

REAL NON-ABELIAN MIXED HODGE STRUCTURES FOR QUASI-PROJECTIVE VARIETIES: FORMALITY AND SPLITTING

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ABSTRACT. We define and construct mixed Hodge structures on real schematic homotopy types of complex quasi-projective varieties, giving mixed Hodge structures on their homotopy groups and pro-algebraic fundamental groups. We also show that these split on tensoring with the ring $\mathbb{R}[x]$ equipped with the Hodge filtration given by powers of $(x - i)$, giving new results even for simply connected varieties. The mixed Hodge structures can thus be recovered from the Gysin spectral sequence of cohomology groups of local systems, together with the monodromy action at the Archimedean place. As the basepoint varies, these structures all become real variations of mixed Hodge structure.

INTRODUCTION

The main aims of this paper are to construct mixed Hodge structures on the real relative Malcev homotopy types of complex varieties, and to investigate how far these can be recovered from the structures on cohomology groups of local systems, and in particular from the Gysin spectral sequence.

In [Mor], Morgan established the existence of natural mixed Hodge structures on the minimal model of the rational homotopy type of a smooth variety X , and used this to define natural mixed Hodge structures on the rational homotopy groups $\pi_*(X \otimes \mathbb{Q})$ of X . This construction was extended to singular varieties by Hain in [Hai2].

When X is also projective, [DGMS] showed that its rational homotopy type is formal; in particular, this means that the rational homotopy groups can be recovered from the cohomology ring $H^*(X, \mathbb{Q})$. However, in [CCM], examples were given to show that the mixed Hodge structure on homotopy groups could not be recovered from that on integral cohomology. We will first describe how formality interacts with the mixed Hodge structure, showing the extent to which the mixed Hodge structure on $\pi_*(X \otimes \mathbb{R}, x_0)$ can be recovered from the pure Hodge structure on $H^*(X, \mathbb{R})$.

This problem was suggested to the author by Carlos Simpson, who asked what happens when we vary the formality quasi-isomorphism. [DGMS] proved formality by using the dd^c Lemma (giving real quasi-isomorphisms), while most subsequent work has used the $\partial\bar{\partial}$ Lemma (giving Hodge-filtered quasi-isomorphisms). The answer (Corollary 2.12) is that, if we define the ring $\mathcal{S} := \mathbb{R}[x]$ to be pure of weight 0, with the Hodge filtration on $\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C}$ given by powers of $(x - i)$, then there is an \mathcal{S} -linear isomorphism

$$\pi_*(X \otimes \mathbb{R}, x_0) \otimes_{\mathbb{R}} \mathcal{S} \cong \pi_*(H^*(X, \mathbb{R})) \otimes_{\mathbb{R}} \mathcal{S},$$

preserving the Hodge and weight filtrations, where the homotopy groups $\pi_*(H^*(X, \mathbb{R}))$ are given the Hodge structure coming from the Hodge structure on the cohomology ring $H^*(X, \mathbb{R})$, regarded as a real homotopy type.

This is proved by replacing d^c with $d^c + xd$ in the proof of [DGMS], so $x \in \mathcal{S}$ is the parameter for varying formality quasi-isomorphisms. In several respects, $\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C}$ behaves like Fontaine's ring B_{st} of semi-stable periods, and the MHS can be recovered from a pro-nilpotent operator on the real homotopy type $H^*(X, \mathbb{R})$, which we regard as

This work was supported by Trinity College, Cambridge; and the Engineering and Physical Sciences Research Council [grant number EP/F043570/1].

monodromy at the Archimedean place. The isomorphism above says that the MHS on $\pi_*(X \otimes \mathbb{R}, x_0)$ has an \mathcal{S} -splitting, and by Proposition 1.26, this is true for all mixed Hodge structures. However, the special feature here is that the splitting is canonical, so preserves the additional structure (such as Whitehead brackets).

For non-nilpotent topological spaces, the rational homotopy type is too crude an invariant to recover much information, so schematic homotopy types were introduced in [Toë], based on ideas from [Gro2]. [Pri3] showed how to recover the groups $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{R}$ from schematic homotopy types for very general topological spaces, and also introduced the intermediate notion of relative Malcev homotopy type, simultaneously generalising both rational and schematic homotopy types. In Corollary 3.11 we will see how relative Malcev homotopy types govern the variation of real homotopy types in a fibration.

Since their inception, one of the main goals of schematic homotopy types has been to define and construct mixed Hodge structures. This programme was initiated in [KPS], and continued in [KPT1]. Although the structures in [KPT1] have important consequences, such as proving that the image of the Hurewicz map is a sub-Hodge structure, they are too weak to give rise to mixed Hodge structures on the homotopy groups, and disagree with the weight filtration on rational homotopy groups defined in [Mor] (see Remark 5.15).

In this paper, we take an alternative approach, giving a new notion of mixed Hodge structures on schematic (and relative Malcev) homotopy types which is compatible with [Mor] (Proposition 5.6). These often yield mixed Hodge structures on the full homotopy groups $\pi_n(X, x_0)$ (rather than just on rational homotopy groups). In Corollaries 5.16 and 6.13 we show not only that the homotopy types of compact Kähler manifolds naturally carry such mixed Hodge structures, but also that they also split and become formal on tensoring with \mathcal{S} . The structure in [KPT1] can then be understood as an invariant of the \mathcal{S} -splitting, rather than of the MHS itself (Remark 6.4). Corollary 7.7 shows that these MHS become variations of mixed Hodge structure as the basepoint varies.

We then adapt this approach to construct mixed Hodge and mixed twistor structures for relative Malcev homotopy types of quasi-projective varieties (Theorems 10.22 and 10.23), but only when the monodromy around the divisor is trivial. Theorem 11.16 addresses a more general case, allowing unitary monodromy around the divisor. Whereas the \mathcal{S} -splittings for projective varieties are realised concretely using the principle of two types, the last part of the paper establishes abstract existence results for \mathcal{S} -splittings of general mixed Hodge and mixed twistor structures (Corollary 13.19). These latter results are then used to construct mixed Hodge and mixed twistor structures on relative Malcev homotopy groups of quasi-projective varieties (Corollaries 13.21 and 13.30).

The structure of the paper is as follows.

In Section 1, we introduce our non-abelian notions of algebraic mixed Hodge and twistor structures. If we define $C^* = (\prod_{\mathbb{C}/\mathbb{R}} \mathbb{A}^1) - \{0\} \cong \mathbb{A}^2 - \{0\}$ and $S = \prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ by Weil restriction of scalars, then our first major observation (Corollary 1.9) is that real vector spaces V equipped with filtrations F on $V \otimes \mathbb{C}$ correspond to flat quasi-coherent modules on the stack $[C^*/S]$, via a Rees module construction, with V being the pullback along $1 \in C^*$. This motivates us to define an algebraic Hodge filtration on a real object Z as an extension of Z over the base stack $[C^*/S]$. This is similar to the approach taken by Kapranov to define mixed Hodge structures in [Kap]; see Remark 1.10 for details. The morphism $\mathrm{SL}_2 \rightarrow C^*$ given by projection of the 1st row corresponds to the Hodge filtration on the ring \mathcal{S} above, and has important universal properties.

Similarly, filtered vector spaces correspond to flat quasi-coherent modules on the stack $[\mathbb{A}^1/\mathbb{G}_m]$, so we define an algebraic mixed Hodge structure on Z to consist of an extension Z_{MHS} over $[\mathbb{A}^1/\mathbb{G}_m] \times [C^*/S]$, with additional data corresponding to an opposedness condition (Definition 1.37). This gives rise to non-abelian mixed Hodge structures in the sense of [KPS], as explained in Remark 1.40. In some cases, a mixed Hodge structure is

too much to expect, and we then give an extension over $[\mathbb{A}^1/\mathbb{G}_m] \times [C^*/\mathbb{G}_m]$: an algebraic mixed twistor structure. For vector bundles, algebraic mixed Hodge and twistor structures coincide with the classical definitions (Propositions 1.41 and 1.49).

Section 2 contains most of the results related to real homotopy types. Corollary 2.12 constructs a non-abelian mixed Hodge structure on the real homotopy type. Moreover, there is an S -equivariant morphism $\text{row}_1: \text{SL}_2 \rightarrow C^*$ corresponding to projection of the first row; all of the structures split on pulling back along row_1 , and these pullbacks can be recovered from cohomology of local systems. This is because the principle of two types (or the dd^c -lemma) holds for any pair $ud + vd^c, xd + yd^c$ of operators, provided $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \in \text{GL}_2$. The pullback row_1 corresponds to tensoring with the algebra \mathcal{S} described above. Proposition 2.18 shows how this pullback to SL_2 can be regarded as an analogue of the limit mixed Hodge structure, while Proposition 2.13, Corollary 2.21 and Proposition 2.14 show how it is closely related to real Deligne cohomology, Consani's Archimedean cohomology and Deninger's Γ -factor of X at the Archimedean place.

Section 3 is mostly a review of the relative Malcev homotopy types introduced in [Pri3], generalising both schematic and real homotopy types, with some new results in §3.3 on homotopy types over general bases (rather than just over fields). Major new results are Theorem 3.10 and Corollary 3.11, which show how relative homotopy types arise naturally in the study of fibrations. Theorem 3.29 adapts the main comparison result of [Pri3] to the case of fixed basepoints.

In Section 4, the constructions of Section 1 are then extended to homotopy types. The main result is Theorem 4.20, showing how non-abelian algebraic mixed Hodge and twistor structures on relative Malcev homotopy types give rise to such structures on homotopy groups, while Proposition 5.6 shows that these are compatible with Morgan's mixed Hodge structures on rational homotopy types and groups.

In the next two sections, we establish the existence of algebraic mixed Hodge structures on various relative Malcev homotopy types of compact Kähler manifolds, giving more information than rational homotopy types when X is not nilpotent (Corollaries 5.16 and 6.13). The starting point is the Hodge structure defined on the reductive complex pro-algebraic fundamental group $\varpi_1(X, x_0)_{\mathbb{C}}^{\text{red}}$ in [Sim3], in the form of a discrete \mathbb{C}^* -action. We only make use of the induced action of $S^1 \subset \mathbb{C}^*$, since this preserves the real form $\varpi_1(X, x_0)_{\mathbb{R}}^{\text{red}}$ and respects the harmonic metric. We regard this as a kind of pure weight 0 Hodge structure on $\varpi_1(X, x_0)_{\mathbb{R}}^{\text{red}}$, since a pure weight 0 Hodge structure is the same as an algebraic S^1 -action. We extend this to a mixed Hodge structure on the schematic (or relative Malcev) homotopy type (Theorem 5.14 and Proposition 6.3).

In some contexts, the unitary action is incompatible with the relative Malcev representation. In these cases, we instead only have mixed twistor structures (as defined in [Sim2]) on the homotopy type (Theorem 6.1) and homotopy groups (Corollary 6.2).

Section 7 shows how representations of $\varpi_1(X, x_0)$ in the category of mixed Hodge structures correspond to variations of mixed Hodge structure (VMHS) on X (Theorem 7.6). This implies (Corollary 7.7) that the relative Malcev homotopy groups become VMHS as the basepoint varies. Taking the case of π_1 , this proves [Ara] Conjecture 5.5 (see Remarks 5.18 and 7.9 for details).

Section 8 is dedicated to describing the mixed Hodge structure on homotopy types in terms of a pro-nilpotent derivation on the split Hodge structure over SL_2 . It provides an explicit description of this derivation in terms of Green's operators on the complex of \mathcal{C}^∞ forms on X , and in particular shows that the real Hodge structure on $\pi_3(X) \otimes \mathbb{R}$ is split whenever X is simply connected (Examples 8.15.2).

In Section 9, we extend the results of Sections 5 and 6 to simplicial compact Kähler manifolds, and hence to singular proper complex varieties.

Section 10 then deals with the Malcev homotopy type $(Y, y)^{\rho, \text{Mal}}$ of a quasi-projective variety $Y = X - D$ with respect to a Zariski-dense representation $\rho: \pi_1(X, y) \rightarrow R(\mathbb{R})$.

When Y is smooth, Theorem 10.22 establishes a non-positively weighted MTS on $(Y, y)^{\rho, \text{Mal}}$, with the associated graded object $\text{gr}^W(Y, y)^{\rho, \text{Mal}}$ corresponding to the R -equivariant DGA

$$\left(\bigoplus_{a,b} H^{a-b}(X, \mathbf{R}^b j_* j^{-1} \mathbb{O}(R))[-a], d_2 \right),$$

where $d_2: H^{a-b}(X, \mathbf{R}^b j_* j^{-1} \mathbb{O}(R)) \rightarrow H^{a-b+2}(X, \mathbf{R}^{b-1} j_* j^{-1} \mathbb{O}(R))$ is the differential on the E_2 sheet of the Leray spectral sequence for $j: Y \rightarrow X$, and $H^{a-b}(X, \mathbf{R}^b j_* j^{-1} \mathbb{O}(R))$ has weight $a + b$. Theorem 10.23 shows that if R -representations underlie variations of Hodge structure, then the MTS above extends to a non-positively weighted MHS on $(Y, y)^{\rho, \text{Mal}}$. Theorem 10.26 gives the corresponding results for singular quasi-projective varieties Y , with $\text{gr}^W(Y, y)^{\rho, \text{Mal}}$ now characterised in terms of cohomology of a smooth simplicial resolution of Y .

In Section 11, these results are extended to Zariski-dense representations $\rho: \pi_1(Y, y) \rightarrow R(\mathbb{R})$ with unitary monodromy around local components of the divisor. The construction of MHS and MTS in these cases is much trickier than for trivial monodromy. The idea behind Theorem 11.16, inspired by [Mor], is to construct the Hodge filtration on the complexified homotopy type, and then to use homotopy limits of diagrams to glue this to give a real form. When R -representations underlie variations of Hodge structure on Y , this gives a non-positively weighted MHS on $(Y, y)^{\rho, \text{Mal}}$, with $\text{gr}^W(Y, y)^{\rho, \text{Mal}}$ corresponding to the R -equivariant DGA

$$\left(\bigoplus_{a,b} H^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2 \right),$$

regarded as a Hodge structure via the VHS structure on $\mathbb{O}(R)$. For more general R , Theorem 11.19 gives a non-positively weighted MTS, with the construction based on homotopy gluing over an affine cover of the analytic space $\mathbb{P}^1(\mathbb{C})$. Simplicial resolutions then extend these results to singular varieties in Theorems 11.21 and 11.22. §11.5 discusses possible extensions to more general monodromy.

Section 12 is concerned with splittings of MHS and MTS on finite-dimensional vector spaces. Every mixed Hodge structure V splits on tensoring with the ring \mathcal{S} defined above, giving an \mathcal{S} -linear isomorphism $V \otimes \mathcal{S} \cong (\text{gr}^W V) \otimes \mathcal{S}$ preserving the Hodge filtration F . Differentiating with respect to V , this gives a map $\beta: (\text{gr}^W V) \rightarrow (\text{gr}^W V) \otimes \Omega(\mathcal{S}/\mathbb{R})$ from which V can be recovered. Theorem 12.6 shows that the \mathcal{S} -splitting can be chosen canonically, corresponding to imposing certain restrictions on β , and this gives an equivalence of categories. In Remark 12.9, β is explicitly related to Deligne's complex splitting of [Del4]. Theorem 12.13 then gives the corresponding results for mixed twistor structures.

The main result in Section 13 is Theorem 13.14, which shows that every non-positively weighted MHS or MTS on a real relative Malcev homotopy type admits a strictification, in the sense that it is represented by an R -equivariant DGA in ind-MHS or ind-MTS. Corollary 13.19 then applies the results of Section 12 to give canonical \mathcal{S} -splittings for such MHS or MTS, while Corollary 13.20 shows that the splittings give equivalences $(Y, y)^{\rho, \text{Mal}} \simeq \text{gr}^W(Y, y)^{\rho, \text{Mal}}$. Corollary 13.21 shows that they give rise to MHS or MTS on homotopy groups, and this is applied to quasi-projective varieties in Corollary 13.30. There are various consequences for deformations of representations (Proposition 13.28). Finally, Theorem 13.33 shows that for projective varieties, the canonical \mathcal{S} -splittings coincide with the explicit Green's operator \mathcal{S} -splittings established in Theorems 5.14 and 6.1.

Acknowledgements. I would like to thank Carlos Simpson for drawing my attention to the questions addressed in this paper, and for helpful discussions, especially concerning

the likely generality for the results in §11. I would also like to thank Jack Morava for suggesting that non-abelian mixed Hodge structures should be related to Archimedean Γ -factors, and Tony Pantev for alerting me to [Del4].

Notation. For any affine scheme Y , write $O(Y) := \Gamma(Y, \mathcal{O}_Y)$.

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1. NON-ABELIAN STRUCTURES

1.1. Hodge filtrations. In this section, we will define algebraic Hodge filtrations on real affine schemes. This construction is essentially that of [Sim1, §5], with the difference that we are working over \mathbb{R} rather than \mathbb{C} .

Definition 1.1. Define C to be the real affine scheme $\prod_{\mathbb{C}/\mathbb{R}} \mathbb{A}^1$ obtained from $\mathbb{A}_{\mathbb{C}}^1$ by restriction of scalars, so for any real algebra A , $C(A) = \mathbb{A}_{\mathbb{C}}^1(A \otimes_{\mathbb{R}} \mathbb{C}) \cong A \otimes_{\mathbb{R}} \mathbb{C}$. Choosing $i \in \mathbb{C}$ gives an isomorphism $C \cong \mathbb{A}_{\mathbb{R}}^2$, and we let C^* be the quasi-affine scheme $C - \{0\}$.

Define S to be the real algebraic group $\prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ obtained as in [Del1, 2.1.2] from $\mathbb{G}_{m,\mathbb{C}}$ by restriction of scalars. Note that there is a canonical inclusion $\mathbb{G}_m \hookrightarrow S$, and that S acts on C and C^* by inverse multiplication, i.e.

$$\begin{aligned} S \times C &\rightarrow C \\ (\lambda, w) &\mapsto (\lambda^{-1}w). \end{aligned}$$

Remark 1.2. A more standard S -action is given by the inclusion $S \hookrightarrow \mathbb{A}^2 \cong C$. However, we wish C to be of weight -1 rather than $+1$.

Remark 1.3. Fix an isomorphism $C \cong \mathbb{A}^2$, with co-ordinates u, v on C so that the isomorphism $C(\mathbb{R}) \cong \mathbb{C}$ is given by $(u, v) \mapsto u + iv$. Thus the algebra $O(C)$ associated to C is the polynomial ring $\mathbb{R}[u, v]$. S is isomorphic to the scheme $\mathbb{A}_{\mathbb{R}}^2 - \{(u, v) : u^2 + v^2 = 0\}$. On $C_{\mathbb{C}}$, we have alternative co-ordinates $w = u + iv$ and $\bar{w} = u - iv$, which give the standard isomorphism $S_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. Note that on C the co-ordinates w and \bar{w} are of types $(-1, 0)$ and $(0, -1)$ respectively.

Definition 1.4. Given an affine scheme X over \mathbb{R} , we define an algebraic Hodge filtration $X_{\mathbb{F}}$ on X to consist of the following data:

- (1) an S -equivariant affine morphism $X_{\mathbb{F}} \rightarrow C^*$,
- (2) an isomorphism $X \cong X_{\mathbb{F},1} := X_{\mathbb{F}} \times_{C^*,1} \text{Spec } \mathbb{R}$.

Definition 1.5. A real splitting of the Hodge filtration $X_{\mathbb{F}}$ consists of an S -action on X , and an S -equivariant isomorphism

$$X \times C^* \cong X_{\mathbb{F}}$$

over C^* .

Remark 1.6. Note that giving $X_{\mathbb{F}}$ as above is equivalent to giving the affine morphism $[X_{\mathbb{F}}/S] \rightarrow [C^*/S]$ of stacks. This fits in with the idea in [KPS] that if $\mathfrak{D}\mathfrak{B}\mathfrak{J}$ is an ∞ -stack parametrising some ∞ -groupoid of objects, then the groupoid of non-abelian filtrations of this object is $\mathcal{H}om([\mathbb{A}^1/\mathbb{G}_m], \mathfrak{D}\mathfrak{B}\mathfrak{J})$.

Now, we may regard a quasi-coherent sheaf \mathcal{F} on a stack \mathfrak{X} as equivalent to the affine cogroup $\mathrm{Spec}(\mathcal{O}_{\mathfrak{X}} \oplus \mathcal{F})$ over \mathfrak{X} . This gives us a notion of an algebraic Hodge filtration on a real vector space. We now show how this is equivalent to the standard definition.

Lemma 1.7. *There is an equivalence of categories between flat quasi-coherent \mathbb{G}_m -equivariant sheaves on \mathbb{A}^1 , and exhaustive filtered vector spaces, where \mathbb{G}_m acts on \mathbb{A}^1 via the standard embedding $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$.*

Proof. Let t be the co-ordinate on \mathbb{A}^1 , and M global sections of a \mathbb{G}_m -equivariant sheaf on \mathbb{A}^1 . Since M is flat, $0 \rightarrow M \xrightarrow{t} M \rightarrow M \otimes_{k[t],0} k \rightarrow 0$ is exact, so t is an injective endomorphism. The \mathbb{G}_m -action is equivalent to giving a decomposition $M = \bigoplus M_n$, and we have $t : M_n \hookrightarrow M_{n+1}$. Thus the images of $\{M_n\}_{n \in \mathbb{Z}}$ give a filtration on $M \otimes_{k[t],1} k$.

Conversely, set M to be the Rees module $\xi(V, F) := \bigoplus F_n V$, with \mathbb{G}_m -action given by setting $F_n V$ to be weight n , and the $k[t]$ -module structure determined by letting t be the inclusion $F_n V \hookrightarrow F_{n+1} V$. If I is a $k[t]$ -ideal, then $I = (f)$, since $k[t]$ is a principal ideal domain. The map $M \otimes I \rightarrow M$ is thus isomorphic to $f : M \rightarrow M$. Writing $f = \sum a_n t^n$, we see that it is injective on $M = \bigoplus M_n$. Thus $M \otimes I \rightarrow M$ is injective, so M is flat by [Mat, Theorem 7.7]. \square

Remark 1.8. We might also ask what happens if we relax the condition that the filtration be flat, since non-flat structures might sometimes arise as quotients.

An arbitrary algebraic filtration on a real vector space V is a system W_r of complex vector spaces with (not necessarily injective) linear maps $s : W_r \rightarrow W_{r+1}$, such that $\lim_{\substack{\rightarrow \\ r \rightarrow \infty}} W_r \cong V$.

Corollary 1.9. *The category of flat algebraic Hodge filtrations on real vector spaces is equivalent to the category of pairs (V, F) , where V is a real vector space and F an exhaustive decreasing filtration on $V \otimes_{\mathbb{R}} \mathbb{C}$. A real splitting of the Hodge filtration is equivalent to giving a real Hodge structure on V (i.e. an S -action).*

Proof. The flat algebraic Hodge filtration on V gives an S -module $\xi(V, \mathbb{F})$ on C^* , with $\xi(V, \mathbb{F})|_1 = V$. Observe that $C^* \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{A}_{\mathbb{C}}^2 - \{0\}$, and $S \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{G}_m \times \mathbb{G}_m$, compatible with the usual actions, the isomorphisms given by $(u, v) \mapsto (u + iv, u - iv)$. Writing $\mathbb{A}_{\mathbb{C}}^2 - \{0\} = (\mathbb{A}^1 \times \mathbb{G}_m) \cup (\mathbb{G}_m \times \mathbb{A}^1)$, we see that giving $\xi(V, \mathbb{F}) \otimes \mathbb{C}$ amounts to giving two filtrations (F, F') on $V \otimes_{\mathbb{R}} \mathbb{C}$, which is the fibre over $(1, 1)$ in the new co-ordinates. The real structure determines behaviour under complex conjugation, with $F' = \bar{F}$. If we set $M \subset \xi(V \otimes \mathbb{C}; F, \bar{F})$ to be the real elements, then $\xi(V, \mathbb{F}) = j^{-1}M$. \square

Remark 1.10. Although flat quasi-coherent sheaves on $[C^*/S]$ also correspond to flat quasi-coherent sheaves on $[C/S]$, we do not follow [Kap] in working over the latter, since many natural non-flat objects arise on $[C/S]$ whose behaviour over $0 \in C$ is pathological. However, our approach has the disadvantage that we cannot simply describe the bigraded vector space $\mathrm{gr}_F \mathrm{gr}_{\bar{F}} V$, which would otherwise be given by pulling back along $[0/S] \rightarrow [C/S]$.

The motivating example comes from the embedding $\mathcal{H}^* \rightarrow A^\bullet$ of real harmonic forms into the real de Rham algebra of a compact Kähler manifold. This gives a quasi-isomorphism of the associated complexes on $[C^*/S]$, since the maps $F^p(\mathcal{H}^* \otimes \mathbb{C}) \rightarrow F^p(A^\bullet \otimes \mathbb{C})$ are quasi-isomorphisms. However, the associated map on $[C/S]$ is not a quasi-isomorphism, as this would force the derived pullbacks to $0 \in C$ to be quasi-isomorphic, implying that the maps $\mathcal{H}^{pq} \rightarrow A^{pq}$ be isomorphisms.

Remark 1.11. We might also ask what happens if we relax the condition that the Hodge filtration be flat.

An arbitrary algebraic Hodge filtration on a real vector space V is a system F^p of complex vector spaces with (not necessarily injective) linear maps $s : F^p \rightarrow F^{p-1}$, such that $\lim_{\rightarrow p \rightarrow -\infty} F^p \cong V \otimes \mathbb{C}$.

Definition 1.12. Let $\widetilde{C}^* \rightarrow C^*$ be the étale covering of C^* given by cutting out the divisor $\{u - iv = 0\}$ from $C^* \otimes_{\mathbb{R}} \mathbb{C}$, for co-ordinates u, v as in Remark 1.3.

Lemma 1.13. *There is an equivalence of categories between flat S -equivariant quasi-coherent sheaves on \widetilde{C}^* , and exhaustive filtrations on complex vector spaces.*

Proof. First, observe that there is an isomorphism $\widetilde{C}^* \cong \mathbb{A}_{\mathbb{C}}^1 \times \mathbb{G}_{m, \mathbb{C}}$, given by $(u, v) \mapsto (u + iv, u - iv)$. As in Corollary 1.9, $S_{\mathbb{C}} \cong \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$ under the same isomorphism. Thus S -equivariant quasi-coherent sheaves on \widetilde{C}^* are equivalent to $\mathbb{G}_{m, \mathbb{C}} \times 1$ -equivariant quasi-coherent sheaves on the scheme $\mathbb{A}_{\mathbb{C}}^1 \subset \widetilde{C}^*_0$ given by $u - iv = 1$. Now apply Lemma 1.7. \square

1.1.1. SL_2 .

Definition 1.14. Define maps $\text{row}_1, \text{row}_2 : GL_2 \rightarrow \mathbb{A}^2$ by projecting onto the first and second rows, respectively. If we make the identification $C = \mathbb{A}^2$ of Definition 1.1, then these are equivariant with respect to the right S -action $GL_2 \times S \rightarrow GL_2$, given by $(A, \lambda) \mapsto A \begin{pmatrix} \Re \lambda & \Im \lambda \\ -\Im \lambda & \Re \lambda \end{pmatrix}^{-1}$.

Definition 1.15. Define an S -action on SL_2 by

$$(\lambda, A) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \lambda \bar{\lambda} \end{pmatrix} A \begin{pmatrix} \Re \lambda & \Im \lambda \\ -\Im \lambda & \Re \lambda \end{pmatrix}^{-1} = \begin{pmatrix} \lambda \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} \Re \lambda & -\Im \lambda \\ \Im \lambda & \Re \lambda \end{pmatrix}.$$

Let $\text{row}_1 : SL_2 \rightarrow C^*$ be the S -equivariant map given by projection onto the first row.

Remark 1.16. Observe that, as an S -equivariant scheme over C^* , we may decompose GL_2 as $GL_2 = \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{G}_m \end{pmatrix} \times SL_2$, where the S -action on \mathbb{G}_m has λ acting as multiplication by $(\lambda \bar{\lambda})^{-1}$.

We may also write $C^* = [SL_2/\mathbb{G}_a]$, where \mathbb{G}_a acts on SL_2 as left multiplication by $\begin{pmatrix} 1 & 0 \\ \mathbb{G}_a & 1 \end{pmatrix}$, where the S -action on \mathbb{G}_a has λ acting as multiplication by $\lambda \bar{\lambda}$.

Lemma 1.17. *The morphism $\text{row}_1 : SL_2 \rightarrow C^*$ is weakly final in the category of S -equivariant affine schemes over C^* .*

Proof. We need to show that for any affine scheme U equipped with an S -equivariant morphism $f : U \rightarrow C^*$, there exists a (not necessarily unique) S -equivariant morphism $g : U \rightarrow SL_2$ such that $f = \text{row}_1 \circ g$.

If $U = \text{Spec } A$, then A is an $O(C) = \mathbb{R}[u, v]$ -algebra, with the ideal $(u, v)_A = A$, so there exist $a, b \in A$ with $ua - vb = 1$. Thus the map factors through $\text{row}_1 : SL_2 \rightarrow C^*$. Complexifying gives an expression $\alpha w + \beta \bar{w} = 1$, for w, \bar{w} as in Remark 1.3. Now splitting α, β into types, we have $\alpha^{10} w + \beta^{01} \bar{w} = 1$. Similarly, $\frac{1}{2}(\alpha^{10} + \beta^{01}) w + \frac{1}{2}(\beta^{01} + \alpha^{10}) \bar{w} = 1$, on conjugating and averaging. Write this as $\alpha' w + \beta' \bar{w} = 1$. Finally, note that $y := \alpha' + \beta'$, $-x := i\alpha' - i\beta'$ are both real, giving $uy - vx = 1$, with x, y having the appropriate S -action to regard A as an $O(SL_2)$ -algebra when SL_2 has co-ordinates $\begin{pmatrix} u & v \\ x & y \end{pmatrix}$. \square

Remark 1.18. Observe that for our action of $\mathbb{G}_m \subset S$ (corresponding to left multiplication by diagonal matrices) on SL_2 , the stack $[SL_2/\mathbb{G}_m]$ is just the affine scheme $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)$. Here, Δ is the diagonal embedding, and the projections to \mathbb{P}^1 correspond to the maps $\text{row}_1, \text{row}_2 : [SL_2/\mathbb{G}_m] \rightarrow [(\mathbb{A}^2 - \{0\})/\mathbb{G}_m]$ (noting that for row_2 this means taking the inverse of our usual \mathbb{G}_m -action on C^*). Lemma 1.17 can then be reformulated to say that $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)$ is weakly final in the category of S^1 -equivariant affine schemes over \mathbb{P}^1 .

Lemma 1.19. *The affine scheme $\mathrm{SL}_2 \xrightarrow{\mathrm{row}_1} C^*$ is a flat algebraic Hodge filtration, corresponding to the algebra*

$$\mathcal{S} := \mathbb{R}[x],$$

with filtration $F^p(\mathcal{S} \otimes \mathbb{C}) = (x - i)^p \mathbb{C}[x]$.

Proof. Since row_1 is flat and equivariant for the inverse right S -action, we know by Corollary 1.9 that we have a filtration on $\mathcal{S} \otimes \mathbb{C}$, for $\mathrm{Spec} \mathcal{S} = \mathrm{SL}_2 \times_{\mathrm{row}_1, C^*, 1} \mathrm{Spec} \mathbb{R}$. $\mathrm{Spec} \mathcal{S}$ consists of invertible matrices $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, giving \mathcal{S} the ring structure claimed.

To describe the filtration, we use Lemma 1.13, considering the pullback of row_1 along $\widetilde{C}^* \rightarrow C^*$. The scheme $\widetilde{\mathrm{SL}}_2 := \mathrm{SL}_2 \times_{\mathrm{row}_1, C^*} \widetilde{C}^*$ is isomorphic to $\widetilde{C}^* \times \mathbb{A}^1$, with projection onto $\mathbb{A}_{\mathbb{C}}^1$ given by $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \mapsto x - iy$. This isomorphism is moreover $S_{\mathbb{C}}$ -equivariant over \widetilde{C}^* , when we set the co-ordinates of \mathbb{A}^1 to be of type $(1, 0)$.

