Derived deformations via bialgebras

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Derived deformation theory

Local version of derived moduli theory.

Λ — complete Noetherian local ring Λ with residue field k

$\mathcal{C}_\Lambda$ (resp. $s\mathcal{C}_\Lambda$) — Artinian (resp. Artinian simplicial) Λ-algebras, residue field k

sSet, sGpd — simplicial sets, simplicial groupoids

Say a derived deformation groupoid is a functor $F : \mathrm{Ho}(s\mathcal{C}_\Lambda) \to \mathrm{Ho}(s\mathrm{Set})$ such that $F(k) = \bullet$, $F(A \times^h_B C) \xrightarrow{\sim} F(A) \times^h_{F(B)} F(C)$ for all $A \twoheadrightarrow B$. These arise as infinitesimal neighbourhoods in geometric derived stacks (cf. Lurie's representability theorem).

These should exist for all deformation problems, recovering the classical deformation groupoid $\mathcal{G}$ by $F(A) \simeq B\mathcal{G}(A)$ for $A \in \mathcal{C}_\Lambda$.

Example. Take a k-algebra R, and let $\mathcal{G}(A)$ be the groupoid of flat A-algebras $\tilde{R}$ with $\tilde{R} \otimes_A k = R$, for $A \in \mathcal{C}_\Lambda$.

For this example, one approach to defining a derived deformation groupoid is to use the equivalence $\mathrm{Ho}(s\mathrm{Set}) \simeq \mathrm{Ho}(s\mathrm{Gpd})$, and to let $F(A)$ correspond to the simplicial groupoid of flat strong homotopy A-algebras deforming R, for $A \in s\mathcal{C}_\Lambda$.

Strong homotopy algebras were introduced by Lada to characterise deformation retracts of topological algebras, but the notion carries over to any simplicial category (in this example, the category of A-modules).

We'll now see how to extend this idea to construct derived deformation groupoids for any classical problem.

Monads and comonads

A monad $(\top, \mu, \eta)$ on a category $\mathcal{B}$ is an endofunctor $\top$, and natural transformations $\mu : \top^2 \implies \top$ (multiplication), $\eta : 1 \implies \top$ (unit) making the following commute

$$
\begin{array}{ccc}
\top^3 & \xRightarrow{\top\mu} & \top^2 \\
{\scriptstyle \mu\top}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mu} \\
\top^2 & \xRightarrow{\mu} & \top
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\top & \xRightarrow{\top\eta} & \top^2 & \xLeftarrow{\eta\top} & \top \\
& {\scriptstyle 1_\top}\searrow & \Big\Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_\top} & \\
& & \top. & &
\end{array}
$$

Examples.

- $\top = UF$, for any adjunction $\mathcal{D} \underset{F}{\overset{U}{\underset{\top}{\rightleftarrows}}} \mathcal{B}$.
- On sets X, take $\top X := k[X]$.
- On A-modules M, take $\top M := R \otimes_A M$, for some fixed A-algebra R.
- Any operad.

Comonads are the dual notion (i.e. monads on the opposite category). Thus a comonad $(\perp, \Delta, \varepsilon)$ has natural transformations $\Delta : \perp \implies \perp^2$ (comultiplication), $\varepsilon : \perp \implies 1$ (counit).

Examples.

- $\perp = VG$, for any adjunction

$$\mathcal{D} \underset{G}{\overset{V}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}.$$

- On A-modules M, take $\perp$ to be the free coalgebra functor.

- Any co-operad.

- For a morphism $f : Y \to X$ of sites, define $\perp$ on the category of sheaves on X by $\perp = f^{-1} f_*$.

$\top$-algebras

A $\top$-algebra is a morphism $\omega : \top X \to X$, with

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \top X \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle\omega} \\ & & E, \end{array} \qquad \begin{array}{ccc} \top^2 X & \xrightarrow{\top\omega} & \top X \\ \downarrow{\scriptstyle\mu_X} & & \downarrow{\scriptstyle\omega} \\ \top X & \xrightarrow{\omega} & X. \end{array}$$

The Eilenberg-Moore category $\mathcal{B}^\top$ has $\top$-algebras as objects, and a morphism

$$g : (\top X \xrightarrow{\theta} X) \to (\top Y \xrightarrow{\phi} Y)$$

in $\mathcal{B}^\top$ is a morphism $g : X \to Y$ in $\mathcal{B}$ such that the following commutes

$$\begin{array}{ccc} \top X & \xrightarrow{\top g} & \top Y \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{g} & Y. \end{array}$$

For example, if $\top X := k[X]$, then $\mathrm{Set}^\top$ is equivalent to k-algebras. If $\top M := R \otimes_A M$, then $\mathrm{Mod}_A^\top$ is equivalent to R-modules.

