Non-abelian reciprocity laws for modular curves

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Recap from Minhyong's talk

- X over number field F, $\mathbb{A}_F = \prod'_v F_v$ (finite adèles); we have

$$X(F) \hookrightarrow \dots \hookrightarrow X(\mathbb{A}_F)_2 \hookrightarrow X(\mathbb{A}_F).$$

- If $\Gamma = \pi_1^{\text{ét}}(\bar{X})$, $T_n = \Gamma^{[n]}/\Gamma^{[n+1]}$, then

$$X(\mathbb{A}_F)_{n+1} = \ker(\mathrm{rec}_n \colon X(\mathbb{A}_F)_n \to \mathrm{H}^2_c(F, T_n)).$$

Sketch of Minhyong's rec_1

- For $T_1 = \mathrm{H}_1(\bar{X})$, $x \in X(F_v)$ splits

$$1 \to T_1 \to \pi_1^{\text{ét}}(X_v)/\Gamma^{[2]} \to G_v \to 1,$$

so $j(x) \in \mathrm{H}^1(G_v, T_1)$, and hence

$$j\colon\ X(\mathbb{A}_F) \to \prod_v{}' \mathrm{H}^1(F_v, T_1) = \mathrm{H}^1(\mathbb{A}_F, T_1).$$

- $\mathrm{rec}_1 = \partial \circ j\colon\ X(\mathbb{A}_F) \to \mathrm{H}^2_c(F, T_1)$, for

$$\to \mathrm{H}^1(F, T_1) \to \mathrm{H}^1(\mathbb{A}_F, T_1) \xrightarrow{\partial} \mathrm{H}^2_c(F, T_1) \to .$$

Modular curves

- congruence subgroup $\Gamma \leqslant \widehat{\mathrm{SL}_2(\mathbb{Z})}$.
- X_Γ (stacky) curve over some F.
- $\mathbb{C}$-points $[\mathfrak{H}/(\Gamma \cap \mathrm{SL}_2(\mathbb{Z}))]$.
- $X_{\widehat{\mathrm{SL}_2(\mathbb{Z})}}$ is moduli stack of elliptic curves.
- $\pi_1^{\text{ét}}(\bar{X}_\Gamma) \cong \Gamma$, so $\pi_1^{\text{ét}}(X_\Gamma) \cong \Gamma \rtimes G_F$.
- $\mathrm{SL}_2(\mathbb{Z})^{\mathrm{ab}} = 0$, so $T_n \otimes \mathbb{Q} = 0 \ \forall n$.
- To get non-zero rec_n, have to take completion relative to $\mathrm{SL}_2(\mathbb{Q}_\ell)$.

Relative reciprocity maps

- Canonical rank 2 $\mathbb{Q}_\ell$-sheaf $\mathbb{V}_1$ on X_Γ; write $\mathbb{V}_n = \mathrm{Symm}^n \mathbb{V}_1$.
- Thus $X_\Gamma(F) \to \mathrm{H}^1(F, \mathrm{GL}_2(\mathbb{Q}_\ell))$.
- Set $X_\Gamma(\mathbb{A}_F)_1 \approx X_\Gamma(\mathbb{A}_F) \times_{\mathrm{H}^1(\mathbb{A}_F, \mathrm{GL}_2(\mathbb{Q}_\ell))} \mathrm{H}^1(F, \mathrm{GL}_2(\mathbb{Q}_\ell))$.

Then

$$X_\Gamma(F) \hookrightarrow \ldots \hookrightarrow X_\Gamma(\mathbb{A}_F)_2 \hookrightarrow X_\Gamma(\mathbb{A}_F)_1.$$

In this setting:

- $T_1^\vee = \bigoplus_{m\geqslant 0} \mathsf{H}^1(\Gamma, \mathbb{V}_m) \otimes V_m^\vee$,
  $T_n^\vee = \mathsf{CoLie}_n(T_1^\vee) \subset (T_1^\vee)^{\otimes n}$.

$$\mathsf{rec}_n\colon\ X_\Gamma(\mathbb{A}_F)_n \to \mathsf{H}^2_c(F, T_n) \overset{PT}{=} \mathsf{H}^1(F, T_n^\vee(1))^\vee$$

- $\mathsf{H}^1(\Gamma, \mathbb{V}_m)$ related to modular forms.

When $\Gamma = \widehat{\mathsf{SL}_2(\mathbb{Z})}$, this says:

- For $G_F \to \mathsf{GL}_2(\mathbb{Q}_\ell)$ everywhere locally given by elliptic curve $E/\mathbb{A}_F$, the maps rec_n all vanish when E defined over F.

