i(\mathcal{P}(A, n)/F^3, \pi) & & \pi_0(\text{PreSp}(A, n)/F^3) \times \pi_0(\mathcal{P}(A, n)/F^3) \\
 \downarrow \mu^{-\sigma} & & \downarrow \mu^{-\sigma} \\
 \pi_i(T\mathcal{P}(A, n)/F^3, 0) & & \pi_0(T_{\omega, \pi}\mathcal{P}(A, n)/F^3)
 \end{array}$$

(A dashed box encloses the right-hand side of the sequence and the bottom-left corner of the left-hand side.)

of groups and sets.

We just have

$$\begin{aligned}
 \pi_i(\text{PreSp}(A, n)/F^3, \omega) &= \text{H}_{i-n}(\Omega_{A/R}^2) \\
 \pi_i(\mathcal{P}(A, n)/F^3) &= \text{H}^{n+2-i}(\text{gr}_F^2 \widehat{\text{Pol}}(A, n)) \\
 \pi_i(T_{\omega, \pi}\mathcal{P}(A, n)/F^3) &= \text{H}^{n+2-i}(\text{gr}_F^2 \widehat{\text{Pol}}(A, n)),
 \end{aligned}$$

with the map $\mu - \sigma$ given as in Definition 1.33 by combining $\mathrm{Symm}^2(\pi^\#): (\Omega_{A/R}^2)_{[2]} \rightarrow \underline{\mathrm{Hom}}_A(\mathrm{CoS}_A^2((\Omega_{A/R}^1)_{[-n-1]}), A)$ with the map $\nu(\omega, \pi) - 1$.

As in Lemma 1.34, when π is non-degenerate, the map $\mathrm{gr}_F^2 \nu(\omega, \pi) - 1$ is a quasi-isomorphism, inducing isomorphisms

$$\pi_i(\mathrm{Comp}(A, n)/F^3, (\omega, \pi)) \cong \pi_i(\mathrm{PreSp}(A, n)/F^3, \omega)$$

for all $i > 0$. For $i = 0$, the argument of Example 1.21 works equally well modulo F^3 ; combined with the exact sequence above, it shows that the locus

$$\pi_0(\mathrm{Comp}(A, n)/F^3)^{\mathrm{nondeg}} \hookrightarrow \pi_0(\mathrm{PreSp}(A, n)/F^3) = \mathrm{H}^n(\Omega_{A/R}^2)$$

corresponds to the non-degenerate elements $\pi_0(\mathrm{Sp}(A, n)/F^3)$. \square

Corollary 1.38. *The canonical maps*

$$\begin{aligned} \mathrm{Comp}(A, n)^{\mathrm{nondeg}} &\rightarrow \mathrm{Sp}(A, n) \\ \mathrm{Comp}(A, n)^{\mathrm{nondeg}} &\rightarrow \mathcal{P}(A, n)^{\mathrm{nondeg}} \end{aligned}$$

are weak equivalences.

Proof. For the first map, just combine Corollary 1.36 with Proposition 1.37. The second map is given by Proposition 1.26. \square

2. COMPATIBLE STRUCTURES ON DERIVED DM STACKS

We will now study symplectic and Poisson structures on derived N -stacks. Rather than studying a single CDGA as in Corollary 1.38, we will establish a similar result for strings of étale morphisms between non-degenerate structures. Following the philosophy of [Pri3], we can represent a derived Artin or DM N -stack \mathfrak{X} as a simplicial derived affine scheme X_\bullet satisfying various conditions. Thus X_\bullet is $\mathrm{Spec} O(X)_\bullet$ for a cosimplicial diagram $O(X)_\bullet$ of CDGAs. For a derived DM stack, the morphisms in this diagram are étale, and descent results will show that Poisson structures are independent of the choice of resolution X_\bullet . A modified construction will be given for derived Artin stacks in §3.

Definition 2.1. Write $dg\mathrm{CAlg}(\mathbb{Q})$ for the category of chain complexes over \mathbb{Q} with graded-commutative multiplication. We will refer to its objects as chain CDGAs (in contrast with the more common, but equivalent, cochain CDGAs). Given $R \in dg\mathrm{CAlg}(\mathbb{Q})$, we write $dg\mathrm{CAlg}(R)$ for the category of graded-commutative R -algebras in chain complexes.

Throughout this section, we fix $R \in dg\mathrm{CAlg}(\mathbb{Q})$.

2.1. Compatible structures on diagrams.

2.1.1. *Definitions.* Given a small category I , an I -diagram A of chain CDGAs over R , and A -modules M, N in I -diagrams of chain complexes, we can define the cochain complex $\underline{\mathrm{Hom}}_A(M, N)$ to be the equaliser of the obvious diagram

$$\prod_{i \in I} \underline{\mathrm{Hom}}_{A(i)}(M(i), N(i)) \rightrightarrows \prod_{f: i \rightarrow j \text{ in } I} \underline{\mathrm{Hom}}_{A(i)}(M(i), f_* N(j)).$$

All the constructions of §1.1 then adapt immediately; in particular, we can define

$$\widehat{\mathrm{Pol}}(A, n) := \underline{\mathrm{Hom}}_A(\mathrm{CoSymm}_A((\Omega_{A/R}^1)_{[-n-1]}), A),$$

leading to a space $\mathcal{P}(A, n)$ of Poisson structures.

In order to ensure that this has the correct homological properties, we now consider categories of the form $[m] = (0 \rightarrow 1 \rightarrow \dots \rightarrow m)$.

Lemma 2.2. *If A is an $[m]$ -diagram in chain CDGAs over R which is cofibrant and fibrant for the injective model structure (i.e. each $A(i)$ is cofibrant and the maps $A(i) \rightarrow A(i+1)$ are surjective), then $\underline{\mathrm{Hom}}_A(\mathrm{CoS}_A^k \Omega_A^1, A)$ is a model for the derived Hom-complex $\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{L}\mathrm{CoS}_A^k \mathbf{L}\Omega_A^1, A)$, and $\underline{\mathrm{Hom}}_A(A, \Omega_A^p) \simeq \mathop{\mathrm{holim}}_{\leftarrow i} \mathbf{L}\Omega_{A(i)}^p$.*

Proof. Because $A(i)$ is cofibrant, $\Omega_{A(i)}^p$ is cofibrant as an $A(i)$ -module, so the complex $\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{L}\mathrm{CoS}_A^k \mathbf{L}\Omega_A^1, A)$ is the homotopy limit of the diagram

$$\underline{\mathrm{Hom}}_{A(0)}(C(0), A(0)) \rightarrow \underline{\mathrm{Hom}}_{A(0)}(C(0), A(1)) \leftarrow \underline{\mathrm{Hom}}_{A(1)}(C(1), A(1)) \rightarrow \dots,$$

where we write $C(i) := \mathrm{CoS}_{A(i)}^k \Omega_{A(i)}^1$. Since the maps $A(i) \rightarrow A(i+1)$ are surjective, the maps $\underline{\mathrm{Hom}}_{A(i)}(\mathrm{CoS}_{A(i)}^k \Omega_{A(i)}^1, A(i)) \rightarrow \underline{\mathrm{Hom}}_{A(i)}(\Omega_{A(i)}^p, A(j))$ are so, and thus the homotopy limit is calculated by the ordinary limit, which is precisely $\underline{\mathrm{Hom}}_A(\mathrm{CoS}_A^k \Omega_A^1, A)$.

Because $[m]$ has initial object 0, $\underline{\mathrm{Hom}}_A(A, \Omega_A^p) \cong \underline{\mathrm{Hom}}_{A(0)}(A(0), \Omega_{A(0)}^p) = \Omega_{A(0)}^p$. For the same reason, $\mathop{\mathrm{holim}}_{\leftarrow i} \Omega_{A(i)}^p \simeq \Omega_{A(0)}^p$. \square

For such diagrams, we can also adapt all the constructions of §1.2 immediately by taking inverse limits, so setting

$$\mathrm{PreSp}(A, n) := \mathrm{PreSp}(A(0), n) = \mathop{\varprojlim}_{i \in [m]} \mathrm{PreSp}(A(i), n)$$

for any $[m]$ -diagram A of chain CDGAs over R .

We then adapt the constructions of §1.3, defining

$$\mu: \mathrm{PreSp}(A, n) \times \mathcal{P}(A, n) \rightarrow T\mathcal{P}(A, n)$$

by setting $\mu(\omega, \Delta)(i) := \mu(\omega(i), \Delta(i)) \in T\mathcal{P}(A(i), n)$ for $i \in [m]$, and letting $\mathrm{Comp}(A, n)$ be the homotopy vanishing locus of

$$(\mu - \sigma): \mathrm{PreSp}(A, n) \times \mathcal{P}(A, n) \rightarrow \mathrm{pr}_2^* T\mathcal{P}(A, n).$$

The obstruction functors and their towers from §1.4 also adapt immediately to $[m]$ -diagrams, giving the obvious analogues of the obstruction spaces defined in terms of

$$\underline{\mathrm{Hom}}_A(\mathrm{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), A) \quad \text{and} \quad \Omega_{A(0)}^p.$$

2.1.2. Functors and descent.

Lemma 2.3. *If $D = (A \rightarrow B)$ is a fibrant cofibrant $[1]$ -diagram in $\mathrm{dgCAlg}(R)$ which is formally étale in the sense that the map*

$$\Omega_A^1 \otimes_A B \rightarrow \Omega_B^1$$

is a quasi-isomorphism, then the map

$$\underline{\mathrm{Hom}}_D(\mathrm{CoS}_D^k \Omega_D^1, D) \rightarrow \underline{\mathrm{Hom}}_A(\mathrm{CoS}_A^k \Omega_A^1, A),$$

is a quasi-isomorphism.

Proof. This follows immediately from the proof of Lemma 2.2, using the quasi-isomorphism

$$\underline{\mathrm{Hom}}_B(\mathrm{CoS}_B^k \Omega_B^1, B) \rightarrow \underline{\mathrm{Hom}}_A(\mathrm{CoS}_A^k \Omega_A^1, B).$$

\square

For a similar result, see [CPT⁺, Lemma 1.4.13].

Definition 2.4. Write $dg\mathcal{CAlg}(R)_{c,\rightarrow} \subset dg\mathcal{CAlg}(R)$ for the subcategory with all cofibrant chain CDGAs over R as objects, and only surjective morphisms.