$\perp$-coalgebras

These are the dual notion, with the Eilenberg-Moore category $\mathcal{B}^{\perp} := ((\mathcal{B}^{\mathrm{opp}})^{\perp^{\mathrm{opp}}})^{\mathrm{opp}}$. Thus a $\perp$-coalgebra is a morphism $\omega : X \to \perp X$, satisfying similar conditions.

If $\perp$ is a co-operad, then $\mathrm{Mod}^{\perp}$ is the category of $\perp$-coalgebras.

For a morphism $f : Y \to X$ of sites, $\mathrm{Sheaves}(Y)^{f^{-1}f_*}$ is the category of sheaves on Y with descent data.

If X^{δ} is the set of points of a topological space (or any site with enough points), and $u : X^{\delta} \to X$ the standard map, then $\mathrm{Sheaves}(X^{\delta})^{u^{-1}u_*}$ is the category of sheaves on X (espace étalé description).

Monadic adjunctions

Given an adjunction

$$\mathcal{D} \underset{F}{\overset{U}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} \mathcal{B},$$

let $\top = UF$ and we define the comparison functor $K : \mathcal{D} \to \mathcal{B}^\top$ by

$$D \mapsto (UFUD \xrightarrow{U\varepsilon_D} UD)$$

on objects, and $K(g) = U(g)$ on morphisms.

The adjunction is said to be monadic (or tripleable) if $K : \mathcal{D} \to \mathcal{B}^\top$ is an equivalence. For example, $\mathrm{Alg}_k \underset{k[-]}{\overset{\rightarrow}{\underset{\leftarrow}{\top}}} \mathrm{Set}$ is monadic.

Comonadic adjunctions

Dually, if we have an adjunction

$$\mathcal{D} \underset{G}{\overset{V}{\rightleftarrows}} \mathcal{B},\quad (V \dashv G)$$

let $\perp = VG$, and can define a natural functor $K : \mathcal{D} \to \mathcal{B}^{\perp}$ similarly. The adjunction is said to be comonadic (or cotripleable) if K is an equivalence.

For example,

$$\text{Sheaves}(X^{\delta}) \underset{u^{-1}}{\overset{u_*}{\rightleftarrows}} \text{Sheaves}(X)$$

$$\text{Coalgebras} \underset{V}{\rightleftarrows} \text{Modules}$$

are comonadic, for G the cofree coalgebra functor.