The filtration F on $\mathcal{S} \otimes \mathbb{C}$ then just comes from the decomposition on $\mathbb{C}[x - iy]$ associated to the action of $\mathbb{G}_{m, \mathbb{C}} \times \{1\} \subset S_{\mathbb{C}}$, giving

$$F^p \mathbb{C}[x - iy] = \bigoplus_{p' \geq p} (x - iy)^{-p'} \mathbb{C}.$$

The filtration on $\mathcal{S} \otimes \mathbb{C}$ is given by evaluating this at $y = 1$, giving $F^p(\mathcal{S} \otimes \mathbb{C}) = (x - i)^p \mathbb{C}[x]$, as required.

For an explicit inverse construction, the complex Rees module $\bigoplus_{p, q \in \mathbb{Z}} w^{-p} \bar{w}^{-q} F^p \bar{F}^q \mathcal{S}$ associated to \mathcal{S} is the $\mathbb{C}[w, \bar{w}]$ -subalgebra of $(\mathcal{S} \otimes \mathbb{C})[w, w^{-1}, \bar{w}, \bar{w}^{-1}]$ generated by $\bar{z} := w^{-1}(x - i)$ and $z := \bar{w}^{-1}(x + i)$. These satisfy the sole relation $w\bar{z} - \bar{w}z = -2i$, so $\begin{pmatrix} u & v \\ \xi & \eta \end{pmatrix} \in \mathrm{SL}_2$, where $z = \xi + i\eta$, $\bar{z} = \xi - i\eta$. \square

Remark 1.20. We may now reinterpret Lemma 1.17 in terms of Hodge filtrations. An S -equivariant affine scheme, flat over C^* , is equivalent to a real algebra A , equipped with an exhaustive decreasing filtration F on $A \otimes_{\mathbb{R}} \mathbb{C}$, such that $\mathrm{gr}_F \mathrm{gr}_{\bar{F}}(A \otimes_{\mathbb{R}} \mathbb{C}) = 0$. This last condition is equivalent to saying that $1 \in F^1 + \bar{F}^1$, or even that there exists $\alpha \in F^1(A \otimes_{\mathbb{R}} \mathbb{C})$ with $\Re \alpha = 1$. We then define a homomorphism $f : \mathcal{S} \rightarrow A$ by setting $f(x) = \Im \alpha$, noting that $f(1 + ix) = \alpha \in F^1(A \otimes_{\mathbb{R}} \mathbb{C})$, so f respects the Hodge filtration.

We may make use of the covering $\mathrm{row}_1 : \mathrm{SL}_2 \rightarrow C^*$ to give an explicit description of the derived direct image $\mathbf{R}j_* \mathcal{O}_{C^*}$ as a DG algebra on C , for $j : C^* \rightarrow C$, as follows.

Definition 1.21. The \mathbb{G}_a -action on SL_2 of Remark 1.16 gives rise to an action of the associated Lie algebra $\mathfrak{g}_a \cong \mathbb{R}$ on $O(\mathrm{SL}_2)$. Explicitly, define the standard generator $N \in \mathfrak{g}_a$ to act as the derivation with $Nx = u, Ny = v, Nu = Nv = 0$, for co-ordinates $\begin{pmatrix} u & v \\ x & y \end{pmatrix}$ on SL_2 .

This is equivalent to the $O(\mathrm{SL}_2)$ -linear isomorphism $\Omega(\mathrm{SL}_2/C) \rightarrow O(\mathrm{SL}_2)$ given by $dx \mapsto u, dy \mapsto v$. This is not S -equivariant, but has type $(-1, -1)$, so we write $\Omega(\mathrm{SL}_2/C) \cong O(\mathrm{SL}_2)(-1)$.

The DG algebra $O(\mathrm{SL}_2) \xrightarrow{N\epsilon} O(\mathrm{SL}_2)(-1)\epsilon$, for ϵ of degree 1, is an algebra over $O(C) = \mathbb{R}[u, v]$, so we may consider the DG algebra $j^{-1}O(\mathrm{SL}_2) \xrightarrow{N\epsilon} j^{-1}O(\mathrm{SL}_2)(-1)\epsilon$ on C^* , for $j : C^* \rightarrow C$. This is an acyclic resolution of the structure sheaf \mathcal{O}_{C^*} , so

$$\mathbf{R}j_* \mathcal{O}_{C^*} \simeq j_*(j^{-1}O(\mathrm{SL}_2) \xrightarrow{N\epsilon} j^{-1}O(\mathrm{SL}_2)(-1)\epsilon) = (O(\mathrm{SL}_2) \xrightarrow{N\epsilon} O(\mathrm{SL}_2)(-1)\epsilon),$$

regarded as an $O(C)$ -algebra. This construction is moreover S -equivariant.

Definition 1.22. From now on, we will denote the DG algebra $O(\mathrm{SL}_2) \xrightarrow{N\epsilon} O(\mathrm{SL}_2)(-1)\epsilon$ by $\mathbf{R}O(C^*)$, thereby making a canonical choice of representative in the equivalence class $\mathbf{R}j_* \mathcal{O}_{C^*}$.

Definition 1.23. Define a (real) quasi-MHS to be a real vector space V , equipped with an exhaustive increasing filtration W on V , and an exhaustive decreasing filtration F on $V \otimes \mathbb{C}$.

We adopt the convention that a (real) MHS is a finite-dimensional quasi-MHS on which W is Hausdorff, satisfying the opposedness condition

$$\mathrm{gr}_n^W \mathrm{gr}_F^i \mathrm{gr}_{\bar{F}}^j (V \otimes \mathbb{C}) = 0$$

for $i + j \neq n$.

Define a (real) ind-MHS to be a filtered direct limit of MHS. Say that an ind-MHS is bounded below if $W_N V = 0$ for $N \ll 0$.

Example 1.24. The ring \mathcal{S} of Lemma 1.19 can be given the structure of a quasi-MHS with the weight filtration $W_0 \mathcal{S} = \mathcal{S}$, $W_{-1} \mathcal{S} = 0$, but is not an ind-MHS.

Definition 1.25. Given a quasi-MHS V , define the decreasing filtration γ^* on V by $\gamma^p V = V \cap F^p(V \otimes \mathbb{C})$.

Proposition 1.26. *Every (finite-dimensional abelian) MHS V admits an \mathcal{S} -splitting, i.e. an \mathcal{S} -linear isomorphism*

$$V \otimes \mathcal{S} \cong (\mathrm{gr}^W V) \otimes \mathcal{S},$$

of quasi-MHS, inducing the identity on the grading associated to W . The set of such splittings is a torsor for the group $\mathrm{id} + W_{-1} \gamma^0 \mathrm{End}((\mathrm{gr}^W V) \otimes \mathcal{S})$.

Proof. We proceed by induction on the weight filtration. \mathcal{S} -linear extensions $0 \rightarrow W_{n-1} V \otimes \mathcal{S} \rightarrow W_n V \otimes \mathcal{S} \rightarrow \mathrm{gr}_n^W V \otimes \mathcal{S} \rightarrow 0$ of quasi-MHS are parametrised by

$$\mathrm{Ext}_{\mathbb{A}^1 \times \mathrm{SL}_2}^1(\mathrm{gr}^W V \otimes O(\mathbb{A}^1) \otimes O(\mathrm{SL}_2), \xi(W_{n-1} V, W, F, \bar{F}) \mathrm{ten}_{O(C)} O(\mathrm{SL}_2))^{\mathbb{G}_m \times S},$$

since $\mathbb{G}_m \times S$ is (linearly) reductive. Now, $\mathrm{gr}^W V \otimes O(\mathbb{A}^1) \otimes O(\mathrm{SL}_2)$ is a projective $O(\mathbb{A}^1) \otimes O(\mathrm{SL}_2)$ -module, so its higher Exts are all 0, and all \mathcal{S} -linear quasi-MHS extensions of $\mathrm{gr}_n^W V \otimes \mathcal{S}$ by $W_{n-1} V \otimes \mathcal{S}$ are isomorphic, so $W_n V \otimes \mathcal{S} \cong W_{n-1} V \otimes \mathcal{S} \oplus \mathrm{gr}_n^W V \otimes \mathcal{S}$.

Finally, observe that any two splittings differ by a unique automorphism of $(\mathrm{gr}^W V) \otimes \mathcal{S}$, preserving the quasi-MHS structure, and inducing the identity on taking gr^W . This group is just $\mathrm{id} + W_{-1} \gamma^0 \mathrm{End}((\mathrm{gr}^W V) \otimes \mathcal{S})$, as required. \square

1.1.2. *Cohomology of Hodge filtrations.* Given a complex \mathcal{F}^\bullet of algebraic Hodge filtrations, we now show how to calculate hypercohomology $\mathbb{H}^*([C^*/S], \mathcal{F}^\bullet)$, and compare this with Beilinson's weak Hodge cohomology.

Considering the étale pushout $C^* = \widetilde{C}^* \cup_{S_{\mathbb{C}}} S$ of affine schemes, $\mathbf{R}\Gamma(C^*, \mathcal{F}^\bullet)$ is the cone of the morphism

$$\mathbf{R}\Gamma(\widetilde{C}^*, \mathcal{F}^\bullet) \oplus \mathbf{R}\Gamma(S, \mathcal{F}^\bullet) \rightarrow \mathbf{R}\Gamma(S_{\mathbb{C}}, \mathcal{F}^\bullet).$$

If \mathcal{F}^\bullet is a flat complex, it corresponds under Corollary 1.9 to a complex V^\bullet of real vector spaces, equipped with an exhaustive filtration F of $V_{\mathbb{C}}^\bullet := V^\bullet \otimes \mathbb{C}$. The expression above then becomes

$$\left(\bigoplus_{n \in \mathbb{Z}} F^n(V_{\mathbb{C}}^\bullet) w^{-n} [\bar{w}, \bar{w}^{-1}] \oplus V_{\mathbb{R}}^\bullet[u, v, (u^2 + v^2)^{-1}] \right) \rightarrow V_{\mathbb{C}}^\bullet[w, w^{-1}, \bar{w}, \bar{w}^{-1}],$$

for co-ordinates u, v and w, \bar{w} on C^* as in Remark 1.3.

Since S is a reductive group, taking S -invariants is an exact functor, so $\mathbf{R}\Gamma([C^*/S], \mathcal{F}^\bullet)$ is the cone of the morphism

$$\mathbf{R}\Gamma(\widetilde{C}^*, \mathcal{F}^\bullet)^S \oplus \mathbf{R}\Gamma(S, \mathcal{F}^\bullet)^S \rightarrow \mathbf{R}\Gamma(S_{\mathbb{C}}, \mathcal{F}^\bullet)^S$$

which is just

$$F^0(V_{\mathbb{C}}^\bullet) \oplus V_{\mathbb{R}}^\bullet \rightarrow V_{\mathbb{C}}^\bullet,$$

which is just the functor $\mathbf{R}\Gamma_{\mathcal{H}w}$ from [Bei].

Therefore

$$\mathbf{R}\Gamma([C^*/S], \mathcal{F}^\bullet) \simeq \mathbf{R}\Gamma_{\mathcal{H}w}(V^\bullet),$$

Likewise, if \mathcal{E}^\bullet is another such complex, coming from a complex U^\bullet of real vector spaces with complex filtrations, then

$$\mathbf{R}\mathrm{Hom}_{[C^*/S]}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{H}w}(U^\bullet, V^\bullet).$$

Remark 1.27. For \mathcal{S} as in Lemma 1.19, and a complex V^\bullet of \mathcal{S} -modules, with compatible filtration F on $V^\bullet \otimes \mathbb{C}$, let \mathcal{F}^\bullet be the associated bundle on $[C^*/S]$. By Lemma 1.19, this is a $\mathrm{row}_{1*}\mathcal{O}_{[\mathrm{SL}_2/S]}$ -module, so $\mathcal{F}^\bullet = \mathrm{row}_{1*}\mathcal{E}^\bullet$, for some quasi-coherent complex \mathcal{E}^\bullet on $[\mathrm{SL}_2/S]$, and

$$\begin{aligned} \mathbf{R}\Gamma([C^*/S], \mathcal{F}^\bullet) &= \mathbf{R}\Gamma([C^*/S], \mathrm{row}_{1*}\mathcal{E}^\bullet) \\ &\simeq \mathbf{R}\Gamma([\mathrm{SL}_2/S], \mathcal{E}^\bullet) \\ &= \Gamma([\mathrm{SL}_2/S], \mathcal{E}^\bullet) \\ &= \Gamma([C^*/S], \mathcal{F}^\bullet), \end{aligned}$$

since SL_2 and row_1 are both affine.

In other words,

$$\mathbf{R}\Gamma_{\mathcal{H}w}(V^\bullet) \simeq \gamma^0 V^\bullet,$$

for γ as in Definition 1.25, which is equivalent to saying that $V \oplus F^0(V \otimes \mathbb{C}) \rightarrow V \otimes \mathbb{C}$ is necessarily surjective for all \mathcal{S} -modules V .

1.2. Twistor filtrations.

Definition 1.28. Given an affine scheme X over \mathbb{R} , we define an algebraic (real) twistor filtration $X_{\mathbb{T}}$ on X to consist of the following data:

- (1) a \mathbb{G}_m -equivariant affine morphism $\mathbb{T} : X_{\mathbb{T}} \rightarrow C^*$,
- (2) an isomorphism $X \cong X_{\mathbb{T},1} := X_{\mathbb{T}} \times_{C^*,1} \mathrm{Spec} \mathbb{R}$.

Definition 1.29. A real splitting of the twistor filtration $X_{\mathbb{T}}$ consists of a \mathbb{G}_m -action on X , and an \mathbb{G}_m -equivariant isomorphism

$$X \times C^* \cong X_{\mathbb{T}}$$

over C^* .

Definition 1.30. Adapting [Sim2, §1] from complex to real structures, say that a twistor structure on a real vector space V consists of a vector bundle \mathcal{E} on $\mathbb{P}_{\mathbb{R}}^1$, with an isomorphism $V \cong \mathcal{E}_1$, the fibre of \mathcal{E} over $1 \in \mathbb{P}^1$.

Proposition 1.31. *The category of finite flat algebraic twistor filtrations on real vector spaces is equivalent to the category of twistor structures.*

Proof. The flat algebraic twistor filtration is a flat \mathbb{G}_m -equivariant quasi-coherent sheaf M on C^* , with $M|_1 = V$. Taking the quotient by the right \mathbb{G}_m -action, M corresponds to a flat quasi-coherent sheaf $M_{\mathbb{G}_m}$ on $[C^*/\mathbb{G}_m]$. Now, $[C^*/\mathbb{G}_m] \cong [(\mathbb{A}^2 - \{0\})/\mathbb{G}_m] = \mathbb{P}^1$, so Lemma 1.7 implies that $M_{\mathbb{G}_m}$ corresponds to a flat quasi-coherent sheaf \mathcal{E} on \mathbb{P}^1 . Note that $\mathcal{E}_1 = (M|_{\mathbb{G}_m})_{\mathbb{G}_m} \cong M_1 \cong V$, as required. \square

Definition 1.32. Define the real algebraic group S^1 to be the circle group, whose A -valued points are given by $\{(a, b) \in A^2 : a^2 + b^2 = 1\}$. Note that $S^1 \hookrightarrow S$, and that $S/\mathbb{G}_m \cong S^1$. This latter S -action gives S^1 a split Hodge filtration.

Lemma 1.33. *There is an equivalence of categories between algebraic twistor filtrations $X_{\mathbb{T}}$ on X , and extensions \check{X} of X over S^1 (with $X = \check{X}_1$) equipped with algebraic Hodge filtrations $\check{X}_{\mathbb{F}}$, compatible with the standard Hodge filtration on S^1 .*

Proof. Given an algebraic Hodge filtration $\check{X}_{\mathbb{F}}$ over $S^1 \times C^*$, take

$$X_{\mathbb{T}} := \check{X}_{\mathbb{F}} \times_{S^1, 1} \text{Spec } \mathbb{R},$$

and observe that this satisfies the axioms of an algebraic twistor filtration. Conversely, given an algebraic twistor filtration $X_{\mathbb{T}}$ (over C^*), set

$$\check{X}_{\mathbb{F}} = (X_{\mathbb{T}} \times S^1)/(-1, -1),$$

with projection $\pi(x, t) = (\text{pr}(x)t^{-1}, t^2) \in C^* \times S^1$. \square

Corollary 1.34. *A flat algebraic twistor filtration on a real vector space V is equivalent to the data of a flat $O(S^1)$ -module \tilde{V}^{S^1} with $\tilde{V}^{S^1} \otimes_{O(S^1)} \mathbb{R} = V$, together with an exhaustive decreasing filtration F on $(\tilde{V}^{S^1}) \otimes \mathbb{C}$, with the morphism $O(S^1) \otimes_{\mathbb{R}} \tilde{V}^{S^1} \rightarrow \tilde{V}^{S^1}$ respecting the filtrations (for the standard Hodge filtration on $O(S^1) \otimes \mathbb{C}$). In particular, the filtration is given by $F^p(\tilde{V}^{S^1} \otimes \mathbb{C}) = (a + ib)^p F^0(\tilde{V}^{S^1} \otimes \mathbb{C})$.*

Definition 1.35. Given a flat algebraic twistor filtration on a real vector space V as above, define $\text{gr}_{\mathbb{F}} \tilde{V}^{S^1}$ to be the real part of $\text{gr}_F \text{gr}_{\mathbb{F}}(\tilde{V}^{S^1} \otimes \mathbb{C})$. Note that this is an $O(S^1)$ -module, and define $\text{gr}_{\mathbb{T}} V := (\text{gr}_{\mathbb{F}} \tilde{V}^{S^1}) \otimes_{O(S^1)} \mathbb{R}$.

These results have the following trivial converse.

Lemma 1.36. *An algebraic Hodge filtration $X_{\mathbb{F}} \rightarrow C^*$ on X is equivalent to an algebraic twistor filtration $\mathbb{T} : X_{\mathbb{T}} \rightarrow C^*$ on X , together with a S^1 -action on $X_{\mathbb{T}}$ with the properties that*

- (1) *the S^1 -action and \mathbb{G}_m -actions on $X_{\mathbb{T}}$ commute,*
- (2) *\mathbb{T} is S^1 -equivariant, and*
- (3) *$-1 \in S^1$ acts as $-1 \in \mathbb{G}_m$.*

Proof. The subgroups S^1 and \mathbb{G}_m of S satisfy $(\mathbb{G}_m \times S^1)/(-1, -1) \cong S$. \square

1.3. Mixed Hodge structures. We now define algebraic mixed Hodge structures on real affine schemes.

Definition 1.37. Given an affine scheme X over \mathbb{R} , we define an algebraic mixed Hodge structure X_{MHS} on X to consist of the following data:

- (1) *an $\mathbb{G}_m \times S$ -equivariant affine morphism $X_{\text{MHS}} \rightarrow \mathbb{A}^1 \times C^*$,*
- (2) *a real affine scheme $\underline{\text{gr}}X_{\text{MHS}}$ equipped with an S -action,*
- (3) *an isomorphism $X \cong X_{\text{MHS}} \times_{(\mathbb{A}^1 \times C^*), (1,1)} \text{Spec } \mathbb{R}$,*
- (4) *a $\mathbb{G}_m \times S$ -equivariant isomorphism $\underline{\text{gr}}X_{\text{MHS}} \times C^* \cong X_{\text{MHS}} \times_{\mathbb{A}^1, 0} \text{Spec } \mathbb{R}$, where \mathbb{G}_m acts on $\underline{\text{gr}}X_{\text{MHS}}$ via the inclusion $\mathbb{G}_m \hookrightarrow S$. This is called the opposedness isomorphism.*

Definition 1.38. Given an algebraic mixed Hodge structure X_{MHS} on X , define $\text{gr}^W X_{\text{MHS}} := X_{\text{MHS}} \times_{\mathbb{A}^1, 0} \text{Spec } \mathbb{R}$, noting that this is isomorphic to $\underline{\text{gr}}X_{\text{MHS}} \times C^*$. We also define $X_{\mathbb{F}} := X_{\text{MHS}} \times_{\mathbb{A}^1, 1} \text{Spec } \mathbb{R}$, noting that this is a Hodge filtration on X .

Definition 1.39. A real splitting of the mixed Hodge structure X_{MHS} is a $\mathbb{G}_m \times S$ -equivariant isomorphism

$$\mathbb{A}^1 \times \underline{\text{gr}}X_{\text{MHS}} \times C^* \cong X_{\text{MHS}},$$

giving the opposedness isomorphism on pulling back along $\{0\} \rightarrow \mathbb{A}^1$.

Remarks 1.40. (1) Note that giving X_{MHS} as above is equivalent to giving the affine morphisms $[X_{\text{MHS}}/\mathbb{G}_m \times S] \rightarrow [\mathbb{A}^1/\mathbb{G}_m] \times [C^*/S]$ and $\underline{\text{gr}}X_{\text{MHS}} \rightarrow BS$ of stacks, satisfying an opposedness condition.

- (2) To compare this with the non-abelian mixed Hodge structures postulated in [KPS], note that pulling back along the morphism $\widetilde{C}^* \rightarrow C^*$ gives an object over $[\mathbb{A}^1/\mathbb{G}_m] \times [\widetilde{C}^*/S_{\mathbb{C}}] \cong [\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m]_{\mathbb{C}}$; this is essentially the stack X_{dR} of [KPS]. The stack $X_{B,\mathbb{R}}$ of [KPS] corresponds to pulling back along $1 : \text{Spec } \mathbb{R} \rightarrow C^*$. Thus our algebraic mixed Hodge structures give rise to pre-non-abelian mixed Hodge structures (pre-NAMHS) in the sense of [KPS]. Our treatment of the opposedness condition is also similar to the linearisation condition for a pre-NAMHS, by introducing additional data corresponding to the associated graded object.

As for Hodge filtrations, this gives us a notion of an algebraic mixed Hodge structure on a real vector space. We now show how this is equivalent to the standard definition.

Proposition 1.41. *The category of flat $\mathbb{G}_m \times S$ -equivariant quasi-coherent sheaves M on $\mathbb{A}^1 \times C^*$ is equivalent to the category of quasi-MHS.*

Under this equivalence, bounded below ind-MHS (V, W, F) correspond to flat algebraic mixed Hodge structures M on V whose weights with respect to the $\mathbb{G}_m \times 1$ -action are bounded below.

A real splitting of the Hodge filtration is equivalent to giving a (real) Hodge structure on V (i.e. an S -action).

Proof. Adapting Corollary 1.9, we see that a flat $\mathbb{G}_m \times S$ -equivariant module M on $\mathbb{A}^1 \times C^*$ corresponds to giving exhaustive filtrations W on $V = M|_{(1,1)}$ and F on $V \otimes \mathbb{C}$, i.e. a quasi-MHS on V . Write $\xi(V, \text{MHS})$ for the $\mathbb{G}_m \times S$ -equivariant quasi-coherent sheaf on $\mathbb{A}^1 \times C^*$ associated to a quasi-MHS (V, W, F) .

A flat algebraic mixed Hodge structure is a flat $\mathbb{G}_m \times S$ -equivariant module M on $\mathbb{A}^1 \times C^*$, with $M|_{(1,1)} = V$, together with a $\mathbb{G}_m \times S$ -equivariant splitting of the algebraic Hodge filtration $M|_{\{0\} \times C^*}$. Under the equivalence above, this gives a quasi-MHS (V, W, F) , with W bounded below, satisfying the split opposedness condition

$$(\text{gr}_n^W V) \otimes \mathbb{C} = \bigoplus_{p+q=n} F^p(\text{gr}_n^W V \otimes \mathbb{C}) \cap \bar{F}^q(\text{gr}_n^W V \otimes \mathbb{C}).$$

When the weights of M are bounded below, we need to express this as a filtered direct limit of MHS. Since W is exhaustive, it will suffice to prove that each $W_r V$ is an ind-MHS. Now $W_N V = 0$ for some N , so split opposedness means that $W_{N+1} V$ is a direct sum of pure Hodge structures (i.e. an S -representation), hence an ind-MHS. Assume inductively that $W_{r-1} V$ is an ind-MHS, and consider the exact sequence

$$0 \rightarrow W_{r-1} V \rightarrow W_r V \rightarrow \text{gr}_r^W V \rightarrow 0.$$

of quasi-MHS. Again, split opposedness shows that $\text{gr}_r^W V$ is an ind-MHS, so we may express it as $\text{gr}_r^W V = \varinjlim_{\alpha} U_{\alpha}$, with each U_{α} a MHS. Thus $W_r V = \varinjlim_{\alpha} W_r V \times_{\text{gr}_r^W V} U_{\alpha}$, so we may assume that $\text{gr}_r^W V$ is finite-dimensional (replacing W_r with $W_r V \times_{\text{gr}_r^W V} U_{\alpha}$). Then quasi-MHS extensions of $\text{gr}_r^W V$ by $W_{r-1} V$ are parametrised by

$$\text{Ext}_{\mathbb{A}^1 \times C^*}^1(\xi(\text{gr}_r^W V, \text{MHS}), \xi(W_{r-1} V, \text{MHS}))^{\mathbb{G}_m \times S}.$$

Express $W_{r-1} V$ as a filtered direct limit $\varinjlim_{\beta} T_{\beta}$ of MHS, and note that

$$\begin{aligned} & \text{Ext}_{\mathbb{A}^1 \times C^*}^1(\xi(\text{gr}_r^W V, \text{MHS}), \xi(W_{r-1} V, \text{MHS}))^{\mathbb{G}_m \times S} \\ &= \varinjlim_{\beta} \text{Ext}_{\mathbb{A}^1 \times C^*}^1(\xi(\text{gr}_r^W V, \text{MHS}), \xi(T_{\beta}, \text{MHS}))^{\mathbb{G}_m \times S}, \end{aligned}$$

since $\xi(\text{gr}_r^W V, \text{MHS})$ is finite and locally free. Thus the extension $W_r V \rightarrow \text{gr}_r^W V$ is a pushout of an extension

$$0 \rightarrow T_{\beta} \rightarrow E \rightarrow \text{gr}_r^W V \rightarrow 0$$

for some β , so $W_r V$ can be expressed as the ind-MHS $W_r V = \varinjlim_{\beta' > \beta} E \oplus_{T_\beta} T_{\beta'}$.

Conversely, any MHS V satisfies the split opposedness condition by [Del1, Proposition 1.2.5], so the same holds for any ind-MHS. Thus every ind-MHS corresponds to a flat algebraic MHS under the equivalence above.

Finally, note that the split opposedness condition determines the data of any real splitting. \square

Remark 1.42. Note that the proof of [Del1, Proposition 1.2.5] does not adapt to infinite filtrations. For instance, the quasi-MHS \mathcal{S} of Example 1.24 satisfies the opposedness condition, but does not give an ind-MHS. Geometrically, this is because the fibre over $\{0\} \in C$ is empty. Algebraically, it is because the Hodge filtration on the ring $\mathcal{S} = \text{gr}_0^W \mathcal{S}$ is not split, but $\text{gr}_F \text{gr}_{\bar{F}}(\mathcal{S} \otimes \mathbb{C}) = 0$, which is a pure Hodge structure of weight 0.

1.3.1. Cohomology of MHS. Given a complex \mathcal{F}^\bullet of algebraic MHS, we now show how to calculate hypercohomology $\mathbb{H}^*([C^*/S] \times [\mathbb{A}^1/\mathbb{G}_m], \mathcal{F}^\bullet)$, and compare this with Beilinson's absolute Hodge cohomology. By Proposition 1.41, \mathcal{F}^\bullet gives rise to a complex V^\bullet of quasi-MHS.

Since \mathbb{A}^1 is affine and \mathbb{G}_m reductive, $\mathbf{R}pr_* = \text{pr}_*$ for the projection $\text{pr} : [C^*/S] \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow [C/S]$. Thus

$$\mathbf{R}\Gamma([C^*/S] \times [\mathbb{A}^1/\mathbb{G}_m], \mathcal{F}^\bullet) \simeq \mathbf{R}\Gamma([C^*/S], \text{pr}_* \mathcal{F}^\bullet),$$

and $\text{pr}_* \mathcal{F}^\bullet$ just corresponds under Corollary 1.9 to the complex $W_0 V_{\mathbb{R}}^\bullet$ with filtration F on $W_0 V_{\mathbb{C}}^\bullet$.

Hence §1.1.2 implies that $\mathbf{R}\Gamma([C^*/S] \times [\mathbb{A}^1/\mathbb{G}_m], \mathcal{F}^\bullet)$ is just the cone of

$$W_0 F^0(V_{\mathbb{C}}^\bullet) \oplus W_0 V_{\mathbb{R}}^\bullet \rightarrow W_0 V_{\mathbb{C}}^\bullet,$$

which is just the absolute Hodge functor $\mathbf{R}\Gamma_{\mathcal{H}}$ from [Bei].

Therefore

$$\mathbf{R}\Gamma([C^*/S] \times [\mathbb{A}^1/\mathbb{G}_m], \mathcal{F}^\bullet) \simeq \mathbf{R}\Gamma_{\mathcal{H}}(V^\bullet),$$

Likewise, if \mathcal{E}^\bullet is another such complex, coming from a complex (U^\bullet, W, F) , then

$$\mathbf{R}\text{Hom}_{[C^*/S] \times [\mathbb{A}^1/\mathbb{G}_m]}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \simeq \mathbf{R}\text{Hom}_{\mathcal{H}}(U^\bullet, V^\bullet).$$

1.4. Mixed twistor structures.

Definition 1.43. Given an affine scheme X over \mathbb{R} , we define an algebraic mixed twistor structure X_{MTS} on X to consist of the following data:

- (1) an $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant affine morphism $X_{\text{MTS}} \rightarrow \mathbb{A}^1 \times C^*$,
- (2) a real affine scheme $\underline{\text{gr}}X_{\text{MTS}}$ equipped with a \mathbb{G}_m -action,
- (3) an isomorphism $X \cong X_{\text{MTS}} \times_{(\mathbb{A}^1 \times C^*), (1,1)} \text{Spec } \mathbb{R}$,
- (4) a $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant isomorphism $\underline{\text{gr}}X_{\text{MTS}} \times C^* \cong X_{\text{MTS}} \times_{\mathbb{A}^1, 0} \text{Spec } \mathbb{R}$. This is called the opposedness isomorphism.

Definition 1.44. Given an algebraic mixed twistor structure X_{MTS} on X , define $\text{gr}^W X_{\text{MTS}} := X_{\text{MTS}} \times_{\mathbb{A}^1, 0} \text{Spec } \mathbb{R}$, noting that this is isomorphic to $\underline{\text{gr}}X_{\text{MTS}} \times C^*$. We also define $X_{\mathbb{T}} := X_{\text{MTS}} \times_{\mathbb{A}^1, 1} \text{Spec } \mathbb{R}$, noting that this is a twistor filtration on X .

Definition 1.45. A real splitting of the mixed twistor structure X_{MTS} is a $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant isomorphism

$$\mathbb{A}^1 \times \underline{\text{gr}}X_{\text{MTS}} \times C^* \cong X_{\text{MTS}},$$

giving the opposedness isomorphism on pulling back along $\{0\} \rightarrow \mathbb{A}^1$.

Remark 1.46. Note that giving X_{MTS} as above is equivalent to giving the affine morphism $[X_{\text{MTS}}/\mathbb{G}_m \times \mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m] \times [C^*/\mathbb{G}_m]$ of stacks, satisfying a split opposedness condition.

Definition 1.47. Adapting [Sim2, §1] from complex to real structures, say that a (real) mixed twistor structure (real MTS) on a real vector space V consists of a finite locally free sheaf \mathcal{E} on $\mathbb{P}_{\mathbb{R}}^1$, equipped with an exhaustive Hausdorff increasing filtration by locally free subsheaves $W_i\mathcal{E}$, such that for all i the graded bundle $\mathrm{gr}_i^W \mathcal{E}$ is semistable of slope i (i.e. a direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(i)$). We also require an isomorphism $V \cong \mathcal{E}_1$, the fibre of \mathcal{E} over $1 \in \mathbb{P}^1$.

Define a quasi-MTS on V to be a flat quasi-coherent sheaf \mathcal{E} on $\mathbb{P}_{\mathbb{R}}^1$, equipped with an exhaustive increasing filtration by quasi-coherent subsheaves $W_i\mathcal{E}$, together with an isomorphism $V \cong \mathcal{E}_1$. Define an ind-MTS to be a filtered direct limit of real MTS, and say that an ind-MTS \mathcal{E} on V is bounded below if $W_N\mathcal{E} = 0$ for $N \ll 0$.

Applying Corollary 1.34 gives the following result.