We already have functors $\mathrm{PreSp}(-, n)$ and $\mathrm{Sp}(-, n)$ from $dg\mathcal{CAlg}(R)$ to $s\mathrm{Set}$, the category of simplicial sets, mapping quasi-isomorphisms in $dg\mathcal{CAlg}(R)_c$ to weak equivalences. Poisson structures are only functorial with respect to formally étale morphisms, in an ∞ -functorial sense which we now make precise.

Observe that, writing F for any of the constructions $\mathcal{P}(-, n)$, $\mathrm{Comp}(-, n)$, $\mathrm{PreSp}(-, n)$, and the associated filtered and graded functors, applied to $[m]$ -diagrams in $dg\mathcal{CAlg}(R)_{c,\rightarrow}$, we have:

- Properties 2.5.* (1) the maps from $F(A(0) \rightarrow \dots \rightarrow A(m))$ to $F(A(0) \rightarrow A(1)) \times_{F(A(1))}^h F(A(1) \rightarrow A(2)) \times_{F(A(2))}^h \dots \times_{F(A(m-1))}^h F(A(m-1) \rightarrow A(m))$ are weak equivalences;
- (2) if the $[1]$ -diagram $A \rightarrow B$ is a quasi-isomorphism, then the natural maps from $F(A \rightarrow B)$ to $F(A)$ and to $F(B)$ are weak equivalences.
- (3) if the $[1]$ -diagram $A \rightarrow B$ is formally étale, then the natural map from $F(A \rightarrow B)$ to $F(A)$ is a weak equivalence.

These properties follow from Lemmas 2.2 and 2.3, together with the obstruction calculus of §1.4 extended to diagrams.

Property 2.5.1 ensures that the simplicial classes $\coprod_{A \in B_m dg\mathcal{CAlg}(R)_{c,\rightarrow}} F(A)$ fit together to give a complete Segal space $\int F$ over the nerve $Bdg\mathcal{CAlg}(R)_{c,\rightarrow}$. Taking complete Segal spaces [Rez, §6] as our preferred model of ∞ -categories:

Definition 2.6. Define $\mathbf{L}dg\mathcal{CAlg}(R)_{c,\rightarrow}$, $\mathbf{L}dg\mathcal{CAlg}(R)$, $\mathbf{L} \int F$, and $\mathbf{L}s\mathrm{Set}$ to be the ∞ -categories obtained by localising the respective categories at quasi-isomorphisms or weak equivalences.

Property 2.5.2 ensures that the homotopy fibre of $\mathbf{L} \int F \rightarrow \mathbf{L}dg\mathcal{CAlg}(R)_{c,\rightarrow}$ over A is still just the simplicial set $F(A)$, regarded as an ∞ -groupoid.

Since the surjections in $dg\mathcal{CAlg}(R)$ are the fibrations, the inclusion $\mathbf{L}dg\mathcal{CAlg}(R)_{c,\rightarrow} \rightarrow \mathbf{L}dg\mathcal{CAlg}(R)$ is a weak equivalence, and we may regard $\mathbf{L} \int F$ as an ∞ -category over $\mathbf{L}dg\mathcal{CAlg}(R)$.

Furthermore, Property 2.5.3 implies that the ∞ -functor $\mathbf{L} \int F \rightarrow \mathbf{L}dg\mathcal{CAlg}(R)$ is a co-Cartesian fibration when we restrict to the 2-sub- ∞ -category $\mathbf{L}dg\mathcal{CAlg}(R)^{\acute{e}t}$ of homotopy formally étale morphisms, giving:

Definition 2.7. When F is any of the constructions above, we define

$$\mathbf{R}F: \mathbf{L}dg\mathcal{CAlg}(R)^{\acute{e}t} \rightarrow \mathbf{L}s\mathrm{Set}$$

to be the ∞ -functor whose Grothendieck construction is $\mathbf{L} \int F$.

In particular, the observations above ensure that

$$(\mathbf{R}F)(A) \simeq F(A)$$

for all cofibrant chain CDGAs A over R .

An immediate consequence of Corollary 1.38 is that the canonical maps

$$\begin{aligned} \mathbf{R}\mathrm{Comp}(-, n)^{\mathrm{nondeg}} &\rightarrow \mathbf{R}\mathrm{Sp}(-, n) \\ \mathbf{R}\mathrm{Comp}(-, n)^{\mathrm{nondeg}} &\rightarrow \mathbf{R}\mathcal{P}(-, n)^{\mathrm{nondeg}} \end{aligned}$$

are natural weak equivalences of ∞ -functors on $\mathbf{LdgCAlg}(R)^{\acute{e}t}$.

2.2. Derived N -hypercgroupoids.

2.2.1. *Background.* We now require our chain CDGA R over \mathbb{Q} to be concentrated in non-negative chain degrees, and write $dg_+ \mathbf{CAlg}(R) \subset dg \mathbf{CAlg}(R)$ for the full subcategory of chain CDGAs which are concentrated in non-negative chain degrees. We denote the opposite category to $dg_+ \mathbf{CAlg}(R)$ by $DG^+ \mathbf{Aff}_R$. Write $sDG^+ \mathbf{Aff}_R$ for the category of simplicial diagrams in $DG^+ \mathbf{Aff}_R$. A morphism in $DG^+ \mathbf{Aff}_R$ is said to be a fibration if it is given by a cofibration in the opposite category $dg_+ \mathbf{CAlg}(R)$ (which in turn just means that it is a retract of a quasi-free map).

As in [TV], a morphism $f : A \rightarrow B$ in $dg_+ \mathbf{CAlg}(R)$ is said to be smooth (resp. étale) if $H_0(f) : H_0 A \rightarrow H_0 B$ is smooth, and the maps $H_n(A) \otimes_{H_0(A)} H_0(B) \rightarrow H_n(B)$ are isomorphisms for all n . The associated map $\mathrm{Spec} B \rightarrow \mathrm{Spec} A$ in $DG^+ \mathbf{Aff}_R$ is said to be surjective if $\mathrm{Spec} H_0 B \rightarrow \mathrm{Spec} H_0 A$ is so.

Definition 2.8. For $m \geq 0$, the combinatorial m -simplex $\Delta^m \in s\mathbf{Set}$ is characterised by the property that $\mathrm{Hom}_{s\mathbf{Set}}(\Delta^m, K) \cong K_m$ for all simplicial sets K . Its boundary $\partial\Delta^m \subset \Delta^m$ is given by $\bigcup_{i=0}^m \partial^i(\Delta^{m-1})$ (taken to include the case $\partial\Delta^0 = \emptyset$), and for $m \geq 1$ the k th horn $\Lambda^{m,k}$ is given by $\bigcup_{\substack{0 \leq i \leq m \\ i \neq k}} \partial^i(\Delta^{m-1})$.

Definition 2.9. Given a simplicial set K and a simplicial object X_\bullet in a complete category \mathcal{C} , we follow [GJ, Proposition VII.1.21] in defining the K -matching object in \mathcal{C} by

$$M_K X := \mathrm{Hom}_{s\mathbf{Set}}(K, X).$$

Note that for finite simplicial sets K , the matching object $M_K X$ still exists even if \mathcal{C} only contains finite limits.

Explicitly, the matching object $M_{\partial\Delta^m}(X)$ is given by the equaliser of a diagram

$$\prod_{0 \leq i \leq m} X_{m-1} \rightrightarrows \prod_{0 \leq i < j \leq m} X_{m-2},$$

while the partial matching object $M_{\Lambda_k^m}(X)$ is given by the equaliser of a diagram

$$\prod_{\substack{0 \leq i \leq m \\ i \neq k}} X_{m-1} \rightrightarrows \prod_{\substack{0 \leq i < j \leq m \\ i, j \neq k}} X_{m-2}.$$

We now recall some results from [Pri3].

Definition 2.10. Given $Y \in sDG^+ \mathbf{Aff}_R$, a DG Artin (resp. DM) N -hypercgroupoid X over Y is a morphism $X \rightarrow Y$ in $sDG^+ \mathbf{Aff}_R$ for which:

- (1) the matching maps

$$X_m \rightarrow M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

are fibrations for all $m \geq 0$;

- (2) the partial matching maps

$$X_m \rightarrow M_{\Lambda_k^m}(X) \times_{M_{\Lambda_k^m}^h(Y)}^h Y_m$$

are smooth (resp. étale) surjections for all $m \geq 1$ and k , and are weak equivalences for all $m > N$ and all k .

A morphism $X \rightarrow Y$ in $sDG^+ \text{Aff}_R$ is a trivial DG Artin (resp. DM) N -hypergroupoid if and only if the matching maps

$$X_m \rightarrow M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

of Definition 2.9 are surjective smooth (resp. étale) fibrations for all m , and are weak equivalences for all $m \geq n$.

The following is [Pri3, Theorem 4.15 and Corollary 6.35], as spelt out in [Pri2, Theorem 5.11]:

Theorem 2.11. *The ∞ -category of strongly quasi-compact N -geometric derived Artin (resp. DM) stacks over R is given by localising the category of DG Artin (resp.) N -hypergroupoids over R at the class of trivial relative DG Artin (resp. DM) N -hypergroupoids.*

Given a DG Artin (resp. DM) N -hypergroupoid X , we denote the associated N -geometric derived Artin (resp. DM) stack by X^\sharp .

Definition 2.12. Given $X \in sDG^+ \text{Aff}_R$, define $cdg\text{Mod}(X)$ to be the category of $O(X)$ -modules in cosimplicial diagrams of chain complexes. We say that a morphism $M \rightarrow N$ in $cdg\text{Mod}(X)$ is a weak equivalence if it induces quasi-isomorphisms $M^i \rightarrow N^i$ of chain complexes in each cosimplicial level.

The following is [Pri3, Proposition 5.12 and Remarks 5.24]:

Proposition 2.13. *For a DG Artin N -hypergroupoid X , the ∞ -category of quasi-coherent complexes (in the sense of [Lur, §5.2]) on the n -geometric derived stack X^\sharp is equivalent to the localisation at weak equivalences of the full subcategory $dg\text{Mod}(X)_{\text{cart}}$ of $cdg\text{Mod}(X)$ consisting of modules M which are homotopy-Cartesian in the sense that the maps*

$$\partial^i: \mathbf{L}\partial_i^* M^{m-1} \rightarrow M^m$$

are quasi-isomorphisms for all i and m .