Bialgebras and distributivity

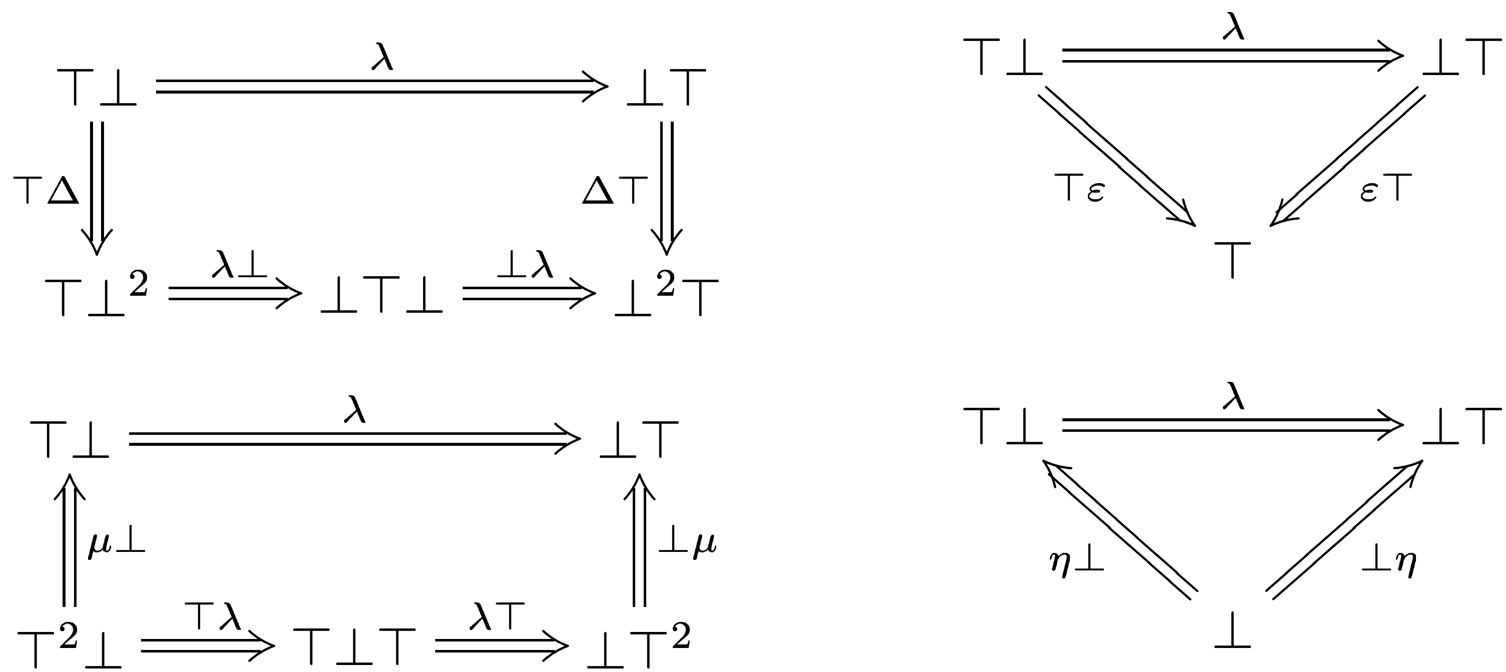
Derived deformations of algebras and coalgebras are generally well understood, with several satisfactory approaches.

Most challenging problems involve both monadic and comonadic structure. Important examples are Hopf algebras, or sheaves of algebras (since deforming a scheme X is equivalent to deforming its structure sheaf $\mathscr{O}_X$ as a sheaf of algebras).

Assume that we have a category $\mathcal{B}$ equipped with both a monad $(\top, \mu, \eta)$ and a comonad $(\bot, \Delta, \varepsilon)$. In order to define a category of $(\top, \bot)$-bialgebras, we need to define a comonad $\bot$ on $\mathcal{B}^{\top}$, and a monad $\top$ on $\mathcal{B}^{\bot}$, such that $(\mathcal{B}^{\top})^{\bot} \simeq (\mathcal{B}^{\bot})^{\top} =: \mathcal{B}^{\bot,\top}$.

For this to be possible, we need a distributivity transformation $\lambda : \top\bot \implies \bot\top$, satisfying various conditions.

§6 of "Combining a monad and a comonad", Power & Watanabe, Theoretical Computer Science 280 (2002) 137–162:



Assume that we have functors

$$\begin{array}{ccc}
\mathcal{D} & \underset{F}{\overset{U}{\underset{\top}{\rightleftarrows}}} & \mathcal{E} \\
V \downarrow \dashv \uparrow G & & V \downarrow \dashv \uparrow G \\
\mathcal{A} & \underset{F}{\overset{U}{\underset{\top}{\rightleftarrows}}} & \mathcal{B},
\end{array}$$

with $F \dashv U$ monadic and $G \vdash V$ comonadic, so $\bot := VG$, $\top := UF$ are a comonad and monad on $\mathcal{B}$. Also assume that one of the forgetful functors U, V commutes with the "orthogonal" adjunction. Then there is a canonical distributivity transformation $\lambda : \top\bot \implies \bot\top$, and

$$\mathcal{D} \simeq \mathcal{B}^{\bot,\top}.$$

Explicit commutativity conditions

For units and counits
$\eta : 1 \Longrightarrow \top$, $\varepsilon : \bot \Longrightarrow 1$, $e : FU \Longrightarrow 1$, $h : 1 \Longrightarrow GV$, the following identities must hold (up to canonical isomorphism):

$$GU = UG \quad \text{or} \quad FV = VF$$

$$UV = VU,$$

$$Ve = eV \quad \text{or} \quad Uh = hU$$

$$V\eta = \eta V \quad \text{or} \quad U\varepsilon = \varepsilon U.$$

(Observe that the adjoint properties ensure that identities on the same line are equivalent.) Then $\bot := VG$, $\top := UF$ are a comonad and monad on $\mathcal{B}$, with distributivity transformation

$$\lambda := UVeGF \circ UVFG\eta = UVGF\varepsilon \circ UVhFG.$$

Examples.