Construction of relative rec_1

For $f\colon X_\Gamma \to \operatorname{Spec} F$ with section x, apply $\mathbf{R}f_*$ to the unit $\mathbb{V}_m \to \mathbf{R}x_* x^* \mathbb{V}_m$ of the adjunction to get a map

$$\mathbf{R}f_* \mathbb{V}_m \to x^* \mathbb{V}_m =: \mathbb{V}_{m,x}$$

in the derived category. Equivalently, we have an element of

$$\mathrm{Ext}^0_F(\mathbf{R}f_* \mathbb{V}_m, \mathbb{V}_{m,x}) \approx \mathrm{H}^1(F, \mathrm{H}^1(\Gamma, \mathbb{V}_m)^\vee \otimes \mathbb{V}_{m,x}).$$

- Similarly, $x \in X_\Gamma(\mathbb{A}_F)$ gives an elt of

$$\mathsf{H}^1(\mathbb{A}_F, \mathsf{H}^1(\Gamma, \mathbb{V}_m)^\vee \otimes \mathbb{V}_{m,x}).$$

- Fix a G_F-rep V and set

$$X_\Gamma(F)_V = \{x \in X(F) \ : \ \mathbb{V}_{1,x} = V\}$$
$$X_\Gamma(\mathbb{A}_F)_V = \{x \in X(\mathbb{A}_F) \ : \ \mathbb{V}_{1,x} = V\}.$$

- Note
$X_\Gamma(F)_V \approx X_\Gamma(F) \times_{\mathsf{H}^1(F, \mathsf{GL}_2(\mathbb{Q}_\ell))} \{V\}$.

- Commutative diagram ($V_m := S^m V$):

$$\begin{array}{ccc}
X_\Gamma(F)_V & \longrightarrow & \mathrm{H}^1(F, \mathrm{H}^1(\Gamma, \mathbb{V}_m)^\vee \otimes V_m) \\
\downarrow & & \downarrow \\
X_\Gamma(\mathbb{A}_F)_V & \longrightarrow & \mathrm{H}^1(\mathbb{A}_F, \mathrm{H}^1(\Gamma, \mathbb{V}_m)^\vee \otimes V_m) \\
& \searrow & \downarrow \\
& & \mathrm{H}^2_c(F, \mathrm{H}^1(\Gamma, \mathbb{V}_m)^\vee \otimes V_m)
\end{array}$$

$$\mathrm{rec}_1 \colon X_\Gamma(\mathbb{A}_F)_V \to \prod_{m \geqslant 0} \mathrm{H}^2_c(F, \mathrm{H}^1(\Gamma, \mathbb{V}_m)^\vee \otimes V_m).$$

Please tell us if you recognise this!

Finer relative completions of $\widehat{\mathrm{SL}_2(\mathbb{Z})}$:

$$
\begin{array}{ccc}
\text{Malcev over } \mathrm{SL}_2(\mathbb{Q}_\ell) & \longleftarrow & \text{Malcev over } \mathrm{SL}_2(\mathbb{Q}_\ell) \times \mathrm{SL}_2(\hat{\mathbb{Z}}) \\
\uparrow & & \uparrow \\
\text{pro-}\ell \text{ over } \mathrm{SL}_2(\mathbb{Z}_\ell) & \longleftarrow & \text{pro-nilpotent over } \mathrm{SL}_2(\hat{\mathbb{Z}}) \\
\| & & \| \\
\text{pro-}\ell \text{ over } \mathrm{SL}_2(\mathbb{F}_\ell) & \longleftarrow & \text{pro-nilpotent over } \prod_p \mathrm{SL}_2(\mathbb{F}_p)
\end{array}
$$

Last two rows give different towers, same limit.

Finer reciprocity maps

- Fix $G_F \to \mathrm{GL}_2(\hat{\mathbb{Z}})$, complete relative to $\mathrm{SL}_2(\mathbb{Q}_\ell) \times \mathrm{SL}_2(\hat{\mathbb{Z}})$. This replaces $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{V}_m)$ with $\varinjlim_\Gamma \mathrm{H}^1(\Gamma, \mathbb{V}_m)$.
- Fix $\rho\colon\ G_F \to \mathrm{GL}_2(\mathbb{F}_\ell)$, take relative pro-$\ell$ completion over $\mathrm{SL}_2(\mathbb{F}_\ell)$. Then

$$\mathrm{rec}_1\colon\ X_\Gamma(\mathbb{A}_F)_\rho \to \mathrm{H}_1(F, \mathrm{H}_1(\Gamma \cap \Gamma(\ell), \mathbb{F}_\ell)(-1))$$

$[\Gamma(\ell) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/\ell))]$.