Definition 2.14. We make $cdg\text{Mod}(X)$ into a simplicial category by setting (for $K \in s\text{Set}$)

$$(M^K)^n := (M^n)^{K_n},$$

as an $O(X)^n$ -module in chain complexes. This has a left adjoint $M \mapsto M \otimes K$.

Given $M \in cdg\text{Mod}(X)$, define $\underline{M} \in (cdg\text{Mod}(X))^\Delta$ to be the cosimplicial diagram given in cosimplicial level n by $M \otimes \Delta^n$.

Combining [Pri3, Definition 7.7 and Corollary 7.19] gives:

Definition 2.15. For a DG Artin N -hypergroupoid X , define the cotangent complex $\mathbb{L}^{X/S} \in cdg\text{Mod}(X)$ by $\mathbb{L}^{X/S} := \text{Tot } N_c^{\leq N} \underline{\Omega}(X/S)$, where N_c denotes cosimplicial conormalisation, and Tot is the direct sum total functor from cochain chain complexes to chain complexes.

By [Pri3, Lemma 7.17 and Corollary 7.14], this is homotopy-Cartesian and recovers the usual cotangent complexes in derived algebraic geometry under the correspondence of Proposition 2.13.

2.2.2. *Poisson and symplectic structures.* Take a DG Deligne–Mumford N -hypergroupoid X over R . Because the face maps $\partial^i: \Delta^{m-1} \rightarrow \Delta^m$ are weak equivalences, the morphisms $\partial_i: X_m \rightarrow X_{m-1}$ are all étale. Since $\sigma_i: X_{m-1} \rightarrow X_m$ has left inverse ∂_i , it follows that σ_i is also étale.

Therefore $O(X)$ is a functor from Δ^{opp} to $dg\text{CAlg}(R)^{\text{ét}}$.

Definition 2.16. For any of the functors F in Definition 2.7, write

$$F(X) := \text{holim}_{j \in \Delta} \mathbf{R}F(O(X_j)).$$

This gives us spaces $\mathcal{P}(X, n)$, $\text{PreSp}(X, n)$ and $\text{Comp}(X, n)$ of Poisson structures, pre-symplectic structures and compatible pairs. Explicitly, strictification theorems then imply that an element of $\mathcal{P}(X, n)$ can be represented by a cosimplicial diagram P of \hat{P}_{n+1} -algebras equipped with a weak equivalence from $O(X)$ to the cosimplicial chain CDGA underlying P .

Proposition 2.17. *If $Y \rightarrow X$ is a trivial DG DM hypergroupoid, then the morphism*

$$F(X) \rightarrow F(Y)$$

is an equivalence for any of the constructions $F = \mathcal{P}, \text{Comp}, \text{PreSp}$.

Proof. The morphism $F(X) \rightarrow F(Y)$ exists because the maps $Y_j \rightarrow X_j$ are all étale. By Propositions 1.32, 1.35 and the corresponding statement for \mathcal{P} , it suffices to prove that the morphisms

$$\Omega_{X_j/R}^p \rightarrow \Omega_{Y_j/R}^p,$$

$$\underline{\text{Hom}}_{O(X_j)}(\text{CoSp}^p((\Omega_{X_j/R}^1)_{[-n-1]}), O(X_j)) \rightarrow \underline{\text{Hom}}_{O(Y_j)}(\text{CoSp}^p((\Omega_{Y_j/R}^1)_{[-n-1]}), O(Y_j))$$

induce quasi-isomorphisms on the homotopy limits over $j \in \Delta$. This follows by faithfully flat descent for quasi-coherent sheaves, since $\mathfrak{X} := X^\sharp \simeq Y^\sharp$, the expressions reducing to $\mathbf{R}\Gamma(\mathfrak{X}, \mathbf{L}\Omega_{\mathfrak{X}/R}^p)$ and $\mathbf{R}\underline{\text{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(\text{CoSp}^p((\mathbb{L}_{\mathfrak{X}/R})_{[-n-1]}), \mathcal{O}_{\mathfrak{X}})$, respectively. \square

Thus the following is well-defined:

Definition 2.18. Given a strongly quasi-compact DG DM N -stack \mathfrak{X} , define the spaces $\mathcal{P}(\mathfrak{X}, n)$, $\text{Comp}(\mathfrak{X}, n)$, $\text{Sp}(\mathfrak{X}, n)$ to be the spaces $\mathcal{P}(X, n)$, $\text{Comp}(X, n)$, $\text{Sp}(X, n)$ for any DG DM N -hypergroupoid X with $X^\sharp \simeq \mathfrak{X}$.

By the results of §2.1.2, an immediate consequence of Corollary 1.38 is then:

Theorem 2.19. *There are natural weak equivalences*

$$\text{Sp}(\mathfrak{X}, n) \leftarrow \text{Comp}(\mathfrak{X}, n)^{\text{nondeg}} \rightarrow \mathcal{P}(\mathfrak{X}, n)^{\text{nondeg}}.$$

3. COMPATIBLE STRUCTURES ON DERIVED ARTIN STACKS

We now show how to extend the comparisons above to derived Artin N -stacks over a fixed chain CDGA $R \in dg_+\text{CAlg}(\mathbb{Q})$. The main difficulty is in establishing an appropriate notion for shifted Poisson structures in this setting. It is tempting just to consider DG Artin N -hypergroupoids equipped with Poisson structures levelwise, with weak equivalences generated by trivial DG Artin N -hypergroupoids. With this approach, it is not easy to compute the obstructions arising, and it seems unlikely that they will be related to the cotangent complex in the desired way.

Remark 3.1. By contrast, deformations of a DG Artin hypergroupoid X do recover all deformations of the associated derived Artin stack, at least on the level of objects (cf. [Pri3, Theorem 8.8]). The key in that case was the comparison between $\Omega^1(X/R)$ and the cotangent complex $\mathbb{L}_{X/R}$ in [Pri3, Lemma 8.4], which relies on strong boundedness properties. The shifts involved in the definition of Poisson structures preclude any analogous result for the obstruction tower associated to a semi-strict Poisson structure.

3.1. Stacky CDGAs. Our solution is to replace Artin N -hypergroupoids in CDGAs with DM hypergroupoids in a suitable category of graded-commutative algebras in double complexes, which we refer to as stacky CDGAs. The idea is to globalise one of the intermediate steps [Pri1, Theorem 4.26] in the proof of the equivalence between formal moduli problems and DGLAs.

Given a stacky CDGA A , the definition of an n -shifted Poisson structure is fairly obvious: it is a Lie bracket of total cochain degree $-n$, or rather an L_∞ -structure in the form of a sequence $[-]_m$ of m -ary operations of cochain degree $1 - (n + 1)(m - 1)$. However, the precise formulation (Definition 3.20) is quite subtle, involving lower bounds on the cochain degrees of the operations.

When working with stacky CDGAs, we write the double complexes as chain cochain complexes, enabling us to distinguish easily between derived (chain) and stacky (cochain) structures:

Definition 3.2. Define a chain cochain complex V over \mathbb{Q} to be a bigraded \mathbb{Q} -vector space $V = \bigoplus_{i,j} V_j^i$, equipped with square-zero linear maps $\partial: V_j^i \rightarrow V_j^{i+1}$ and $\delta: V_j^i \rightarrow V_{j-1}^i$ such that $\partial\delta + \delta\partial = 0$.

There is an obvious tensor product operation \otimes on this category, and a stacky CDGA is then defined to be a chain cochain complex A equipped with a commutative product $A \otimes A \rightarrow A$ and unit $\mathbb{Q} \rightarrow A$.

We regard all chain complexes as chain cochain complexes $V = V_\bullet^0$. Given a chain CDGA R , a stacky CDGA over R is then a morphism $R \rightarrow A$ of stacky CDGAs. We write $DGdg\text{CAlg}(R)$ for the category of stacky CDGAs over R , and $DG^+dg\text{CAlg}(R)$ for the full subcategory consisting of objects A concentrated in non-negative cochain degrees.

Definition 3.3. Say that a morphism $U \rightarrow V$ of chain cochain complexes is a levelwise quasi-isomorphism if $U^i \rightarrow V^i$ is a quasi-isomorphism for all $i \in \mathbb{Z}$. Say that a morphism of stacky CDGAs is a levelwise quasi-isomorphism if the underlying morphism of chain cochain complexes is so.

Lemma 3.4. *There is a cofibrantly generated model structure on stacky CDGAs over R in which fibrations are surjections and weak equivalences are levelwise quasi-isomorphisms.*

Proof. We first prove the corresponding statement for chain cochain complexes over \mathbb{Q} . Let D^i be the chain complex $k_{[-i]} \rightarrow k_{[1-i]}$ and $S^i := k_{[-i]}$, so there is an obvious map $S^i \rightarrow D^i$. Write D_i for the cochain complex $k^{[1-i]} \rightarrow k^{[-i]}$. We can then take the set I of generating cofibrations to consist of the morphisms

$$\{D_i \otimes S^j \rightarrow D_i \otimes D^j\}_{i,j \in \mathbb{Z}},$$

with the set J of generating trivial cofibrations given by

$$\{0 \rightarrow D_i \otimes D^j\}_{i,j \in \mathbb{Z}}.$$

It is straightforward to verify that this satisfies the conditions of [Hov, Theorem 2.1.19]. The forgetful functor from stacky CDGAs to chain cochain complexes has a left adjoint $V \mapsto R \otimes \text{Sym}_{\mathbb{Q}} V$, which transfers the model structure by [Hir, Theorem 11.3.2]. \square

As described for instance in [Pri1, Definition 4.20], there is a denormalisation functor D from non-negatively graded cochain CDGAs to cosimplicial algebras, with left adjoint D^* . Given a cosimplicial chain CDGA A , D^*A is then a stacky CDGA, with $(D^*A)_j^i = 0$ for $i < 0$. The functor D satisfies $(DB)^m \cong \bigoplus_{i=0}^m (B^i)^{\binom{m}{i}}$, with multiplication coming from the shuffle product, so in particular the iterated codegeneracy map $(DB)^m \rightarrow B^0$ is always an m -nilpotent extension. As a result, the left adjoint D^* factors through the functor sending A to its pro-nilpotent completion over A^0 .