$$\begin{array}{ccc}
\text{Hopf Algebras} & \underset{\text{Symm}}{\overset{\longrightarrow}{\underset{\longleftarrow}{\top}}} & \text{Coalgebras} \\
\big\downarrow \dashv \big\uparrow G & & \big\downarrow \dashv \big\uparrow G \\
\text{Algebras} & \underset{\text{Symm}}{\overset{\longrightarrow}{\underset{\longleftarrow}{\top}}} & \text{Modules},
\end{array}
\qquad
\begin{array}{ccc}
\text{AlgSheaves}(X) & \underset{\text{Symm}}{\overset{\longrightarrow}{\underset{\longleftarrow}{\top}}} & \text{ModSheaves}(X) \\
u^{-1} \big\downarrow \dashv \big\uparrow u_* & & u^{-1} \big\downarrow \dashv \big\uparrow u_* \\
\text{AlgSheaves}(X^\delta) & \underset{\text{Symm}}{\overset{\longrightarrow}{\underset{\longleftarrow}{\top}}} & \text{ModSheaves}(X^\delta),
\end{array}$$

for Symm the free algebra functor, and G the cofree coalgebra functor.

for X^δ the set of points of X, and $u : X^\delta \to X$.

The general scenario

For a functor $\mathcal{B} : \mathcal{C}_\Lambda \to \mathrm{Cat}$ with trivial deformation theory (e.g. $A \rightsquigarrow$ free A-modules), we wish to describe derived deformations of $D \in \mathcal{B}^{\top,\perp}(k)$, given distributive $\top, \perp$.

For any monad $(\top, \mu, \eta)$ on a category $\mathcal{B}$, there is a natural enrichment $\bar{\mathcal{B}}$ of $\mathcal{B}$ given by $\mathscr{H}om^n_{\bar{\mathcal{B}}}(B, B') := \mathrm{Hom}_{\mathcal{B}}(\top^n B, B')$, with operations

$$g * f = g \circ \top^n f, \quad \partial^i(f) = f \circ \top^{i-1}\mu_{\top^{m-i}B}, \quad \sigma^i(g) = f \circ \top^i \eta_{\top^{m-i-1}B'},$$

for $g \in \mathscr{H}om^n(B', B''), f \in \mathscr{H}om^m(B, B')$.

More generally, for any monad-comonad pair $(\top, \mu, \eta)$, $(\perp, \Delta, \varepsilon)$ on $\mathcal{B}$, with distributivity transformation $\lambda : \top\perp \implies \perp\top$, we may set

$$\begin{aligned}
\mathscr{H}om^n_{\bar{\mathcal{B}}}(B, B') &:= \mathrm{Hom}_{\mathcal{B}}(\top^n B, \perp^n B'); \\
g * f &= \perp^m g \circ \lambda^{m,n} \circ \top^n f, \\
\partial^i(f) &= \perp^{m-i}\Delta_{\perp^{i-1}B} \circ f \circ \top^{i-1}\mu_{\top^{m-i}B}, \\
\sigma^i(f) &= \perp^{m-i-1}\varepsilon_{\perp^i B} \circ f \circ \top^i \eta_{\top^{m-i-1}B},
\end{aligned}$$

for $\lambda^{m,n} : \top^n\perp^m \implies \perp^m\top^n$ formed from λ.

This is an enrichment of $\mathcal{B}$ in the monoidal category $\mathrm{Set}^{\Delta_{**}}$:

Definition. Define Δ_{**} to be the subcategory of the ordinal number category Δ containing only those morphisms $f : \mathbf{m} \to \mathbf{n}$ with $f(0) = 0, f(m) = n$. Given a category $\mathcal{C}$, a functor $X : \Delta_{**} \to \mathcal{C}$ consists of objects $X^n \in \mathcal{C}$, with all of the operations ∂^i, σ^i of a cosimplicial complex except $\partial^0, \partial^{n+1} : X^n \to X^{n+1}$.