Topological obstruction theory

Central extension $\Gamma'' \twoheadrightarrow \Gamma'$ of profinite groups, kernel T. Then fibration sequence

$$B\Gamma'' \to B\Gamma' \xrightarrow{\text{ob}} B^2 T$$

(delooping) and hence on mapping spaces:

$$M(Y, B\Gamma'') \to M(Y, B\Gamma') \xrightarrow{\text{ob}} M(Y, B^2 T),$$

giving l.e.s. of cohomology on homotopy groups

$$\pi_n M(Y, B\Gamma) = \mathrm{H}^{1-n}(Y, \Gamma),$$
$$\pi_n M(Y, B^2 T) = \mathrm{H}^{2-n}(Y, T).$$

Variant: $\Gamma'' \twoheadrightarrow \Gamma'$, abelian kernel T, quotients by BT-action give homotopy limit

$$B\Gamma'' \longrightarrow B\Gamma' \underset{0}{\overset{\text{ob}}{\rightrightarrows}} B(\Gamma' \ltimes BT) \longrightarrow B\Gamma'.$$

- So $M(Y, B(\Gamma' \ltimes BT)) \to M(Y, B\Gamma')$,
- ob a section of this bundle,
- vanishing locus $M(Y, B\Gamma'')$.
- $\rho \in \mathrm{H}^1(Y, \Gamma')$ gives $\mathrm{ob}(\rho) \in \mathrm{H}^2(Y, T_\rho)$, vanishing iff ρ lifts to $\mathrm{H}^1(Y, \Gamma'')$.
- Reciprocity maps compare ob for $Y = BG_F$ and $Y = \coprod'_v BG_v$.

X/F & $X_{\text{ét}}$ a $K(\Delta, 1)$; $x \in X(\mathbb{A}_F)$ gives:

$$\begin{array}{ccc} \coprod'_v BG_v & \xrightarrow{x} & B\Delta \\ \downarrow & \nearrow & \downarrow \\ BG_F & = & BG_F. \end{array}$$

Minhyong's reciprocity maps look at:

$$\begin{array}{ccc} \coprod'_v BG_v & \longrightarrow & B(\Delta_{n+1}) \\ \downarrow & \nearrow & \downarrow \\ BG_F & \longrightarrow & B(\Delta_n), \end{array}$$

with $\Delta_n = \Delta/\Gamma^{[n]}$, for $\Gamma = \ker(\Delta \to G_F)$.

Alternative pro-finite towers

Take any tower $\ldots \to \Delta_2 \to \Delta_1 \to G_F$ of quotients of Δ with $\ker(\Delta_{n+1} \to \Delta_n)$ abelian Δ_1-reps, and fix a section of $\Delta_1 \to G_F$.

- When $\Delta = \pi_1^{\text{ét}}(X_\Gamma)$, Δ_1 can be $G_F \times_{A^*} \mathrm{GL}_2(A)$ for $A = \mathbb{F}_\ell, \mathbb{Z}_\ell, \prod_p \mathbb{F}_p, \hat{\mathbb{Z}}$

Let $K = \ker(\Delta \to \Delta_1)$ and $\Delta_n = \Delta/K_n$ for:

- $K_n = K^{[n]}$ (pro-nilpotent completion);
- or $K_{n+1} = \langle [K, K_n], K_n^\ell \rangle$ (relative pro-ℓ completion).

Let $X_\Gamma(\mathbb{A})_n$ be the homotopy fibre product

$$\begin{array}{ccc} X_\Gamma(\mathbb{A})_n & \longrightarrow & M(BG_F, B\Delta_n) \\ \downarrow & & \downarrow \\ X_\Gamma(\mathbb{A}) & \longrightarrow & \prod'_v M(BG_v, B\Delta_n), \end{array}$$

so main example gives

$$X_\Gamma(\mathbb{A})_1 \approx X_\Gamma(\mathbb{A}) \times_{\mathrm{H}^1(\mathbb{A}_F, \mathrm{GL}_2(\mathbb{Z}_\ell))} \mathrm{H}^1(F, \mathrm{GL}_2(\mathbb{Z}_\ell)),$$

and $X_\Gamma(F) \to \ldots \to X_\Gamma(\mathbb{A})_2 \to X_\Gamma(\mathbb{A})_1$.

Obstruction bundles

Compare obstruction theories: write $\mathrm{OB}(F, T)$ for the homotopy kernel of

$$M(BG_F, B(\Delta_1 \ltimes BT)) \to \prod\nolimits_v' M(BG_v, B(\Delta_1 \ltimes BT))$$

pulled back to $X(\mathbb{A})_1$.