Lemma 3.5. *The functor D^* is a left Quillen functor from the Reedy model structure on cosimplicial chain CDGAs to the model structure of Lemma 3.4.*

Proof. The denormalisation functor D on non-negatively graded cochain complexes extends to all cochain complexes by composing with brutal truncation. The right adjoint to D^* is given by applying D to the cochain index of a stacky CDGA A . Since fibrations and weak equivalences in the Reedy model structure are levelwise surjections and levelwise quasi-isomorphisms, it follows immediately that D is right Quillen, so D^* is left Quillen. \square

To any DG Artin N -hypergroupoid X over R , we can then associate the stacky CDGA $D^*O(X)$. This behaves well because DG Artin hypergroupoids are Reedy fibrant, so there is no need to replace D^* with an associated left-derived functor. Because DA is always a nilpotent extension of A^0 , we cannot recover X from the levelwise quasi-isomorphism class of $D^*O(X)$, but we can recover X_0 and the formal completion of X along X_0 . Thus $D^*O(X)$ will play the same role as the formal affine derived stacks of [CPT⁺].

For any DG Artin N -hypergroupoid X , the diagram X^{Δ^m} is another DG Artin N -hypergroupoid resolving the same stack, and we will then consider the cosimplicial stacky CDGA $m \mapsto D^*O(X^{\Delta^m})$, which we can think of as a kind of DM N -hypergroupoid in stacky CDGAs.

Example 3.6. If Y is a derived affine scheme equipped with an action of a smooth affine group scheme G , then the nerve $X := B[Y/G]$ is a hypergroupoid resolving the derived Artin 1-stack $[Y/G]$. Since $B[Y/G]_i = Y \times G^i$, the simplicial derived scheme X^{Δ^m} is given by $B[Y \times G^m/G^{m+1}]$, with action

$$(y, h_1, \dots, h_m)(g_0, \dots, g_m) = (yg_0, g_0^{-1}h_1g_1, g_1^{-1}h_2g_2, \dots, g_{m-1}^{-1}h_mg_m).$$

Taking Reedy fibrant replacement of X gives us a DG Artin hypergroupoid X' and a levelwise quasi-isomorphism $X \rightarrow X'$. Since the face maps of X are smooth, the map $D^*O((X')^{\Delta^\bullet}) \rightarrow D^*O(X^{\Delta^\bullet})$ is a levelwise quasi-isomorphism of cosimplicial stacky CDGAs. Thus Poisson structures on the stack $[Y/G]$ can be defined in terms of polyvectors on the cosimplicial stacky CDGA $D^*O(X^{\Delta^\bullet})$ when Y is fibrant.

The completion $O(\hat{X})$ of $O(X)$ over $O(X_0)$ is a cosimplicial CDGA with $O(\hat{X})^i = O(Y)[[\mathfrak{g}^\vee]^{\oplus i}]$, for \mathfrak{g} the Lie algebra of G , where we write $A[[M]]$ for the (M) -adic

completion of $A \otimes_R \text{Symm}_R(M)$. Thus the stacky CDGA $D^*O(X)$ is the Chevalley–Eilenberg double complex

$$O([Y/\mathfrak{g}]) := (O(Y) \xrightarrow{\partial} O(Y) \otimes \mathfrak{g}^\vee \xrightarrow{\partial} O(Y) \otimes \Lambda^2 \mathfrak{g}^\vee \xrightarrow{\partial} \dots)$$

of \mathfrak{g} with coefficients in the chain \mathfrak{g} -module $O(Y)$.

The same calculation for $[Y \times G^m/G^{m+1}]$ shows that the cosimplicial stacky CDGA $D^*O(X^{\Delta^\bullet}) = D^*O((B[Y/G])^{\Delta^\bullet})$ is given by a diagram

$$O([Y/\mathfrak{g}]) \begin{array}{c} \xrightarrow{\dots} \\ \xleftarrow{\dots} \end{array} O([Y \times G/\mathfrak{g}^{\oplus 2}]) \begin{array}{c} \xrightarrow{\dots} \\ \xleftarrow{\dots} \end{array} O([Y \times G^2/\mathfrak{g}^{\oplus 3}]) \quad \vdots \quad \dots$$

Beware that maps $(Y', G') \rightarrow (Y, G)$ only induce levelwise quasi-isomorphisms $D^*O(B[Y'/G']) \rightarrow D^*O(B[Y/G])$ when Y' is weakly equivalent to Y and \mathfrak{g}' isomorphic to \mathfrak{g} — it is not enough for the Lie algebra cohomology groups to be isomorphic.

3.1.1. Modules over stacky CDGAs. For now, we fix a cofibrant stacky CDGA A over a \mathbb{Q} -CDGA R .

Definition 3.7. Given a chain cochain complex V , define the cochain complex $\hat{\text{Tot}} V \subset \text{Tot}^{\text{II}} V$ by

$$(\hat{\text{Tot}} V)^m := \left(\bigoplus_{i < 0} V_{i-m}^i \right) \oplus \left(\prod_{i \geq 0} V_{i-m}^i \right)$$

with differential $\partial \pm \delta$.

An alternative description of $\hat{\text{Tot}} V$ is as the completion of $\text{Tot} V$ with respect to the filtration $\{\text{Tot} \sigma^{\geq m} V\}_m$, where $\sigma^{\geq m}$ denotes brutal truncation in the cochain direction. In fact, we can write

$$\varprojlim_m \varinjlim_n \text{Tot} ((\sigma^{\geq -n} V)/(\sigma^{\geq m} V)) = \hat{\text{Tot}} V = \varinjlim_n \varprojlim_m \text{Tot} ((\sigma^{\geq -n} V)/(\sigma^{\geq m} V)).$$

The latter description also shows that there is a canonical map $(\hat{\text{Tot}} U) \otimes (\hat{\text{Tot}} V) \rightarrow \hat{\text{Tot}}(U \otimes V)$ — the same is not true of Tot^{II} in general.

Write $DGdg\text{Mod}(A)$ for the category of A -modules in chain cochain complexes. When $A \in DG^+ dg\text{CAlg}(\mathbb{Q})$, write $DG^+ dg\text{Mod}(A) \subset DGdg\text{Mod}(A)$ for the full subcategory of objects concentrated in non-negative cochain degrees.

Definition 3.8. Given A -modules M, N in chain cochain complexes, we define internal Homs $\mathcal{H}om_A(M, N)$ by

$$\mathcal{H}om_A(M, N)_j^i = \text{Hom}_{A^\#} (M^\#_{i-1}, N^\#_{j-1}),$$

with differentials $\partial f := \partial_N \circ f \pm f \circ \partial_M$ and $\delta f := \delta_N \circ f \pm f \circ \delta_M$, where $V^\#$ denotes the bigraded vector space underlying a chain cochain complex V .

We then define the Hom complex $\hat{\mathcal{H}om}_A(M, N)$ by

$$\hat{\mathcal{H}om}_A(M, N) := \hat{\text{Tot}} \mathcal{H}om_A(M, N).$$

Observe that there is a multiplication $\hat{\mathcal{H}om}_A(M, N) \otimes \hat{\mathcal{H}om}_A(N, P) \rightarrow \hat{\mathcal{H}om}_A(M, P)$ — the same is not true for $\text{Tot}^{\text{II}} \mathcal{H}om_A(M, N)$ in general.

Lemma 3.9. For $A \in DG^+ dg\text{CAlg}(\mathbb{Q})$, the denormalisation functor $D: DG^+ dg\text{Mod}(A) \rightarrow cdg\text{Mod}(DA)$ to the category of DA -modules in cosimplicial chain complexes has a left adjoint D_{Mod}^* . If, for $M \in cdg\text{Mod}(DA)$, the

underlying almost cosimplicial (see Definition 3.12) graded module $M_{\#}^{\#}$ is isomorphic to $D^{\#}A_{\#} \otimes_{A_{\#}^0} L_{\#}$ for an almost cosimplicial graded $A_{\#}^0$ -module L , then

$$D_{\text{Mod}}^*(M)_{\#}^{\#} \cong A_{\#}^{\#} \otimes_{A_{\#}^0} N_c L_{\#}$$

as graded $A_{\#}^{\#}$ -modules, where N_c denotes Dold–Kan conormalisation.

Proof. Existence of the functor essentially follows because the functor D preserves all limits. The description for objects of the form $D^{\#}A_{\#} \otimes_{A_{\#}^0} L_{\#}$ follows immediately from the Dold–Kan correspondence for almost cosimplicial A^0 -modules. \square

In fact, $D_{\text{Mod}}^*(M)$ is always a quotient of $N_c M$ with A -module structure defined by the Alexander–Whitney cup product. For Cartesian DA -modules M , Lemma 3.9 (taking $L = M^0$) then implies that $D_{\text{Mod}}^* M \cong N_c M$, with $DD_{\text{Mod}}^* M \cong M$.

We now write $D: DGdg\text{Mod}(A) \rightarrow cdg\text{Mod}(DA)$ for the composition of D with the brutal truncation functor $\sigma^{\geq 0}$. We call a DA -module M levelwise cofibrant if each M^i is cofibrant in $dg\text{Mod}(D^i A)$. \mathbf{L} in \mathbf{LD}^* .

Lemma 3.10. *For $A \in DG^+ dg\text{CAlg}(\mathbb{Q})$, a levelwise cofibrant DA -module M in $cdg\text{Mod}(DA)$, and $P \in DGdg\text{Mod}(A)$, there is a canonical quasi-isomorphism*

$$\mathbf{R}\underline{\text{Hom}}_{DA}(M, DP) \simeq \text{Tot}^{\Pi} \sigma^{\geq 0} \mathcal{H}om_A(D_{\text{Mod}}^* M, P).$$

Proof. The adjunction of Lemma 3.9 gives us an isomorphism

$$\underline{\text{Hom}}_{DA}(M, DP) \cong Z^0 \mathcal{H}om_A(D_{\text{Mod}}^* M, P).$$

There is a model structure on $cdg\text{Mod}(DA)$ in which cofibrations and weak equivalences are defined levelwise in cosimplicial degrees. Similarly, there is a model structure on $DGdg\text{Mod}(A)$ in which weak equivalences are defined levelwise in cochain degrees, and a morphism $M \rightarrow N$ is a cofibration if $M^{\#} \rightarrow N^{\#}$ has the left lifting property with respect to all surjections of $A^{\#}$ -modules in graded chain complexes. These are essentially the injective diagram model structures (somewhat confusingly, the diagrams take values in the projective model structure on chain complexes). The functors $D_{\text{Mod}}^* \dashv D$ then form a Quillen adjunction, so to calculate $\mathbf{R}\underline{\text{Hom}}_{DA}$ it suffices to take a fibrant replacement for P .