Definition. Given a monoidal category $\mathcal{C}$ containing finite coproducts, define a monoidal structure on the category $\mathcal{C}^{\Delta_{**}}$ by

$$(X \otimes Y)^n = \coprod_{i+j=n} X^i \otimes Y^j,$$

where for $x \in X^m, y \in Y^n$, we define

$$\partial^i(x \otimes y) = \begin{cases} (\partial^i x) \otimes y & i \leq m, \\ x \otimes (\partial^{i-m} y) & i > m, \end{cases} \quad \sigma^i(x \otimes y) = \begin{cases} (\sigma^i x) \otimes y & i < m, \\ x \otimes (\sigma^{i-m} y) & i \geq m. \end{cases}$$

(like the Alexander-Whitney cup product).

The Maurer-Cartan equations

Given a monoid $E \in \mathrm{Set}^{\Delta_{**}}$, define the Maurer-Cartan space

$$\mathrm{MC}(E) \subset E^1,$$

to consist of ω satisfying the associativity and unit laws

$$\omega * \omega = \partial^1 \omega \in E^2, \quad \sigma^0 \omega = 1 \in E^0.$$

Definition. Given a category $\bar{\mathcal{B}}$ enriched in $\mathrm{Set}^{\Delta_{**}}$, let $\pi^1 \bar{\mathcal{B}}$ be the category with objects (B, ω), for $\omega \in \mathrm{MC}(\mathscr{H}om_{\bar{\mathcal{B}}}(B, B))$. Morphisms in $\pi^1 \bar{\mathcal{B}}$ from (B, ω) to (B', ω') are given by

$$\{f \in \mathscr{H}om^0_{\bar{\mathcal{B}}}(B, B') \, : \, \omega' * f = f * \omega\}.$$

Lemma. *For the enrichment $\bar{\mathcal{B}}$ of $\mathcal{B}$ coming from $\top, \bot$, we have*

$$\pi^1 \bar{\mathcal{B}} \cong \mathcal{B}^{\bot, \top}.$$

Strong homotopy $\top$-algebras

Given a monoid $E \in (s\mathrm{Set})^{\Delta_{**}}$,we relax the Maurer-Cartan equations to hold up to homotopy, so we set $I = \Delta^1$ and take $\mathrm{MC}(E) \subset \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{S}}(I^n, E^{n+1})$ to consist of $\underline{\omega}$ satisfying the higher Maurer-Cartan equations
$\omega_m(s_1, \ldots, s_m) * \omega_n(t_1, \ldots, t_n) = \omega_{m+n+1}(s_1, \ldots, s_m, 0, t_1, \ldots, t_n)$;

$$\begin{aligned}
\partial^i \omega_n(t_1, \ldots, t_n) &= \omega_{n+1}(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_n); \\
\sigma^0 \omega_0 &= 1; \\
\sigma^i \omega_n(t_1, \ldots, t_n) &= \omega_{n-1}(t_1, \ldots, t_{i-1}, \min\{t_i, t_{i+1}\}, t_{i+2}, \ldots, t_n); \\
\sigma^0 \omega_n(t_1, \ldots, t_n) &= \omega_{n-1}(t_2, \ldots, t_n); \\
\sigma^{n-1} \omega_n(t_1, \ldots, t_n) &= \omega_{n-1}(t_1, \ldots, t_{n-1}).
\end{aligned}$$

If $E \in \mathrm{Set}^{\Delta_{**}}$, this recovers the usual definition. The last three conditions aren't used by Lada, but won't affect our final answers.

Strong homotopy bialgebras

Given a simplicial category $\mathcal{E}$ equipped with a distributive monad-comonad pair $\top, \bot$, and an object $E \in \mathscr{E}$, we regard $\mathrm{MC}(\mathscr{H}om_{\bar{\mathcal{E}}}(E,E))$ as being the set of strong homotopy $\top, \bot$-bialgebras over E.

Take a functor $\mathcal{B} : \mathcal{C}_\Lambda \to \mathrm{Cat}$ with trivial deformation theory, equipped with a distributive monad-comonad pair $\top, \bot$, and $D = (B, \omega_0) \in \mathcal{B}^{\top,\bot}(k)$. Given $A \in s\mathcal{C}_\Lambda$, $\mathcal{B}(A)$ is a simplicial category, with an enrichment $\bar{\mathcal{B}}(A)$ in $(s\mathrm{Set})^{\Delta_{**}}$, and we define $\mathrm{MC}(D, A) := \mathrm{MC}(\mathscr{H}om_{\bar{\mathcal{B}}(A)}(B,B))_{\omega_0}$, giving a functor $\mathrm{MC}(D, -) : s\mathcal{C}_\Lambda \to \mathrm{Set}$.