Lemma 1.48. *A flat algebraic mixed twistor structure on a real vector space V is equivalent to giving an $O(S^1)$ -module V' , equipped with a mixed Hodge structure (compatible with the weight 0 real Hodge structure on $O(S^1)$), together with an isomorphism $V' \otimes_{O(S^1)} \mathbb{R} \cong V$.*

Proposition 1.49. *The category of flat $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant quasi-coherent sheaves on $\mathbb{A}^1 \times C^*$ is equivalent to the category of quasi-MTS.*

Under this equivalence, bounded below ind-MTS on V correspond to flat algebraic mixed twistor structures $\xi(V, \mathrm{MTS})$ on V whose weights with respect to the $\mathbb{G}_m \times 1$ -action are bounded below.

Proof. The first statement follows by combining Lemma 1.31 with Lemma 1.7.

Now, given a flat algebraic mixed twistor structure $\xi(V, \mathrm{MTS})$ on V whose weights with respect to the $\mathbb{G}_m \times 1$ -action are bounded below, the proof of Proposition 1.41 adapts (replacing S with \mathbb{G}_m) to show that $\xi(V, \mathrm{MTS})$ is a filtered direct limit of finite flat algebraic mixed twistor structures. It therefore suffices to show that finite flat algebraic mixed twistor structures correspond to MTS.

A finite flat algebraic mixed twistor structure is a finite locally free $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant module M on $\mathbb{A}^1 \times C^*$, with $M|_{(1,1)} = V$, together with a $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant splitting of the algebraic twistor filtration $M|_{\{0\} \times C^*}$. Taking the quotient by the right \mathbb{G}_m -action, M corresponds to a finite locally free \mathbb{G}_m -equivariant module $M_{\mathbb{G}_m}$ on $\mathbb{A}^1 \times [C^*/\mathbb{G}_m]$. Note that $[C^*/\mathbb{G}_m] \cong [(\mathbb{A}^2 - \{0\})/\mathbb{G}_m] = \mathbb{P}^1$, so Lemma 1.7 implies that $M_{\mathbb{G}_m}$ corresponds to a finite locally free module on \mathcal{E} on \mathbb{P}^1 , equipped with a finite filtration W .

Now, $\mathrm{gr} X_{\mathrm{MTS}}$ corresponds to a \mathbb{G}_m -representation V , or equivalently a graded vector space $V = \bigoplus V^n$. If π denotes the projection $\pi : C^* \rightarrow \mathbb{P}^1$, then the opposedness isomorphism is equivalent to a \mathbb{G}_m -equivariant isomorphism

$$\mathrm{gr}^W \mathcal{E} \cong V \otimes^{\mathbb{G}_m} (\pi_* \mathcal{O}_{C^*}) = \bigoplus_n V^n \otimes_{\mathbb{R}} \mathcal{O}_{\mathbb{P}^1}(n),$$

so $\mathrm{gr}_n^W \mathcal{E} \cong V^n \otimes_{\mathbb{R}} \mathcal{O}_{\mathbb{P}^1}(n)$, as required. \square

Remark 1.50. Note that every MHS (V, W, F) has an underlying MTS \mathcal{E} on V , given by forming the S -equivariant Rees module $\xi(V, F)$ on C^* as in Corollary 1.9, and setting \mathcal{E} to be the quotient $\xi(V, F)_{\mathbb{G}_m}$ by the action of $\mathbb{G}_m \subset S$. Beware that if \mathcal{E} is the MTS underlying V , then $\mathcal{E}(-2n)$ is the MTS underlying the MHS $V(n)$.

2. \mathcal{S} -SPLITTINGS FOR REAL HOMOTOPY TYPES

Fix a compact Kähler manifold X .

In [Mor, Theorem 9.1], a mixed Hodge structure was given on the rational homotopy groups of a smooth complex variety X . Here, we study the consequences of formality quasi-isomorphisms for this Hodge filtration when X is a connected compact Kähler manifold.

Let $A^\bullet(X)$ be the differential graded algebra of real C^∞ forms on X . As in [DGMS], this is the real (nilpotent) homotopy type of X . If we write J for the complex structure on $A^\bullet(X)$, then there is a differential $d^c := J^{-1}dJ$ on the underlying graded algebra $A^*(X)$. Note that $dd^c + d^cd = 0$.

2.1. The mixed Hodge structure.

Definition 2.1. Define the DGA $\tilde{A}^\bullet(X)$ on C by

$$\tilde{A}^\bullet(X) = (A^*(X) \otimes_{\mathbb{R}} O(C), ud + vd^c),$$

for co-ordinates u, v as in Remark 1.3. We denote the differential by $\tilde{d} := ud + vd^c$. Note that \tilde{d} is indeed flat:

$$\tilde{d}^2 = u^2d^2 + uv(dd^c + d^cd) + v^2(d^c)^2 = 0.$$

Definition 2.2. There is an action of S on $A^*(X)$, which we will denote by $a \mapsto \lambda \diamond a$, for $\lambda \in \mathbb{C}^* = S(\mathbb{R})$. For $a \in (A^*(X) \otimes \mathbb{C})^{p,q}$, it is given by

$$\lambda \diamond a := \lambda^p \bar{\lambda}^q a.$$

Lemma 2.3. *There is a natural algebraic S -action on $\tilde{A}^\bullet(X)$ over C .*

Proof. For $\lambda \in S(\mathbb{R}) = \mathbb{C}^*$, this action is given on $A^*(X)$ by $a \mapsto \lambda \diamond a$, extending to $\tilde{A}^\bullet(X)$ by tensoring with the action on C from Definition 1.1. We need to verify that this action respects the differential \tilde{d} .

Taking the co-ordinates (u, v) on C from Remark 1.3, we will consider the co-ordinates $w = u + iv, \bar{w} = u - iv$ on $C_{\mathbb{C}}$. Now, we may decompose d and d^c into types (over \mathbb{C}) as $d = \partial + \bar{\partial}$ and $d^c = i\partial - i\bar{\partial}$. Thus $\tilde{d} = w\partial + \bar{w}\bar{\partial}$, so

$$\tilde{d} : (A^*(X) \otimes \mathbb{C})^{p,q} \rightarrow w(A^*(X) \otimes \mathbb{C})^{p+1,q} \oplus \bar{w}(A^*(X) \otimes \mathbb{C})^{p,q+1},$$

which is equivariant under the S -action given, with λ acting as multiplication by $\lambda^p \bar{\lambda}^q$ on both sides. \square

As in [Mor], there is a natural quasi-MHS on $A^\bullet(X)$. The weight filtration is given by the good truncation $W_i A^\bullet(X) = \tau^{\leq i} A^\bullet(X)$, and Hodge filtration on $A^\bullet(X) \otimes_{\mathbb{R}} \mathbb{C} = F^p(A^\bullet(X) \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{p' \geq p} A^{p',q}(X, \mathbb{C})$.

Lemma 2.4. *The S -equivariant C^* -bundle $j^* \tilde{A}^\bullet(X)$ corresponds under Corollary 1.9 to the Hodge filtration on $A^\bullet(X, \mathbb{C})$.*

Proof. We just need to verify that $\tilde{A}^\bullet(X) \otimes \mathbb{C}$ is isomorphic to the Rees algebra $\xi(A^\bullet(X), F, \bar{F})$ (for F the Hodge filtration), with the same complex conjugation.

Now,

$$\xi(A^\bullet(X), F, \bar{F}) = \bigoplus_{pq} F^p \cap \bar{F}^q,$$

with $\lambda \in S(\mathbb{R}) \cong \mathbb{C}^*$ acting as $\lambda^p \bar{\lambda}^q$ on $F^p \cap \bar{F}^q$, and inclusion $F^p \rightarrow F^{p-1}$ corresponding to multiplication by $w = u + iv$. We therefore define an $O(C)$ -linear map $f : \tilde{A}^\bullet(X) \rightarrow \xi(A^\bullet(X), F, \bar{F})$ by mapping $(A(X) \otimes \mathbb{C})^{p,q}$ to $F^p \cap \bar{F}^q$. It only remains to check that this respects the differentials.

For $a \in (A(X) \otimes \mathbb{C})^{p,q}$,

$$f(\tilde{d}a) = f(w\partial a + \bar{w}\bar{\partial}a) = w(\partial a) + \bar{w}(\bar{\partial}a) \in w(F^{p+1} \cap \bar{F}^q) + \bar{w}(F^p \cap \bar{F}^{q+1}).$$

But $w(F^{p+1} \cap \bar{F}^q) = F^p \cap \bar{F}^q = \bar{w}(F^p \cap \bar{F}^{q+1})$, so this is just $\partial a + \bar{\partial}a = da$ in $F^p \cap \bar{F}^q$, which is just $df(a)$, as required. \square

Combining this with the weight filtration means that the bundle $\xi(A^\bullet(X), \text{MHS})$ on $[\mathbb{A}^1/\mathbb{G}_m] \times [C^*/S]$ associated to the quasi-MHS $A^\bullet(X)$ is just the Rees algebra $\xi(j^* \tilde{A}^\bullet(X), W)$, regarded as a $\mathbb{G}_m \times S$ -equivariant $\mathbb{A}^1 \times C^*$ -bundle.

2.2. The family of formality quasi-isomorphisms.

Lemma 2.5. *Given a graded module V^* over a ring B , equipped with operators d, d^c of degree 1 such that $[d, d^c] = d^2 = (d^c)^2 = 0$, then for $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \in \mathrm{GL}_2(B)$,*

$$\begin{aligned} \ker d \cap \ker d^c &= \ker(ud + vd^c) \cap \ker(xd + yd^c), \\ \mathrm{Im}(ud + vd^c) + \mathrm{Im}(xd + yd^c) &= \mathrm{Im} d + \mathrm{Im} d^c, \\ \mathrm{Im}(ud + vd^c)(xd + yd^c) &= \mathrm{Im} dd^c. \end{aligned}$$

Proof. Observe that if we take any matrix, the corresponding inequalities (with \leq replacing $=$) all hold. For invertible matrices, we may express d, d^c in terms of $(ud + vd^c), (xd + yd^c)$ to give the reverse inequalities. \square

Proposition 2.6. *If the pair (d, d^c) of Lemma 2.5 satisfies the principle of two types, then so does $(ud + vd^c), (xd + yd^c)$.*

Proof. The principle of two types states that

$$\ker d \cap \ker d^c \cap (\mathrm{Im} d + \mathrm{Im} d^c) = \mathrm{Im} dd^c.$$

\square

Corollary 2.7. *On the graded algebra*

$$A_{\mathbb{R}}^*(X) \otimes O(\mathrm{SL}_2),$$

for X compact Kähler, the operators $(ud + xd^c), (xd + yd^c)$ satisfy the principle of two types, where

$$O(\mathrm{SL}_2) = \mathbb{Z}[u, v, x, y]/(uy - vx - 1)$$

is the ring associated to the affine group scheme SL_2 .

Definition 2.8. We therefore set $\tilde{d}^c := xd + yd^c$.

The principle of two types now gives us a family of quasi-isomorphisms:

Corollary 2.9. *We have the following S -equivariant quasi-isomorphisms of DGAs over SL_2 , with notation from Definition 1.15:*

$$\mathrm{row}_1^* j^* \tilde{A}^\bullet(X) \xleftarrow{i} \ker(\tilde{d}^c) \xrightarrow{p} \mathrm{row}_2^* H^*(j^* \tilde{A}^\bullet(X)) \cong H^*(A^\bullet(X)) \otimes_{\mathbb{R}} O(\mathrm{SL}_2),$$

where $\ker(\tilde{d}^c)$ means $\ker(\tilde{d}^c) \cap \mathrm{row}_1^ j^* \tilde{A}^\bullet(X)$, with differential \tilde{d} .*

Proof. The principle of two types implies that i is a quasi-isomorphism, and that we may define p as projection onto $H_{\tilde{d}^c}^*(A^*(X) \otimes O(\mathrm{SL}_2))$, on which the differential \tilde{d} is 0. The final isomorphism now follows from the description $H^*(A^\bullet(X)) \cong \frac{\ker d \cap \ker d^c}{\mathrm{Im} dd^c}$, which clearly maps to $H^*(j^* \tilde{A}^\bullet(X))$, the principle of two types showing it to be an isomorphism. \square

Since the weight filtration is just defined in terms of good truncation, this also implies that

$$\xi(\mathrm{row}_1^* j^* \tilde{A}^\bullet(X), W) \simeq \xi(H^*(X, \mathbb{R}), W) \otimes \mathcal{O}_{\mathrm{SL}_2}$$

as $\mathbb{G}_m \times S$ -equivariant dg algebras over $\mathbb{A}^1 \times \mathrm{SL}_2$.

Corollary 2.10. *For \mathcal{S} as in Example 1.24, we have the following W -filtered quasi-isomorphisms of DGAs*

$$A^\bullet(X) \otimes \mathcal{S} \xleftarrow{i'} \ker(d^c + xd) \xrightarrow{p'} H^*(X, \mathbb{R}) \otimes \mathcal{S},$$

where $\ker(d^c + xd) := \ker(d^c + xd) \cap (A^\bullet(X) \otimes \mathcal{S})$, with differential d . Moreover, on tensoring with \mathbb{C} , these become (W, F) -bifiltered quasi-isomorphisms.

Proof. Under the equivalence of Lemma 1.19, $\text{Spec } \mathcal{S}$ corresponds to $\begin{pmatrix} 1 & 0 \\ \mathbb{A}^1 & 1 \end{pmatrix} \subset \text{SL}_2$, equipped with a Hodge filtration. Then Corollary 2.9 is equivalent to the statement that i' and p' are quasi-isomorphisms which become F -filtered quasi-isomorphisms on tensoring with \mathbb{C} .

That these are also W -filtered quasi-isomorphisms is immediate, since W is defined as good truncation. \square

Remarks 2.11. Note that we cannot deduce Corollary 2.10 directly from Lemma 2.5 for the pair $d, d^c + xd$, since that would only establish that i', p' are quasi-isomorphisms preserving the filtrations, rather than filtered quasi-isomorphisms.

Setting $x = 0$ recovers the real formality quasi-isomorphism of [DGMS], while $x = i$ gives the complex filtered quasi-isomorphism used in [Mor].

Since $A^*(X, \mathbb{R}) \otimes \mathcal{S}$ is not a mixed Hodge structure in the classical sense (as F is not bounded on \mathcal{S}), we cannot now apply the theory of mixed Hodge structures on real homotopy types from [Mor] to infer consequences for Hodge structures on homotopy groups. In §§4 and 5, we will develop theory allowing us to deduce the following result on the interaction between formality and the mixed Hodge structure.

Corollary 2.12. *For $x_0 \in X$, \mathcal{S} as in Example 1.24, and for all n , there are \mathcal{S} -linear isomorphisms*

$$\pi_*(X \otimes \mathbb{R}, x_0) \otimes_{\mathbb{R}} \mathcal{S} \cong \pi_*(H^*(X, \mathbb{R})) \otimes_{\mathbb{R}} \mathcal{S},$$

of inverse systems of quasi-MHS, compatible with Whitehead brackets and Hurewicz maps. The associated graded map from the weight filtration is just the pullback of the standard isomorphism $\text{gr}_W \pi_(X \otimes \mathbb{R}, x_0) \cong \pi_*(H^*(X, \mathbb{R}))$ (coming from the opposedness isomorphism).*

Here, $\pi_(H^*(X, \mathbb{R}))$ are the real homotopy groups $H_{*-1} \tilde{G}(H^*(X, \mathbb{R}))$ (see Definition 3.23) associated to the formal homotopy type $H^*(X, \mathbb{R})$, with a real Hodge structure coming from the Hodge structure on $H^*(X, \mathbb{R})$.*

Proof. Corollary 5.4 will show how $j^* \tilde{A}^\bullet$ determines an ind-MHS on $\pi_*(X \otimes \mathbb{R})$, and we will see in Proposition 5.6 that this is the same as the Hodge structure of [Mor]. The \mathcal{S} -splitting of ind-MHS is then proved as Corollary 5.5. \square

In §7, we will see how these MHS become variations of mixed Hodge structure as the basepoint $x_0 \in X$ varies, while §8 will show how to recover the MHS explicitly from the formality quasi-isomorphisms.

2.3. Real Deligne cohomology. Now, consider the derived direct image of $j^* \tilde{A}^\bullet(X)$ under the morphism $q : [C^*/S] \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ given by $u, v \mapsto u^2 + v^2$. This is equivalent to $(\mathbf{R}j_* j^* \tilde{A}^\bullet(X))^{S^1}$ for $j : C^* \rightarrow C$, since S^1 is reductive, $\mathbb{G}_m = S/S^1$ and $\mathbb{A}^1 = C^*/S^1$.

Proposition 2.13. *There are canonical isomorphisms*

$$(\mathbf{R}^m j_* j^* \tilde{A}^\bullet(X))^{S^1} \cong \left(\bigoplus_{a < 0} H^m(X, \mathbb{R}) \right) \oplus \left(\bigoplus_{a \geq 0} (2\pi i)^{-a} H_{\mathcal{D}}^m(X, \mathbb{R}(a)) \right),$$

where a is the weight under the action of $S/S^1 \cong \mathbb{G}_m$, and $H_{\mathcal{D}}^m(X, \mathbb{R}(a))$ is real Deligne cohomology.

Proof. The isomorphism $\mathbb{G}_m = S/S^1$ allows us to regard $O(\mathbb{G}_m)$ as an S -representation, and

$$(\mathbf{R}q_* j^* \tilde{A}^\bullet(X))^{S^1} \simeq \mathbf{R}\Gamma([C^*/S], j^* \tilde{A}^\bullet(X) \otimes O(\mathbb{G}_m)).$$

Now, $O(\mathbb{G}_m) = \mathbb{R}[s, s^{-1}]$, with s of type $(-1, -1)$, so $O(\mathbb{G}_m) \cong \bigoplus_a (2\pi i)^{-a} \mathbb{R}(a)$, giving (by §1.1.2)

$$(\mathbf{R}q_* j^* \tilde{A}^\bullet(X))^{S^1} \simeq \bigoplus_a (2\pi i)^{-a} \mathbf{R}\Gamma_{\mathcal{H}w}(A^\bullet(X)(a)),$$

which is just real Deligne cohomology by [Bei]. \square

We may also compare these cohomology groups with the groups considered in [Den1] and [Den2] for defining Γ -factors of smooth projective varieties at Archimedean places.

Proposition 2.14. *The torsion-free quotient of the \mathbb{G}_m -equivariant \mathbb{A}^1 -module $(\mathbf{R}^m j_* j^* \tilde{A}^\bullet(X))^{S^1}$ is the Rees module of $H^m(X, \mathbb{R})$ with respect to the filtration γ .*

Proof. The results of §1.1.2 give a long exact sequence

$$\dots \rightarrow (\mathbf{R}^m j_* j^* \tilde{A}^\bullet(X))^{S^1} \rightarrow \bigoplus_{a \in \mathbb{Z}} (F^a H^m(X, \mathbb{C}) \oplus H^m(X, \mathbb{R})) \rightarrow \bigoplus_{a \in \mathbb{Z}} H^m(X, \mathbb{C}) \rightarrow \dots,$$

and hence

$$0 \rightarrow \bigoplus_{a \in \mathbb{Z}} \frac{H^{m-1}(X, \mathbb{C})}{F^a H^{m-1}(X, \mathbb{C}) + H^{m-1}(X, \mathbb{R})} \rightarrow (\mathbf{R}^m j_* j^* \tilde{A}^\bullet(X))^{S^1} \rightarrow \bigoplus_{a \in \mathbb{Z}} \gamma^a H^m(X, \mathbb{R}) \rightarrow 0.$$

Since multiplication by the standard co-ordinate of \mathbb{A}^1 corresponds to the embedding $F^{a+1} \hookrightarrow F^a$, the left-hand module is torsion, giving the required result. \square

Remark 2.15. In [Den1] and [Con], Γ -factors of real varieties were also considered. If we let σ denote the de Rham conjugation of the associated complex variety, then we may replace S throughout this paper by $S \rtimes \langle \sigma \rangle$, with σ acting on $S(\mathbb{R})$ by $\lambda \mapsto \bar{\lambda}$, and on SL_2 by $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \mapsto \begin{pmatrix} u & -v \\ -x & y \end{pmatrix}$ (i.e. conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$), noting that $\sigma(d^c) = -d^c$. In that case, the cohomology group considered in [Den1] is the torsion-free quotient of $(\mathbf{R}^m j_* j^* \tilde{A}^\bullet(X))^{S^1 \rtimes \langle \sigma \rangle}$.

Lemma 2.16. *There is a canonical S -equivariant quasi-isomorphism*

$$\mathbf{R}j_* j^* \tilde{A}^\bullet(X) \simeq H^*(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbf{R}O(C^*)$$

of C -modules, where $H^*(X, \mathbb{R})$ is equipped with its standard S -action (the real Hodge structure), and $\mathbf{R}O(C^*)$ is from Definition 1.22.

Proof. The natural inclusion $\mathcal{H}^* \otimes O(C) \rightarrow \tilde{A}^\bullet$ of real harmonic forms gives rise to a morphism

$$\mathcal{H}^* \otimes \mathcal{O}(C^*) \rightarrow j^* \tilde{A}^\bullet$$

of S -equivariant cochain complexes over C^* , which is a quasi-isomorphism by Lemma 2.5, and hence

$$\mathcal{H}^* \otimes \mathbf{R}O(C^*) \simeq \mathbf{R}j_* j^* \tilde{A}^\bullet,$$

as required. \square

Corollary 2.17. *As an S -representation, the summand of $\mathbb{H}^n(C^*, j^* \tilde{A}^\bullet) \otimes_{\mathbb{R}} \mathbb{C}$ of type (p, q) is given by*

$$\bigoplus_{\substack{p' \geq p \\ q' \geq q \\ p' + q' = n}} \mathcal{H}^{p'q'} \oplus \bigoplus_{\substack{p' < p \\ q' < q \\ p' + q' = n-1}} \mathcal{H}^{p'q'}.$$

In particular, this describes Deligne cohomology by taking invariants under complex conjugation when $p = q$.

Proof. This follows from Lemma 2.16, since $H^*(C, \mathcal{O}_{C^*}) \cong \bigoplus_n H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$. \square

2.4. Analogies with limit Hodge structures. If Δ is the open unit disc, and $f : X \rightarrow \Delta$ a proper surjective morphism of complex Kähler manifolds, smooth over the punctured disc Δ^* , then Steenbrink ([Ste]) defined a limit mixed Hodge structure at 0. Take the universal covering space $\widetilde{\Delta}^*$ of Δ^* , and let $\widetilde{X}^* := X \times_{\Delta} \widetilde{\Delta}^*$. Then the limit Hodge structure is defined as a Hodge structure on

$$\lim_{t \rightarrow 0} H^*(X_t) := H^*(\widetilde{X}^*)$$

[Ste, (2.19)] gives an exact sequence

$$\dots \rightarrow H^n(X^*) \rightarrow H^n(\widetilde{X}^*) \xrightarrow{N} H^n(\widetilde{X}^*)(-1) \rightarrow \dots,$$

where N is the monodromy operator associated to the deck transformation of $\widetilde{\Delta}^*$.

Since we are working with quasi-coherent cohomology, connected affine schemes replace contractible topological spaces, and Lemma 1.17 implies that we may then regard SL_2 as the universal cover of C^* , with deck transformations \mathbb{G}_a . We then substitute C for Δ , C^* for Δ^* and SL_2 for $\widetilde{\Delta}^*$. We also replace \mathcal{O}_{X^*} with $j^* \tilde{A}^\bullet(X)$, so $\mathcal{O}_{\widetilde{X}^*}$ becomes $\mathrm{row}_1^* j^* \tilde{A}^\bullet(X)$. This suggests that we should think of $\mathrm{row}_1^* j^* \tilde{A}^\bullet(X)$ (with its natural S -action) as the limit mixed Hodge structure at the Archimedean special fibre.

The derivation N of Definition 1.21 then acts as the monodromy transformation. Since N is of type $(-1, -1)$ with respect to the S -action, the weight decomposition given by the action of $\mathbb{G}_m \subset S$ splits the monodromy-weight filtration. The following result allows us to regard $\mathrm{row}_1^* j^* \tilde{A}^\bullet(X)$ as the limit Hodge structure at the special fibre corresponding to the Archimedean place.

Proposition 2.18. *$\mathrm{R}j_* j^* \tilde{A}^\bullet(X)$ is naturally isomorphic to the cone complex of the diagram $\mathrm{row}_1^* j^* \tilde{A}^\bullet(X) \xrightarrow{N} \mathrm{row}_1^* j^* \tilde{A}^\bullet(X)(-1)$, where N is the locally nilpotent derivation given by differentiating the \mathbb{G}_a -action on SL_2 .*

Proof. This follows from the description of $\mathbf{RO}(C^*)$ in §1.1.1. \square

2.5. Archimedean cohomology. As in §2.4, the S -action gives a real (split) Hodge structure on the cohomology groups $H^q(\mathrm{row}_1^* j^* \tilde{A}^\bullet(X))$. In order to avoid confusion with the weight filtration on $j^* \tilde{A}^\bullet(X)$, we will denote the associated weight filtration by M_* .

Corollary 2.19. *There are canonical isomorphisms*

$$\mathrm{gr}_{q+r}^M H^q(\mathrm{row}_1^* j^* \tilde{A}^\bullet(X)) \cong H^q(X, \mathbb{R}) \otimes \mathrm{gr}_r^M O(\mathrm{SL}_2)$$

Proof. This is an immediate consequence of the splitting in Corollary 2.9. \square

Lemma 2.20. *$\ker N \cap H^q(\mathrm{row}_1^* j^* \tilde{A}^\bullet(X)) \cong H^q(X, \mathbb{R}) \otimes \mathbb{R}[u, v]$, and $\mathrm{coker} N \cap H^q(\mathrm{row}_1^* j^* \tilde{A}^\bullet(X)(-1)) \cong H^q(X, \mathbb{R}) \otimes \mathbb{R}[x, y](-1)$, for N as in Definition 1.21.*

Proof. This is a direct consequence of Corollary 2.19, since $\mathbb{R}[u, v] = \ker N|_{O(\mathrm{SL}_2)}$ and the map $\mathbb{R}[x, y] \rightarrow \mathrm{coker} N|_{O(\mathrm{SL}_2)}$ is an isomorphism. \square

Corollary 2.21. *The S^1 -invariant subspace $H^q(\mathrm{row}_1^* j^* \tilde{A}^\bullet(X))^{S^1}$ is canonically isomorphic as an N -representation to the Archimedean cohomology group $H^q(\tilde{X}^*)$ defined in [Con].*

Proof. First observe that N acts on $O(\mathrm{SL}_2)$ as the derivation $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2$ acting on the left, and that differentiating the action of $\mathbb{G}_m \subset S$ on $O(\mathrm{SL}_2)$ gives the derivation $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{sl}_2$, also acting on the left. Therefore decomposition by the weights of the \mathbb{G}_m -action gives a splitting of the filtration M associated to the locally nilpotent operator N .

By Proposition 2.13 and [Con, Proposition 4.1], we know that Deligne cohomology arises as the cone of $N : H^* \rightarrow H^*(-1)$, for both cohomology theories H^* .

It now follows from Corollary 2.19 and Lemma 2.20 that the graded N -module $H^q(\text{row}_1^* j^* \tilde{A}^\bullet(X))^{S^1}$ shares all the properties of [Con, Corollary 4.4, Proposition 4.8 and Corollary 4.10], which combined are sufficient to determine the graded N -module $H^q(\tilde{X}^*)$ up to isomorphism. \square

Note that under the formality isomorphism of Corollary 2.9, this becomes $H^q(\tilde{X}^*) \cong H^q(X, \mathbb{R}) \otimes^{S^1} O(\text{SL}_2)$.

2.5.1. *Archimedean periods.* We can construct the ring \mathcal{S} without choosing co-ordinates as follows. Given any \mathbb{R} -algebra A , let UA be the underlying \mathbb{R} -module. Then

$$\mathcal{S} = \mathbb{R}[UC] \otimes_{\mathbb{R}[U\mathbb{R}]} \mathbb{R},$$

For an explicit comparison, write $UC = \mathbb{R}x_1 \oplus \mathbb{R}x_i$, so the right-hand side is $\mathbb{R}[x_1, x_i] \otimes_{\mathbb{R}[x_1]} \mathbb{R} = \mathbb{R}[x_i]$.

The filtration F is then given by powers of the augmentation ideal of the canonical map $\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathbb{C}$, since the ideal is $(x_i - i)$. The derivation N (from Definition 1.21) is differentiation $\mathcal{S} \rightarrow \Omega(\mathcal{S}/\mathbb{R})$, and $\Omega(\mathcal{S}/\mathbb{R}) = \mathcal{S} \otimes_{\mathbb{R}} (\mathbb{C}/\mathbb{R})$. There is also an action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on \mathcal{S} , determined by the action on UC , which corresponds to the generator $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$ acting \mathbb{C} -linearly as $\sigma(x) = -x$.

For $K = \mathbb{R}, \mathbb{C}$, we therefore define $B(K) := \mathcal{S} \otimes_{\mathbb{R}} \mathbb{C}$, with Frobenius ϕ acting as complex conjugation, and the Hodge filtration, $\text{Gal}(\mathbb{C}/K)$ -action and N defined as above. However, beware that $B(K)$ differs from the ring B_{ar} from [Den1].

We think of $B(K)$ as analogous to the ring B_{st} of semi-stable periods (see e.g. [Ill]) used in crystalline cohomology. For a p -adic field K , recall that $B_{\text{st}}(K)$ is a \mathbb{Q}_p -algebra equipped with a $\text{Gal}(\bar{K}/K)$ -action, a Frobenius-linear automorphism ϕ , a decreasing filtration $F^i B_{\text{st}}(K)$, and a nilpotent derivation

$$N : B_{\text{st}}(K) \rightarrow B_{\text{st}}(K)(-1).$$

Thus we think of $X \otimes \mathbb{R}$ as being of semi-stable reduction at ∞ , with nilpotent monodromy operator N on the Archimedean fibre $X \otimes \mathcal{S}$. The comparison with B_{st} is further justified by comparison with [Pri6], where the crystalline comparison from [Ols] is used to show that for a variety of good reduction, the p -adic étale homotopy type $(X_{\text{ét}} \otimes \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{st}}^\phi$ is formal as Galois representation in homotopy types, and that formality preserves N (since good reduction means that N acts trivially on $(X_{\text{ét}} \otimes \mathbb{Q}_p)$, while $B_{\text{st}}^\phi \cap \ker N = B_{\text{cris}}^\phi$).

In our case, $B(K)^\phi = \mathcal{S}$, so $(X \otimes \mathbb{R}) \otimes_{\mathbb{R}} B(K)^\phi$ is formal as a $\text{Gal}(\bar{K}/K)$ -representation in non-abelian MHS. However, formality does not preserve N since we only have semi-stable, not good, reduction at ∞ .

In keeping with the philosophy of Arakelov theory, there should be a norm $\langle -, - \rangle$ on $B(K)$ to compensate for the finiteness of $\text{Gal}(\mathbb{C}/K)$. In order to ensure that $\tilde{d}^* = -[\Lambda, \tilde{d}^c]$, we define a semilinear involution $*$ on $O(\text{SL}_2) \otimes \mathbb{C}$ by $u^* = y, v^* = -x$. This corresponds to the involution $A \mapsto (A^\dagger)^{-1}$ on $\text{SL}_2(\mathbb{C})$, so the most natural metric on $O(\text{SL}_2)$ comes via Haar measure on $\text{SU}_2(\mathbb{C})$ (the unit quaternions). However, the ring homomorphism $O(\text{SL}_2) \rightarrow B(K)$ (corresponding to $\begin{pmatrix} 1 & 0 \\ \mathbb{G}_a & 1 \end{pmatrix} \leq \text{SL}_2$) is not then bounded for any possible norm on $B(K)$, suggesting that we should think of SL_2 as being more fundamental than \mathcal{S} .

Remark 2.22. If we wanted to work with k -MHS for a subfield $k \subset \mathbb{R}$, we could replace \mathcal{S} with the ring

$$\mathcal{S}_k = k[U_k \mathbb{C}] \otimes_{k[U_k k]} k,$$

where $U_k B$ is the k -module underlying a k -algebra B . The results of §§1.1.2, 1.3.1 then carry over, including Remark 1.27.

We can use this to find the analogue of Definitions 1.4 and 1.37 for k -MHS. First, note that S -equivariant SL_2 -modules are quasi-coherent sheaves on $[\text{SL}_2/S]$, and that

$[\mathrm{SL}_2/S] = [\mathrm{SSym}_2/\mathbb{G}_m]$, where $\mathrm{SSym}_2 \subset \mathrm{SL}_2$ consist of symmetric matrices, and the identification $\mathrm{SL}_2/S^1 = \mathrm{SSym}_2$ is given by $A \mapsto AA^t$ (noting that S^1 acts on SL_2 as right multiplication by O_2). The action of $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on SSym_2 is conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, while the involution $*$ is given by $B \mapsto \bar{B}^{-1}$.