Write S for the chain cochain complex $S_{\#}^{\#} = \bigoplus_{i \geq 0} (\mathbb{Q}_{[-i]}^{[-i]} \oplus \mathbb{Q}_{[-i]}^{[1-i]})$, with all possible differentials in S being the identity. Thus S looks like a staircase in the second quadrant, with all columns acyclic. A fibrant replacement of P is given by $\mathcal{H}om_{\mathbb{Q}}(S, P)$, so

$$\begin{aligned} \mathbf{R}\underline{\text{Hom}}_{DA}(M, DP) &\simeq \underline{\text{Hom}}_A(M, D\mathcal{H}om_{\mathbb{Q}}(S, P)) \\ &\cong Z^0 \mathcal{H}om_A(D_{\text{Mod}}^* M, D\mathcal{H}om_{\mathbb{Q}}(S, P)) \\ &\cong Z^0 \mathcal{H}om_{\mathbb{Q}}(S, \mathcal{H}om_A(D_{\text{Mod}}^* M, P)) \\ &\cong \text{Tot}^{\Pi} \sigma^{\geq 0} \mathcal{H}om_A(D_{\text{Mod}}^* M, P). \end{aligned}$$

\square

Definition 3.11. Given a stacky CDGA A , say that an A -module M in chain cochain complexes is homotopy-Cartesian if the maps

$$A^i \otimes_{A^0}^{\mathbf{L}} M^0 \rightarrow M^i$$

are quasi-isomorphisms for all i .

3.2. Comparing DG Artin hypergroupoids and stacky CDGAs. Given a DG Artin N -hypergroupoid X , and an $O(X)$ -module M , we may pull M back along the unit $\eta: O(X) \rightarrow DD^*O(X)$ of the adjunction $D^* \dashv D$, and then apply Lemma 3.9 to obtain a $D^*O(X)$ -module $D_{\text{Mod}}^*\eta^*M$.

Definition 3.12. As for instance in [Pri3, Definition 2.15], define almost cosimplicial diagrams to be functors on the subcategory Δ_* of the ordinal number category Δ containing only those morphisms f with $f(0) = 0$; define almost simplicial diagrams dually. Thus an almost simplicial diagram X_* in \mathcal{C} consists of objects $X_n \in \mathcal{C}$, with all of the operations ∂_i, σ_i of a simplicial diagram except ∂_0 , satisfying the usual relations.

Given a simplicial (resp. cosimplicial) diagram X , we write $X_{\#}$ (resp. $X^{\#}$) for the underlying almost simplicial diagram (resp. almost cosimplicial) diagram.

The denormalisation functor D descends to a functor from graded-commutative algebras to almost cosimplicial algebras, with D^* thus descending to a functor in the opposite direction. In other words, $(D^*B)^{\#}$ does not depend on ∂_B^0 , and ∂_{DA}^0 is the only part of the structure on DA to depend on ∂_A . The same is true for D_{Mod}^* applied to cosimplicial DA -modules.

From this, it can be seen that for any DG Artin N -hypergroupoid X , the graded-commutative algebra $H_0D^*O(X)^{\#}$ is freely generated over $H_0O(X)^0 = H_0D^*O(X)^0$ by a graded projective module, and that

$$H_i^*O(X)^{\#} \cong H_0D^*O(X)^{\#} \otimes_{H_0O(X)^0} H_iO(X)^0.$$

If M is a homotopy-Cartesian $O(X)$ -module, the map

$$(\mathbf{L}\eta^*M)^0 \otimes_{D^*O(X)^0}^{\mathbf{L}} D^{\#}D^*O(X) \rightarrow (\mathbf{L}\eta^*M)^{\#}$$

of almost cosimplicial chain complexes is a levelwise quasi-isomorphism. Applying $\mathbf{L}D_{\text{Mod}}^*$, it then follows from Lemma 3.9 that the map $(\mathbf{L}\eta^*M)^0 \otimes_{D^*O(X)^0}^{\mathbf{L}} D^*O(X)^{\#} \rightarrow \mathbf{L}D_{\text{Mod}}^*(\eta^*M)^{\#}$ is a levelwise quasi-isomorphism, so $\mathbf{L}D_{\text{Mod}}^*(\eta^*M)^{\#}$ is also homotopy-Cartesian.

3.2.1. Equivalences of hypersheaves. Given a simplicial presheaf F on $DGAff(R)$, there is an induced simplicial presheaf FD on $DGdgAlg(R)$ given by

$$(FD)(A) := \text{holim}_{i \in \Delta} F(D^i A).$$

In the case of a simplicial derived affine X , regarded as a functor from $dgAlg(R)$ to simplicial sets, XD is represented by the cosimplicial chain CDGA $D^*O(X^{\Delta^\bullet})$, given in cosimplicial level m by $D^*O(X^{\Delta^m})$. Applying D then gives a bisimplicial derived affine $\text{Spec} DD^*O(X^{\Delta^\bullet})$, and it is natural to consider the diagonal simplicial object $\text{diag} \text{Spec} DD^*O(X^{\Delta^\bullet})$.

Proposition 3.13. *For any Reedy fibrant simplicial derived affine X , the morphism*

$$X \rightarrow \text{diag} \text{Spec} DD^*O(X^{\Delta^\bullet})$$

*of simplicial derived affines, coming from the isomorphisms $D^*O(X^{\Delta^m})^0 = O(X_m)$, induces a weak equivalence on the associated simplicial presheaves on $DGAff(R)$.*

Proof. We can rewrite the morphism above as

$$\text{holim}_{i \in \Delta} X_i \rightarrow \text{holim}_{i, j \in \Delta \times \Delta} \text{Spec} D^j D^*O(X^{\Delta^i}).$$

It will therefore suffice to show that for all j , the maps $\eta^j: X \rightarrow \text{Spec } D^j D^* O(X^{\Delta^\bullet})$ give weak equivalences of the associated simplicial presheaves. For this it is enough to show that each η^j is a simplicial deformation retract.

Now, the bisimplicial derived affine $j \mapsto X^{\Delta^j}$ admits a map from the bisimplicial object $j \mapsto X$, which we denote by cX . Forgetting the coface maps $\partial^0: \Delta^j \rightarrow \Delta^{j+1}$ in Δ^\bullet gives us an almost cosimplicial simplicial set $\Delta^\#$, and hence an almost simplicial simplicial derived affine $X^{\Delta^\#}$. Inclusion of the 0th vertex makes Δ^0 a deformation retract of Δ^j , with obvious contracting homotopy $\Delta^1 \times \Delta^j \rightarrow \Delta^j$. These homotopies combine to give a homotopy $\Delta^1 \times \Delta^\# \rightarrow \Delta^\#$ of almost cosimplicial simplicial sets.

This homotopy makes $O(cX_\#)$ a cosimplicial deformation retract of $O(X^{\Delta^\#})$ (as cosimplicial almost cosimplicial rings). Since DD^* descends to a functor on almost cosimplicial algebras, applying DD^* then makes $O(cX_\#)$ a cosimplicial deformation retract of $DD^*O(X^{\Delta^\bullet})^\#$, so in particular the maps $\eta^j: X \rightarrow \text{Spec } D^j D^* O(X^{\Delta^\bullet})$ are all simplicial deformation retracts. \square

If X is a DG Artin N -hypergroupoid, with associated N -stack \mathfrak{X} , this means that we can recover \mathfrak{X} from the cosimplicial stacky CDGA $D^*O(X^{\Delta^\bullet})$, just as well as from the cosimplicial CDGA $O(X)$.

Proposition 3.13 has the following immediate consequence:

Corollary 3.14. *For any simplicial presheaf F on $DGAff(R)$ and any Reedy fibrant simplicial derived affine X , there is a canonical weak equivalence*

$$\text{holim}_{\overleftarrow{j \in \Delta}} \text{map}(\text{Spec } DD^*O(X^{\Delta^j}), F) \rightarrow \text{map}(X, F).$$

3.2.2. Tangent and cotangent complexes. Now consider a DG Artin N -hypergroupoid X over R . Combining the adjunction $D^* \dashv D$ with Lemma 3.9 and the universal property of Kähler differentials, observe that there is an isomorphism

$$D_{\text{Mod}}^*(\Omega_{O(X)/R}^1 \otimes_{O(X)} DD^*O(X)) \cong \Omega_{D^*O(X)/R}^1.$$

Since $O(X)$ is Reedy cofibrant, Lemma 3.5 ensures that $D^*O(X)$ is cofibrant in the model structure of Lemma 3.4. If we write cone_h and cone_v for cones in the chain and cochain directions respectively, then for any $D^*O(X)$ -module M , the map $\text{cone}_h \text{cone}_v(M)_{[i]}^{[m-1]} \rightarrow \text{cone}_h(M)_{[i]}^{[m]}$ is levelwise acyclic, so cofibrancy of $D^*O(X)$ implies lifting of the respective spaces of derivations, or equivalently surjectivity of

$$\mathcal{H}om_{D^*O(X)}(\Omega_{D^*O(X)}^1, M)_i^{m-1} \rightarrow \mathbb{Z}^m \mathcal{H}om_{D^*O(X)}(\Omega_{D^*O(X)}^1, M)_i.$$

In other words, the columns of $\mathcal{H}om_{D^*O(X)}(\Omega_{D^*O(X)}^1, M)$ are acyclic.

Because X is an Artin N -hypergroupoid, there are in fact other restrictions on $\Omega_{D^*O(X)}^1$. The results below (or an argument adapted from [Pri3, Lemma 2.16]) show that for $M \in DG^+ dg\text{Mod}(D^*O(X))$, the rows $\mathcal{H}om_{D^*O(X)}(\Omega_{D^*O(X)}^1, M)^i$ are acyclic for $i < -N$. Moreover, the argument of [Pri1, Corollary 1.46] shows that when M is concentrated in degree $(0, 0)$, we have $H_j \mathcal{H}om_{D^*O(X)}(\Omega_{D^*O(X)}^1, M)^i = 0$ for $i, j \neq 0$, although we will not need this.

Lemma 3.15. *If, for $A \in DG^+ dg\text{CAlg}(R)$, an A -module $M \in DG^+ dg\text{Mod}(R)$ admits an A -linear contracting homotopy, then the map*

$$\text{holim}_{\overleftarrow{i \in \Delta}} \text{Spec } D^i A \rightarrow \text{holim}_{\overleftarrow{i \in \Delta}} \text{Spec } D^i (A \oplus M)$$

of simplicial presheaves on $DG^+ \text{Aff}(R)$ is a weak equivalence.