Unlike the strong homotopy $\top$-algebras, there is no natural bar construction, so we cannot define morphisms of s.h. $\top, \bot$-bialgebras, so we cannot easily extend this to a sGpd-valued functor. However, we will construct a sSet-valued functor analogous to the nerve.

For any $X \in s\mathrm{Set}$, and $\mathrm{Set}^{\Delta**}$-enriched category $\bar{\mathcal{B}}$, we have an enrichment $\bar{\mathcal{B}}^X$ of $\mathcal{B}^{X_0}$ in $\mathrm{Set}^{\Delta**}$, given by

$$\mathcal{H}om^n_{\bar{\mathcal{B}}^X}(\underline{B}, \underline{B}') := \prod_{x \in X_n} \mathcal{H}om^n_{\bar{\mathcal{B}}^X}(\underline{B}(x_n), \underline{B}'(x_0)),$$

where $x_0 := (\partial_1)^n x, x_n := (\partial_0)^n x$ the 0th and nth vertices of x.

For $x \in X_{n+1}$, $y \in X_{n-1}$, $z \in X_{m+n}$, $1 \leq i \leq n$, $0 \leq j < n$, $e \in \mathcal{H}om^n_{\bar{\mathcal{B}}^X}$ and $f \in \mathcal{H}om^m_{\bar{\mathcal{B}}^X}$, we define the operations by

$$\begin{aligned} \partial^i(e)(x) &:= \partial^i(e(\partial_i x)) \\ \sigma^j(e)(y) &:= \sigma^j(e(\sigma_j y)), \\ (f * e)(z) &:= f((\partial_{m+1})^n z) * e((\partial_0)^m z). \end{aligned}$$

Lemma. *If $\mathbb{I}$ is a small category, with nerve $B\mathbb{I} \in s\mathrm{Set}$, then $\pi^1(\bar{\mathcal{B}}^{B\mathbb{I}}) \simeq (\pi^1 \bar{\mathcal{B}})^{\mathbb{I}}$, the category of $\mathbb{I}$-diagrams, (which equals $(\mathcal{B}^{\top,\perp})^{\mathbb{I}}$ in the motivating case).*

The derived deformation groupoid

Define $\mathfrak{MC}(D, -) : s\mathcal{C}_\Lambda \to \mathrm{Set}$ by setting

$$\mathfrak{MC}(D, A)_n := \mathrm{MC}(\mathscr{H}om_{\bar{\mathcal{B}}^{\mathbf{n}}(A)}(B, B))_{\omega_0},$$

where $\mathbf{n}$ is the category $0 \to 1 \to \ldots \to n$.

- $\mathfrak{MC}(D, A)_0 = \mathrm{MC}(D, A)$
- $\mathfrak{MC}(D, -)_n$ describes derived deformations of the diagram $\overbrace{D \xrightarrow{\mathrm{id}} D \xrightarrow{\mathrm{id}} \ldots \xrightarrow{\mathrm{id}} D}^{n+1}$, so for $A \in \mathcal{C}_\Lambda$, $\mathrm{MC}(D, A)$ is just the nerve of the classical deformation groupoid
- This functor is left exact, maps surjections to fibrations, and maps acyclic surjections to acyclic fibrations. Call this property "quasi-smoothness".
- There is a similar construction if we start with $\mathcal{B}$ simplicial.

The model structure

Let $[\mathcal{C},\mathcal{D}]_{\mathrm{cts}}$ be the category of continuous (i.e. limit-preserving, or left exact) functors from $\mathcal{C}$ to $\mathcal{D}$.

Then $[s\mathcal{C}_\Lambda, s\mathrm{Set}]_{\mathrm{cts}}$ has a natural model structure, for which fibrant means quasi-smooth, and a morphism $X \to Y$ of quasi-smooth objects is a weak equivalence if and only if $X(A) \to Y(A)$ is a weak equivalence for all A.