Note that $\mathrm{SSym}_2 = \mathrm{Spec} \xi(\mathcal{S}, \gamma)$, for $\xi(\mathcal{S}, \gamma)$ the Rees algebra with respect to the filtration γ .

Now, algebraic Hodge filtrations on real complexes correspond to S -equivariant $\mathbf{RO}(C^*)$ -complexes (for $\mathbf{RO}(C^*)$ as in Definition 1.22). The identifications above and Remark 1.27 ensure that these are equivalent to \mathbb{G}_m -equivariant $\mathbf{RO}(C^*)^{S^1}$ -complexes, where $\mathbf{RO}(C^*)^{S^1}$ is the cone of $\xi(\mathcal{S}, \gamma) \xrightarrow{N} \xi(\Omega(\mathcal{S}/\mathbb{R}), \gamma)$.

Therefore we could define algebraic Hodge filtrations on k -complexes to be \mathbb{G}_m -equivariant $\xi(\Omega^\bullet(\mathcal{S}_k/k), \gamma)$ -complexes, where $\gamma^p V = V \cap F^p(V \otimes_k \mathbb{C})$.

To complete the analogy with étale and crystalline homotopy types, there should be a graded homotopy type $\xi(X_{\mathrm{st}}, \gamma)$ over the generalised ring (in the sense of [Har]) $\xi(B, \gamma)^{(-, -), \mathrm{Gal}(\mathbb{C}/K)}$ of norm 1 Galois-invariant elements in the Rees algebra $\xi(B, \gamma) = O(\mathrm{SSym}_2)$, equipped with a monodromy operator N and complex conjugation ϕ .

The generalised tensor product $\xi(X_{\mathrm{st}}, \gamma) \otimes_{\xi(B, \gamma)^{(-, -), \mathrm{Gal}(\mathbb{C}/K)}} \xi(B, \gamma)$ should then be equivalent to $\xi((X \otimes \mathbb{R}) \otimes_{\mathbb{R}} B, \gamma)$, and then we could recover the rational homotopy type as the subalgebra

$$(X \otimes \mathbb{R}) = \xi(X_{\mathrm{st}} \otimes B, \gamma)^{\mathbb{G}_m, \phi=1, N=0} = F^0(X_{\mathrm{st}} \otimes B)^{\phi=1, N=0}.$$

By comparison with étale cohomology, the existence of a Hodge filtration on $X \otimes \mathbb{R}$ seems anomalous, but it survives this process because (unlike the crystalline case) $F^0 B = B$.

3. RELATIVE MALCEV HOMOTOPY TYPES

Given a reductive pro-algebraic group R , a topological space X , and a Zariski-dense morphism $\rho : \pi_1(X, x) \rightarrow R(k)$, [Pri3] introduced the Malcev homotopy type $X^{\rho, \mathrm{Mal}}$ of X relative to ρ . If $R = 1$ and $k = \mathbb{Q}$ (resp. $k = \mathbb{R}$), then this is just the rational (resp. real) homotopy type of X . If R is the reductive pro-algebraic fundamental group of X , then $X^{\rho, \mathrm{Mal}}$ is the schematic homotopy type of X .

Readers uninterested in non-nilpotent topological spaces or homotopy fibres can skip straight to §3.3, restricting to the case $R = 1$.

3.1. Review of pro-algebraic homotopy types. Here we give a summary of the results from [Pri3] which will be needed in this paper. Fix a field k of characteristic zero.

Definition 3.1. Given a pro-algebraic group G (i.e. an affine group scheme over k), define the reductive quotient G^{red} of G by

$$G^{\mathrm{red}} = G/R_{\mathrm{u}}(G),$$

where $R_{\mathrm{u}}(G)$ is the pro-unipotent radical of G . Observe that G^{red} is then a reductive pro-algebraic group, and that representations of G^{red} correspond to semisimple representations of G .

Proposition 3.2. *For any pro-algebraic group G , there is a Levi decomposition $G = G^{\mathrm{red}} \ltimes R_{\mathrm{u}}(G)$, unique up to conjugation by $R_{\mathrm{u}}(G)$.*

Proof. See [HM]. □

3.1.1. *The pointed pro-algebraic homotopy type of a topological space.* We now recall the results from [Pri3, §1].

Definition 3.3. Let \mathbb{S} be the category of simplicial sets, let \mathbb{S}_0 be the category of reduced simplicial sets, i.e. simplicial sets with one vertex, and let $s\text{Gp}$ be the category of simplicial groups. Let Top_0 denote the category of pointed connected compactly generated Hausdorff topological spaces.

Note that there is a functor from Top_0 to \mathbb{S}_0 which sends (X, x) to the simplicial set

$$\text{Sing}(X, x)_n := \{f \in \text{Hom}_{\text{Top}}(|\Delta^n|, X) : f(v) = x \quad \forall v \in \Delta_0^n\}.$$

this is a right Quillen equivalence, the corresponding left equivalence being geometric realisation. For the rest of this section, we will therefore restrict our attention to reduced simplicial sets.

As in [GJ, Ch.V], there is a classifying space functor $\bar{W}: s\text{Gp} \rightarrow \mathbb{S}_0$. This has a left adjoint $G: \mathbb{S}_0 \rightarrow s\text{Gp}$, Kan's loop group functor ([Kan]), and these give a Quillen equivalence of model categories. In particular, $\pi_i(G(X)) = \pi_{i+1}(X)$. This allows us to study simplicial groups instead of pointed topological spaces.

Definition 3.4. Given a simplicial object G_\bullet in the category of pro-algebraic groups, define $\pi_0(G_\bullet)$ to be the coequaliser

$$G_1 \begin{array}{c} \xrightarrow{\partial_1} \\ \xrightarrow{\partial_0} \end{array} G_0 \longrightarrow \pi_0(G)$$

in the category of pro-algebraic groups.

Definition 3.5. Define a pro-algebraic simplicial group to consist of a simplicial diagram G_\bullet of pro-algebraic groups, such that the maps $G_n \rightarrow \pi_0(G)$ are pro-unipotent extensions of pro-algebraic groups, i.e. $\ker(G_n \rightarrow \pi_0(G))$ is pro-unipotent. We denote the category of pro-algebraic simplicial groups by $s\text{AGp}$.

There is a forgetful functor $(k): s\text{AGp} \rightarrow s\text{Gp}$, given by sending G_\bullet to $G_\bullet(k)$. This functor clearly commutes with all limits, so has a left adjoint $G_\bullet \mapsto (G_\bullet)^{\text{alg}}$. We can describe $(G_\bullet)^{\text{alg}}$ explicitly. First let $(\pi_0(G))^{\text{alg}}$ be the pro-algebraic completion of the abstract group $\pi_0(G)$, then let $(G^{\text{alg}})_n$ be the relative Malcev completion (in the sense of [Hai4]) of the morphism

$$G_n \rightarrow (\pi_0(G))^{\text{alg}}.$$

In other words, $G_n \rightarrow (G^{\text{alg}})_n \xrightarrow{f} (\pi_0(G))^{\text{alg}}$ is the universal diagram with f a pro-unipotent extension.

Proposition 3.6. *The functors (k) and $^{\text{alg}}$ give rise to a pair of adjoint functors*

$$\text{Ho}(s\text{Gp}) \begin{array}{c} \xrightarrow{\mathbf{L}^{\text{alg}}} \\ \xleftarrow{(k)} \end{array} \text{Ho}(s\text{AGp})$$

on the homotopy categories, with $\mathbf{L}^{\text{alg}}G(X) = G(X)^{\text{alg}}$, for any $X \in \mathbb{S}_0$.

Proof. As in [Pri3, Proposition 1.36], it suffices to observe that (k) preserves fibrations and trivial fibrations. \square

Definition 3.7. Given a reduced simplicial set (or equivalently a pointed, connected topological space) (X, x) , define the pro-algebraic homotopy type $(X, x)^{\text{alg}}$ of (X, x) over k to be the object

$$G(X, x)^{\text{alg}}$$

in $\text{Ho}(s\text{AGp})$. Define the pro-algebraic fundamental group by $\varpi_1(X, x) := \pi_0(G(X, x)^{\text{alg}})$. Note that $\pi_0(G^{\text{alg}})$ is the pro-algebraic completion of the group $\pi_0(G)$.

We then define the higher pro-algebraic homotopy groups $\varpi_n(X, x)$ by

$$\varpi_n(X, x) := \pi_{n-1}(G(X, x)^{\text{alg}}).$$

3.1.2. Pointed relative Malcev homotopy types.

Definition 3.8. Assume we have an abstract group G , a reductive pro-algebraic group R , and a representation $\rho : G \rightarrow R(k)$ which is Zariski-dense on morphisms. Define the Malcev completion $(G, \rho)^{\text{Mal}}$ (or $G^{\rho, \text{Mal}}$, or $G^{R, \text{Mal}}$) of G relative to ρ to be the universal diagram

$$G \rightarrow (G, \rho)^{\text{Mal}} \xrightarrow{p} R,$$

with p a pro-unipotent extension, and the composition equal to ρ .

Note that finite-dimensional representations of $(G, \rho)^{\text{Mal}}$ correspond to G -representations which are Artinian extensions of R -representations.

Definition 3.9. Given a Zariski-dense morphism $\rho : \pi_1(X, x) \rightarrow R(k)$, let the Malcev completion $G(X, x)^{\rho, \text{Mal}}$ (or $G(X, x)^{R, \text{Mal}}$) of (X, x) relative to ρ be the pro-algebraic simplicial group $(G(X, x), \rho)^{\text{Mal}}$. Observe that the Malcev completion of (X, x) relative to $(\pi_1(X, x))^{\text{red}}$ is just $G(X, x)^{\text{alg}}$. Let $\varpi_1(X^{\rho, \text{Mal}}, x) = \pi_0(G(X, x), \rho)^{\text{Mal}}$ and $\varpi_n(X^{\rho, \text{Mal}}, x) = \pi_{n-1}(G(X, x), \rho)^{\text{Mal}}$. Observe that $\varpi_1(X^{\rho, \text{Mal}}, x)$ is just the relative Malcev completion $\varpi_1(X, x)^{\rho, \text{Mal}}$ of $\rho : \pi_1(X, x) \rightarrow R(k)$.

Note that for any cosimplicial $G(X, x)^{R, \text{Mal}}$ -representation (i.e. $O(G(X, x)^{R, \text{Mal}})$ -comodule, and in particular any $\varpi_1(X^{\rho, \text{Mal}}, x)$ -representation) V , the canonical map $\mathbb{H}^*(G(X, x)^{\rho, \text{Mal}}, V) \rightarrow \mathbb{H}^*(X, V)$ is an isomorphism.

Theorem 3.10. Take a fibration $f : (X, x) \rightarrow (Y, y)$ (of pointed connected topological spaces) with connected fibres, and set $F := f^{-1}(y)$. Take a Zariski-dense representation $\rho : \pi_1(X, x) \rightarrow R(k)$ to a reductive pro-algebraic group R , let K be the closure of $\rho(\pi_1(F, x))$, and $T := R/K$. If the monodromy action of $\pi_1(Y, y)$ on $\mathbb{H}^*(F, V)$ factors through $\varpi_1(Y, y)^{T, \text{Mal}}$ for all K -representations V , then $G(F, x)^{K, \text{Mal}}$ is the homotopy fibre of $G(X, x)^{R, \text{Mal}} \rightarrow G(Y, y)^{T, \text{Mal}}$.

In particular, there is a long exact sequence

$$\begin{aligned} \dots \rightarrow \varpi_n(F, x)^{K, \text{Mal}} \rightarrow \varpi_n(X, x)^{R, \text{Mal}} \rightarrow \varpi_n(Y, y)^{T, \text{Mal}} \rightarrow \varpi_{n-1}(F, x)^{K, \text{Mal}} \rightarrow \\ \dots \rightarrow \varpi_1(F, x)^{K, \text{Mal}} \rightarrow \varpi_1(X, x)^{R, \text{Mal}} \rightarrow \varpi_1(Y, y)^{T, \text{Mal}} \rightarrow 1. \end{aligned}$$

Proof. First observe that $\rho(\pi_1(F, x))$ is normal in $\pi_1(X, x)$, so K is normal in R , and T is therefore a reductive pro-algebraic group, so $(Y, y)^{T, \text{Mal}}$ is well-defined. Next, observe that since K is normal in R , $R_u(K)$ is also normal in R , and is therefore 1, ensuring that K is reductive, so $(F, x)^{K, \text{Mal}}$ is also well-defined.

Consider the complex $O(R) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})$ of $G(X, x)^{R, \text{Mal}}$ -representations, regarded as a complex of sheaves on X . The Leray spectral sequence for f with coefficients in this complex is

$$E_2^{i,j} = \mathbb{H}^i(Y, \mathbb{H}^j(F, O(R)) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})) \implies \mathbb{H}^{i+j}(X, O(R) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})).$$

Regarding $O(R)$ as a K -representation, $\mathbb{H}^*(F, O(R))$ is a $\varpi_1(Y, y)^{T, \text{Mal}}$ -representation by hypothesis. Hence $\mathbb{H}^*(F, O(R)) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})$ is a cosimplicial $G(Y)^{T, \text{Mal}}$ -representation, so

$$\begin{aligned} \mathbb{H}^i(Y, \mathbb{H}^j(F, O(R)) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})) \\ \cong \mathbb{H}^i(G(Y)^{T, \text{Mal}}, \mathbb{H}^j(F, O(R)) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})). \end{aligned}$$

Now, $\mathbb{H}^*(F, O(R)) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})$ is a fibrant cosimplicial $G(Y)^{T, \text{Mal}}$ -representation, so

$$\begin{aligned} & \mathbb{H}^i(G(Y)^{T, \text{Mal}}, \mathbb{H}^j(F, O(R)) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})) \\ & \cong \mathbb{H}^i \Gamma(G(Y)^{T, \text{Mal}}, \mathbb{H}^j(F, O(R)) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})) \\ & = \begin{cases} \mathbb{H}^j(F, O(R)) \otimes_{O(T)} k = \mathbb{H}^j(F, O(K)) & i = 0 \\ 0 & i \neq 0, \end{cases} \end{aligned}$$

so

$$\mathbb{H}^j(X, O(R) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})) \cong \mathbb{H}^j(F, O(K)).$$

Now, let \mathcal{F} be the homotopy fibre of $G(X, x)^{R, \text{Mal}} \rightarrow G(Y, y)^{T, \text{Mal}}$, noting that there is a natural map $G(F, x)^{K, \text{Mal}} \rightarrow \mathcal{F}$. We have

$$\mathbb{H}^j(X, O(R) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})) = \mathbb{H}^j(G(X)^{R, \text{Mal}}, O(R) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})),$$

and a Leray-Serre spectral sequence

$$\mathbb{H}^i(G(Y)^{T, \text{Mal}}, \mathbb{H}^j(\mathcal{F}, O(R)) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})) \implies \mathbb{H}^{i+j}(G(X)^{R, \text{Mal}}, O(R) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})).$$

The reasoning above adapts to show that this spectral sequence also collapses, yielding

$$\mathbb{H}^j(\mathcal{F}, O(K)) = \mathbb{H}^j(X, O(R) \otimes_{O(T)} O(G(Y)^{T, \text{Mal}})).$$

We have therefore shown that the map $G(F, x)^{K, \text{Mal}} \rightarrow \mathcal{F}$ gives an isomorphism

$$\mathbb{H}^*(\mathcal{F}, O(K)) \rightarrow \mathbb{H}^*(G(F, x)^{K, \text{Mal}}, O(K)),$$

and hence isomorphisms $\mathbb{H}^*(\mathcal{F}, V) \rightarrow \mathbb{H}^*(G(F, x)^{K, \text{Mal}}, V)$ for all K -representations V . Since this is a morphism of simplicial pro-unipotent extensions of K , [Pri3, Corollary 1.55] implies that $G(F, x)^{K, \text{Mal}} \rightarrow \mathcal{F}$ is a weak equivalence. \square

A special case of Theorem 3.10 has appeared in [KPT2, Proposition 4.20], when F is simply connected and of finite type, and $T = \varpi_1(Y, y)^{\text{red}}$.

Corollary 3.11. *Given a fibration $f : (X, x) \rightarrow (Y, y)$ with connected fibres, assume that the fibre $F := f^{-1}(y)$ has finite-dimensional cohomology groups $\mathbb{H}^i(F, k)$ and let R be the reductive quotient of the Zariski closure of the homomorphism $\pi_1(Y, y) \rightarrow \prod_i \text{GL}(H^i(F, k))$. Then the Malcev homotopy type $(F \otimes k, x)$ is the homotopy fibre of*

$$G(X, x)^{R, \text{Mal}} \rightarrow G(Y, y)^{R, \text{Mal}}.$$

Proof. This is just Theorem 3.10, with $R = T$ and $K = 1$. \square

Note that for a morphism $f : (X, x) \rightarrow (Y, y)$ which is not a fibration, we can apply Theorem 3.10 to a weakly equivalent fibration, replacing F with the homotopy fibre of f over y .

Remark 3.12. Beware that even when Y is a $K(\pi, 1)$, the relative completion $Y^{R, \text{Mal}}$ need not be so. For instance, [Hai3] and [Hai1] are concerned with studying the exact sequence $1 \rightarrow T_g \rightarrow \Gamma_g \rightarrow \text{Sp}_g(\mathbb{Z}) \rightarrow 1$, where Γ_g is the mapping class group and T_g the Torelli group. Taking $R = \text{Sp}_g/\mathbb{Q}$, we get $\mathbb{H}^1(\text{Sp}_g(\mathbb{Z}), O(R)) = 0$, but $\mathbb{H}^2(\text{Sp}_g(\mathbb{Z}), O(R)) \cong \mathbb{Q}$. Therefore $\varpi_1(B\text{Sp}_g(\mathbb{Z}))^{R, \text{Mal}} = R$, but the Hurewicz theorem gives $\varpi_2(B\text{Sp}_g(\mathbb{Z}))^{R, \text{Mal}} = \mathbb{Q}$. Thus the long exact sequence for homotopy has

$$\mathbb{Q} \rightarrow T_g \otimes \mathbb{Q} \rightarrow \Gamma_g^{R, \text{Mal}} \rightarrow \text{Sp}_g(\mathbb{Z})^{R, \text{Mal}} \rightarrow 1.$$

This is consistent with [Hai3, Proposition 7.1] and [Hai1, Theorem 3.4], which show that $T_g \otimes \mathbb{Q} \rightarrow \Gamma_g^{R, \text{Mal}}$ is a central extension by \mathbb{Q} .

Definition 3.13. Define a group Γ to be good with respect to a Zariski-dense representation $\rho : \Gamma \rightarrow R(k)$ to a reductive pro-algebraic group if the homotopy groups $\varpi_n(B\Gamma)^{R, \text{Mal}}$ are 0 for all $n \geq 2$.

By [Pri3, Examples 3.20], the fundamental group of a compact Riemann surface is algebraically good with respect to all representations, as are finite groups, free groups and finitely generated nilpotent groups.

Lemma 3.14. *A group Γ is good relative to $\rho : \Gamma \rightarrow R(k)$ if and only if the map*

$$H^n(\Gamma^{\rho, \text{Mal}}, V) \rightarrow H^n(\Gamma, V)$$

is an isomorphism for all n and all finite-dimensional R -representations V .

Proof. This follows by looking at the map $f : G(B\Gamma)^{R, \text{Mal}} \rightarrow \Gamma^{R, \text{Mal}}$ of simplicial pro-algebraic groups, which is a weak equivalence if and only if $\varpi_n(B\Gamma)^{R, \text{Mal}} = 0$ for all $n \geq 2$. By [Pri3, Corollary 1.55], f is a weak equivalence if and only if the morphisms

$$H^*(\Gamma^{R, \text{Mal}}, V) \rightarrow H^*(G(B\Gamma)^{R, \text{Mal}}, V)$$

are isomorphisms for all R -representations V . Since $H^*(G(B\Gamma)^{R, \text{Mal}}, V) = H^*(B\Gamma, V) = H^*(\Gamma, V)$, the result follows. \square

Lemma 3.15. *Assume that Γ is finitely presented, with $H^n(\Gamma, -)$ commuting with filtered direct limits of $\Gamma^{\rho, \text{Mal}}$ -representations, and $H^n(\Gamma, V)$ finite-dimensional for all finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representations V .*

Then Γ is good with respect to ρ if and only if for any finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representation V , and $\alpha \in H^n(\Gamma, V)$, there exists an injection $f : V \rightarrow W_\alpha$ of finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representations, with $f(\alpha) = 0 \in H^n(\Gamma, W_\alpha)$.

Proof. The proof of [KPT2, Lemma 4.15] adapts to this generality. \square

Definition 3.16. Say that a group Γ is n -good with respect to a Zariski-dense representation $\rho : \Gamma \rightarrow R(k)$ to a reductive pro-algebraic group if for all finite-dimensional $\Gamma^{\rho, \text{Mal}}$ -representations V , the map

$$H^i(\Gamma^{\rho, \text{Mal}}, V) \rightarrow H^i(\Gamma, V)$$

is an isomorphism for all $i \leq n$ and an inclusion for $i = n + 1$.

The following is [Pri6, Theorem 2.25], which strengthens [Pri3, Theorem 3.21]:

Theorem 3.17. *If (X, x) is a pointed connected topological space with fundamental group Γ , equipped with a Zariski-dense representation $\rho : \Gamma \rightarrow R(\mathbb{R})$ to a reductive pro-algebraic groupoid for which:*

- (1) Γ is $(N + 1)$ -good with respect to ρ ,
- (2) $\pi_n(X, x)$ is of finite rank for all $1 < n \leq N$, and
- (3) the Γ -representation $\pi_n(X, x) \otimes_{\mathbb{Z}} \mathbb{R}$ is an extension of R -representations (i.e. a $\Gamma^{\rho, \text{Mal}}$ -representation) for all $1 < n \leq N$,

then the canonical map

$$\pi_n(X, x) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \varpi_n(X^{\rho, \text{Mal}}, x)(\mathbb{R})$$

is an isomorphism for all $1 < n \leq N$.

To see how to compare homotopy groups when the goodness hypotheses are not satisfied, apply Theorem 3.10 to the fibration $(X, x) \rightarrow B\Gamma$.

3.2. Equivalent formulations.

Definition 3.18. Define $\mathcal{E}(R)$ to be the full subcategory of $\mathrm{AGp} \downarrow R$ consisting of those morphisms $\rho : G \rightarrow R$ of pro-algebraic groups which are pro-unipotent extensions. Similarly, define $s\mathcal{E}(R)$ to consist of the pro-unipotent extensions in $s\mathrm{AGp} \downarrow R$, and $\mathrm{Ho}_*(s\mathcal{E}(R))$ to be full subcategory of $\mathrm{Ho}(s\mathrm{AGp})$ on objects $s\mathcal{E}(R)$.

Definition 3.19. Let $c\mathrm{Alg}(R)_*$ be the category of R -representations in cosimplicial \mathbb{R} -algebras, equipped with an augmentation to the structure sheaf $O(R)$ of R . A weak equivalence in $c\mathrm{Alg}(R)_*$ is a map which induces isomorphisms on cohomology groups. We denote by $\mathrm{Ho}(c\mathrm{Alg}(R)_*)$ the localisation of $c\mathrm{Alg}(R)_*$ at weak equivalences. Denote the respective opposite categories by $s\mathrm{Aff}(R)_* = R \downarrow s\mathrm{Aff}(R)$ and $\mathrm{Ho}(s\mathrm{Aff}(R)_*)$.

Definition 3.20. Define $D\mathrm{GAlg}(R)_*$ to be the category of R -representations in non-negatively graded cochain \mathbb{R} -algebras, equipped with an augmentation to $O(R)$. Here, a cochain algebra is a cochain complex $A = \bigoplus_{i \in \mathbb{N}_0} A^i$ over k , equipped with a graded-commutative associative product $A^i \times A^j \rightarrow A^{i+j}$, and unit $1 \in A^0$.

A weak equivalence in $D\mathrm{GAlg}(R)_*$ is a map which induces isomorphisms on cohomology groups. We denote by $\mathrm{Ho}(D\mathrm{GAlg}(R)_*)$ the localisation of $D\mathrm{GAlg}(R)_*$ at weak equivalences. Define $dg\mathrm{Aff}(R)_*$ to be the category opposite to $D\mathrm{GAlg}(R)_*$, and $\mathrm{Ho}(dg\mathrm{Aff}(R)_*)$ opposite to $\mathrm{Ho}(D\mathrm{GAlg}(R)_*)$.

Let $D\mathrm{GAlg}(R)_{0*}$ be the full subcategory of $D\mathrm{GAlg}(R)_*$ whose objects A satisfy $H^0(A) = k$. Let $\mathrm{Ho}(D\mathrm{GAlg}(R)_{0*})_0$ be the full subcategory of $\mathrm{Ho}(D\mathrm{GAlg}(R)_*)$ on the objects of $D\mathrm{GAlg}(R)_{0*}$. Let $dg\mathrm{Aff}(R)_{0*}$ and $\mathrm{Ho}(dg\mathrm{Aff}(R)_{0*})_0$ be the opposite categories to $D\mathrm{GAlg}(R)_{0*}$ and $\mathrm{Ho}(D\mathrm{GAlg}(R)_{0*})_0$, respectively. Given $A \in D\mathrm{GAlg}(R)_*$, write $\mathrm{Spec} A \in dg\mathrm{Aff}(R)_*$ for the corresponding object of the opposite category.

Definition 3.21. Define $dg\mathcal{N}(R)$ to be the category of R -representations in finite-dimensional nilpotent non-negatively graded chain Lie algebras. Let $dg\hat{\mathcal{N}}(R)$ be the category of pro-objects in the Artinian category $dg\mathcal{N}(R)$.

Let $dg\mathcal{P}(R)$ be the category with the same objects as $dg\hat{\mathcal{N}}(R)$, and morphisms given by

$$\mathrm{Hom}_{dg\mathcal{P}(R)}(\mathfrak{g}, \mathfrak{h}) = \exp(H_0\mathfrak{h}) \times^{\exp(\mathfrak{h}_0^R)} \mathrm{Hom}_{\mathrm{Ho}(dg\hat{\mathcal{N}}(R))}(\mathfrak{g}, \mathfrak{h}).$$

where \mathfrak{h}_0^R (the Lie subalgebra of R -invariants in \mathfrak{h}_0) acts by conjugation on the set of homomorphisms. Composition of morphisms is given by $(u, f) \circ (v, g) = (u \circ f(v), f \circ g)$.

Definition 3.22. Let $s\hat{\mathcal{N}}(R)$ be the category of simplicial objects in $\hat{\mathcal{N}}(R)$, and let $s\mathcal{P}(R)$ be the category with the same objects as $s\hat{\mathcal{N}}(R)$, and morphisms given by

$$\mathrm{Hom}_{s\mathcal{P}(R)}(\mathfrak{g}, \mathfrak{h}) = \exp(\pi_0\mathfrak{h}) \times^{\exp(\mathfrak{h}_0^R)} \mathrm{Hom}_{\mathrm{Ho}(s\hat{\mathcal{N}}(R))}(\mathfrak{g}, \mathfrak{h}),$$

where composition of morphisms is given by $(u, f) \circ (v, g) = (u \circ f(v), f \circ g)$.

Definition 3.23. Define a functor $\bar{W} : dg\hat{\mathcal{N}}(R) \rightarrow dg\mathrm{Aff}(R)$ by $O(\bar{W}\mathfrak{g}) := \mathrm{Sym}(\mathfrak{g}^\vee[-1])$ the graded polynomial ring on generators $\mathfrak{g}^\vee[-1]$, with derivation defined on generators by $d_{\mathfrak{g}} + \Delta$, for Δ the Lie cobracket on \mathfrak{g}^\vee .

\bar{W} has a left adjoint G , given by writing $\sigma A^\vee[1]$ for the brutal truncation (in non-negative degrees) of $A^\vee[1]$, and setting

$$G(A) = \mathrm{Lie}(\sigma A^\vee[1]),$$

the free graded Lie algebra, with differential similarly defined on generators by $d_A + \Delta$, with Δ here being the coproduct on A^\vee .

We may then define $\bar{G} : \mathrm{Ho}(dg\mathrm{Aff}(R)_{0*}) \rightarrow dg\mathcal{P}(R)$ on objects by choosing, for $A \in D\mathrm{GAlg}(R)_{0*}$, a quasi-isomorphism $B \rightarrow A$ with $B^0 = k$ (for an explicit construction of B , see [Pri3, Remark 4.35]) and setting $\bar{G}(A) := G(B)$.

Definition 3.24. Given a cochain algebra $A \in DG\text{Alg}(R)$, and a chain Lie algebra $\mathfrak{g} \in dg\hat{\mathcal{N}}(R)$, define the Maurer-Cartan space by

$$\text{MC}(A \hat{\otimes}^R \mathfrak{g}) := \{\omega \in \prod_n A^{n+1} \hat{\otimes}^R \mathfrak{g}_n \mid d\omega + \frac{1}{2}[\omega, \omega] = 0\},$$

where, for an inverse system $\{V_i\}$, $\{V_i\} \hat{\otimes}^R A := \varprojlim (V_i \otimes A)$, and $\{V_i\} \hat{\otimes}^R A$ consists of R -invariants in this. Note that

$$\text{Hom}_{dg\text{Aff}(R)}(\text{Spec } A, \bar{W}\mathfrak{g}) \cong \text{MC}(A \hat{\otimes}^R \mathfrak{g}).$$

Definition 3.25. Given $A \in DG\text{Alg}(R)$ and $\mathfrak{g} \in dg\hat{\mathcal{N}}(R)$, we define the gauge group by

$$\text{Gg}(A \hat{\otimes}^R \mathfrak{g}) := \exp\left(\prod_n A^n \hat{\otimes}^R \mathfrak{g}_n\right).$$

Define a gauge action of $\text{Gg}(A \hat{\otimes}^R \mathfrak{g})$ on $\text{MC}(A \hat{\otimes}^R \mathfrak{g})$ by

$$g(\omega) := g \cdot \omega \cdot g^{-1} - (dg) \cdot g^{-1}.$$

Here, $a \cdot b$ denotes multiplication in the universal enveloping algebra $\mathcal{U}(A \hat{\otimes}^R \mathfrak{g})$ of the differential graded Lie algebra (DGLA) $A \hat{\otimes}^R \mathfrak{g}$. That $g(\omega) \in \text{MC}(A \hat{\otimes}^R \mathfrak{g})$ is a standard calculation (see [Kon] or [Man]).

Proposition 3.26. For $A \in DG\text{Alg}(R)_*$ and $\mathfrak{g} \in dg\hat{\mathcal{N}}(R)$,

$$\text{Hom}_{\text{Ho}(dg\text{Aff}(R)_*)}(\text{Spec } A, \bar{W}\mathfrak{g}) \cong \exp(\text{H}_0\mathfrak{g}) \times^{\text{Gg}(A \hat{\otimes}^R \mathfrak{g})} \text{MC}(A \hat{\otimes}^R \mathfrak{g}),$$

where $\bar{W}\mathfrak{g} \in dg\text{Aff}_*$ is the composition $R \rightarrow \text{Spec } k \rightarrow \bar{W}\mathfrak{g}$, and the morphism $\text{Gg}(A \hat{\otimes}^R \mathfrak{g}) \rightarrow \exp(\text{H}_0\mathfrak{g})$ factors through $\text{Gg}(O(R) \hat{\otimes}^R \mathfrak{g}) = \mathfrak{g}_0$.

Proof. The derived Hom space $\mathbf{R}\underline{\text{Hom}}_{dg\text{Aff}(R)_*}(\text{Spec } A, \bar{W}\mathfrak{g})$ is the homotopy fibre of

$$\mathbf{R}\underline{\text{Hom}}_{dg\text{Aff}(R)}(\text{Spec } A, \bar{W}\mathfrak{g}) \rightarrow \mathbf{R}\underline{\text{Hom}}_{dg\text{Aff}(R)}(R, \bar{W}\mathfrak{g}),$$

over the unique element 0 of $\text{MC}(O(R) \hat{\otimes}^R \mathfrak{g})$. For a morphism $f : X \rightarrow Y$ of simplicial sets (or topological spaces), path components $\pi_0 F$ of the homotopy fibre over $0 \in Y$ are given by pairs (x, γ) , for $x \in X$ and γ a homotopy class of paths from 0 to fx , modulo the equivalence relation $(x, \gamma) \sim (x', \gamma')$ if there exists a path $\delta : x \rightarrow x'$ in X with $\gamma * f\delta = \gamma'$. If Y has a unique vertex 0, this reduces to pairs (x, γ) , for $x \in X$ and $\gamma \in \pi_1(Y, 0)$, with δ acting as before.