Proof. A contracting homotopy is the same as a section of the canonical map $\text{cocone}_v(M) \rightarrow M$ of A -modules, where the vertical cocone $\text{cocone}_v(M)$ is given by $M \otimes \text{cocone}_v(\mathbb{Q})$ for $\text{cocone}_v(\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}^{[-1]}$ with ∂ the identity.

On applying D , we then have a section s of $D\text{cocone}_v(M) \rightarrow DM$ as DA -modules in cosimplicial chain complexes. The Alexander–Whitney cup product gives a canonical morphism $D\text{cocone}_v(M) \rightarrow (DM) \otimes D\text{cocone}_v(\mathbb{Q})$. Since $\mathbb{Q}^{\Delta^1} = \mathbb{Q} \oplus D\text{cocone}_v(\mathbb{Q})$, our section s then gives a contracting homotopy $DM \rightarrow (DM)^{\Delta^1}$, so evaluation at $0 \in \Delta^1$ is the zero map, and evaluation at $1 \in \Delta^1$ the identity.

We therefore have a homotopy $\Delta^1 \times \text{Spec } D(A \oplus M) \rightarrow \text{Spec } D(A \oplus M)$ of simplicial derived affine schemes (and hence of simplicial presheaves) realising $\text{Spec } DA$ as a deformation retract of $\text{Spec } D(A \oplus M)$. \square

From now on, we will simply write $(\text{Spec } DA)^\sharp := \text{holim}_{i \in \Delta} (\text{Spec } D^i A)^\sharp$ for the étale hypersheafification of the simplicial derived affine $\text{Spec } DA$.

Corollary 3.16. *For $A \in DG^+ dg\text{CAlg}(R)$, $M \in DG^+ dg\text{Mod}(A)$ and $m > 0$, there is a weak equivalence*

$$(\text{Spec } D(A \oplus M^{[-m]}))^\sharp \simeq (\text{Spec } D(A \oplus M) \times S^m)^\sharp \cup_{\text{Spec } D(A)^\sharp \times S^m}^{\mathbf{L}} (\text{Spec } DA)^\sharp$$

of étale hypersheaves on $DGAff(R)$, where $S^m \simeq \Delta^m / \partial \Delta^m$ is the m -sphere.

Proof. We have an exact sequence $0 \rightarrow M[-1] \rightarrow \text{cocone}_v(M) \rightarrow M \rightarrow 0$. Since $\text{cocone}_v(M)$ has a contracting homotopy, Lemma 3.15 shows that $(\text{Spec } D(A \oplus \text{cocone}_v(M)))^\sharp \simeq (\text{Spec } DA)^\sharp$. Thus $(\text{Spec } D(A \oplus M[-1]))^\sharp$ is the homotopy pushout of the nilpotent maps

$$(\text{Spec } DA)^\sharp \leftarrow (\text{Spec } D(A \oplus M))^\sharp \rightarrow (\text{Spec } DA)^\sharp,$$

so is the suspension of $(\text{Spec } D(A \oplus M))^\sharp$ over $(\text{Spec } DA)^\sharp$. Replacing M with $M[-j]$ for $0 \leq j \leq m$ then gives the desired result by induction. \square

Definition 3.17. For any stacky CDGA A over R , the module $\Omega_{A/R}^1$ of Kähler differentials is an A -module in chain cochain complexes, and we define $\Omega_{A/R}^p := \Lambda_A^p \Omega_{A/R}^1$, denoting its differentials (inherited from A) by (∂, δ) . There is also a de Rham differential $d: \Omega_{A/R}^p \rightarrow \Omega_{A/R}^{p+1}$.

Proposition 3.18. *For a DG Artin N -hypergroupoid X over R , and any $M \in DG^+ dg\text{Mod}(D^*O(X))$, the cotangent complex $\mathbb{L}^{X/R}$ satisfies*

$$\begin{aligned} \mathbf{R}\underline{\text{Hom}}_{O(X)}(\mathbb{L}^{X/R}, DM) &\simeq \text{Tot}^{\Pi_{\sigma \geq -N}} \mathcal{H}om_{D^*O(X)}(\Omega_{D^*O(X)/R}^1, M) \\ &\simeq \widehat{\text{Hom}}_{D^*O(X)}(\Omega_{D^*O(X)/R}^1, M). \end{aligned}$$

The same is true for any homotopy-Cartesian module $M \in DGdg\text{Mod}(D^*O(X))$.

Proof. If $\mathfrak{X} = X^\sharp$ is the derived N -stack associated to X , then for $f: X \rightarrow \mathfrak{X}$, Lemma 3.16 gives an equivalence

$$\mathbf{R}\underline{\text{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(\mathbf{L}c^* \mathbb{L}^{\mathfrak{X}^{S^N}}, \mathbf{R}f_* DM) \simeq \mathbf{R}\underline{\text{Hom}}_{O(X)}(\Omega_{X/R}^1, D(M^{[-N]})),$$

for the map $c: \mathfrak{X} \rightarrow \mathfrak{X}^{S^N}$. Since \mathfrak{X} is an N -stack, there is a canonical equivalence $\mathbb{L}^{\mathfrak{X}} \simeq \mathbb{L}_{[N]}^{\mathfrak{X}^{S^N}}$ (cf. [Pri3, Corollary 7.19]), so Lemma 3.10 gives

$$\mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{O(X)}(\mathbb{L}^{X/R}, DM) \simeq \mathrm{Tot}^{\Pi} \sigma^{\geq -N} \mathcal{H}om_{D^*O(X)}(\Omega_{D^*O(X)/R}^1, M).$$

Since X is *a fortiori* an $(N+r)$ -hypergroupoid for all $r \geq 0$, we may replace $\sigma^{\geq -N}$ with $\sigma^{\geq -N-r}$. Taking the filtered colimit over all r then replaces $\mathrm{Tot}^{\Pi} \sigma^{\geq -N} \mathcal{H}om$ with $\hat{\mathrm{Tot}} \mathcal{H}om = \underline{\mathbf{H}}\mathbf{om}$, yielding the second quasi-isomorphism. Finally, observe that for M homotopy-Cartesian, we must have M^i acyclic for $i < 0$, because $A^i = 0$. Thus we can replace M with $\sigma^{\geq 0}M$, and the results apply. \square

Proposition 3.19. *For a DG Artin N -hypergroupoid X over R , any $M \in DG^+ dg\mathrm{Mod}(D^*O(X))$, and any $r \geq 0$, we have*

$$\mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{O(X)}((\mathbb{L}^{X/R})^{\otimes r}, DM) \simeq \underline{\mathbf{H}}\mathbf{om}_{D^*O(X)}((\Omega_{D^*O(X)/R}^1)^{\otimes r}, M).$$

Proof. There is a natural map $\mathbb{L}^{X/R} \rightarrow \Omega_{X/R}^1$ coming from Definition 2.15. Corollary 3.10 thus ensures that we have a natural map from right to left.

First consider the case where $M = M^i[-i]$ is an $O(X_0)$ -module, since $O(X)^0 = D^*O(X)^0$. Writing L for the pullback of $\mathbb{L}^{X/R}$ to X_0 , and $\bar{\Omega}$ for the pullback of $\Omega_{D^*O(X)/R}^1$ to $O(X_0)$, we therefore wish to show that the map

$$\hat{\mathrm{Tot}} \mathcal{H}om_{O(X_0)}(\bar{\Omega}^{\otimes r}, M) \rightarrow \mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{O(X_0)}(L^{\otimes r}, M)$$

is a quasi-isomorphism.

Proposition 3.18 implies that $\bar{\Omega}$ is levelwise quasi-isomorphic to the brutal cotruncation $\sigma^{\leq N} \bar{\Omega} := \bar{\Omega} / \sigma^{\geq N+1}$, so we may replace $\bar{\Omega}^{\otimes r}$ with a bicomplex in cochain degrees $[0, Nr]$. The resulting $\mathcal{H}om$ bicomplex is bounded below, so $\hat{\mathrm{Tot}}$ is equivalent to Tot^{Π} , and we need only show a quasi-isomorphism

$$\mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{O(X_0)}((\mathrm{Tot} \sigma^{\leq N} \bar{\Omega})^{\otimes r}, M) \simeq \mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{O(X_0)}(L^{\otimes r}, M),$$

which follows because $L \simeq \mathrm{Tot} \sigma^{\leq N} \bar{\Omega}$ by Proposition 3.18. (We could also take the limit over all N to give $L \simeq \mathrm{Tot}^{\Pi} \bar{\Omega}$.)

For general M , we have

$$\mathcal{H}om_{D^*O(X)}((\Omega_{D^*O(X)/R}^1)^{\otimes r}, M) = \varprojlim_i \mathcal{H}om_{D^*O(X)}((\Omega_{D^*O(X)/R}^1)^{\otimes r}, \sigma^{\leq i} M),$$

and the spectral sequence for the cochain brutal truncation filtration for M combines with boundedness of $\bar{\Omega}^{\otimes r}$ to show that the right-hand complex is bounded below in cochain degrees, allowing us to replace $\underline{\mathbf{H}}\mathbf{om}$ with $\mathrm{Tot}^{\Pi} \sigma^{\geq -Nr} \mathcal{H}om$. The spectral sequence therefore satisfies complete convergence, and the general quasi-isomorphism follows from the quasi-isomorphism above for the graded pieces $M^i[-i]$. \square

3.3. Poisson and symplectic structures. To allow for some flexibility in computations and resolutions, we will not just consider stacky CDGAs of the form $D^*O(X)$ for DG Artin N -hypergroupoids X , but will fix $A \in DG^+ dg\mathrm{CAlg}(R)$ satisfying:

- (1) for any cofibrant replacement $\tilde{A} \rightarrow A$ in the model structure of Lemma 3.4, the morphism $\Omega_{\tilde{A}/R}^1 \rightarrow \Omega_{A/R}^1$ is a levelwise quasi-isomorphism,

- (2) the $A^\#$ -module $(\Omega_{A/R}^1)^\#$ in graded chain complexes is cofibrant (i.e. it has the left lifting property with respect to all surjections of $A^\#$ -modules in graded chain complexes),
- (3) there exists N for which the chain complexes $(\Omega_{A/R}^1 \otimes_A A^0)^i$ are acyclic for all $i > N$.