The quasi-smooth objects give rise to derived deformation groupoids (i.e. functors $F : \mathrm{Ho}(s\mathcal{C}_\Lambda) \to \mathrm{Ho}(s\mathrm{Set})$ such that $F(k) = \bullet$, $F(A \times^h_B C) \xrightarrow{\sim} F(A) \times^h_{F(B)} F(C)$ for all $A \twoheadrightarrow B$).

The homotopy category $\mathrm{Ho}([s\mathcal{C}_\Lambda, s\mathrm{Set}]_{\mathrm{cts}})$ is equivalent to the category of derived deformation groupoids (using Heller's theorem).

Equivalent formulations in char. 0

If $\Lambda = k$, a field of characteristic 0, let $dg\mathcal{C}_k$ (resp. $dg_{\geq 0}\mathcal{C}_k$) consist of $\mathbb{Z}$-graded (resp. non-negatively graded) chain complexes $A_\bullet$ with Artinian local graded-commutative multiplication. Then Dold-Kan normalisation $N : s\mathcal{C}_k \to dg_{\geq 0}\mathcal{C}_k$ gives an equivalence of homotopy categories.

The following are then right Quillen equivalences of model categories (for suitable model structures), so give equivalences of homotopy categories:

$$\begin{array}{ccccc} [s\mathcal{C}_\Lambda, s\mathrm{Set}]_{\mathrm{cts}} & \xleftarrow{N^*} & [dg_{\geq 0}\mathcal{C}_\Lambda, s\mathrm{Set}]_{\mathrm{cts}} & = & [cdg_{\geq 0}\mathcal{C}_\Lambda, \mathrm{Set}]_{\mathrm{cts}} \\ & & & & \downarrow D^* \\ DG\mathrm{LA} & \xrightarrow{\mathrm{MC}} & [dg\mathcal{C}_\Lambda, \mathrm{Set}]_{\mathrm{cts}} & \xrightarrow{\mathrm{Tot}^*} & [DG^{\geq 0}dg_{\geq 0}\mathcal{C}_\Lambda, \mathrm{Set}]_{\mathrm{cts}} \end{array}$$

Here, $cdg_{\geq 0}\mathcal{C}_\Lambda$ consists of cosimplicial objects in $dg_{\geq 0}\mathcal{C}_\Lambda$, D is cosimplicial denormalisation, and Tot is the total functor. MC is the classical Maurer-Cartan functor

$$\mathrm{MC}(L)(A) = \{\omega \in \prod_{n \in \mathbb{Z}} L^{n+1} \otimes A_n \,:\, d\omega + \frac{1}{2}[\omega, \omega] = 0\}.$$

By Grothendieck pro-representability, $[dg\mathcal{C}_\Lambda, \mathrm{Set}]_{\mathrm{cts}}$ is opposite to the category $\mathrm{pro}(dg\mathcal{C}_\Lambda)$ of pro-objects. Taking duals of maximal ideals shows that $[dg\mathcal{C}_\Lambda, \mathrm{Set}]_{\mathrm{cts}}$ is equivalent to the category of ind-conilpotent cocommutative dg coalgebras (without counit). Fibrant objects in this category are those whose (ind-conilpotent) coalgebra structure is cofree — these are precisely SHLAs (L_∞-algebras).

Deforming group schemes

For deformations of a group scheme G, with multiplication $m : G \times G \to G$ and identity $e : \operatorname{Spec} k \to G$, we let $\operatorname{HopfAlg}(G)$ (resp. $\operatorname{CoAlg}(G)$) be the category of algebras (resp. modules) $\mathscr{F}$ on G, equipped with compatible coidentity $\mathscr{F} \to e_* k$ and coassociative comultiplication $\mathscr{F} \to m_*(\mathscr{F} \otimes \mathscr{F})$.

The following diagram gives derived deformations:

$$\begin{array}{ccc}
\operatorname{HopfAlg}(G) & \underset{\text{Symm}}{\overset{\longrightarrow}{\underset{\longleftarrow}{\top}}} & \operatorname{CoAlg}(G) \\
{\scriptstyle u^{-1}\circ V}\downarrow \dashv \uparrow{\scriptstyle Q\circ u_*} & & {\scriptstyle u^{-1}\circ V}\downarrow \dashv \uparrow{\scriptstyle Q\circ u_*} \\
\operatorname{Alg}(G') & \underset{\text{Symm}}{\overset{\longrightarrow}{\underset{\longleftarrow}{\top}}} & \operatorname{Mod}(G'),
\end{array}$$

for Q right adjoint to the functor V forgetting the comultiplication and coidentity.