Now, we can define an object $V\mathfrak{g} \in dg\text{Aff}(R)$ by

$$\text{Hom}_{dg\text{Aff}(R)}(\text{Spec } A, V\mathfrak{g}) \cong \text{Gg}(A \hat{\otimes}^R \mathfrak{g}),$$

and by [Pri3, Lemma 4.33], $V\mathfrak{g} \times \bar{W}\mathfrak{g}$ is a path object for $\bar{W}\mathfrak{g}$ in $dg\text{Aff}(R)$ via the maps

$$\bar{W}\mathfrak{g} \xrightarrow{(\text{id}, 1)} \bar{W}\mathfrak{g} \times V\mathfrak{g} \xrightarrow[\phi]{\text{pr}_1} \bar{W}\mathfrak{g},$$

where ϕ is the gauge action.

Thus the loop object $\Omega(\bar{W}\mathfrak{g}, 0)$ for $0 \in \text{MC}(A \hat{\otimes}^R \mathfrak{g})$ is given by

$$\text{Hom}_{dg\text{Aff}(R)}(\text{Spec } A, \Omega(\bar{W}\mathfrak{g}, 0)) = \{g \in \text{Gg}(A \hat{\otimes}^R \mathfrak{g}) : g(0) = 0\} = \exp(\ker d \cap \prod_n A^n \hat{\otimes}^R \mathfrak{g}_n)$$

Hence

$$\pi_i \mathbf{R}\underline{\text{Hom}}_{dg\text{Aff}(R)}(\text{Spec } A, \Omega(\bar{W}\mathfrak{g}, 0)) \cong \text{H}^{-i}\left(\prod_n A^n \hat{\otimes}^R \mathfrak{g}_n\right),$$

and in particular,

$$\pi_1(\mathbf{R}\underline{\text{Hom}}_{dg\text{Aff}(R)}(R, \bar{W}\mathfrak{g}), 0) = \pi_0 \mathbf{R}\underline{\text{Hom}}_{dg\text{Aff}(R)}(\text{Spec } A, \Omega(\bar{W}\mathfrak{g}, 0)) \cong \exp(\text{H}_0\mathfrak{g}).$$

This gives us a description of

$$\mathrm{Hom}_{\mathrm{Ho}(dg\mathrm{Aff}(R)_*)}(\mathrm{Spec} A, \bar{W}\mathfrak{g}) = \pi_0 \mathbf{R}\underline{\mathrm{Hom}}_{dg\mathrm{Aff}(R)_*}(\mathrm{Spec} A, \bar{W}\mathfrak{g})$$

as consisting of pairs (x, γ) for $x \in \mathrm{MC}(A \hat{\otimes}^R \mathfrak{g})$ and $\gamma \in \exp(\mathrm{H}_0 \mathfrak{g})$, modulo the equivalence $(x, \gamma) \sim (\delta(x), \delta * \gamma)$ for $\delta \in \mathrm{Gg}(A \hat{\otimes}^R \mathfrak{g})$. In other words,

$$\mathrm{Hom}_{\mathrm{Ho}(dg\mathrm{Aff}(R)_*)}(\mathrm{Spec} A, \bar{W}\mathfrak{g}) \cong \mathrm{MC}(A \hat{\otimes}^R \mathfrak{g}) \times^{\mathrm{Gg}(A \hat{\otimes}^R \mathfrak{g})} \exp(\mathrm{H}_0 \mathfrak{g}),$$

as required. \square

Corollary 3.27. \bar{W} defines a functor $\bar{W} : dg\mathcal{P}(R) \rightarrow \mathrm{Ho}(dg\mathrm{Aff}(R)_*)$.

Proof. On objects, we map \mathfrak{g} to $\bar{W}\mathfrak{g}$. Given a morphism $f : \mathfrak{g} \rightarrow \mathfrak{h}$ in $dg\hat{\mathcal{N}}(R)$ and $h \in \mathrm{H}_0 \mathfrak{h}$, we can define an element $\bar{W}(h, f)$ of

$$\mathrm{Hom}_{\mathrm{Ho}(dg\mathrm{Aff}(R)_*)}(\bar{W}\mathfrak{g}, \bar{W}\mathfrak{h})$$

by $[(\exp(h), \bar{W}f)] \in \exp(\mathrm{H}_0 \mathfrak{g}) \times^{\mathrm{Gg}(A \hat{\otimes}^R \mathfrak{g})} \mathrm{Hom}_{dg\mathrm{Aff}(R)}(\bar{W}\mathfrak{g}, \bar{W}\mathfrak{h})$.

If f is a weak equivalence then $\bar{W}(0, f)$ is a weak equivalence in $dg\mathrm{Aff}(R)_*$, which implies that \bar{W} must descend to a functor

$$\bar{W} : dg\mathcal{P}(R) \rightarrow \mathrm{Ho}(dg\mathrm{Aff}(R)_*),$$

since $\bar{W}(h, f)$ is a function of the homotopy class of f . \square

Definition 3.28. Recall that the Thom–Sullivan (or Thom–Whitney) functor Th from cosimplicial algebras to DG algebras is defined as follows. Let $\Omega(|\Delta^n|)$ be the DG algebra of rational polynomial forms on the n -simplex, so

$$\Omega(|\Delta^n|) = \mathbb{Q}[t_0, \dots, t_n, dt_0, \dots, dt_n] / (1 - \sum_i t_i),$$

for t_i of degree 0. The usual face and degeneracy maps for simplices yield $\partial_i : \Omega(|\Delta^n|) \rightarrow \Omega(|\Delta^{n-1}|)$ and $\sigma_i : \Omega(|\Delta^n|) \rightarrow \Omega(|\Delta^{n-1}|)$, giving a simplicial DGA. Given a cosimplicial algebra A , we then set

$$\mathrm{Th}(A) := \{a \in \prod_n A^n \otimes \Omega(|\Delta^n|) : \partial_A^i a_n = \partial_i a_{n+1}, \sigma_A^j a_n = \sigma_j a_{n-1} \forall i, j\}.$$

Theorem 3.29. We have the following diagram of equivalences of categories:

$$\begin{array}{ccc} \mathrm{Ho}(dg\mathrm{Aff}(R)_*)_0 & \xrightarrow{\mathrm{Spec} D} & \mathrm{Ho}(s\mathrm{Aff}(R)_*)_0 \\ \bar{W} \uparrow \downarrow \bar{G} & & \bar{W} \exp \uparrow \\ dg\mathcal{P}(R) & \xleftarrow{N} s\mathcal{P}(R) \xrightarrow{R \times \exp(-)} & \mathrm{Ho}_*(s\mathcal{E}(R)), \end{array}$$

where N denotes Dold–Kan normalisation ([Pri3, Definition 4.11]), D denormalisation ([Pri3, Definition 4.26]), and $\bar{W} \exp(\mathfrak{g})$ is the classifying space of the simplicial group $\exp(\mathfrak{g})$. A homotopy inverse to D is given by the functor Th of Thom–Sullivan cochains.

Proof. First, [Pri3, Propositions 4.27 and 4.12] ensure that $\mathrm{Spec} D$ and N are equivalences, while [Pri3, Theorem 4.39] implies that $(\mathrm{Spec} D) \circ \bar{W} = \bar{W} \circ N$. [HS, 4.1] shows that D and Th are homotopy inverses. We now adapt the proof of [Pri3, Corollary 4.41].

The functor $R \times \exp : s\mathcal{P}(R) \rightarrow \mathrm{Ho}_*(s\mathcal{E}(R))$ maps \mathfrak{g} to the simplicial pro-algebraic group given in level n by $R \times \exp(\mathfrak{g}_n)$. Given a morphism $(f, u) \in \exp(\pi_0 \mathfrak{h}) \times^{\exp(\mathfrak{h}^R)} \mathrm{Hom}_{\mathrm{Ho}(s\mathcal{N}(R))}(\mathfrak{g}, \mathfrak{h})$, lift u to $\tilde{u} \in \exp(\mathfrak{h}_0)$, and construct a morphism

$$\mathrm{ad}_{\tilde{u}} \circ (R \times \exp(f))$$

in $s\mathcal{E}(R)$, where $\text{ad}_g(x) = gxg^{-1}$. Another choice \tilde{u}' of \tilde{u} amounts to giving $v \in \exp(\mathfrak{h}_1)$ with $\partial_0 v = \tilde{u}$ and $\partial_1 v = \tilde{u}'$. Thus $\text{ad}_v \circ (R \times \exp(f))$ gives a homotopy from $\text{ad}_{\tilde{u}} \circ (R \times \exp(f))$ to $\text{ad}_{\tilde{u}'} \circ (R \times \exp(f))$. This means that $R \times \exp(-) : s\mathcal{P}(R) \rightarrow \text{Ho}_*(s\mathcal{E}(R))$ is a well-defined functor.

The existence of Levi decompositions ensures that $R \times \exp(-)$ is essentially surjective and full (since every morphism in $s\mathcal{E}(R)$ is the composite of an inner automorphism and a morphism preserving Levi decompositions). Since the choice of inner automorphism on $R \times U$ is unique up to R -invariants U^R , $R \times \exp(-) : s\mathcal{P}(R) \rightarrow \text{Ho}_*(s\mathcal{E}(R))$ is an equivalence (see [Pri3, Proposition 3.15] for a similar result).

Now, by [Pri3, Proposition 3.48], there is a canonical quasi-isomorphism $\bar{W}G(X) \rightarrow X$, for all $X \in dg\text{Aff}(R)$ with $X_0 = \text{Spec } k$, and hence for all $X \in dg\text{Aff}(R)_*$ with $X_0 = \text{Spec } k$. With reasoning as in Definition 3.23, this means that

$$\bar{W} : dg\mathcal{P}(R) \rightarrow \text{Ho}(dg\text{Aff}(R)_*)_0$$

is essentially surjective, with $\bar{G}(Y)$ in the essential pre-image of Y , although it does not guarantee that we can define \bar{G} consistently on morphisms.

To establish that \bar{W} is full and faithful, it will suffice to show that for all $A \in DGA\text{lg}(R)$ with $A^0 = k$, the transformation

$$\text{Hom}_{dg\mathcal{P}(R)}(G(A), \mathfrak{h}) \rightarrow \text{Hom}_{\text{Ho}(dg\text{Aff}(R)_*)}(\text{Spec } A, \bar{W}\mathfrak{h})$$

is an isomorphism. For $A = k$, this is certainly true, since in both cases we get $\exp(\text{H}_0\mathfrak{h})/\exp(\text{H}_0\mathfrak{h})^R$ for both Hom-sets (using Proposition 3.26). The morphism $k \rightarrow A$ gives surjective maps from both Hom-sets above to $\exp(\text{H}_0\mathfrak{h})/\exp(\text{H}_0\mathfrak{h})^R$, and by Proposition 3.26, the map on any fibre is just

$$\text{Hom}_{\text{Ho}(dg\hat{\mathcal{N}}(R))}(G(A), \mathfrak{h})/\exp(\ker(\mathfrak{h}_0^R \rightarrow \text{H}_0\mathfrak{h}^R)) \xrightarrow{\theta} \text{MC}(A \hat{\otimes}^R \mathfrak{h})/\ker(\text{Gg}(A \hat{\otimes}^R \mathfrak{h}) \rightarrow \exp(\text{H}_0\mathfrak{h}^R)).$$

Now, $G(A)$ is a hull for both functors on $dg\mathcal{N}(R)$ (in the sense of [Pri3, Proposition 3.43]), so by the argument of [Pri3, Proposition 3.47], it suffices to show that θ is an isomorphism whenever $\mathfrak{h} \in \mathcal{N}(R)$ (i.e. whenever $\mathfrak{h}_i = 0$ for all $i > 0$). In that case,

$$\text{Hom}_{\text{Ho}(dg\hat{\mathcal{N}}(R))}(G(A), \mathfrak{h}) = \text{Hom}_{dg\hat{\mathcal{N}}(R)}(G(A), \mathfrak{h}) = \text{MC}(A \hat{\otimes}^R \mathfrak{h}),$$

and

$$\text{Gg}(A \hat{\otimes}^R \mathfrak{h}) = \exp(A^0 \hat{\otimes}^R \mathfrak{h}) = \exp(k \hat{\otimes}^R \mathfrak{h}) = \exp(\mathfrak{h}^R) = \exp(\mathfrak{h}_0^R),$$

so θ is indeed an isomorphism. Hence \bar{W} is an equivalence, with quasi-inverse \bar{G} on objects. \square

Remark 3.30. If we take a set T of points in X , then the groupoid $\Gamma := T \times_{|X|} \pi_f X$ has objects T , with morphisms $\Gamma(x, y)$ corresponding to homotopy classes of paths from x to y in X . If $T = \{x\}$, note that Γ is just $\pi_1(X, x)$.

Take a reductive pro-algebraic groupoid R (as in [Pri3, §2]) on objects T , and a morphism $\rho : \Gamma \rightarrow R$ preserving T . The relative Malcev completion $G(X; T)^{\rho, \text{Mal}}$ is then a pro-unipotent extension of R (as a simplicial pro-algebraic groupoid — see [Pri3, §2.4]). Then $\varpi_1(X; T)^{\rho, \text{Mal}} := \pi_0 G(X; T)^{\rho, \text{Mal}}$ is a groupoid on objects T , with $\varpi_1(X; T)^{\rho, \text{Mal}}(x, x) = \varpi_1(X, x)^{\rho_x, \text{Mal}}$. Likewise, $\varpi_n(X; T)^{\rho, \text{Mal}} := \pi_{n-1} G(X; T)^{\rho, \text{Mal}}$ is a $\varpi_1(X; T)^{\rho, \text{Mal}}$ -representation, with $\varpi_1(X; T)^{\rho, \text{Mal}}(x) = \varpi_1(X, x)^{\rho_x, \text{Mal}}$. Here, $\rho_x : \pi_1(X, x) \rightarrow R(x, x)$ is defined by restricting ρ to $x \in T$.

If we set $dg\text{Aff}(R)_* := (\coprod_{x \in T} R(x, -)) \downarrow dg\text{Aff}(R)$ and $s\text{Aff}(R)_* := (\coprod_{x \in T} R(x, -)) \downarrow s\text{Aff}(R)$, where $R(x, -)$ is the R -representation $y \mapsto R(x, y)$, then Theorem 3.29 adapts

to give equivalences

$$\begin{array}{ccc}
\mathrm{Ho}(dg\mathrm{Aff}(R)_*)_0 & \xrightarrow{\mathrm{Spec} D} & \mathrm{Ho}(s\mathrm{Aff}(R)_*)_0 \\
\bar{W} \uparrow \downarrow \bar{G} & & \bar{W} \uparrow \\
dg\mathcal{P}(R) & \xleftarrow{N} & s\mathcal{P}(R) \xrightarrow{R \times \exp(-)} \mathrm{Ho}_*(s\mathcal{E}(R)),
\end{array}$$

where $\mathrm{Ho}_*(s\mathcal{E}(R))$ is the full subcategory of the homotopy category $\mathrm{Ho}(T \downarrow s\mathrm{AGpd} \downarrow R)$ (of simplicial pro-algebraic groupoids under T and over R) whose objects are pro-unipotent extensions of R . The objects of $s\mathcal{P}(R)$ are R -representations in $s\mathcal{N}$, with morphisms given by

$$\mathrm{Hom}_{s\mathcal{P}(R)}(\mathfrak{g}, \mathfrak{h}) = \left(\prod_{x \in T} \exp(\pi_0 \mathfrak{h}(x)) \right) \times^{\exp(\mathfrak{h}_0^R)} \mathrm{Hom}_{\mathrm{Ho}(s\mathcal{N}(R))}(\mathfrak{g}, \mathfrak{h}),$$

where \mathfrak{h}_0^R is the Lie algebra $\mathrm{Hom}_R(\mathbb{R}, \mathfrak{h}_0)$ (with \mathbb{R} regarded as a constant R -representation). The category $dg\mathcal{P}(R)$ is defined similarly.

Definition 3.31. Recall that $O(R)$ has the natural structure of an $R \times R$ -representation, with the R -actions given by left and right multiplication.

Definition 3.32. Let \mathbb{B}_ρ be the R -torsor on X corresponding to the representation $\rho : \pi_1(X, x) \rightarrow R(\mathbb{R})$, and let $O(\mathbb{B}_\rho)$ be the R -representation $\mathbb{B}_\rho \times^R O(R)$ in local systems of \mathbb{R} -algebras on X (with the R -representation structure given by the right action on $O(R)$).

Proposition 3.33. *Under the equivalences of Theorem 3.29, the relative Malcev homotopy type $G(X, x)^{\rho, \mathrm{Mal}}$ of a pointed topological space (X, x) corresponds to the complex*

$$(C^\bullet(X, O(\mathbb{B}_\rho)) \xrightarrow{x^*} O(R)) \in c\mathrm{Alg}(R)_{0*}$$

of $O(\mathbb{B}_\rho)$ -valued chains on X .

Proof. This is essentially [Pri3, Theorem 3.55]. \square

Definition 3.34. Given a manifold X , denote the sheaf of real C^∞ n -forms on X by \mathcal{A}^n . Given a real sheaf \mathcal{F} on X , write

$$A^n(X, \mathcal{F}) := \Gamma(X, \mathcal{F} \otimes_{\mathbb{R}} \mathcal{A}^n).$$

Proposition 3.35. *If $k = \mathbb{R}$, then the relative Malcev homotopy type of a pointed manifold (X, x) relative to $\rho : \pi_1(X, x) \rightarrow R(\mathbb{R})$ is given in $DG\mathrm{Alg}(R)_*$ by $A^\bullet(X, O(\mathbb{B}_\rho)) \xrightarrow{x^*} O(R)$.*

Proof. This is essentially [Pri3, Proposition 4.50]. \square

Remark 3.36. If we take a set T of points in X and ρ as in Remark 3.30, then Proposition 3.33 adapts to say that the relative Malcev homotopy type $G(X; T)^{\rho, \mathrm{Mal}}$ corresponds to the complex

$$(C^\bullet(X, O(\mathbb{B}_\rho)) \xrightarrow{\prod_{x \in T} x^*} \prod_{x \in T} O(R)(x, -)) \in c\mathrm{Alg}(R)_{0*}.$$

Proposition 3.35 adapts to show that $(X; T)^{\rho, \mathrm{Mal}}$ is given by

$$A^\bullet(X, O(\mathbb{B}_\rho)) \xrightarrow{\prod_{x \in T} x^*} \prod_{x \in T} O(R)(x, -) \in DG\mathrm{Alg}(R)_*.$$

3.3. General homotopy types.

Lemma 3.37. *For an R -representation A in DG algebras, there is a cofibrantly generated model structure on the category $DG_{\mathbb{Z}}\text{Mod}_A(R)$ of R -representations in \mathbb{Z} -graded cochain A -modules, in which fibrations are surjections, and weak equivalences are isomorphisms on cohomology.*

Proof. Let $S(n)$ denote the cochain complex $A[-n]$. Let $D(n)$ be the cone complex of $\text{id} : A[1-n] \rightarrow A[1-n]$, so the underlying graded vector space is just $A[1-n] \oplus A[-n]$.

Define I to be the set of canonical maps $S(n) \otimes V \rightarrow D(n) \otimes V$, for $n \in \mathbb{Z}$ and V ranging over all finite-dimensional R -representations. Define J to be the set of morphisms $0 \rightarrow D(n) \otimes V$, for $n \in \mathbb{Z}$ and V ranging over all finite-dimensional R -representations. Then we have a cofibrantly generated model structure, with I the generating cofibrations and J the generating trivial cofibrations, by verifying the conditions of [Hov, Theorem 2.1.19]. \square

Definition 3.38. Let $DG_{\mathbb{Z}}\text{Alg}(R)$ be the category of R -representations in \mathbb{Z} -graded cochain \mathbb{R} -algebras. For an R -representation A in algebras, we define $DG_{\mathbb{Z}}\text{Alg}_A(R)$ to be the comma category $A \downarrow DG_{\mathbb{Z}}\text{Alg}(R)$. Denote the opposite category by $dg_{\mathbb{Z}}\text{Aff}_A(R)$. We will also sometimes write this as $dg_{\mathbb{Z}}\text{Aff}_{\text{Spec } A}(R)$.

Lemma 3.39. *There is a cofibrantly generated model structure on $DG_{\mathbb{Z}}\text{Alg}_A(R)$, in which fibrations are surjections, and weak equivalences are quasi-isomorphisms.*

Proof. This follows by applying [Hir, Theorem 11.3.2] to the forgetful functor $DG_{\mathbb{Z}}\text{Alg}_A(R) \rightarrow DG_{\mathbb{Z}}\text{Mod}_{\mathbb{Q}}(R)$. \square

3.3.1. Derived pullbacks and base change.

Definition 3.40. Given a morphism $f : X \rightarrow Y$ in $dg\text{Aff}(R)$, the pullback functor $f^* : DG_{\mathbb{Z}}\text{Alg}_Y(R) \rightarrow DG_{\mathbb{Z}}\text{Alg}_X(R)$ is left Quillen, with right adjoint f_* . Denote the derived left Quillen functor by $\mathbf{L}f^* : \text{Ho}(DG_{\mathbb{Z}}\text{Alg}_Y(R)) \rightarrow \text{Ho}(DG_{\mathbb{Z}}\text{Alg}_X(R))$. Observe that f_* preserves weak equivalences, so the derived right Quillen functor is just $\mathbf{R}f_* = f_*$. Denote the functor opposite to $\mathbf{L}f^*$ by $\times_Y^{\mathbf{R}}X : \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_Y(R)) \rightarrow \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_X(R))$.

Lemma 3.41. *If $f : \text{Spec } B \rightarrow \text{Spec } A$ is a flat morphism in $\text{Aff}(R)$, then $\mathbf{L}f^* = f^*$.*

Proof. This is just the observation that flat pullback preserves weak equivalences. $\mathbf{L}f^*C$ is defined to be $f^*\tilde{C}$, for $\tilde{C} \rightarrow C$ a cofibrant approximation, but we then have $f^*\tilde{C} \rightarrow f^*C$ a weak equivalence, so $\mathbf{L}f^*C = f^*C$. \square

Proposition 3.42. *If $S \in DG_{\mathbb{Z}}\text{Alg}_A(R)$, and $f : A \rightarrow B$ is any morphism in $DG\text{Alg}(R)$, then cohomology of $\mathbf{L}f^*S$ is given by the hypertor groups*

$$\mathbf{H}^i(\mathbf{L}f^*S) = \mathbf{Tor}_{-i}^A(S, B).$$

Proof. Take a cofibrant approximation $C \rightarrow S$, so $\mathbf{L}f^*S \cong f^*C$. Thus $A \rightarrow C$ is a retraction of a transfinite composition of pushouts of generating cofibrations. The generating cofibrations are filtered direct limits of projective bounded complexes, so C is a retraction of a filtered direct limit of projective bounded cochain complexes. Since cohomology and hypertor both commute with filtered direct limits (the latter following since we may choose a Cartan-Eilenberg resolution of the colimit in such a way that it is a colimit of Cartan-Eilenberg resolutions of the direct system), we may apply [Wei, Application 5.7.8] to see that C is a resolution computing the hypertor groups of S . \square

Proposition 3.43. *If $S \in DG_{\mathbb{Z}}\text{Alg}_A(R)$ is flat, and $f : A \rightarrow B$ is any morphism in $\text{Alg}(R)$, with either S bounded or f of finite flat dimension, then*

$$\mathbf{L}f^*S \simeq f^*S.$$

Proof. If S is bounded, then $\mathbf{L}f^*S \simeq S \otimes_A^{\mathbf{L}} B$, which is just $S \otimes_A B$ when S is also flat. If instead f is of finite flat dimension, then [Wei, Corollary 10.5.11] implies that $\mathbf{H}^*(S \otimes_A B) = \mathbf{Tor}_{-*}^A(S, B)$, as required. \square

Definition 3.44. Given an R -representation Y in schemes, define $DG_{\mathbb{Z}}\text{Alg}_Y(R)$ to be the category of R -equivariant quasi-coherent \mathbb{Z} -graded cochain algebras on Y . Define a weak equivalence in this category to be a map giving isomorphisms on cohomology sheaves (over Y), and define $\text{Ho}(DG_{\mathbb{Z}}\text{Alg}_Y(R))$ to be the homotopy category obtained by localising at weak equivalences. Define the categories $dg_{\mathbb{Z}}\text{Aff}_Y(R)$, $\text{Ho}(dg_{\mathbb{Z}}\text{Aff}_Y(R))$ to be the respective opposite categories.

Definition 3.45. Given a quasi-compact, quasi-affine scheme X , let $j : X \rightarrow \bar{X}$ be the open immersion $X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$. Take a resolution $\mathcal{O}_X \rightarrow \mathcal{C}_X^\bullet$ of \mathcal{O}_X in $DG_{\mathbb{Z}}\text{Alg}_X(R)$, flabby with respect to Zariski cohomology (for instance by applying the Thom-Sullivan functor Th to the cosimplicial algebra $\mathcal{C}^\bullet(\mathcal{O}_X)$ arising from a Čech resolution). Define $\mathbf{R}j_*\mathcal{O}_X$ to be $j_*\mathcal{C}_X^\bullet \in DG_{\mathbb{Z}}\text{Alg}_{\bar{X}}(R)$.

Proposition 3.46. *The functor $j^* : DG_{\mathbb{Z}}\text{Alg}_{\mathbf{R}j_*\mathcal{O}_X}(R) \rightarrow DG_{\mathbb{Z}}\text{Alg}_X(R)$ induces an equivalence $\text{Ho}(DG_{\mathbb{Z}}\text{Alg}_{\mathbf{R}j_*\mathcal{O}_X}(R)) \rightarrow \text{Ho}(DG_{\mathbb{Z}}\text{Alg}_X(R))$.*

For any R -representation B in algebras, this extends to an equivalence $\text{Ho}(DG_{\mathbb{Z}}\text{Alg}_{\mathbf{R}j_\mathcal{O}_X}(R) \downarrow \mathbf{R}j_*\mathcal{O}_X \otimes B) \rightarrow \text{Ho}(DG_{\mathbb{Z}}\text{Alg}_X(R) \downarrow \mathcal{O}_X \otimes B)$.*

Proof. Since j is flat, j^* preserves quasi-isomorphisms, so j^* descends to a morphism of homotopy categories. If $\mathcal{C}_X^\bullet = \text{Th } \mathcal{C}^\bullet(\mathcal{O}_X)$, then a quasi-inverse functor will be given by $\mathcal{A} \mapsto j_*\text{Th } \check{\mathcal{C}}^\bullet(\mathcal{A})$. The inclusion $\mathcal{A} \rightarrow \text{Th } \check{\mathcal{C}}^\bullet(\mathcal{A})$ is a quasi-isomorphism, as is the map $j^*j_*\text{Th } \check{\mathcal{C}}^\bullet(\mathcal{A}) \rightarrow \text{Th } \check{\mathcal{C}}^\bullet(\mathcal{A})$, since

$$\mathcal{H}^i(j^*j_*\text{Th } \check{\mathcal{C}}^\bullet(\mathcal{A})) = j^*\mathbf{R}^i j_*(\mathcal{A}) = \mathcal{H}^i(\mathcal{A}),$$

as $j^*\mathbf{R}^i j_*\mathcal{F} = 0$ for $i > 0$ and \mathcal{F} a quasi-coherent sheaf (concentrated in degree 0), X being quasi-affine.

Now, the composite morphism

$$\mathbf{R}j_*\mathcal{O}_X \rightarrow j_*j^*(\mathbf{R}j_*\mathcal{O}_X) \rightarrow j_*\text{Th } \check{\mathcal{C}}^\bullet(j^*(\mathbf{R}j_*\mathcal{O}_X))$$

is a quasi-isomorphism, since $j^*(\mathbf{R}j_*\mathcal{O}_X) \rightarrow \mathcal{O}_X$ is a quasi-isomorphism. Cofibrant objects $\mathcal{M} \in DG_{\mathbb{Z}}\text{Mod}_{\mathbf{R}j_*\mathcal{O}_X}(R)$ are retracts of I -cells, which admit (ordinal-indexed) filtrations whose graded pieces are copies of $(\mathbf{R}j_*\mathcal{O}_X)[i]$, so we deduce that for cofibrant modules \mathcal{M} , the map

$$\mathcal{M} \rightarrow j_*\text{Th } \check{\mathcal{C}}^\bullet(j^*\mathcal{M})$$

is a quasi-isomorphism. Since cofibrant algebras are a fortiori cofibrant modules, $\mathcal{B} \rightarrow j_*\text{Th } \check{\mathcal{C}}^\bullet(j^*\mathcal{B})$ is a quasi-isomorphism for all cofibrant $\mathcal{B} \in DG_{\mathbb{Z}}\text{Alg}_{\mathbf{R}j_*\mathcal{O}_X}(R)$, which completes the proof in the case when $\mathcal{C}_X^\bullet = \text{Th } \check{\mathcal{C}}^\bullet(\mathcal{O}_X)$.

For the general case, note that we have quasi-isomorphisms $\text{Th } \check{\mathcal{C}}^\bullet(\mathcal{O}_X) \rightarrow \text{Th } \check{\mathcal{C}}^\bullet(\mathcal{C}_X^\bullet) \leftarrow \mathcal{C}_X^\bullet$, giving quasi-isomorphisms $j_*\text{Th } \check{\mathcal{C}}^\bullet(\mathcal{O}_X) \rightarrow j_*\text{Th } \check{\mathcal{C}}^\bullet(\mathcal{C}_X^\bullet) \leftarrow j_*\mathcal{C}_X^\bullet$, and hence right Quillen equivalences

$$DG_{\mathbb{Z}}\text{Alg}_{j_*\text{Th } \check{\mathcal{C}}^\bullet(\mathcal{O}_X)}(R) \leftarrow DG_{\mathbb{Z}}\text{Alg}_{j_*\text{Th } \check{\mathcal{C}}^\bullet(\mathcal{C}_X^\bullet)}(R) \rightarrow DG_{\mathbb{Z}}\text{Alg}_{j_*\mathcal{C}_X^\bullet}(R).$$

\square

Lemma 3.47. *Let G be an affine group scheme, with a reductive subgroup scheme H acting on a reductive pro-algebraic group R . Then the model categories $dg_{\mathbb{Z}}\text{Aff}_G(R \rtimes H)$ and $dg_{\mathbb{Z}}\text{Aff}_{G/H}(R)$ are equivalent.*

Proof. This is essentially the observation that H -equivariant quasi-coherent sheaves on G are equivalent to quasi-coherent sheaves on G/H . Explicitly, define $U : dg_{\mathbb{Z}}\text{Aff}_{G/H}(R) \rightarrow$

$dg_{\mathbb{Z}}\text{Aff}_G(R \rtimes H)$ by $U(Z) = Z \times_{G/H} G$. This has a left adjoint $F(Y) = Y/H$. We need to show that the unit and co-unit of this adjunction are isomorphisms.

The co-unit is given on $Z \in dg_{\mathbb{Z}}\text{Aff}_{G/H}(R)$ by

$$Z \leftarrow FU(Z) = (Z \times_{G/H} G)/H \cong Z \times_{G/H} (G/H) \cong Z,$$

so is an isomorphism.

The unit is $Y \rightarrow UF(Y) = (Y/H) \times_{G/H} G$, for $Y \in dg_{\mathbb{Z}}\text{Aff}_G(R \rtimes H)$. Now, there is an isomorphism $Y \times_{G/H} G \cong Y \times H$, given by $(y, \pi(y) \cdot h^{-1}) \leftrightarrow (y, h)$, for $\pi : Y \rightarrow G$. This map is H -equivariant for the left H -action on $Y \times_{G/H} G$, and the diagonal H -action on $Y \times H$. Thus

$$UF(Y) = (Y \times_{G/H} G)/(H \times 1) \cong (Y \times H)/H \cong Y,$$

with the final isomorphism given by $(y, h) \mapsto y \cdot h^{-1}$. \square

3.3.2. Extensions.

Definition 3.48. Given $B \in DG_{\mathbb{Z}}\text{Alg}_A(R)$, define the cotangent complex

$$\mathbb{L}_{B/A}^{\bullet} \in \text{Ho}(DG_{\mathbb{Z}}\text{Mod}_B(R))$$

by taking a factorisation $A \rightarrow C \rightarrow B$, with $A \rightarrow C$ a cofibration and $C \rightarrow B$ a trivial fibration. Then set $\mathbb{L}_{B/A}^{\bullet} := \Omega_{C/A}^{\bullet} \otimes_C B = I/I^2$, where $I = \ker(C \otimes_A B \rightarrow B)$. Note that $\mathbb{L}_{B/A}^{\bullet}$ is independent of the choices made, as it can be characterised as the evaluation at B of the derived left adjoint to the functor $M \mapsto B \oplus M$ from DG B -modules to B -augmented DG algebras over A .