Observe that these conditions are satisfied by $D^*O(X)$, by the results above, but they are also satisfied by more general stacky CDGAs. The first two conditions do not require A to be cofibrant in the model structure of Lemma 3.4; it is enough for $A^\#$ to be cofibrant as a graded chain CDGA over R , or for $A \in DG^+dg_+ \text{CAlg}(R)$ with $A^\#$ ind-smooth as a bigraded commutative algebra over $R_\#$.

3.3.1. *Polyvectors.* All the definitions and properties of §1 now carry over. In particular:

Definition 3.20. Given a stacky CDGA A over R as above, define the complex of n -shifted polyvector fields on A by

$$\widehat{\text{Pol}}(A, n) := \prod_{j \geq 0} \underline{\text{Hom}}_A(\text{CoS}_A^j((\Omega_{A/R}^1)_{[-n-1]}), A).$$

This has a filtration by complexes

$$F^p \widehat{\text{Pol}}(A, n) := \prod_{j \geq p} \underline{\text{Hom}}_A(\text{CoS}_A^j((\Omega_{A/R}^1)_{[-n-1]}), A),$$

with $[F^i, F^j] \subset F^{i+j-1}$ and $F^i F^j \subset F^{i+j}$, where the commutative product and Schouten–Nijenhuis bracket are defined as before.

We now define the space $\mathcal{P}(A, n)$ of Poisson structures and its tangent space $T\mathcal{P}(A, n)$ by the formulae of Definitions 1.5, 1.8. As in Definition 1.11, there is a canonical tangent vector $\sigma: \mathcal{P}(A, n) \rightarrow T\mathcal{P}(A, n)$.

As well as composition for internal $\mathcal{H}om$'s, we have maps $\mathcal{H}om(M_1, P_1) \otimes \mathcal{H}om(M_2, P_2) \rightarrow \mathcal{H}om(M_1 \otimes M_2, P_1 \otimes P_2)$, and hence

$$\begin{aligned} \mathcal{H}om_A(\Omega_A^1, \mathcal{H}om_A(M, A))^{\otimes p} &\rightarrow \mathcal{H}om_A((\Omega_A^1)^{\otimes p}, \mathcal{H}om_A(M, A)^{\otimes p}) \\ &\rightarrow \mathcal{H}om_A((\Omega_A^1)^{\otimes p}, \mathcal{H}om_A((M^{\otimes p}, A))). \end{aligned}$$

Substituting $M = \bigoplus_{j=2}^{r-1} \text{CoS}_A^j((\Omega_{A/R}^1)_{[-n-1]})$, taking shifts and S_p -coinvariants, and applying $\widehat{\text{Tot}}$, we see that an element

$$\pi \in F^2 \widehat{\text{Pol}}(A, n)^{n+2} / F^r$$

thus defines a contraction morphism

$$\mu(-, \pi): \text{Tot}^\Pi \Omega_A^p \rightarrow F^p \widehat{\text{Pol}}(A, n)^{n+2} / F^{p(r-1)}$$

of bigraded vector spaces, noting that $\text{Tot}^\Pi \Omega_A^p = \widehat{\text{Tot}} \Omega_A^p$. When $r = 3$ and $p = 1$, we simply denote this morphism by $\pi_2^\# : \text{Tot}^\Pi \Omega_A^1 \rightarrow \underline{\text{Hom}}_A(\Omega_A^1, A)[-n]$.

Definition 3.21. We say that a Poisson structure $\pi \in \mathcal{P}(A, n)/F^p$ is non-degenerate if $\text{Tot}^\Pi(\Omega_{A/R}^1 \otimes_A A^0)$ is a perfect complex over A^0 and the map

$$\pi_2^\#: \text{Tot}^\Pi(\Omega_{A/R}^1 \otimes_A A^0) \rightarrow \underline{\text{Hom}}_A(\Omega_A^1, A^0)[-n]$$

is a quasi-isomorphism.

Lemma 3.22. *If $\pi_2 \in \mathcal{P}(A, n)/F^3$ is non-degenerate, then the maps*

$$\mu(-, \pi_2): \text{Tot}^\Pi(\Omega_A^p \otimes_A M) \rightarrow \underline{\widehat{\text{Hom}}}_A(\text{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), M)$$

are quasi-isomorphisms for all $M \in DG^+ dg\text{Mod}(A)$.

Proof. We may proceed as in Proposition 3.19. By hypothesis, $(\Omega_{A/R}^1 \otimes_A A^0)^i$ is acyclic for all $i > N$, so $\mathcal{H}om_A(\text{CoS}_A^p((\Omega_{A/R}^1)_{[-n-1]}), A)^i$ is acyclic for all $i < -Np$, meaning that we may replace $\underline{\widehat{\text{Hom}}}$ with $\text{Tot}^\Pi \mathcal{H}om$. Since $M = \varprojlim_i (M/\sigma^{\geq i} M)$, a spectral sequence argument allows us to reduce to the case where $M \in dg\text{Mod}(A^0)$.

We may also replace $(\Omega_{A/R}^1 \otimes_A A^0)$ with its brutal truncation $\sigma^{\leq N}(\Omega_{A/R}^1 \otimes_A A^0)$, so all complexes are bounded in the cochain direction, and then observe that non-degeneracy of π gives us quasi-isomorphisms

$$(\text{Tot } \sigma^{\leq N}(\Omega_{A/R}^1 \otimes_A A^0))^{\otimes_{(A^0)}^p} \otimes_{A^0} M \rightarrow \underline{\text{Hom}}_{A^0}(\sigma^{\leq N}(\Omega_{A/R}^1 \otimes_A A^0)^{\otimes_{(A^0)}^p}, M)[-np];$$

the required result follows on taking S_p -coinvariants. \square

3.3.2. The de Rham complex.

Definition 3.23. Define the de Rham complex $\text{DR}(A/R)$ to be the product total complex of the double cochain complex

$$\text{Tot}^\Pi A \xrightarrow{d} \text{Tot}^\Pi \Omega_{A/R}^1 \xrightarrow{d} \text{Tot}^\Pi \Omega_{A/R}^2 \xrightarrow{d} \dots,$$

so the total differential is $d \pm \delta \pm \partial$.

We define the Hodge filtration F on $\text{DR}(A/R)$ by setting $F^p \text{DR}(A/R) \subset \text{DR}(A/R)$ to consist of terms $\text{Tot}^\Pi \Omega_{A/R}^i$ with $i \geq p$.

The definitions of shifted symplectic structures from §1.2 now carry over:

Definition 3.24. Define the space $\text{PreSp}(A, n)$ of n -shifted pre-symplectic structures on A/R by writing $\text{PreSp}(A, n)/F^{i+2} := \underline{\text{MC}}(F^2 \text{DR}(A/R)[n+1]/F^{i+2})$, and setting $\text{PreSp}(A, n) := \varprojlim_i \text{PreSp}(A, n)/F^{i+2}$.

Say that an n -shifted pre-symplectic structure ω is symplectic if $\text{Tot}^\Pi(\Omega_{A/R}^1 \otimes_A A^0)$ is a perfect complex over A^0 and the map

$$\omega_2^\sharp: \underline{\widehat{\text{Hom}}}_A(\Omega_A^1, A^0)[-n] \rightarrow \text{Tot}^\Pi(\Omega_{A/R}^1 \otimes_A A^0)$$

is a quasi-isomorphism. Let $\text{Sp}(A, n) \subset \text{PreSp}(A, n)$ consist of the symplectic structures — this is a union of path-components.

Lemma 3.25. *For a DG Artin N -hypergroupoid X , the space of closed p -forms of degree n from [PTVV] is given by*

$$\mathcal{A}_R^{p, cl}(X^\sharp, n) \simeq \text{holim}_{j \in \Delta} \underline{\text{MC}}(F^p \text{DR}(D^* O(X^{\Delta^j})/R)^{[n+p-1]}).$$

Proof. This follows by combining Corollary 3.14 and Proposition 3.19. \square

3.3.3. Compatible pairs.

The proof of Lemma 1.18 adapts to give maps

$$(\text{pr}_2 + \mu\epsilon): \text{PreSp}(A, n)/F^p \times \mathcal{P}(A, n)/F^p \rightarrow T\mathcal{P}(A, n)/F^p$$

over $\mathcal{P}(A, n)/F^p$ for all p , compatible with each other.

We may now define the space $\text{Comp}(A, n)$ of compatible pairs as in Definition 1.24, with the results of §1 all adapting to show that the maps $\text{Sp}(A, n) \leftarrow \text{Comp}(A, n)^{\text{nondeg}} \rightarrow \mathcal{P}(A, n)^{\text{nondeg}}$ are weak equivalences.

3.4. Diagrams and functoriality. We now extend the constructions of §2.1 to stacky CDGAs.

3.4.1. *Definitions.* Given a small category I , an I -diagram A of stacky CDGAs over R , and A -modules M, N in I -diagrams of chain cochain complexes, we can define the cochain complex $\underline{\widehat{\text{Hom}}}_A(M, N)$ to be the equaliser of the obvious diagram

$$\prod_{i \in I} \widehat{\text{Hom}}_{A(i)}(M(i), N(i)) \implies \prod_{f: i \rightarrow j \text{ in } I} \widehat{\text{Hom}}_{A(i)}(M(i), f_*N(j)).$$

All the constructions of §3.3 then adapt immediately; in particular, we can define

$$\widehat{\text{Pol}}(A, n) := \prod_{j \geq 0} \widehat{\text{Hom}}_A(\text{CoS}_A^j((\Omega_{A/R}^1)_{[-n-1]}), A),$$

leading to a space $\mathcal{P}(A, n)$ of Poisson structures.

We can now adapt the formulae of §2.1 to this setting, defining pre-symplectic structures by

$$\text{PreSp}(A, n) := \text{PreSp}(A(0), n) = \varprojlim_{i \in [m]} \text{PreSp}(A(i), n)$$

for any $[m]$ -diagram A of stacky CDGAs over R , and setting $\text{Comp}(A, n)$ be the homotopy vanishing locus of the obvious maps

$$(\mu - \sigma): \text{PreSp}(A, -1) \times \mathcal{P}(A, n) \rightarrow T\mathcal{P}(A, n).$$

over $\mathcal{P}(A, n)$.