For $x \in X(\mathbb{A})_1$ over $\rho\colon G_F \to \mathrm{GL}_2(A)$, this gives $\pi_i(\mathrm{OB}(F, T)_x) = \mathrm{H}_c^{2-i}(F, T_\rho)$.

Non-abelian reciprocity maps

Simultaneously applying obstruction maps with $T_n := \ker(\Delta_{n+1} \to \Delta_n)$ gives

$$\mathrm{rec}_n \colon X_\Gamma(\mathbb{A})_n \to \mathrm{OB}(F, T_n)$$

over $X_\Gamma(\mathbb{A})_1$; homotopy kernel $X_\Gamma(\mathbb{A})_{n+1}$, so

$$\ldots \longrightarrow \pi_1 X_\Gamma(\mathbb{A})_n \longrightarrow \mathrm{H}^1_c(F, T_n) \longrightarrow \pi_0 X_\Gamma(\mathbb{A})_{n+1} \longrightarrow \pi_0 X_\Gamma(\mathbb{A})_n \xrightarrow{\mathrm{rec}} \mathrm{H}^2_c(F, T_n).$$

Note the final term!

(not just l.e.s. of π_* for $X_\Gamma(\mathbb{A})_{n+1} \to X_\Gamma(\mathbb{A})_n$)

Relative (Malcev) completion

Idea: deformation theory for groups instead of rings.

- reductive gp R , $H \twoheadrightarrow R$ unipotent kernel (vs. field k, $A \twoheadrightarrow k$ nilpotent).

$$\begin{array}{ccc} \Gamma & \xrightarrow{\text{universal}} & \Gamma^{R,\mathrm{Mal}}(\mathbb{Q}_\ell) \\ & \searrow^{\rho} & \downarrow{\scriptstyle \text{unipotent}} \\ & & R(\mathbb{Q}_\ell). \end{array}$$

- (cf. relative pro-ℓ, also by Hain.)

- We have $\Delta = \pi_1^{\text{ét}}(X_\Gamma)$, so

$$1 \to \Gamma \to \Delta \to G_F \to 1.$$

- $\mathbb{V}_1$ on X_Γ gives $\Delta \to \mathrm{GL}_2(\mathbb{Q}_\ell)$.

Then

$$\begin{array}{ccccc} \Gamma & \longrightarrow & \Delta & \longrightarrow & G_F \\ \downarrow & & \downarrow{\scriptstyle \mathbb{V}_1} & & \downarrow{\scriptstyle \mathbb{Q}_\ell(1)} \\ \mathrm{SL}_2(\mathbb{Q}_\ell) & \longrightarrow & \mathrm{GL}_2(\mathbb{Q}_\ell) & \xrightarrow{\det} & \mathbb{G}_m(\mathbb{Q}_\ell). \end{array}$$

$$1 \to \Gamma^{\mathrm{SL}_2,\mathrm{Mal}} \to \Delta^{\mathrm{GL}_2,\mathrm{Mal}} \to G_F^{\mathbb{G}_m,\mathrm{Mal}} \to 1$$

Let $D := \Delta^{\mathrm{GL}_2,\mathrm{Mal}}$, and write U for pro-unipotent radical $\mathrm{R_u}\Gamma^{\mathrm{SL}_2,\mathrm{Mal}}$, so

$$1 \to U \to D \to \mathrm{GL}_2 \times_{\mathbb{G}_m} (G_F^{\mathbb{G}_m,\mathrm{Mal}}) \to 1.$$

Write $D_n := D/U^{[n]}$, $T_n := U^{[n]}/U^{[n+1]}$.

Standard theory & vanishing of H^2 give:

$$T_n = \widehat{\mathrm{Lie}}(U_1), \quad T_1 = \prod_m \mathrm{H}^1(\Gamma, \mathbb{V}_m)^\vee \otimes V_m.$$

- Let $\Delta_n := D_n(\mathbb{Q}_\ell) \times_{\mathsf{GL}_2(\mathbb{Q}_\ell)} \mathsf{GL}_2(\mathbb{Z}_\ell)$; beware $\Delta \to \Delta_n$ not onto.
- $T_n = \ker(\Delta_{n+1} \to \Delta_n)$.
- $U(\mathbb{Q}_\ell)$ is pro-ind-profinite, so Δ_n is.
- Careful limits give $M(-, B\Delta_n)$ and $\ldots \to X(\mathbb{A})_2 \to X(\mathbb{A})_1$ using earlier formulae.
- $\mathrm{rec}_n \colon X(\mathbb{A})_n \to \mathrm{H}^2_c(F, T_n)$ as before.
- T_n related to modular forms.