Lemma 3.49. *Given a surjection $A \rightarrow B$ in $DG_{\mathbb{Z}}\text{Alg}(R)$, with square-zero kernel I , and a morphism $f : T \rightarrow C$ in $DG_{\mathbb{Z}}\text{Alg}_A(R)$, the hyperext group*

$$\text{Ext}_{T,R}^1(\mathbb{L}_{T/A}^{\bullet}, T \otimes_A^{\mathbb{L}} I \xrightarrow{f} C \otimes_A^{\mathbb{L}} I)$$

of the cone complex is naturally isomorphic to the weak equivalence class of triples (θ, f', γ) , where $\theta : T' \otimes_A^{\mathbb{L}} B \rightarrow T \otimes_A^{\mathbb{L}} B$ is a weak equivalence, $f' : T' \rightarrow C$ a morphism, and γ a homotopy between the morphisms $(f \otimes_A B) \circ \theta, (f' \otimes_A B) : T' \otimes_A^{\mathbb{L}} B \rightarrow C \otimes_A^{\mathbb{L}} B$.

Proof. This is a slight generalisation of a standard result, and we now sketch a proof. Assume that $A \rightarrow T$ is a cofibration, and that $T \rightarrow C$ is a fibration (i.e. surjective). We first consider the case $\gamma = 0$, considering objects T' (flat over A) such that $\theta : T' \otimes_A B \rightarrow T \otimes_A B$ is an isomorphism and $(f \otimes_A B) \circ \theta = (f' \otimes_A B)$.

Since T is cofibrant over A , the underlying graded ring UT is a retract of a polynomial ring, so $UT' \cong UT$. The problem thus reduces to deforming the differential d on T . If we denote the differential of T' by d' , then fixing an identification $UT = UT'$ gives $d' = d + \alpha$, for $\alpha : UT \rightarrow UT \otimes_A I[1]$ a derivation with $d\alpha + \alpha d = 0$. In order for $f : T' \rightarrow C$ to be a chain map, we also need $f\alpha = 0$. Thus

$$\alpha \in Z^1 \text{HOM}_{T,R}(\Omega_{T/A}, \ker(f) \otimes_A I),$$

where $\text{HOM}(U, V)$ is the \mathbb{Z} -graded cochain complex given by setting $\text{HOM}(U, V)^n$ to be the space of graded morphisms $U \rightarrow V[n]$ (not necessarily respecting the differential).

Another choice of isomorphism $UT \cong UT'$ (fixing $T \otimes_A B$) amounts to giving a derivation $\beta : UT \rightarrow UT \otimes_A I$, with $\text{id} + \beta$ the corresponding automorphism of UT . In order to respect the augmentation f , we need $f\beta = 0$. This new choice of isomorphism sends α to $\alpha + d\beta$, so the isomorphism class is

$$[\alpha] \in \text{Ext}_{T,R}^1(\Omega_{T/A}, \ker(f) \otimes_A I).$$

Since $A \rightarrow T$ is a cofibration and f a fibration, this is just hyperext

$$\text{Ext}_{T,R}^1(\mathbb{L}_{T/A}^{\bullet}, T \otimes_A^{\mathbb{L}} I \xrightarrow{f} C \otimes_A^{\mathbb{L}} I)$$

of the cone complex. Since this expression is invariant under weak equivalences, it follows that it gives the weak equivalence class required. \square

4. STRUCTURES ON RELATIVE MALCEV HOMOTOPY TYPES

Now, fix a real reductive pro-algebraic group R , a pointed connected topological space (X, x) , and a Zariski-dense morphism $\rho : \pi_1(X, x) \rightarrow R(\mathbb{R})$.

Definition 4.1. Given a pro-algebraic group K acting on R and on a scheme Y , define $dg_{\mathbb{Z}}\text{Aff}_Y(R)_*(K)$ to be the category $(Y \times R) \downarrow dg_{\mathbb{Z}}\text{Aff}_Y(R \times K)$ of objects under $R \times Y$. Note that this is not the same as $dg_{\mathbb{Z}}\text{Aff}_Y(R \times K)_* = (Y \times R \times K) \downarrow dg_{\mathbb{Z}}\text{Aff}_Y(R \times K)$.

4.1. Homotopy types. Motivated by Definitions 1.4, 1.28, 1.37 and 1.43, we make the following definitions:

Definition 4.2. An algebraic Hodge filtration on a pointed Malcev homotopy type $(X, x)^{\rho, \text{Mal}}$ consists of the following data:

- (1) an algebraic action of S^1 on R ,
- (2) an object $(X, x)_{\mathbb{F}}^{\rho, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^*}(R)_*(S))$, where the S -action on R is defined via the isomorphism $S/\mathbb{G}_m \cong S^1$, while the $R \rtimes S$ -action on R combines multiplication by R with conjugation by S .
- (3) an isomorphism $(X, x)^{\rho, \text{Mal}} \cong (X, x)_{\mathbb{F}}^{\rho, \text{Mal}} \times_{C^*, 1}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)$.

Note that under the equivalence $dg_{\mathbb{Z}}\text{Aff}(R) \simeq dg_{\mathbb{Z}}\text{Aff}_S(R \rtimes S)$ of Lemma 3.47, $(X, x)^{\rho, \text{Mal}}$ corresponds to the flat pullback $(X, x)_{\mathbb{F}} \times_{C^*} S$.

Definition 4.3. An algebraic twistor filtration on a pointed Malcev homotopy type $(X, x)^{\rho, \text{Mal}}$ consists of the following data:

- (1) an object $(X, x)_{\mathbb{T}}^{\rho, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^*}(R)_*(\mathbb{G}_m))$,
- (2) an isomorphism $(X, x)^{\rho, \text{Mal}} \cong (X, x)_{\mathbb{T}}^{\rho, \text{Mal}} \times_{C^*, 1}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)$.

Note that under the equivalence $dg_{\mathbb{Z}}\text{Aff}(R) \simeq dg_{\mathbb{Z}}\text{Aff}_{\mathbb{G}_m}(R \times \mathbb{G}_m)$ of Lemma 3.47, $(X, x)^{\rho, \text{Mal}}$ corresponds to the derived pullback $(X, x)_{\mathbb{T}}^{\rho, \text{Mal}} \times_{C^*}^{\mathbf{R}} \mathbb{G}_m$.

Definition 4.4. An algebraic mixed Hodge structure $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ on a pointed Malcev homotopy type $(X, x)^{\rho, \text{Mal}}$ consists of the following data:

- (1) an algebraic action of S^1 on R ,
- (2) an object

$$(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathbb{G}_m \times S)),$$

where S acts on R via the S^1 -action, using the canonical isomorphism $S^1 \cong S/\mathbb{G}_m$,

- (3) an object

$$\underline{\text{gr}}(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*(S)),$$

- (4) an isomorphism $(X, x)^{\rho, \text{Mal}} \cong (X, x)_{\text{MHS}}^{\rho, \text{Mal}} \times_{(\mathbb{A}^1 \times C^*), (1,1)}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)$,
- (5) an isomorphism (called the opposedness isomorphism)

$$\theta^{\sharp}(\underline{\text{gr}}(X, x)_{\text{MHS}}^{\rho, \text{Mal}}) \times C^* \cong (X, x)_{\text{MHS}}^{\rho, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^*}(R)_*(\mathbb{G}_m \times S)),$$

for the canonical map $\theta : \mathbb{G}_m \times S \rightarrow S$ given by combining the inclusion $\mathbb{G}_m \hookrightarrow S$ with the identity on S .

Definition 4.5. Given an algebraic mixed Hodge structure $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ on $(X, x)^{\rho, \text{Mal}}$, define $\text{gr}^W(X, x)_{\text{MHS}}^{\rho, \text{Mal}} := (X, x)_{\text{MHS}}^{\rho, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^*}(\mathbb{G}_m \times R)_*(S))$, noting that this is isomorphic to $\theta^{\sharp}(\underline{\text{gr}}(X, x)_{\text{MHS}}^{\rho, \text{Mal}}) \times C^*$. We also define $(X, x)_{\mathbb{F}}^{\rho, \text{Mal}} := (X, x)_{\text{MHS}}^{\rho, \text{Mal}} \times_{\mathbb{A}^1, 1}^{\mathbf{R}} \text{Spec } \mathbb{R}$, noting that this is an algebraic Hodge filtration on $(X, x)^{\rho, \text{Mal}}$.

Definition 4.6. A real splitting of the mixed Hodge structure $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ is a $\mathbb{G}_m \times S$ -equivariant isomorphism

$$\mathbb{A}^1 \times \underline{\text{gr}}(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \times C^* \cong (X, x)_{\text{MHS}}^{\rho, \text{Mal}},$$

in $\text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathbb{G}_m \times S))$, giving the opposedness isomorphism on pulling back along $\{0\} \rightarrow \mathbb{A}^1$.

Definition 4.7. An algebraic mixed twistor structure $(X, x)_{\text{MTS}}^{\rho, \text{Mal}}$ on a pointed Malcev homotopy type $(X, x)^{\rho, \text{Mal}}$ consists of the following data:

- (1) an object

$$(X, x)_{\text{MTS}}^{\rho, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathbb{G}_m \times \mathbb{G}_m)),$$

- (2) an object $\underline{\text{gr}}(X, x)_{\text{MTS}}^{\rho, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*(\mathbb{G}_m))$,

- (3) an isomorphism $(X, x)^{\rho, \text{Mal}} \cong (X, x)_{\text{MTS}}^{\rho, \text{Mal}} \times_{(\mathbb{A}^1 \times C^*), (1,1)}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)$,

- (4) an isomorphism (called the opposedness isomorphism)

$$\theta^{\sharp}(\underline{\text{gr}}(X, x)_{\text{MTS}}^{\rho, \text{Mal}}) \times C^* \cong (X, x)_{\text{MTS}}^{\rho, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^*}(R)_*(\mathbb{G}_m \times \mathbb{G}_m)),$$

for the canonical diagonal map $\theta : \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m$.

Definition 4.8. Given an algebraic mixed twistor structure $(X, x)_{\text{MTS}}^{\rho, \text{Mal}}$ on $(X, x)^{\rho, \text{Mal}}$, define $\text{gr}^W(X, x)_{\text{MTS}}^{\rho, \text{Mal}} := (X, x)_{\text{MTS}}^{\rho, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(R \times C^* dg_{\mathbb{Z}}\text{Aff}_{C^*}(\mathbb{G}_m \times R \times \mathbb{G}_m))$, noting that this is isomorphic to $\theta^{\sharp}(\underline{\text{gr}}(X, x)_{\text{MTS}}^{\rho, \text{Mal}}) \times C^*$. We also define $(X, x)_{\mathbb{T}}^{\rho, \text{Mal}} := (X, x)_{\text{MTS}}^{\rho, \text{Mal}} \times_{\mathbb{A}^1, 1}^{\mathbf{R}} \text{Spec } \mathbb{R}$, noting that this is an algebraic twistor filtration on $(X, x)^{\rho, \text{Mal}}$.

Remark 4.9. As in Remark 3.30, we might want to consider many basepoints, or none. The definitions above then have analogues $(X; T)_{\mathbb{F}}^{\rho, \text{Mal}}$, $(X; T)_{\mathbb{T}}^{\rho, \text{Mal}}$, $(X; T)_{\text{MHS}}^{\rho, \text{Mal}}$, $(X; T)_{\text{MTS}}^{\rho, \text{Mal}}$, given by replacing the R -representation R with the representation $\coprod_{x \in T} R(x, -)$, as in Remark 3.36.

4.2. Splittings over \mathcal{S} . We now work with the S -equivariant map $\text{row}_1 : \text{SL}_2 \rightarrow C^*$ as defined in §1.1.1.

Definition 4.10. An \mathcal{S} -splitting (or SL_2 -splitting) of a mixed Hodge structure $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ on a relative Malcev homotopy type is a $\mathbb{G}_m \times S$ -equivariant isomorphism

$$\mathbb{A}^1 \times \underline{\text{gr}}(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \times \text{SL}_2 \cong \text{row}_1^*(X, x)_{\text{MHS}}^{\rho, \text{Mal}},$$

in $\text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times \text{SL}_2}(R)_*(\mathbb{G}_m \times S))$, giving row_1^* of the opposedness isomorphism on pulling back along $\{0\} \rightarrow \mathbb{A}^1$.

An \mathcal{S} -splitting (or SL_2 -splitting) of a mixed twistor structure $(X, x)_{\text{MTS}}^{\rho, \text{Mal}}$ on a relative Malcev homotopy type is a $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant isomorphism

$$\mathbb{A}^1 \times \underline{\text{gr}}(X, x)_{\text{MTS}}^{\rho, \text{Mal}} \times \text{SL}_2 \cong \text{row}_1^*(X, x)_{\text{MTS}}^{\rho, \text{Mal}},$$

in $\text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times \text{SL}_2}(R)_*(\mathbb{G}_m \times \mathbb{G}_m))$, giving row_1^* of the opposedness isomorphism on pulling back along $\{0\} \rightarrow \mathbb{A}^1$.

Lemma 4.11. *Let S' be S or \mathbb{G}_m . Take flat fibrant objects*

$$Y \in dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times \text{SL}_2}(R)_*(\mathbb{G}_m \times S') \text{ and } Z \in dg_{\mathbb{Z}}\text{Aff}(R)_*(\mathbb{G}_m \times S'),$$

together with a surjective quasi-isomorphism $\phi^{\sharp} : 0^ \mathcal{O}_Y \rightarrow \mathcal{O}_Z \otimes \mathcal{O}_{\text{SL}_2}$ in $dg_{\mathbb{Z}}\text{Aff}_{\text{SL}_2}(R)_*(\mathbb{G}_m \times S')$. Then the weak equivalence class of objects $X \in dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathbb{G}_m \times S')$ equipped with weak equivalences $f : \text{row}_1^* X \rightarrow Y$ and $g :$*

$0^*X \rightarrow Z \times C^*$ with $\phi \circ \text{row}_1^*g = 0^*f$ is either \emptyset or a principal homogeneous space for the group

$$\text{Ext}^0(L^\bullet(1), \ker(\phi^\sharp : \mathcal{O}_Y \rightarrow \mathcal{O}_Z \otimes \mathcal{O}_{\text{SL}_2}) \rightarrow (W_{-1}\mathcal{O}_{\mathbb{A}^1}) \otimes (y_*\mathcal{O}(R)) \otimes \mathcal{O}_{\text{SL}_2})^{\mathbb{G}_m \times R \times S'},$$

where L^\bullet is the cotangent complex of $Y \cup_{(Z \times \text{SL}_2)}^{\mathbb{L}} (Z \times C^*)$ over $(\mathbb{A}^1 \times \text{SL}_2) \cup_{(\{0\} \times \text{SL}_2)} (\{0\} \times C^*)$, and Ext is taken over $Y \cup_{(Z \times \text{SL}_2)}^{\mathbb{L}} (Z \times C^*)$.

Proof. The data Y, Z, ϕ determine the pullback of X to

$$(\mathbb{A}^1 \times \text{SL}_2) \cup_{(\{0\} \times \text{SL}_2)} (\{0\} \times C^*).$$

Since ϕ^\sharp is surjective, we may define

$$\mathcal{O}_Y \times_{\phi^\sharp, (\mathcal{O}_Z \otimes \mathcal{O}_{\text{SL}_2})} (\mathcal{O}_Z \otimes \mathbf{RO}(C^*)) \rightarrow \mathcal{O}(R) \otimes ((\mathcal{O}_{\mathbb{A}^1} \otimes \mathcal{O}_{\text{SL}_2}) \times_{\mathcal{O}_{\text{SL}_2}} \mathbf{RO}(C^*))$$

over

$$(O(\mathbb{A}^1) \otimes O(\text{SL}_2)) \times_{O(\text{SL}_2)} \mathbf{RO}(C^*),$$

which we wish to lift to $\mathbf{RO}(C^*)$, making use of Proposition 3.46.

Now, the morphism $\mathbf{RO}(C^*) \rightarrow (O(\mathbb{A}^1) \otimes O(\text{SL}_2)) \times_{O(\text{SL}_2)} \mathbf{RO}(C^*)$ is surjective, with square-zero kernel $(W_{-1}O(\mathbb{A}^1)) \otimes O(\text{SL}_2)(-1)[-1]$, where $W_{-1}O(\mathbb{A}^1) = \ker(O(\mathbb{A}^1) \xrightarrow{0^*} \mathbb{R})$, so Proposition 3.49 gives the required result. \square

Corollary 4.12. *The weak equivalence class of \mathcal{S} -split mixed Hodge structures $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ with $\underline{\text{gr}}(X, x)_{\text{MHS}}^{\rho, \text{Mal}} = (R \xrightarrow{z} Z)$ is canonically isomorphic to*

$$\text{Ext}_Z^0(\mathbb{L}_Z^\bullet, (W_{-1}O(\mathbb{A}^1)) \otimes (\mathcal{O}_Z \rightarrow z_*\mathcal{O}(R)) \otimes O(\text{SL}_2)(-1))^{\mathbb{G}_m \times R \times S}.$$

The weak equivalence class of \mathcal{S} -split mixed twistor structures $(X, x)_{\text{MTS}}^{\rho, \text{Mal}}$ with $\underline{\text{gr}}(X, x)_{\text{MTS}}^{\rho, \text{Mal}} = (R \xrightarrow{z} Z)$ is canonically isomorphic to

$$\text{Ext}_Z^0(\mathbb{L}_Z^\bullet, (W_{-1}O(\mathbb{A}^1)) \otimes (\mathcal{O}_Z \rightarrow z_*\mathcal{O}(R)) \otimes O(\text{SL}_2)(-1))^{\mathbb{G}_m \times R \times \mathbb{G}_m}.$$

Proof. Set $Y = \mathbb{A}^1 \times Z \times \text{SL}_2$ in Lemma 4.11, and note that the cone of $O(\mathbb{A}^1) \xrightarrow{0^*} \mathbb{R}$ is quasi-isomorphic to $W_{-1}O(\mathbb{A}^1)$. The class of possible extensions is non-empty, since $\mathbb{A}^1 \times Z \times C^*$ is one possibility for $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ (resp. $(X, x)_{\text{MTS}}^{\rho, \text{Mal}}$). This gives a canonical basepoint for the principal homogeneous space, and hence the canonical isomorphism. \square

4.3. Grouplike structures.

Definition 4.13. Given $A \in \text{DGalg}(R)$, define the category of R -equivariant dg pro-algebraic groups G_\bullet over A to be opposite to the category of R -equivariant DG Hopf algebras over A . Explicitly, this consists of objects $Q \in \text{DGalg}_A(R)$ equipped with morphisms $Q \rightarrow Q \otimes_A Q$, (comultiplication), $Q \rightarrow A$ (coidentity) and $Q \rightarrow Q$ (coinverse), satisfying the usual axioms.

A morphism $f : G_\bullet \rightarrow K_\bullet$ of dg pro-algebraic groups is said to be a quasi-isomorphism if it induces an isomorphism $H^*O(K) \rightarrow H^*O(G)$ on cohomology of the associated DG Hopf algebras.

Definition 4.14. Given $G \in s\text{AGp}$, define the dg pro-algebraic group NG over \mathbb{R} by setting $O(NG) = D^*O(G)$, where D^* is left adjoint to the denormalisation functor for algebras. The comultiplication on $O(NG)$ is then defined using the fact that D^* preserves coproducts, so $D^*(O(G) \otimes O(G)) = O(NG) \otimes O(NG)$, where $(O(G) \otimes O(G))^n = O(G)^n \otimes O(G)^n$, but $(O(NG) \otimes O(NG))^n = \bigoplus_{i+j=n} O(NG)^i \otimes O(NG)^j$.

Examples 4.15. Given $\mathfrak{g} \in dg\hat{\mathcal{N}}(R \rtimes S)$, we may form an S -equivariant dg pro-algebraic group $\exp(\mathfrak{g})$ over \mathbb{R} by letting $O(\exp(\mathfrak{g}))$ represent the functor

$$\exp(\mathfrak{g})(A) := \exp(\{g \in \prod_n \mathfrak{g}_n \hat{\otimes} A^n : (d \otimes 1)g_n = (1 \otimes d)g_{n-1}\}),$$

for DG algebras A . Note that the underlying dg algebra is given by $O(\exp(\mathfrak{g})) = \mathbb{R}[\mathfrak{g}^\vee]$, with comultiplication dual to the Campbell-Baker-Hausdorff formula.

For any DG algebra B , observe that the R -action on \mathfrak{g} provides an $R(H^0B)$ -action on $\exp(\mathfrak{g})(B)$, so we can then define the S -equivariant dg pro-algebraic group $R \times \exp(\mathfrak{g})$ to represent the functor

$$A \mapsto R(H^0A) \times \exp(\mathfrak{g})(A),$$

noting that $O(R \times \exp(\mathfrak{g})) \cong O(R) \otimes O(\exp(\mathfrak{g}))$ as a DG algebra.

If $\mathfrak{g} \in s\hat{\mathcal{N}}(R \rtimes S)$, note that $N(R \times \exp(\mathfrak{g})) \cong R \times \exp(N\mathfrak{g})$, since both represent the same functor.

Definition 4.16. Define a grouplike mixed Hodge structure on a pointed Malcev homotopy type $(X, x)^{\rho, \text{Mal}}$ to consist of the following data:

- (1) an algebraic action of S^1 on R ,
- (2) a flat $\mathbb{G}_m \times S$ -equivariant dg pro-algebraic group $G(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ over $O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)$, equipped with an S -equivariant map $G(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \rightarrow \mathbb{A}^1 \times R \times \text{Spec } \mathbf{RO}(C^*)$ of dg pro-algebraic groups over $\mathbb{A}^1 \times \text{Spec } \mathbf{RO}(C^*)$, where S acts on R via the S^1 -action.
- (3) an object $\underline{\text{gr}}\mathfrak{g}(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \in dg\hat{\mathcal{N}}(R \rtimes S)$.
- (4) a weak equivalence $NG(X, x)^{\rho, \text{Mal}} \simeq G(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \times_{(\mathbb{A}^1 \times \text{Spec } \mathbf{RO}(C^*)), (1, I)} \text{Spec } \mathbb{R}$ of pro-algebraic dg groups on $\text{Spec } \mathbb{R}$, respecting the R -augmentations, where $I : \text{Spec } \mathbb{R} \rightarrow \text{SL}_2 \rightarrow \text{Spec } \mathbf{RO}(C^*)$ is the identity matrix.
- (5) a weak equivalence

$$\theta^\sharp(R \times \exp(\underline{\text{gr}}\mathfrak{g}(X, x)_{\text{MHS}}^{\rho, \text{Mal}})) \times \text{Spec } \mathbf{RO}(C^*) \simeq G(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \times_{\mathbb{A}^1, 0} \text{Spec } \mathbb{R}$$

of pro-algebraic dg groups on $B(\mathbb{G}_m \times S)$, for the canonical map $\theta : \mathbb{G}_m \times S \rightarrow S$ given by combining the inclusion $\mathbb{G}_m \hookrightarrow S$ with the identity on S .

Definition 4.17. Define a grouplike mixed twistor structure similarly, dispensing with the S^1 -action on R , and replacing S with \mathbb{G}_m .

Remark 4.18. We can adapt Definition 4.13 in the spirit of Remark 3.30 by defining an R -equivariant dg pro-algebraic groupoid G over A to consist of a set $\text{Ob } G$ of objects, together with $O(G)(x, y) \in D\text{GAlg}_A(R)$ for all $x, y \in \text{Ob}$, equipped with morphisms $O(G)(x, z) \rightarrow O(G)(x, y) \otimes_A O(G)(y, z)$ (comultiplication), $O(G)(x, x) \rightarrow A$ (coidentity) and $O(G)(x, y) \rightarrow O(G)(y, x)$ (coinverse), satisfying the usual axioms.

Given a reductive pro-algebraic groupoid with an S -action, and $\mathfrak{g} \in dg\hat{\mathcal{N}}(R \rtimes S)$, we then define the S -equivariant dg pro-algebraic group $R \times \exp(\mathfrak{g})$ to have objects $\text{Ob } R$, with

$$(R \times \exp(\mathfrak{g}))(x, y) = R(x, y) \times \exp(\mathfrak{g}(y)),$$

and multiplication as in [Pri3, Definition 2.15].

Definitions 4.16 and 4.17 then adapt to multipointed Malcev homotopy types $(X; T)^{\rho, \text{Mal}}$, replacing dg pro-algebraic groups with dg pro-algebraic groupoids on objects T , noting that $\text{Ob } R = T$.

Proposition 4.19. *Take an \mathcal{S} -split MHS $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ (resp. an \mathcal{S} -split MTS $(X, x)_{\text{MTS}}^{\rho, \text{Mal}}$) on a relative Malcev homotopy type, and assume that $\underline{\text{gr}}(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*(S)_0)$ (resp. $\underline{\text{gr}}(X, x)_{\text{MTS}}^{\rho, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*(\mathbb{G}_m)_0)$).*

Then there is a canonical grouplike MHS (resp. grouplike MTS) on $(X, x)^{\rho, \text{Mal}}$, independent of the choice of \mathcal{S} -splitting.

Moreover, the induced pro-MHS $\mathbb{R}_u(G(X, x)_{\text{MHS}}^{\rho, \text{Mal}})^{\text{ab}}$ (resp. MTS $\mathbb{R}_u(G(X, x)_{\text{MTS}}^{\rho, \text{Mal}})^{\text{ab}}$) on the abelianisation of the pro-unipotent radical of $G(X, x)^{\rho, \text{Mal}}$ is dual to the complex given by the cokernel of

$$\mathbb{R}[-1] \rightarrow O(X)_{\text{MHS}}^{\rho, \text{Mal}}[-1] \text{ resp. } \mathbb{R}[-1] \rightarrow O(X)_{\text{MTS}}^{\rho, \text{Mal}}[-1],$$

where $X_{\text{MHS}}^{\rho, \text{Mal}} = \text{Spec } O(X)_{\text{MHS}}^{\rho, \text{Mal}}$.

Proof. We will prove this for mixed Hodge structure; the case of mixed twistor structures is entirely similar.

Choose a representative Z for $\underline{\text{gr}}(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ with $Z_0 = \text{Spec } \mathbb{R}$, and set $\mathfrak{g} = \underline{\text{gr}}\mathfrak{g}(X, x)_{\text{MHS}}^{\rho, \text{Mal}} := G(Z)$ (for G as in Definition 3.23). Then $Z \rightarrow \bar{W}\mathfrak{g}$ is a weak equivalence, making $O(\bar{W}\mathfrak{g})$ into a cofibrant representative for Z , so by Corollary 4.12, $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ corresponds to a class

$$\nu \in \mathbb{E}x\text{t}^0(\Omega(O(\bar{W}\mathfrak{g})/\mathbb{R}), (W_{-1}O(\mathbb{A}^1)) \otimes (O(Z) \rightarrow z_*O(R)) \otimes O(\text{SL}_2)(-1))^{\mathbb{G}_m \times R \rtimes S}.$$

Now, $\Omega(O(\bar{W}\mathfrak{g})/\mathbb{R}) \cong \mathfrak{g}^\vee[-1]$, so we may choose a representative

$$(\alpha', \gamma') : \mathfrak{g}^\vee[-1] \rightarrow (W_{-1}O(\mathbb{A}^1)) \otimes (O(Y) \times O(R)[-1]) \otimes O(\text{SL}_2)(-1)$$

for ν , with $[d, \alpha'] = 0$, $[d, \gamma'] = z^*\alpha'$.

Studying the adjunction $\bar{W} \vdash G$, we see that α' is equivalent to an $R \times S$ -equivariant Lie coalgebra derivation $\alpha : \mathfrak{g}^\vee \rightarrow W_{-1}O(\mathbb{A}^1) \otimes \mathfrak{g}^\vee \otimes O(\text{SL}_2)(-1)$ with $[d, \alpha] = 0$. This generates a derivation $\alpha : O(R \rtimes \exp(\mathfrak{g})) \rightarrow (W_{-1}O(\mathbb{A}^1)) \otimes O(R \rtimes \exp(\mathfrak{g})) \otimes O(\text{SL}_2)(-1)$. γ' corresponds to an element $\gamma \in (\mathfrak{g}_0 \hat{\otimes} (W_{-1}O(\mathbb{A}^1) \otimes O(\text{SL}_2)(-1)))^{\mathbb{G}_m \times S}$, and conjugation by this gives another such derivation $[\gamma, -]$, so we then set $O(G(X, x)_{\text{MHS}}^{\rho, \text{Mal}})$ to be the quasi-isomorphism class of the dg Hopf algebra over $O(\mathbb{A}^1) \otimes \mathbf{R}O(C^*)$ given by the graded Hopf algebra

$$O(\mathbb{A}^1) \otimes O(R \rtimes \exp(\mathfrak{g})) \otimes (O(\text{SL}_2) \oplus O(\text{SL}_2)(-1)\epsilon)$$

(where ϵ is of degree 1 and $\epsilon^2 = 0$), with differential $d_{(\alpha, \gamma)} := d_{O(R \rtimes \exp(\mathfrak{g}))} + (\text{id} \otimes \text{id} \otimes N + \alpha + [\gamma, -])\epsilon$.

Explicitly, $G(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ represents the group-valued functor on $DG\text{Alg}_{\mathbb{A}^1 \times \mathbf{R}O(C^*)}(\mathbb{G}_m \times S)$ given by mapping A to the subgroup of $(R(A^0) \rtimes \exp(\mathfrak{g} \hat{\otimes} A^0))^S$, consisting of (r, g) such that

$$d_A \circ (r, g) = (r, g) \circ d_{(\alpha, \gamma)} : O(R \rtimes \exp(\mathfrak{g})) \rightarrow A[1]$$

or equivalently $(d_A \circ (r, g) - (r, g) \circ d_{(\alpha, \gamma)}) \cdot (r, g)^{-1} = 0$, so

$$d(rg) \cdot g^{-1} = r\alpha(g)g^{-1}r^{-1} + (\gamma rg - rg\gamma)g^{-1}r^{-1} \in (\text{Lie } R) \hat{\otimes} A^1 \oplus (\mathfrak{g} \hat{\otimes} A)^1,$$

where d is the total differential $d_A - d_{\mathfrak{g}}$. This reduces to

$$dg \cdot g^{-1} + r^{-1} \cdot dr = \alpha(g) \cdot g^{-1} + \text{ad}_{r^{-1}}\gamma - \text{ad}_g(\gamma).$$

To see that this is well-defined, another choice of representative for ν would be of the form $(\alpha + [d, h], \gamma + dk)$, for a Lie coalgebra derivation $h : \mathfrak{g}^\vee \rightarrow (W_{-1}O(\mathbb{A}^1)) \otimes \mathfrak{g}^\vee \otimes O(\text{SL}_2)(-1)[1]$, and $k \in (\mathfrak{g}_1 \hat{\otimes} (W_{-1}O(\mathbb{A}^1)) \otimes O(\text{SL}_2)(-1))^{\mathbb{G}_m \times S}$. The morphism $\text{id} + h\epsilon + [k, -]\epsilon$ then provides a quasi-isomorphism between the two representatives of $O(G(X, x)_{\text{MHS}}^{\rho, \text{Mal}})$.

The evaluations of α and γ at $0 \in \mathbb{A}^1$ are both 0 (since $W_{-1}O(\mathbb{A}^1) = \ker 0^*$), so there is a canonical isomorphism

$$\psi : 0^*G(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \cong (R \rtimes \exp(\mathfrak{g})) \times \text{Spec } \mathbf{R}O(C^*).$$

Meanwhile, pulling back along the canonical morphism $r_1 : \mathrm{SL}_2 \rightarrow \mathrm{Spec} \mathbf{RO}(C^*)$ gives an isomorphism $r_1^* G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \cong \mathbb{A}^1 \times (R \ltimes \exp(\mathfrak{g})) \times \mathrm{SL}_2$, so

$$(1, I)^* G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \cong (R \ltimes \exp(\mathfrak{g}))$$

and combining this with the pullback along $I \rightarrow \mathrm{SL}_2$ of our choice of SL_2 -splitting gives a quasi-isomorphism

$$\phi : (1, I)^* G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \simeq G(X, x)^{\rho, \mathrm{Mal}}.$$

Now, $(R_u G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})^{\mathrm{ab}}$ is dual to the complex

$$O(\mathbb{A}^1) \otimes \mathrm{coker}(\mathbb{R} \rightarrow O(Z))[-1] \otimes (O(\mathrm{SL}_2) \oplus O(\mathrm{SL}_2)(-1)\epsilon)$$

with differential $d_Z + (\mathrm{id} \otimes \mathrm{id} \otimes N + \alpha')\epsilon$. Under the characterisation of Lemma 4.12, this is quasi-isomorphic to the cokernel $\mathrm{coker}(\mathbb{R} \rightarrow O(X)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})$ of complexes, with ϕ and ψ recovering the structure maps of the ind-MHS.