The obstruction functors and their towers from §1.4 also adapt immediately, giving the obvious analogues of the obstruction spaces defined in terms of

$$\widehat{\text{Hom}}_A(\text{CoS}_A^p(\Omega_{A/R}^1[n+1]), A), \quad \text{Tot}^\Pi \Omega_{A(0)}^p.$$

3.4.2. *Functors and descent.* For $[m]$ -diagrams in $DG^+ dg\text{CAlg}(R)$, we will consider the injective model structure, so an $[m]$ -diagram A is cofibrant if each $A(i)$ is cofibrant for the model structure of Lemma 3.4, and is fibrant if the maps $A(i) \rightarrow A(i+1)$ are all surjective.

Lemma 3.26. *If $D = (A \rightarrow B)$ is a fibrant cofibrant $[1]$ -diagram in $DG^+ dg\text{CAlg}(R)$ which is formally étale in the sense that the map*

$$\{\text{Tot } \sigma^{\leq q}(\Omega_A^1 \otimes_A B^0)\}_q \rightarrow \{\text{Tot } \sigma^{\leq q}(\Omega_B^1 \otimes_B B^0)\}_q$$

is a pro-quasi-isomorphism, then the map

$$\widehat{\text{Hom}}_D(\text{CoS}_D^k \Omega_D^1, D) \rightarrow \widehat{\text{Hom}}_A(\text{CoS}_A^k \Omega_A^1, A),$$

is a quasi-isomorphism for all k .

Proof. This follows by reasoning as in the proof of Proposition 3.19. \square

Definition 3.27. Write $DG^+ dg\text{CAlg}(R)_{c, \rightarrow} \subset DG^+ dg\text{CAlg}(R)$ for the subcategory with all cofibrant stacky CDGAs (in the model structure of Lemma 3.4) over R as objects, and only surjections as morphisms.

For the notion of being formally étale from Lemma 3.26, we may extend the conditions of Properties 2.5 to constructions on $DG^+ dg\text{CAlg}(R)_{c, \rightarrow}$, with quasi-isomorphisms taken levelwise. The constructions $\mathcal{P}(-, n)$, $\text{Comp}(-, n)$, $\text{PreSp}(-, n)$, and their associated filtered and graded functors, all satisfy these properties. The first two properties

follow from the right lifting property for fibrations (in the injective model structure on diagrams), and the third from Lemma 3.26.

Thus the simplicial classes $\coprod_{A \in B_m DG^+ dg\mathcal{CAlg}(R)_{c, \rightarrow}} F(A)$ fit together to give a complete Segal space $\int F$ over the nerve $B DG^+ dg\mathcal{CAlg}(R)_{c, \rightarrow}$.

Definition 2.6 then adapts to give us an ∞ -category $\mathbf{L} \int F$, and Definition 2.7 adapts to give an ∞ -functor

$$\mathbf{R}F: \mathbf{LDG}^+ dg\mathcal{CAlg}(R)^{\acute{e}t} \rightarrow \mathbf{LsSet}$$

with $(\mathbf{R}F)(A) \simeq F(A)$ for all cofibrant stacky CDGAs A over R , where $DG^+ dg\mathcal{CAlg}(R)^{\acute{e}t} \subset DG^+ dg\mathcal{CAlg}(R)$ is the 2-sub- ∞ -category of homotopy formally étale morphisms.

An immediate consequence of §3.3.3 is that the canonical maps

$$\begin{aligned} \mathbf{RComp}(-, n)^{\text{nondeg}} &\rightarrow \mathbf{RSp}(-, n) \\ \mathbf{RComp}(-, n)^{\text{nondeg}} &\rightarrow \mathbf{RP}(-, n)^{\text{nondeg}} \end{aligned}$$

are weak equivalences of ∞ -functors on the full subcategory of $\mathbf{LDG}^+ dg\mathcal{CAlg}(R)^{\acute{e}t}$ consisting of objects satisfying the conditions of §3.3.

Corollary 3.14 and Proposition 3.19 ensure that if a morphism $X \rightarrow Y$ of DG Artin N -hypergroupoids becomes an equivalence on hypersheafifying, then $D^*O(Y) \rightarrow D^*O(X)$ is formally étale in the sense of Lemma 3.26. In particular this means that the maps $\partial^i: D^*O(X^{\Delta^j}) \rightarrow D^*O(X^{\Delta^{j+1}})$ and $\sigma^i: D^*O(X^{\Delta^{j+1}}) \rightarrow D^*O(X^{\Delta^j})$ are formally étale. Thus $D^*O(X^{\Delta^\bullet})$ can be thought of as a DM hypergroupoid in stacky CDGAs, and we may make the following definition:

Definition 3.28. Given a DG Artin N -hypergroupoid X over R and any of the functors F above, write

$$F(X) := \operatorname{holim}_{j \in \Delta} \mathbf{R}F(D^*O(X^{\Delta^j})).$$

Proposition 3.29. *If $Y \rightarrow X$ is a trivial DG Artin hypergroupoid, then the morphism*

$$F(X) \rightarrow F(Y)$$

is an equivalence for any of the constructions $F = \mathcal{P}, \text{Comp}, \text{PreSp}$.

Proof. The proof of Proposition 2.17 adapts, replacing Propositions 1.32 and 1.35 with Corollary 3.14 and Proposition 3.19. \square

Thus the following is well-defined:

Definition 3.30. Given a strongly quasi-compact DG Artin N -stack \mathfrak{X} , define the spaces $\mathcal{P}(\mathfrak{X}, n)$, $\text{Comp}(\mathfrak{X}, n)$, $\text{Sp}(\mathfrak{X}, n)$ to be the spaces $\mathcal{P}(X, n)$, $\text{Comp}(X, n)$, $\text{Sp}(X, n)$ for any DG Artin N -hypergroupoid X with $X^\sharp \simeq \mathfrak{X}$.

Examples 3.31. If $R = H_0 R$, with Y/R a smooth affine scheme equipped with an action of a Lie algebra \mathfrak{g} , we start by considering 2-shifted Poisson structures on the Chevalley–Eilenberg complex $O([Y/\mathfrak{g}])$ of Example 3.6 (a cochain CDGA). In this case, the DGLA $F^i \widehat{\text{Pol}}([Y/\mathfrak{g}], 2)[3]$ is concentrated in cochain degrees $[2i-3, \infty)$. In particular, this means that $\mathcal{P}([Y/\mathfrak{g}], 2) \cong Z^4(\text{gr}_F^2 \widehat{\text{Pol}}([Y/\mathfrak{g}], 2))$ is a discrete space (so all homotopy groups are trivial). Explicitly,

$$\mathcal{P}([Y/\mathfrak{g}], 2) \cong \{\pi \in (S^2 \mathfrak{g} \otimes O(Y))^\mathfrak{g} : [\pi, a] = 0 \in \mathfrak{g} \otimes O(Y) \forall a \in O(Y)\}.$$

Specialising to the case $Y = \text{Spec } R$, we have $\mathcal{P}(B\mathfrak{g}, 2) \cong (S^2\mathfrak{g})^{\mathfrak{g}}$, the set of quadratic Casimir elements.

If \mathfrak{g} is the Lie algebra of a linear algebraic group G as in Example 3.6, then for $X := B[* / G]$, the spaces $\mathcal{P}(D^*O(X^{\Delta^j}), 2)$ are all discrete sets, so the space $\mathcal{P}(BG, 2)$ is just the equaliser of the maps $\mathcal{P}(B\mathfrak{g}, 2) \rightrightarrows \mathcal{P}([G/\mathfrak{g}^{\oplus 2}], 2)$ coming from the vertex maps of the cosimplicial CDGA $D^*O(X^{\Delta^\bullet})$ given by

$$O(B\mathfrak{g}) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} O([G/\mathfrak{g}^{\oplus 2}]) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} O([G^2/\mathfrak{g}^{\oplus 3}]) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} O([G^3/\mathfrak{g}^{\oplus 4}]) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \cdots$$

These étale vertex maps are induced by applying S^2 to the maps $\mathfrak{g} \rightrightarrows \mathfrak{g}^{\oplus 2} \otimes O(G)$ which when evaluated on $g \in G$ send $v \in \mathfrak{g}$ to (v, gvg^{-1}) and (gvg^{-1}, v) respectively. The equaliser $\mathcal{P}(BG, 2) \subset (S^2\mathfrak{g})^{\mathfrak{g}}$ is thus

$$\mathcal{P}(BG, 2) \cong (S^2\mathfrak{g})^G,$$

with each element corresponding to a cosimplicial P_3 -algebra structure on $D^*O((B[* / G])^{\Delta^\bullet})$. Taking G reductive and restricting to path components recovers an example in [CPT⁺, §3.1].

Remark 3.32. To relate Definition 3.30 with the Poisson structures of [CPT⁺], first note that reindexation gives an equivalence of categories between double complexes and the “graded mixed complexes” of [PTVV, CPT⁺] (but beware [CPT⁺, Remark 1.1.2]: in the latter paper, “graded mixed complexes” do not have mixed differentials). Thus “graded mixed cdgas” are just stacky CDGAs, and to a derived stack \mathfrak{X} , [CPT⁺, Definition 4.2.11] associates a sheaf $\mathbb{D}_{\mathfrak{X}/\mathfrak{X}_{\text{dR}}}$ of stacky CDGAs on the de Rham stack \mathfrak{X}_{dR} , defining Poisson structures in terms of polyvectors on $\mathbb{D}_{\mathfrak{X}/\mathfrak{X}_{\text{dR}}}$. A comparison with our definition should then involve the observation that $\text{Spec } DD^*O(X^{\Delta^j})$ is a model for the relative de Rham stack $(X_j/\mathfrak{X})_{\text{dR}} = (X_j)_{\text{dR}} \times_{\mathfrak{X}_{\text{dR}}}^h \mathfrak{X}$, possibly with $D^*O(X^{\Delta^j}) \simeq \mathbb{D}_{X_j/\mathfrak{X}}$.

Combined with the results above, an immediate consequence of §3.3.3 is:

Theorem 3.33. *For any strongly quasi-compact DG Artin N -stack \mathfrak{X} over R , there are natural weak equivalences*

$$\text{Sp}(\mathfrak{X}, n) \leftarrow \text{Comp}(\mathfrak{X}, n)^{\text{nondeg}} \rightarrow \mathcal{P}(\mathfrak{X}, n)^{\text{nondeg}}.$$

Modulo the comparison suggested in Remark 3.32, an alternative proof of Theorem 3.33 is given as [CPT⁺, Theorem 3.2.4].

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