Deforming a complex of quasi-coherent sheaves

For a chain complex of $\mathscr{O}_X$-modules (simplicial categories):

$$\begin{array}{ccc}
dg_{\geq 0}\mathscr{O}_X\mathrm{Mod}(X) & \underset{\mathscr{O}_X\otimes -}{\overset{\longrightarrow}{\underset{\longleftarrow}{\top}}} & dg_{\geq 0}\mathrm{Mod}(X) \\
u^{-1}\downarrow \dashv \uparrow u_* G & & u^{-1}\downarrow \dashv \uparrow u_* G \\
g_{\geq 0}(u^{-1}\mathscr{O}_X)\mathrm{Mod}(X') & \underset{(u^{-1}\mathscr{O}_X)\otimes -}{\overset{\longrightarrow}{\underset{\longleftarrow}{\top}}} & g_{\geq 0}\mathrm{Mod}(X'),
\end{array}$$

for $(GV_*)_n := V_n \oplus V_{n-1}$, with $d(v,w) = (w,0)$, and $g_{\geq 0}\mathrm{Mod}$ is the category of graded modules, with simplicial structure determined by the normalisation

$$N^s_n \underline{\mathrm{Hom}}_{dg_{\geq 0}\mathrm{Mod}}(U_\bullet, V_\bullet) := \begin{cases} \mathrm{Hom}_{dg_{\geq 0}\mathrm{Mod}}(U_\bullet, V_\bullet) & n = 0 \\ \prod_{i \geq 0} \mathrm{Hom}_{\mathrm{Mod}}(U_i, V_{i+n}) & n > 0. \end{cases}$$

Deforming a stack

For deformations of a quasi-compact stack $\mathfrak{X}$, we take a smooth simplicial hypercovering $X_\bullet \to \mathfrak{X}$, with each X_n an affine scheme.

Let $c\mathcal{C}$ be the category of cosimplicial objects of $\mathcal{C}$, and $c_+\mathcal{C}$ similar, but without ∂^0 (so $c_+\mathrm{Mod} \simeq g_{\geq 0}\mathrm{Mod}$, by normalisation).

The simplicial affine scheme $X_\bullet$ is opposite to a cosimplicial algebra, giving the following monadic adjunction of simplicial categories:

$$c\mathrm{Alg} \underset{\mathrm{Symm} F_\partial}{\overset{U_\partial U_{\mathrm{alg}}}{\underset{\top}{\rightleftarrows}}} c_+\mathrm{Mod}\,,$$

where $F_\partial : c_+\mathrm{FMod}(A) \to c\mathrm{FMod}(A)$ is left adjoint to the functor U_∂ forgetting ∂^0, and $\underline{\mathrm{Hom}}_{c_+\mathcal{C}}(B^*, C^*)_n := \mathrm{Hom}_{c_+\mathcal{C}}(B^*, (C^*)^{\Delta^n_*})$.

This allows us to define derived deformations of $\mathfrak{X}$.

Cohomology and obstruction

For $F : s\mathcal{C}_\Lambda \to s\mathrm{Set}$ quasi-smooth, there exist k-vector spaces $\mathrm{H}^j(F)$ such that for any simplicial vector space V,

$$\pi_n F(k \oplus V\epsilon) \cong H^{-n}(F \otimes V) := \bigoplus_{i \geq 0} \mathrm{H}^{i-n}(F) \otimes \mathrm{H}_i(V).$$

For any extension $e : I \to A \xrightarrow{f} B$ in $s\mathcal{C}_\Lambda$ with $I \cdot \mathfrak{m}_A = 0$, there is a long exact sequence

$$\cdots \xrightarrow{e_*} \pi_n(FA, y) \xrightarrow{f_*} \pi_n(FB, x) \xrightarrow{o_e} \mathrm{H}^{1-n}(F \otimes I) \xrightarrow{e_*} \cdots$$

$$\cdots \xrightarrow{e_*} \pi_1(FA, y) \xrightarrow{f_*} \pi_1(FB) \xrightarrow{o_e} \mathrm{H}^0(F \otimes I) \xrightarrow{e_*}$$

$$\pi_0(FA) \xrightarrow{f_*} \pi_0(FB) \xrightarrow{o_e} \mathrm{H}^1(F \otimes I).$$