Finally, another choice of \mathcal{S} -splitting amounts to giving a homotopy class of automorphisms u of $\mathbb{A}^1 \times \underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \times \mathrm{SL}_2$, giving the identity on pulling back along $0 \rightarrow \mathbb{A}^1$. Since $Z \simeq \underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$ and $\eta : Z \rightarrow \bar{W}G(Z)$ is a fibrant replacement for Z , u gives rise to a homotopy class of morphisms $v : \mathbb{A}^1 \times Z \times \mathrm{SL}_2 \rightarrow \bar{W}G(Z)$, with $0^*v = \eta$. Via the adjunction $G \dashv \bar{W}$ this gives a homotopy automorphism $U : \mathbb{A}^1 \times G(Z) \times \mathrm{SL}_2 \rightarrow \mathbb{A}^1 \times G(Z) \times \mathrm{SL}_2$ with $0^*U = \mathrm{id}$. This gives a quasi-isomorphism between the respective constructions of $(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}, \psi, \phi)$. \square

Theorem 4.20. *If the S -action (resp. the \mathbb{G}_m -action) on $\mathrm{H}^*O(\underline{\mathrm{gr}}X^{\rho, \mathrm{Mal}})$ is of non-negative weights, then the grouplike MHS (resp. grouplike MTS) of Proposition 4.19 gives rise to ind-MHS (resp. ind-MTS) on the duals $(\varpi_n(X, x)^{\rho, \mathrm{Mal}})^\vee$ of the relative Malcev homotopy groups for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(X, x)^{\rho, \mathrm{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n , the Whitehead bracket and the Hurewicz maps $\varpi_n(X^{\rho, \mathrm{Mal}}) \rightarrow \mathrm{H}^n(X, O(\mathbb{B}_\rho))^\vee$ ($n \geq 2$) and $R_u\varpi_1(X^{\rho, \mathrm{Mal}}) \rightarrow \mathrm{H}^1(X, O(\mathbb{B}_\rho))$, for \mathbb{B}_ρ as in Definition 3.32.

Proof. Again we give the proof for MHS only, as the MTS case follows by replacing S with \mathbb{G}_m and Proposition 1.41 with Proposition 1.49.

Choose a representative Z for $\underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$ with $Z_0 = \mathrm{Spec} \mathbb{R}$, and $O(Z)$ of non-negative weights. [To see that this is possible, take a minimal model \mathfrak{m} for $\bar{G}(\underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})$ as in [Pri3, Proposition 4.7], and note that $\mathfrak{m}/[\mathfrak{m}, \mathfrak{m}] \cong \mathrm{H}^*O(\underline{\mathrm{gr}}X^{\rho, \mathrm{Mal}})^\vee$ is of non-positive weights, so \mathfrak{m} is of non-negative weights, and therefore $O(\bar{W}\mathfrak{m})$ is of non-negative weights, so $\bar{W}\mathfrak{m}$ is a possible choice for Z .] Set $\mathfrak{g} = \underline{\mathrm{gr}}\mathfrak{g}(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} := G(Z)$.

Since $\mathbf{RO}(C^*)$ is a dg algebra over $O(C)$, we may regard it as a quasi-coherent sheaf on C , and consider the quasi-coherent dg algebra $j^{-1}\mathbf{RO}(C^*)$ on C^* , for $j : C^* \rightarrow C$.

Define

$$\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} := \mathbf{Spec} \mathcal{H}^0(j^*O(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})),$$

which is an affine group object over $\mathbb{A}^1 \times C^*$, as $j^{-1}\mathbf{RO}(C^*)$ is a resolution of \mathcal{O}_{C^*} . Since $\mathrm{row}_1 : \mathrm{SL}_2 \rightarrow C^*$ is flat,

$$\mathrm{row}_1^*\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} = \mathbf{Spec} \mathcal{H}^0(\mathrm{row}_1^*O(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})).$$

Now the choice of \mathcal{S} -splitting gives

$$\chi : \mathrm{row}_1^*O(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}) \cong \mathbb{A}^1 \times (R \ltimes \exp(\mathfrak{g})) \times \mathrm{Spec}(\mathrm{row}_1^*\mathbf{RO}(C^*)),$$

and $\mathrm{row}_1^*\mathbf{RO}(C^*)$ is a resolution of $O(\mathrm{SL}_2)$, so

$$\mathrm{row}_1^*\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \cong \mathbb{A}^1 \times (R \ltimes \exp(\mathfrak{g})) \times \mathrm{SL}_2,$$

whose structure sheaf is flat on $\mathbb{A}^1 \times \mathrm{SL}_2$, and has non-negative weights with respect to the $\mathbb{G}_m \times 1$ -action. Lemma 1.17 then implies that the structure sheaf of $\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$ is flat over $\mathbb{A}^1 \times C^*$, with non-negative weights.

Set $\underline{\mathrm{gr}}\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} := (R \times \exp(\mathrm{H}_0\mathfrak{g}))$. The morphisms ϕ and ψ from the proof of Proposition 4.19 now induce an S -equivariant isomorphism

$$\mathrm{Spec} \mathbb{R} \times_{\mathbb{A}^1, 0} \varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \cong \underline{\mathrm{gr}}\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \times C^*,$$

and an isomorphism

$$\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \times_{\mathbb{A}^1 \times C^*, (1,1)} \mathrm{Spec} \mathbb{R} \cong \varpi_1(X, x)^{\rho, \mathrm{Mal}},$$

giving the data of a flat algebraic MHS on $O(\varpi_1(X, x)^{\rho, \mathrm{Mal}})$, of non-negative weights. By Proposition 1.41, this is the same as an ind-MHS of non-negative weights.

Next, we consider the dg Lie coalgebra over $O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)$ given by

$$\mathcal{C}(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}) := \Omega(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} / \mathbf{RO}(C^*)) \otimes_{O(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}), 1} \mathbf{RO}(C^*),$$

which has non-negative weights with respect to the $\mathbb{G}_m \times 1$ -action. Pulling back along j gives a dg Lie coalgebra $j^{-1}\mathcal{C}(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})$ over $O(\mathbb{A}^1) \otimes j^{-1}\mathbf{RO}(C^*)$, so the cohomology sheaves $\mathcal{H}^*(j^{-1}\mathcal{C}(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}))$ form a graded Lie coalgebra over $O(\mathbb{A}^1) \otimes \mathcal{O}_{C^*}$.

The isomorphism χ above implies that these sheaves are flat over $\mathbb{A}^1 \times C^*$, and therefore that $\mathcal{H}^0(j^{-1}\mathcal{C}(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}))$ is just the Lie coalgebra of $\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$. For $n \geq 2$, we set

$$\varpi_n(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}\vee} := \mathcal{H}^{n-1}(\mathrm{row}_{1*}\mathcal{C}(G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})),$$

noting that these have a conjugation action by $\varpi_1(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$ and a natural Lie bracket.

Setting $\underline{\mathrm{gr}}\varpi_n(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}\vee} := (\mathrm{H}_{n-1}\mathfrak{g})^\vee$, the isomorphisms ϕ and ψ induce S -equivariant isomorphisms

$$\mathrm{Spec} \mathbb{R} \times_{\mathbb{A}^1, 0} \varpi_n(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}\vee} \cong \underline{\mathrm{gr}}\varpi_n(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}\vee} \times C^*,$$

and isomorphisms

$$\varpi_n(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}\vee} \times_{\mathbb{A}^1 \times C^*, (1,1)} \mathrm{Spec} \mathbb{R} \cong \varpi_n(X, x)^{\rho, \mathrm{Mal}\vee},$$

so Proposition 1.41 gives the data of a non-negatively weighted ind-MHS on $(\varpi_n(X, x)^{\rho, \mathrm{Mal}\vee})^\vee$, compatible with the ϖ_1 -action and Whitehead bracket.

Finally, the Hurewicz map comes from

$$\mathrm{R}_u G(X, x)^{\rho, \mathrm{Mal}} \rightarrow (\mathrm{R}_u G(X, x)^{\rho, \mathrm{Mal}})^{\mathrm{ab}} \simeq \mathrm{coker}(\mathbb{R} \rightarrow O(X^{\rho, \mathrm{Mal}}))[-1]^\vee,$$

which is compatible with the ind-MHS, by the final part of Proposition 4.19. Thus the Hurewicz maps

$$\varpi_n(X, x)^{\rho, \mathrm{Mal}\vee} \rightarrow \mathrm{H}^n(X, O(\mathbb{B}_\rho))^\vee \quad \mathrm{R}_u \varpi_1(X^{\rho, \mathrm{Mal}}) \rightarrow \mathrm{H}^1(X, O(\mathbb{B}_\rho))$$

preserve the ind-MHS. \square

In Proposition 4.19 and Theorem 4.20, the only rôle of the \mathcal{S} -splitting is to ensure that the algebraic MHS is flat. We now show how a choice of \mathcal{S} -splitting gives additional data.

Theorem 4.21. *A choice of \mathcal{S} -splitting for $(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$ (resp. $(X, x)_{\mathrm{MTS}}^{\rho, \mathrm{Mal}}$) gives an isomorphism*

$$O(\varpi_1(X, x)^{\rho, \mathrm{Mal}}) \otimes \mathcal{S} \cong \mathrm{gr}^W O(\varpi_1(X, x)^{\rho, \mathrm{Mal}}) \otimes \mathcal{S}$$

of (real) quasi-MHS (resp. quasi-MTS) in Hopf algebras, and isomorphisms

$$(\varpi_n(X, x)^{\rho, \mathrm{Mal}\vee})^\vee \otimes \mathcal{S} \cong \mathrm{gr}^W (\varpi_n(X, x)^{\rho, \mathrm{Mal}\vee})^\vee \otimes \mathcal{S}$$

of (real) quasi-MHS (resp. quasi-MTS), inducing the identity on gr^W , and compatible with the Whitehead bracket.

Proof. The choice of \mathcal{S} -splitting gives an isomorphism

$$\mathrm{row}_1^* G(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}} \cong \mathbb{A}^1 \times (R \rtimes \exp(\mathfrak{g})) \times \mathrm{row}_1^* \mathrm{Spec} \mathbf{RO}(C^*)$$

in Proposition 4.19. The isomorphisms now follow from Lemma 1.19 and the constructions of Theorem 4.20. \square

Remark 4.22. This leads us to ask what additional data are required to describe the ind-MHS on homotopy groups in terms of the Hodge structure $\underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$. If we set $\mathfrak{g} = \tilde{G}(\underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})$, then we can let $\mathcal{D}^\bullet := \mathcal{D}er_{\mathbb{R}}(R \rtimes \exp(\mathfrak{g}), R \rtimes \exp(\mathfrak{g}))$ be the complex of DG Hopf algebra derivations on $O(R \rtimes \exp(\mathfrak{g}))$. This has a canonical \mathcal{S} -action (inherited from R and \mathfrak{g}), and the proof of Proposition 4.19 gives

$$[\beta] := [\alpha + [\gamma, -]] \in H^0(W_{-1}\gamma^0(\mathcal{D}^\bullet \otimes \mathcal{S}(-1))),$$

for γ^0 as in Definition 1.25. This determines the mixed Hodge structure, by Corollary 4.12, and $\gamma^0(\mathcal{D}^\bullet \otimes \mathcal{S}(-1)) \simeq \mathbf{R}\Gamma_{w\mathcal{H}}(\mathcal{D}^\bullet \otimes \mathcal{S}(-1))$, by Remark 1.27.

This gives a derivation $N + \beta : O(R \rtimes \exp(\mathfrak{g})) \otimes \mathcal{S} \rightarrow O(R \rtimes \exp(\mathfrak{g})) \otimes \mathcal{S}(-1)$, and this diagram is a resolution of $D^*O(G(X, x))$, making $O(G(X, x))$ into a mixed Hodge complex. As in §2.4, we think of $N + \beta$ as the monodromy operator at the Archimedean place. This will be constructed explicitly in §8.

Moreover, for any \mathcal{S} -split MHS V arising as an invariant of $O(G(X, x))$, the induced map $N + \bar{\beta} : (\mathrm{gr}^W V) \otimes \mathcal{S} \rightarrow (\mathrm{gr}^W V) \otimes \mathcal{S}(-1)$ just comes from conjugating the surjective map $\mathrm{id} \otimes N : V \otimes \mathcal{S} \rightarrow V \otimes \mathcal{S}(-1)$ with respect to the splitting isomorphism $(\mathrm{gr}^W V) \otimes \mathcal{S} \cong V \otimes \mathcal{S}$. Therefore $N + \bar{\beta}$ is surjective, and $V = \ker(N + \bar{\beta})$.

All these results have analogues for mixed twistor structures.

Remark 4.23. If we have a multipointed MHS (resp. MTS) $(X; T)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$ (resp. $(X; T)_{\mathrm{MTS}}^{\rho, \mathrm{Mal}}$) as in Remark 4.9, then Proposition 4.19 and Theorems 4.20 and 4.21 adapt to give \mathcal{S} -split multipointed grouplike MHS (resp. MTS) as in Remark 4.18, together with \mathcal{S} -split ind-MHS (resp. ind-MTS) on the algebras $O(\varpi_1(X; x, y)^{\rho, \mathrm{Mal}})$, compatible with the pro-algebraic groupoid structure. The \mathcal{S} -split ind-MHS (resp. ind-MTS) on $(\varpi_n(X, x)^{\rho, \mathrm{Mal}})^\vee$ are then compatible with the co-action

$$(\varpi_n(X, x)^{\rho, \mathrm{Mal}})^\vee \rightarrow O(\varpi_1(X; x, y)^{\rho, \mathrm{Mal}}) \otimes (\varpi_n(X, y)^{\rho, \mathrm{Mal}})^\vee.$$

In the proof of Proposition 4.19, $\mathfrak{g} = \underline{\mathrm{gr}}\mathfrak{g}(X; T)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$ becomes an R -representation, giving \mathfrak{g}_x for all $x \in T$. For objects $G(X; T)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}$ is then defined on $\mathbb{G}_m \times S$ -equivariant DGAs A over $O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)$ by setting, for $x, y \in T$, $G(X; x, y)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}(A)$ to be the subset of $(R(x, y)(A^0) \rtimes \exp(\mathfrak{g}_y \hat{\otimes} A)^0)^S$, consisting of (r, g) such that

$$dg \cdot g^{-1} + r^{-1} \cdot dr = \alpha(g) \cdot g^{-1} + \mathrm{ad}_{r^{-1}}\gamma_x - \mathrm{ad}_g(\gamma_y).$$

4.4. MHS representations. Take a pro-unipotent extension $G \rightarrow R$ of pro-algebraic groups with kernel U , together with a compatible ind-MHS on the Hopf algebra $O(G)$. This gives rise to $\mathbb{G}_m \times S$ -equivariant affine group objects $U_{\mathrm{MHS}} \triangleleft G_{\mathrm{MHS}}$ over $\mathbb{A}^1 \times C^*$, given by

$$G_{\mathrm{MHS}} = \mathbf{Spec} \xi(O(G), \mathrm{MHS}), \quad U_{\mathrm{MHS}} = \mathbf{Spec} \xi(O(U), \mathrm{MHS}),$$

and this gives a morphism $G_{\mathrm{MHS}} \rightarrow \mathbb{A}^1 \times R \times C^*$ with kernel U_{MHS} .

Now, since $U = \exp(\mathfrak{u})$ is pro-unipotent, we can express $G_{\mathrm{MHS}} \rightarrow \mathbb{A}^1 \times R \times C^*$ as a composition of extensions by locally free abelian groups. On pulling back to the affine scheme $\mathbb{A}^1 \times \mathrm{SL}_2$, the argument of [Pri3, Proposition 2.17] adapts to give a $\mathbb{G}_m \times S$ -equivariant section

$$\sigma_G : \mathbb{A}^1 \times R \times \mathrm{SL}_2 \rightarrow \mathrm{row}_1^* G_{\mathrm{MHS}},$$

since R and $\mathbb{G}_m \times S$ (linearly) reductive. This section is unique up to conjugation by $\Gamma(\mathbb{A}^1 \times \mathrm{SL}_2, U_{\mathrm{MHS}})^{\mathbb{G}_m \times S}$.

This is equivalent to giving a retraction $\sigma_G^\sharp : O(G) \otimes \mathcal{S} \rightarrow O(R) \otimes \mathcal{S}$ of quasi-MHS in Hopf algebras over \mathcal{S} , unique up to conjugation by $\exp(W_0\gamma^0(\mathfrak{u} \otimes \mathcal{S}))$.

Now, applying the derivation N gives a morphism

$$(\sigma_G^\sharp + N\sigma_G^\sharp\epsilon) : O(G) \otimes (\mathcal{S} \oplus \mathcal{S}(-1)\epsilon) \rightarrow O(R) \otimes (\mathcal{S} \oplus \mathcal{S}(-1)\epsilon)$$

of quasi-MHS in Hopf algebras over $\mathcal{S} \oplus \mathcal{S}(-1)\epsilon$, where $\epsilon^2 = 0$. The argument above (considering affine group schemes over $\mathbb{A}^1 \times \text{Spec}(O(\text{SL}_2) \oplus O(\text{SL}_2)(-1)\epsilon)$) adapts to show that there exists $\gamma_G \in \Gamma(\mathbb{A}^1 \times \text{SL}_2, \mathfrak{u}_{\text{MHS}}(-1))^{\mathbb{G}_m \times S} = W_0\gamma^0(\mathfrak{u} \otimes \mathcal{S}(-1))$ with

$$\sigma_G + N\sigma_G\epsilon = \text{ad}_{(1+\gamma_G\epsilon)} \circ \sigma_G : \mathbb{A}^1 \times R \times \text{Spec}(O(\text{SL}_2) \oplus O(\text{SL}_2)(-1)\epsilon) \rightarrow G_{\text{MHS}}.$$

Then observe that $N - [\gamma_G, -] : \mathfrak{u} \otimes \mathcal{S} \rightarrow \mathfrak{u} \otimes \mathcal{S}(-1)$ is R -equivariant, and denote this derivation by α_G .

If instead we started with an ind-MTS on $O(G)$, then the construction above would give corresponding data for $G_{\text{MTS}} = \mathbf{Spec} \xi(O(G), \text{MTS})$, replacing S with \mathbb{G}_m throughout.

Definition 4.24. For $G \rightarrow R$ as above, let $(\mathfrak{u}, \alpha_G)_{\text{MHS}}$ (resp. $(\mathfrak{u}, \alpha_G)_{\text{MTS}}$) be the Lie algebra $\text{row}_1^*\xi(\mathfrak{u}, \text{MHS}) \xrightarrow{\alpha_G} \text{row}_1^*\xi(\mathfrak{u}, \text{MHS})(-1)$ (resp. $\text{row}_1^*\xi(\mathfrak{u}, \text{MTS}) \xrightarrow{\alpha_G} \text{row}_1^*\xi(\mathfrak{u}, \text{MTS})(-1)$) over $O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)$.

Definition 4.25. Given a pro-nilpotent DG Lie algebra L^\bullet in non-negative cochain degrees, define the Deligne groupoid $\mathfrak{Del}(L)$ to have objects $\text{MC}(L) \subset L^1$ (see Definition 3.24), with morphisms $\omega \rightarrow \omega'$ consisting of $g \in \text{Gg}(L) = \exp(L^0)$ with $g * \omega = \omega'$, for the gauge action of Definition 3.25.

Since Theorem 4.20 gives the Hopf algebra $O(\varpi_1(X, x)^{\rho, \text{Mal}})$ an ind-MHS or ind-MTS independent of the choice of \mathcal{S} -splitting, we now show how to describe MHS and MTS representations of $\varpi_1(X, x)^{\rho, \text{Mal}}$ in terms of $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$.

Proposition 4.26. For $G \rightarrow R$ and $(X, x)^{\rho, \text{Mal}}$ as above, the set $\text{Hom}(\varpi_1(X, x)^{\rho, \text{Mal}}, G)_\rho^{\text{MHS}}$ (resp. $\text{Hom}(\varpi_1(X, x)^{\rho, \text{Mal}}, G)_\rho^{\text{MTS}}$) of morphisms

$$O(G) \rightarrow O(\varpi_1(X, x)^{\rho, \text{Mal}})$$

of ind-MHS (resp. ind-MTS) in Hopf algebras extending ρ is isomorphic to the fibre of the morphism

$$\mathfrak{Del}(O(X_{\text{MHS}}^{\rho, \text{Mal}}) \hat{\otimes}_{\mathbb{A}^1 \times C^*}^{\mathbb{G}_m \times R \times S} (\mathfrak{u}, \alpha_G)_{\text{MHS}}) \xrightarrow{x^*} \mathfrak{Del}((\mathfrak{u}, \alpha_G)_{\text{MHS}}^{\mathbb{G}_m \times S})$$

(resp.

$$\mathfrak{Del}(O(X_{\text{MTS}}^{\rho, \text{Mal}}) \hat{\otimes}_{\mathbb{A}^1 \times C^*}^{\mathbb{G}_m \times R \times \mathbb{G}_m} (\mathfrak{u}, \alpha_G)_{\text{MTS}}) \xrightarrow{x^*} \mathfrak{Del}((\mathfrak{u}, \alpha_G)_{\text{MTS}}^{\mathbb{G}_m \times \mathbb{G}_m}))$$

over γ_G . Here, the morphism of Deligne groupoids is induced by $x : \mathbb{A}^1 \times R \times C^* \rightarrow X_{\text{MHS}}^{\rho, \text{Mal}}$ (resp. $x : \mathbb{A}^1 \times R \times C^* \rightarrow X_{\text{MTS}}^{\rho, \text{Mal}}$).

Proof. We will prove this for the MHS case only; the MTS case can be recovered by replacing S with \mathbb{G}_m .

An element of $\text{Hom}(\varpi_1(X, x)^{\rho, \text{Mal}}, G)_\rho^{\text{MHS}}$ is just a $\mathbb{G}_m \times S$ -equivariant morphism $\psi : G(X, x)_{\text{MHS}}^{\rho, \text{Mal}} \rightarrow G_{\text{MHS}}$ of pro-unipotent extensions of $\mathbb{A}^1 \times R \times C^*$. The proof of Proposition 4.19 gives a choice $\sigma : R \times \text{SL}_2 \rightarrow G(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ of section, and the argument above shows that there must exist $u \in \Gamma(\mathbb{A}^1 \times \text{SL}_2, U_{\text{MHS}})^{\mathbb{G}_m \times S} = W_0\gamma^0 U(\mathcal{S})$ with $\psi \circ \sigma = \text{ad}_u \circ \sigma_G$. Then $\text{ad}_u^{-1} \circ \psi$ preserves the Levi decompositions, giving an R -equivariant morphism $f : \mathfrak{g} \otimes \mathcal{S} \rightarrow \mathfrak{u} \otimes \mathcal{S}$ of pro-Lie algebras in quasi-MHS over \mathcal{S} , with

$$\psi(r \cdot g) = u \cdot r \cdot f(g) \cdot u^{-1},$$

for $r \in R, g \in \exp(\mathfrak{g})$.

We also need ψ to commute with N . Looking at $N \circ \psi = \psi \circ N$ restricted to R , we need

$$[uf(\gamma)u^{-1}, uru^{-1}] = [\gamma_G + \alpha_G(u)u^{-1}, uru^{-1}]$$

for $r \in R$, and γ as in the proof of Proposition 4.19. Equivalently, there exists $b \in W_0\gamma^0(\mathbf{u}^R \otimes \mathcal{S}(-1))$ with $u^{-1}\gamma_G u + u^{-1}\alpha_G(u) = f(\gamma) + b$. Looking at ψ on $g \in \exp(\mathfrak{g})$ then gives the condition that $\alpha_G \circ f(g) - f \circ \alpha(g) = [b, f(g)]$.

A different choice of u would be of the form uv , for $v \in \Gamma(\mathbb{A}^1 \times \mathrm{SL}_2, U_{\mathrm{MHS}})^{\mathbb{G}_m \times R \times S}$, and we then have to replace (f, b) with $(\mathrm{ad}_{v^{-1}}f, v^{-1}bv + v^{-1}\alpha_G(v))$.

We now proceed by developing an equivalent description of the Deligne groupoids. Define a $\mathbb{G}_m \times R \times S$ -equivariant DG algebra A over $O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)$ by

$$A^n = (O(\mathbb{A}^1) \otimes O(\bar{W}\mathfrak{g})^n \otimes O(\mathrm{SL}_2)) \oplus (O(\mathbb{A}^1) \otimes O(\bar{W}\mathfrak{g})^{n-1} \otimes O(\mathrm{SL}_2)(-1)\epsilon),$$

with differential $d_{\bar{W}} \pm (N + \alpha)$. Then

$$\mathrm{MC}(A \otimes_{O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)}^{\mathbb{G}_m \times R \times S} ((\mathbf{u}, \alpha_G)_{\mathrm{MHS}}))$$

consists of pairs

$$(f, b) \in (\mathrm{Hom}(\mathfrak{g}, \mathrm{row}_1^*\xi(\mathbf{u}, \mathrm{MHS})) \times \Gamma(\mathbb{A}^1 \times \mathrm{SL}_2, \mathrm{row}_1^*\xi(\mathbf{u}, \mathrm{MHS})(-1)))^{\mathbb{G}_m \times R \times S},$$

satisfying the the Maurer-Cartan conditions. These are equivalent to saying that f is a Lie algebra homomorphism, and that $\alpha_G \circ f(g) - f \circ \alpha(g) = [b, f(g)]$.

Meanwhile,

$$\begin{aligned} \mathrm{MC}((\mathbf{u}, \alpha_G)_{\mathrm{MHS}})^{\mathbb{G}_m \times S} &= W_0\gamma^0(\mathbf{u} \otimes \mathcal{S}(-1)), \\ \mathrm{Gg}((\mathbf{u}, \alpha_G)_{\mathrm{MHS}})^{\mathbb{G}_m \times S} &= W_0\gamma^0 U(\mathcal{S}), \\ \mathrm{Gg}(A \otimes_{O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)}^{\mathbb{G}_m \times R \times S} (\mathbf{u}, \alpha_G)_{\mathrm{MHS}}) &= W_0\gamma^0 U^R(\mathcal{S}). \end{aligned}$$

There is a morphism $A \rightarrow O(\mathbb{A}^1) \otimes O(R) \otimes \mathbf{RO}(C^*)$ determined on generators by $\gamma : \mathfrak{g}^\vee \rightarrow O(\mathbb{A}^1) \otimes O(\mathrm{SL}_2)(-1)$ in level 1, for γ as in the proof of Proposition 4.19.

Thus the fibre of the Deligne groupoids

$$\mathfrak{Del}(A \otimes_{O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)}^{\mathbb{G}_m \times R \times S} (\mathbf{u}, \alpha_G)_{\mathrm{MHS}}) \rightarrow \mathfrak{Del}((\mathbf{u}, \alpha_G)_{\mathrm{MHS}})^{\mathbb{G}_m \times S}$$

over γ_G consists of (f, b) as above, together with $u \in W_0\gamma^0 U(\mathcal{S})$ mapping γ_G to $f(\gamma) + b$ under the gauge action of Definition 3.25. Morphisms in this groupoid are given by $v \in W_0\gamma^0 U^R(\mathcal{S})$, mapping (f, b, u) to $(\mathrm{ad}_{v^{-1}}f, v^{-1}bv + v^{-1}\alpha_G(v), vu)$. Taking $v = u^{-1}$, we see that fibre is therefore equivalent to the groupoid with objects $\mathrm{Hom}(\varpi_1(X, x)^{\rho, \mathrm{Mal}}, G)_\rho^{\mathrm{MHS}}$ and trivial morphisms.

Finally, observe that Corollary 4.12 combines with the proof of Proposition 4.19 to give a quasi-isomorphism

$$A \simeq O(X_{\mathrm{MHS}}^{\rho, \mathrm{Mal}})$$

in $DG\mathrm{Alg}_{O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)}(\mathbb{G}_m \times R \times S) \downarrow (O(\mathbb{A}^1) \otimes O(R) \otimes \mathbf{RO}(C^*))$, where the augmentation map $O(X_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}) \rightarrow O(\mathbb{A}^1) \otimes O(R) \otimes \mathbf{RO}(C^*)$ is given by x . Therefore there is an equivalence

$$\mathfrak{Del}(A \otimes_{O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)}^{\mathbb{G}_m \times R \times S} (\mathbf{u}, \alpha_G)_{\mathrm{MHS}}) \simeq \mathfrak{Del}(O(X) \otimes_{O(\mathbb{A}^1) \otimes \mathbf{RO}(C^*)}^{\mathbb{G}_m \times R \times S} (\mathbf{u}, \alpha_G)_{\mathrm{MHS}})$$

of groupoids over $\mathfrak{Del}((\mathbf{u}, \alpha_G)_{\mathrm{MHS}})^{\mathbb{G}_m \times S}$ (by [GM, Theorem 2.4]), giving the required result. \square

5. MHS ON RELATIVE MALCEV HOMOTOPY TYPES OF COMPACT KÄHLER MANIFOLDS

Fix a compact Kähler manifold X and a point $x \in X$.

5.1. Real homotopy types.

Definition 5.1. Define the Hodge filtration on the real homotopy type $(X \otimes \mathbb{R}, x)$ by $(X \otimes \mathbb{R}, x)_{\mathbb{F}} := (\mathrm{Spec} \mathbb{R} \times C^* \xrightarrow{x} \mathbf{Spec} j^* \tilde{A}^{\bullet}(X)) \in \mathrm{Ho}(C^* \downarrow dg_{\mathbb{Z}} \mathrm{Aff}_{C^*}(S))$, for $j : C^* \rightarrow C$ and $\tilde{A}^{\bullet}(X)$ as in Definition 2.1.

Definition 5.2. Define the algebraic mixed Hodge structure $(X \otimes \mathbb{R}, x)_{\mathrm{MHS}}$ on $(X \otimes \mathbb{R}, x)$ to be \mathbf{Spec} of the Rees algebra associated to the good truncation filtration $W_r = \tau^{\leq r} j^* \tilde{A}^{\bullet}(X)$, equipped with the augmentation $\tilde{A}^{\bullet}(X) \xrightarrow{x^*} O(C)$.

Define $(\underline{\mathrm{gr}}(X \otimes \mathbb{R})_{\mathrm{MHS}}, 0)$ to be the unique morphism $\mathrm{Spec} \mathbb{R} \rightarrow \mathrm{Spec} H^*(X, \mathbb{R})$, determined by the isomorphism $H^0(X, \mathbb{R}) \cong \mathbb{R}$. Now

$$(X \otimes \mathbb{R}, x)_{\mathrm{MHS}} \times_{\mathbb{A}^1}^h \{0\} = (C^* \xrightarrow{x} \mathbf{Spec} \underline{\mathrm{gr}}^W j^* \tilde{A}^{\bullet}(X)),$$

and there is a canonical quasi-isomorphism $\underline{\mathrm{gr}}^W j^* \tilde{A}^{\bullet}(X) \rightarrow \mathcal{H}^*(j^* \tilde{A}^{\bullet}(X))$. As in the proof of Corollary 2.9, this is S -equivariantly isomorphic to $H^*(X, \mathbb{R}) \otimes \mathcal{O}(C^*)$, giving the opposedness quasi-isomorphism

$$(X \otimes \mathbb{R}, x) \times_{\mathbb{A}^1}^h \{0\} \xleftarrow{\sim} (\underline{\mathrm{gr}}(X \otimes \mathbb{R})_{\mathrm{MHS}}, 0) \times C^*.$$

Proposition 5.3. *The algebraic MHS $(X \otimes \mathbb{R}, x)_{\mathrm{MHS}}$ splits on pulling back along $\mathrm{row}_1 : \mathrm{SL}_2 \rightarrow C^*$. Explicitly, there is an isomorphism*

$$(X \otimes \mathbb{R}, x)_{\mathrm{MHS}} \times_{C^*, \mathrm{row}_1}^{\mathbf{R}} \mathrm{SL}_2 \cong \mathbb{A}^1 \times (\underline{\mathrm{gr}}(X \otimes \mathbb{R})_{\mathrm{MHS}}, 0) \times C^*,$$

in $\mathrm{Ho}(\mathbb{A}^1 \times \mathrm{SL}_2 \downarrow dg_{\mathbb{Z}} \mathrm{Aff}_{\mathbb{A}^1 \times \mathrm{SL}_2}(\mathbb{G}_m \times S))$, whose pullback to $0 \in \mathbb{A}^1$ is given by the opposedness isomorphism.

Proof. Corollary 2.9 establishes the corresponding splitting for the Hodge filtration $(X \otimes \mathbb{R}, x)_{\mathbb{F}}$, and good truncation commutes with everything, giving the splitting for $(X \otimes \mathbb{R}, x)_{\mathrm{MHS}}$. The proof of Corollary 2.9 ensures that pulling the \mathcal{S} -splitting back to $0 \in \mathbb{A}^1$ gives row_1^* applied to the opposedness isomorphism. \square

Corollary 5.4. *There are natural pro-MHS on the homotopy groups $\pi_n(X \otimes \mathbb{R}, x)$.*

Proof. Apply Theorem 4.20 in the case $R = 1$, noting that Proposition 5.3 gives the requisite \mathcal{S} -splitting. \square

Corollary 5.5. *For \mathcal{S} as in Example 1.24, and for all $n \geq 1$, there are \mathcal{S} -linear isomorphisms*

$$\pi_n(X \otimes \mathbb{R}, x)^{\vee} \otimes_{\mathbb{R}} \mathcal{S} \cong \pi_n(H^*(X, \mathbb{R}))^{\vee} \otimes_{\mathbb{R}} \mathcal{S},$$

of quasi-MHS, compatible with Whitehead brackets and Hurewicz maps. The graded map associated to the weight filtration is just the pullback of the standard isomorphism $\underline{\mathrm{gr}}_W \pi_n(X \otimes \mathbb{R}, x) \cong \pi_n(H^*(X, \mathbb{R}))$ (coming from the opposedness isomorphism).

Proof. The \mathcal{S} -splitting of Proposition 5.3 allows us to apply Theorem 4.21, giving isomorphisms

$$\pi_n(X \otimes \mathbb{R}, x)^{\vee} \otimes_{\mathbb{R}} \mathcal{S} \cong \varpi_n(\underline{\mathrm{gr}}(X \otimes \mathbb{R})_{\mathrm{MHS}}, 0)^{\vee} \otimes_{\mathbb{R}} \mathcal{S}$$

of quasi-MHS.

The definition of $\underline{\mathrm{gr}}(X \otimes \mathbb{R})_{\mathrm{MHS}}$ implies that $\varpi_n(\underline{\mathrm{gr}}(X \otimes \mathbb{R})_{\mathrm{MHS}}, 0) = \pi_{n-1} \tilde{G}(H^*(X, \mathbb{R}))$, giving the required result. \square

5.1.1. *Comparison with Morgan.* We now show that our mixed Hodge structure on homotopy groups agrees with the mixed Hodge structure given in [Mor] for simply connected varieties.

Proposition 5.6. *The mixed Hodge structures on homotopy groups given in Corollary 5.4 and [Mor, Theorem 9.1] agree.*

Proof. In [Mor, §6], a minimal model \mathcal{M} was constructed for $\mathcal{A}^\bullet(X, \mathbb{C})$, equipped with a bigrading (i.e. a $\mathbb{G}_m \times \mathbb{G}_m$ -action). The associated quasi-isomorphism $\psi : \mathcal{M} \rightarrow \mathcal{A}^\bullet(X, \mathbb{C})$ satisfies $\psi(\mathcal{M}^{pq}) \subset \tau^{\leq p+q} F^p \mathcal{A}^\bullet(X, \mathbb{C})$. Thus ψ is a map of bifiltered DGAs. It is also a quasi-isomorphism of DGAs, but we need to show that it is a quasi-isomorphism of bifiltered DGAs. By [Mor, Lemma 6.2b], ψ maps $H^*(\mathcal{M}^{pq})$ isomorphically to $H^{pq}(X, \mathbb{C})$, so the associated Rees algebras are quasi-isomorphic.

Equivalently, this says that we have a $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant quasi-isomorphism

$$\xi(\tilde{A}^\bullet(X) \otimes_{O(C)} O(\mathbb{A}_{\mathbb{C}}^1); \tau) \simeq \xi(\mathcal{M}; F, W).$$

over the subscheme $\mathbb{A}^1 \times \mathbb{A}_{\mathbb{C}}^1 \subset \mathbb{A}^1 \times \widetilde{C}^*$ given by $u - iv = 1$ as in Lemma 1.13. Now, Lemma 3.47 gives equivalences

$$\begin{aligned} DG_{\mathbb{Z}}\text{Alg}_{\mathbb{A}^1 \times \widetilde{C}^*}(\mathbb{G}_m \times S) &\cong DG_{\mathbb{Z}}\text{Alg}_{\mathbb{A}^1 \times \widetilde{C}^*}(\mathbb{G}_m \times S_{\mathbb{C}}) \\ &\cong DG_{\mathbb{Z}}\text{Alg}_{\mathbb{A}^1 \times \mathbb{A}_{\mathbb{C}}^1 \times \mathbb{G}_{m, \mathbb{C}}}(\mathbb{G}_m \times \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}) \\ &\simeq DG_{\mathbb{Z}}\text{Alg}_{\mathbb{A}^1 \times \mathbb{A}_{\mathbb{C}}^1}(\mathbb{G}_m \times \mathbb{G}_{m, \mathbb{C}}), \end{aligned}$$

so $\xi(\mathcal{M}; F, W) \otimes O(\mathbb{G}_{m, \mathbb{C}})$ is quasi-isomorphic to $\xi(\tilde{A}^\bullet(X) \otimes_{O(C)} O(\widetilde{C}^*), \tau)$, which is just the pullback $p^*O((X \otimes \mathbb{R})_{\text{MHS}})$ along $p : \widetilde{C}^* \rightarrow C^*$. Equivalently, \mathcal{M} is a \widetilde{C}^* -splitting (rather than an SL_2 -splitting) of the MHS on $O(X \otimes \mathbb{R})$.

Note that $\mathcal{M}^0 = \mathbb{C}$, so there is a unique map $\mathcal{M} \rightarrow \mathbb{C}$, and thus $\xi(\mathcal{M}; F, W) \otimes O(\mathbb{G}_{m, \mathbb{C}})$ is quasi-isomorphic to $p^*O((X \otimes \mathbb{R})_{\text{MHS}})$ in $DG\text{Alg}_{\mathbb{A}^1 \times \widetilde{C}^*}(\mathbb{G}_m \times S) \downarrow O(\mathbb{A}^1 \times \widetilde{C}^*)$. Since p factors through $\text{row}_1 : \text{SL}_2 \rightarrow C^*$ by Lemma 1.17, we have a morphism $q : \mathbf{RO}(C^*) \rightarrow O(\widetilde{C}^*)$, and the construction of Proposition 4.19 then gives a quasi-isomorphism

$$q^*G(X \otimes \mathbb{R}, x)_{\text{MHS}} \simeq \xi(\exp(G(\mathcal{M})); W, F)$$

of $\mathbb{G}_m \times S$ -equivariant pro-nilpotent Lie algebras over $\mathbb{A}^1 \times \widetilde{C}^*$.

Taking homotopy groups as in the proof of Theorem 4.20, we see that

$$q^*\varpi_n(X \otimes \mathbb{R}, x)_{\text{MHS}} \cong \xi(H_{n-1}(G(\mathcal{M})); W, F).$$

Now, under the equivalences of Theorem 3.29, $H_{n-1}(G(\mathcal{M}))^\vee = H^n(\mathbb{L}^{\mathcal{M}/\mathbb{R}} \otimes_{\mathcal{M}}^{\mathbf{L}} \mathbb{R})$. Since \mathcal{M} is cofibrant, this is just $H^n(\Omega(\mathcal{M}/\mathbb{R}) \otimes_{\mathcal{M}} \mathbb{R})$. Finally, \mathcal{M} is minimal, so the complex $\Omega(\mathcal{M}/\mathbb{R}) \otimes_{\mathcal{M}} \mathbb{R}$ is isomorphic to the indecomposables I of \mathcal{M} , with trivial differential. This means that $H_{n-1}(G(\mathcal{M}))^\vee \cong I^n$, and

$$\xi(\varpi_n(X \otimes \mathbb{C}, x)^\vee; W, F) = p^*\xi(\varpi_n(X \otimes \mathbb{R}, x)^\vee, \text{MHS}) \cong \xi(I^n; W, F),$$

so the Hodge and weight filtrations from Theorem 4.20 and [Mor] agree. \square

5.2. Relative Malcev homotopy types.

5.2.1. The reductive fundamental groupoid is pure of weight 0.

Lemma 5.7. *There is a canonical action of the discrete group $(S^1)^\delta$ on the real reductive pro-algebraic completion $\varpi_1(X, x)^{\text{red}}$ of the fundamental group $\pi_1(X, x)$.*

Proof. By Tannakian duality, this is equivalent to establishing a $(S^1)^\delta$ -action on the category of real semisimple local systems on X . This is just the unitary part of the \mathbb{C}^* -action on complex local systems from [Sim3]. Given a real \mathcal{C}^∞ vector bundle \mathcal{V} with a flat connection D , there is an essentially unique pluriharmonic metric, giving a unique decomposition $D = d^+ + \vartheta$ of D into antisymmetric and symmetric parts. In the notation of [Sim3], $d^+ = \partial + \bar{\partial}$ and $\vartheta = \theta + \bar{\theta}$. Given $t \in (S^1)^\delta$, we define $t \diamond D$ by $d^+ + t \diamond \vartheta = \partial + \bar{\partial} + t\theta + t^{-1}\bar{\theta}$ (for \diamond as in Definition 2.2), which preserves the metric. \square

5.2.2. *Variations of Hodge structure.* The following results are taken from [Pri1, §2.3].

Definition 5.8. Given a discrete group Γ acting on a pro-algebraic group G , define ${}^\Gamma G$ to be the maximal quotient of G on which Γ acts algebraically. This is the inverse limit $\varprojlim_\alpha G_\alpha$ over those surjective maps

$$G \rightarrow G_\alpha,$$

with G_α algebraic (i.e. of finite type), for which the Γ -action descends to G_α . Equivalently, $O({}^\Gamma G)$ is the sum of those finite-dimensional Γ -representations of $O(G)$ which are closed under comultiplication.

Definition 5.9. Define the quotient group ${}^{\text{VHS}}\varpi_1(X, x)$ of $\varpi_1(X, x)$ by

$${}^{\text{VHS}}\varpi_1(X, x) := (S^1)^\delta \varpi_1(X, x)^{\text{red}}.$$

Remarks 5.10. This notion is analogous to the definition given in [Pri4] of the maximal quotient of the l -adic pro-algebraic fundamental group on which Frobenius acts algebraically. In the same way that representations of that group corresponded to semisimple subsystems of local systems underlying Weil sheaves, representations of ${}^{\text{VHS}}\varpi_1(X, x)$ will correspond to local systems underlying variations of Hodge structure (Proposition 5.12).

Proposition 5.11. *The action of S^1 on ${}^{\text{VHS}}\varpi_1(X, x)$ is algebraic, in the sense that*

$$S^1 \times {}^{\text{VHS}}\varpi_1(X, x) \rightarrow {}^{\text{VHS}}\varpi_1(X, x)$$

is a morphism of schemes.

It is also an inner action, coming from a morphism

$$S^1 \rightarrow ({}^{\text{VHS}}\varpi_1(X, x))/Z({}^{\text{VHS}}\varpi_1(X, x))$$

of pro-algebraic groups, where Z denotes the centre of the group.

Proof. In the notation of Definition 5.8, write $\varpi_1(X, x) = \varprojlim_\alpha G_\alpha$. As in [Sim3, Lemma 5.1], the map

$$\text{Aut}(G_\alpha) \rightarrow \text{Hom}(\pi_1(X, x), G_\alpha)$$

is a closed immersion of schemes, so the map

$$(S^1)^\delta \rightarrow \text{Aut}(G_\alpha)$$

is continuous. This means that it defines a one-parameter subgroup, so is algebraic. Therefore the map

$$S^1 \times {}^{\text{VHS}}\varpi_1(X, x) \rightarrow {}^{\text{VHS}}\varpi_1(X, x)$$

is algebraic, as ${}^{\text{VHS}}\varpi_1(X, x) = \varprojlim_\alpha G_\alpha$.

Since $\varpi_1(X, x)^{\text{red}}$ is a reductive pro-algebraic group, G_α is a reductive algebraic group. This implies that the connected component $\text{Aut}(G_\alpha)^0$ of the identity in $\text{Aut}(G_\alpha)$ is given by

$$\text{Aut}(G_\alpha)^0 = G_\alpha(x, x)/Z(G_\alpha).$$

Since

$${}^{\text{VHS}}\varpi_1(X, x)/Z({}^{\text{VHS}}\varpi_1(X, x)) = \varprojlim_\alpha G_\alpha/Z(G_\alpha),$$

we have an algebraic map

$$S^1 \rightarrow {}^{\text{VHS}}\varpi_1(X, x)/Z({}^{\text{VHS}}\varpi_1(X, x)),$$

as required. \square

Proposition 5.12. *The following conditions are equivalent:*

- (1) V is a representation of ${}^{\text{VHS}}\varpi_1(X, x)$;
- (2) V is a representation of $\varpi_1(X, x)^{\text{red}}$ such that $t \otimes V \cong V$ for all $t \in (S^1)^\delta$;
- (3) V is a representation of $\varpi_1(X, x)^{\text{red}}$ such that $t \otimes V \cong V$ for some non-torsion $t \in (S^1)^\delta$.

Moreover, representations of ${}^{\text{VHS}}\varpi_1(X, x) \rtimes S^1$ correspond to weight 0 variations of Hodge structure on X .

Proof.

1. \implies 2. If V is a representation of ${}^{\text{VHS}}\varpi_1(X, x)$, then it is a representation of $\varpi_1(X, x)^{\text{red}}$, so is a semisimple representation of $\varpi_1(X, x)$. By Lemma 5.11, $t \in (S^1)^\delta$ is an inner automorphism of ${}^{\text{VHS}}\varpi_1(X, x)$, coming from $g \in {}^{\text{VHS}}\varpi_1(X, x)$, say. Then multiplication by g gives the isomorphism $t \otimes V \cong V$.
2. \implies 3. Trivial.
3. \implies 1. Let M be the monodromy group of V ; this is a quotient of $\varpi_1(X, x)^{\text{red}}$. The isomorphism $t \otimes V \cong V$ gives an element $g \in \text{Aut}(M)$, such that g is the image of t in $\text{Hom}(\pi_1(X, x), M)$, using the standard embedding of $\text{Aut}(M)$ as a closed subscheme of $\text{Hom}(\pi_1(X, x), M)$. The same is true of g^n, t^n , so the image of S^1 in $\text{Hom}(\pi_1(X, x), M)$ is just the closure of $\{g^n\}_{n \in \mathbb{Z}}$, which is contained in $\text{Aut}(M)$, as $\text{Aut}(M)$ is closed. For any $s \in (S^1)^\delta$, this gives us an isomorphism $s \otimes V \cong V$, as required.

Finally, a representation of ${}^{\text{VHS}}\varpi_1(X, x) \rtimes S^1$ gives a semisimple local system $\mathbb{V} = \ker(D : \mathcal{V} \rightarrow \mathcal{V} \otimes_{\mathcal{A}^0} \mathcal{A}^1)$ (satisfying one of the equivalent conditions above), together with a coassociative coaction $\mu : (\mathcal{V}, D) \rightarrow (\mathcal{V} \otimes O(S^1), t \otimes D)$ of ind-finite-dimensional local systems, for $t = a + ib \in O(S^1) \otimes \mathbb{C}$. This is equivalent to giving a decomposition $\mathcal{V} \otimes \mathbb{C} = \bigoplus_{p+q=0} \mathcal{V}^{pq}$ with $\overline{\mathcal{V}^{pq}} = \mathcal{V}^{qp}$, and with the decomposition $D = \partial + \bar{\partial} + t\theta + t^{-1}\bar{\theta}$ (as in Lemma 5.7) satisfying

$$\partial : \mathcal{V}^{pq} \rightarrow \mathcal{V}^{pq} \otimes \mathcal{A}^{10}, \quad \bar{\theta} : \mathcal{V}^{pq} \rightarrow \mathcal{V}^{p+1, q-1} \otimes \mathcal{A}^{01},$$

which is precisely the condition for \mathbb{V} to be a VHS. Note that if we had chosen \mathbb{V} not satisfying one of the equivalent conditions, then $(\mathcal{V} \otimes O(S^1), t \otimes D)$ would not yield an ind-finite-dimensional local system. \square

Lemma 5.13. *The obstruction φ to a surjective map $\alpha : \varpi_1(X, x)^{\text{red}} \rightarrow R$, for R algebraic, factoring through ${}^{\text{VHS}}\varpi_1(X, x)$ lies in $H^1(X, \text{ad}\mathbb{B}_\alpha)$, for $\text{ad}\mathbb{B}_\alpha$ the vector bundle associated to the adjoint representation of α on the Lie algebra of R . Explicitly, φ is given by $\varphi = [i\theta - i\bar{\theta}]$, for $\theta \in A^{10}(X, \text{ad}\mathbb{B}_\alpha)$ the Higgs form associated to α .*

Proof. We have a continuous

$$S^1 \times \pi_1(X, x) \rightarrow R,$$

and α will factor through ${}^{\text{VHS}}\varpi_1(X, x)$ if and only if the induced map

$$S^1 \xrightarrow{\phi} \text{Hom}(\pi_1(X, x), R)/\text{Aut}(R)$$

is constant. Since R is reductive and S^1 connected, it suffices to replace $\text{Aut}(R)$ by the group of inner automorphisms. On tangent spaces, we then have a map

$$i\mathbb{R} \xrightarrow{D_1\phi} H^1(X, \text{ad}\mathbb{B}_\alpha);$$

let $\varphi \in H^1(X, \text{ad}\mathbb{B}_\alpha)$ be the image of i . The description $\varphi = [i\theta - i\bar{\theta}]$ comes from differentiating $e^{ir} \otimes D = \partial + \bar{\partial} + e^{ir}\theta + e^{-ir}\bar{\theta}$ with respect to r .

If ϕ is constant, then $\varphi = 0$. Conversely, observe that for $t \in S^1(R)$, $D_t\phi = tD_1\phi t^{-1}$, making use of the action of $(S^1)^\delta$ on $\text{Hom}(\pi_1(X, x), G)$. If $\varphi = 0$, this implies that $D_t\phi = 0$ for all $t \in (S^1)^\delta$, so ϕ is constant, as required. \square

5.2.3. Mixed Hodge structures.

Theorem 5.14. *If R is any quotient of ${}^{\text{VHS}}\varpi_1(X, x)_{\mathbb{R}}^{\text{red}}$, then there is an algebraic mixed Hodge structure $(X, x)_{\text{MHS}}^{\rho, \text{Mal}}$ on the relative Malcev homotopy type $(X, x)^{\rho, \text{Mal}}$, where ρ denotes the quotient map to R .*

There is also an S -equivariant splitting

$$\mathbb{A}^1 \times (\underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}, 0) \times \mathrm{SL}_2 \simeq (X^{\rho, \mathrm{Mal}}, x)_{\mathrm{MHS}} \times_{C^*, \mathrm{row}_1}^{\mathbf{R}} \mathrm{SL}_2$$

on pulling back along $\mathrm{row}_1 : \mathrm{SL}_2 \rightarrow C^*$, whose pullback over $0 \in \mathbb{A}^1$ is given by the opposedness isomorphism.

Proof. By Proposition 5.11, we know that representations of R all correspond to local systems underlying polarised variations of Hodge structure, and that the $(S^1)^\delta$ -action on ${}^{\mathrm{VHS}}\varpi_1(X, x)_{\mathbb{R}}^{\mathrm{red}}$ descends to an inner algebraic action on R , via $S^1 \rightarrow R/Z(R)$. This allows us to consider the semi-direct products $R \rtimes S^1$ and $R \rtimes S$ of pro-algebraic groups, making use of the isomorphism $S^1 \cong S/\mathbb{G}_m$.

The R -representation $O(\mathbb{B}_\rho) = \mathbb{B}_\rho \times^R O(R)$ in local systems of \mathbb{R} -algebras on X thus has an algebraic S^1 -action, denoted by $(t, v) \mapsto t \otimes v$ for $t \in S^1, v \in O(\mathbb{B}_\rho)$, and we define an S -action on the de Rham complex

$$\mathcal{A}^*(X, O(\mathbb{B}_\rho)) = \mathcal{A}^*(X, \mathbb{R}) \otimes_{\mathbb{R}} O(\mathbb{B}_\rho)$$

by $\lambda \boxtimes (a \otimes v) := (\lambda \diamond a) \otimes (\bar{\lambda} \otimes v)$, noting that the \diamond and \otimes actions commute. This gives an action on the global sections

$$A^*(X, O(\mathbb{B}_\rho)) := \Gamma(X, \mathcal{A}^*(X, O(\mathbb{B}_\rho))).$$

It follows from [Sim3, Theorem 1] that there exists a harmonic metric on every semisimple local system \mathbb{V} , and hence on $O(\mathbb{B}_\rho)$. We then decompose the connection D as $D = d^+ + \vartheta$ into antisymmetric and symmetric parts, and let $D^c := i \diamond d^+ - i \diamond \vartheta$. To see that this is independent of the choice of metric, observe that for $C = -1 \in S^1$ acting on $\varpi_1(X, x)^{\mathrm{red}}$, antisymmetric and symmetric parts are the 1- and -1 -eigenvectors.

Now, we define the DGA $\tilde{A}^\bullet(X, O(\mathbb{B}_\rho))$ on C by

$$\tilde{A}^\bullet(X, O(\mathbb{B}_\rho)) := (A^*(X, O(\mathbb{B}_\rho)) \otimes_{\mathbb{R}} O(C), uD + vD^c),$$

and we denote the differential by $\tilde{D} := uD + vD^c$. Note that the \boxtimes S -action makes this S -equivariant over C . Thus $\tilde{A}(X, O(\mathbb{B}_\rho)) \in \mathrm{DGAlg}_C(R \times S)$, and we define the Hodge filtration by

$$(X_{\mathbb{F}}^{\rho, \mathrm{Mal}}, x) := (R \times C^* \xrightarrow{x} (\mathrm{Spec} \tilde{A}(X, O(\mathbb{B}_\rho))) \times_C C^*) \in \mathrm{dgZAff}_{C^*}(R)_*(S),$$

making use of the isomorphism $O(\mathbb{B}_\rho)_x \cong O(R)$.

We then define the mixed Hodge structure $(X_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}, x)$ by

$$(\mathbb{A}^1 \times R \times C^* \xrightarrow{x} (\mathrm{Spec} \xi(\tilde{A}(X, O(\mathbb{B}_\rho)), \tau)) \times_C C^*) \in \mathrm{dgZAff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathbb{G}_m \times S),$$

with $(\underline{\mathrm{gr}}X_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}, 0)$ given by

$$(R \rightarrow \mathrm{Spec} H^*(X, O(\mathbb{B}_\rho))) \in \mathrm{dgZAff}(R)_*(S).$$

The rest of the proof is now the same as in §5.1, using the principle of two types from [Sim3, Lemmas 2.1 and 2.2]. Corollary 2.9 adapts to give the quasi-isomorphism

$$(X_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}, x) \times_{C^*, \mathrm{row}_1}^{\mathbf{R}} \mathrm{SL}_2 \simeq (\underline{\mathrm{gr}}X_{\mathrm{MHS}}^{\rho, \mathrm{Mal}}, 0) \times \mathrm{SL}_2,$$

which gives the splitting. \square

Observe that this theorem easily adapts to multiple basepoints, as considered in Remark 4.9.

Remark 5.15. Note that the filtration W here and later is not related to the weight tower W^*F^0 of [KPT1, §3], which does not agree with the weight filtration of [Mor]. W^*F^0 corresponded to the lower central series filtration $\Gamma_n \mathfrak{g}$ on $\mathfrak{g} := \mathrm{R}_u(G(X)^{\mathrm{alg}})$, given by $\Gamma_1 \mathfrak{g} = \mathfrak{g}$ and $\Gamma_n \mathfrak{g} = [\Gamma_{n-1} \mathfrak{g}, \mathfrak{g}]$, by the formula $W^i F^0 = \mathfrak{g}/\Gamma_{n+1} \mathfrak{g}$. Since this is just the filtration $\bar{G}(\mathrm{Fil})$ coming from the filtration $\mathrm{Fil}_{-1} A^\bullet = 0, \mathrm{Fil}_0 A^\bullet = \mathbb{R}, \mathrm{Fil}_1 A^\bullet = A^\bullet$ on A^\bullet , it

amounts to setting higher cohomology groups to be pure of weight 1; [KPT1, Proposition 3.2.6(4)] follows from this observation, as the graded pieces $\mathrm{gr}_W^i F^0$ defined in [KPT1, Definition 3.2.3] are just $\mathrm{gr}_{G(\mathbb{F}_i)}^{i+1} \mathfrak{g}$.

Corollary 5.16. *In the scenario of Theorem 5.14, the homotopy groups $\varpi_n(X^{\rho, \mathrm{Mal}}, x)$ for $n \geq 2$, and the Hopf algebra $O(\varpi_1(X^{\rho, \mathrm{Mal}}, x))$ carry natural ind-MHS, functorial in (X, x) , and compatible with the action of ϖ_1 on ϖ_n , the Whitehead bracket and the Hurewicz maps $\varpi_n(X^{\rho, \mathrm{Mal}}, x) \rightarrow \mathbf{H}^n(X, O(\mathbb{B}_\rho))^\vee$.*

Moreover, there are \mathcal{S} -linear isomorphisms

$$\begin{aligned} \varpi_n(X^{\rho, \mathrm{Mal}}, x)^\vee \otimes \mathcal{S} &\cong \pi_n(H^*(X, O(\mathbb{B}_\rho)))^\vee \otimes \mathcal{S} \\ O(\varpi_1(X^{\rho, \mathrm{Mal}}, x)) \otimes \mathcal{S} &\cong O(R \rtimes \pi_1(H^*(X, O(\mathbb{B}_\rho)))) \otimes \mathcal{S} \end{aligned}$$

of quasi-MHS. The associated graded map from the weight filtration is just the pullback of the standard isomorphism $\mathrm{gr}_W \varpi_*(X^{\rho, \mathrm{Mal}}) \cong \pi_*(H^*(X, O(\mathbb{B}_\rho)))$.

Here, $\pi_*(H^*(X, O(\mathbb{B}_\rho)))$ are the homotopy groups $\mathbf{H}_{*-1} \bar{G}(H^*(X, O(\mathbb{B}_\rho)))$ associated to the $R \rtimes S$ -equivariant DGA $H^*(X, O(\mathbb{B}_\rho))$ (as constructed in Definition 3.23), with the induced real Hodge structure.

Proof. Theorem 5.14 provides the data required by Theorems 4.20 and 4.21 to construct \mathcal{S} -split ind-MHS on homotopy groups. \square

Remark 5.17. If we have a set T of several basepoints, then Remark 4.23 gives \mathcal{S} -split ind-MHS on the algebras $O(\varpi_1(X; x, y)^{\rho, \mathrm{Mal}})$, compatible with the pro-algebraic groupoid structure. The \mathcal{S} -split ind-MHS on $(\varpi_n(X, x)^{\rho, \mathrm{Mal}})^\vee$ are then compatible with the co-action

$$(\varpi_n(X, x)^{\rho, \mathrm{Mal}})^\vee \rightarrow O(\varpi_1(X; x, y)^{\rho, \mathrm{Mal}}) \otimes (\varpi_n(X, y)^{\rho, \mathrm{Mal}})^\vee.$$

Remark 5.18. Corollary 5.16 confirms the first part of [Ara, Conjecture 5.5]. If \mathbb{V} is a k -variation of Hodge structure on X , for a field $k \subset \mathbb{R}$, and R is the Zariski closure of $\pi_1(X, x) \rightarrow \mathrm{GL}(\mathbb{V}_x)$, the conjecture states that there is a natural ind- k -MHS on the k -Hopf algebra $O(\varpi_1(X^{\rho, \mathrm{Mal}}, x)^{\rho, \mathrm{Mal}})$. Applying Corollary 5.16 to the Zariski-dense real representation $\rho_{\mathbb{R}} : \pi_1(X, x) \rightarrow R(\mathbb{R})$ gives a real ind-MHS on the real Hopf algebra $O(\varpi_1(X^{\rho, \mathrm{Mal}}, x)^{\rho_{\mathbb{R}}, \mathrm{Mal}}) = O(\varpi_1(X^{\rho, \mathrm{Mal}}, x)^{\rho, \mathrm{Mal}}) \otimes_k \mathbb{R}$. The weight filtration is just given by the lower central series on the pro-unipotent radical, so descends to k , giving an ind- k -MHS on the k -Hopf algebra $O(\varpi_1(X^{\rho, \mathrm{Mal}}, x)^{\rho, \mathrm{Mal}})$.

If \mathbb{V} is a variation of Hodge structure on X , and $R = \mathrm{GL}(\mathbb{V}_x)$, then Corollary 5.16 recovers the ind-MHS on $O(\varpi_1(X^{\rho, \mathrm{Mal}}, x))$ first described in [Hai4, Theorem 13.1]. If T is a set of basepoints, and R is the algebraic groupoid $R(x, y) = \mathrm{Iso}(\mathbb{V}_x, \mathbb{V}_y)$ on objects T , then Remark 5.17 recovers the ind-MHS on $\varpi_1(X^{\rho, \mathrm{Mal}}; T)$ first described in [Hai4, Theorem 13.3].

Corollary 5.19. *If $\pi_1(X, x)$ is algebraically good with respect to R and the homotopy groups $\pi_n(X, x)$ have finite rank for all $n \geq 2$, with each $\pi_1(X, x)$ -representation $\pi_n(X, x) \otimes_{\mathbb{Z}} \mathbb{R}$ an extension of R -representations, then Theorem 4.20 gives mixed Hodge structures on $\pi_n(X, x) \otimes \mathbb{R}$ for all $n \geq 2$, by Theorem 3.17.*

Before stating the next proposition, we need to observe that for any morphism $f : X \rightarrow Y$ of compact Kähler manifolds, the induced map $\pi_1(X, x) \rightarrow \pi_1(Y, fx)$ gives rise to a map $\varpi_1(X, x)^{\mathrm{red}} \rightarrow \varpi_1(Y, fx)^{\mathrm{red}}$ of reductive pro-algebraic fundamental groups. This is not true for arbitrary topological spaces, but holds in this case because semisimplicity is preserved by pullbacks between compact Kähler manifolds, since Higgs bundles pull back to Higgs bundles.

Proposition 5.20. *If we have a morphism $f : X \rightarrow Y$ of compact Kähler manifolds, and a commutative diagram*

$$\begin{array}{ccc} \pi_1(X, x) & \xrightarrow{f} & \pi_1(Y, fx) \\ \rho \downarrow & & \downarrow \varrho \\ R & \xrightarrow{\theta} & R' \end{array}$$

of groups, with R, R' real reductive pro-algebraic groups to which the $(S^1)^\delta$ -actions descend and act algebraically, and ρ, ϱ Zariski-dense, then the natural map $(X^{\rho, \text{Mal}}, x) \rightarrow \theta^\#(Y^{\varrho, \text{Mal}}, fx) = (Y^{\varrho, \text{Mal}}, fx) \times_{BR'} BR$ extends to a natural map

$$(X_{\text{MHS}}^{\rho, \text{Mal}}, x) \rightarrow \theta^\#(Y_{\text{MHS}}^{\varrho, \text{Mal}}, fx)$$

of algebraic mixed Hodge structures.

Proof. This is really just the observation that the construction $\tilde{A}^\bullet(X, \mathbb{V})$ is functorial in X . \square

Note that, combined with Theorem 3.10, this gives canonical MHS on homotopy types of homotopy fibres.

6. MTS ON RELATIVE MALCEV HOMOTOPY TYPES OF COMPACT KÄHLER MANIFOLDS

Let X be a compact Kähler manifold.

Theorem 6.1. *If $\rho : (\pi_1(X, x))_{\mathbb{R}}^{\text{red}} \rightarrow R$ is any quotient, then there is an algebraic mixed twistor structure on the relative Malcev homotopy type $(X, x)^{\rho, \text{Mal}}$, functorial in (X, x) , which splits on pulling back along $\text{row}_1 : \text{SL}_2 \rightarrow C^*$, with the pullback of the splitting over $0 \in \mathbb{A}^1$ given by the opposedness isomorphism.*

Proof. For $O(\mathbb{B}_\rho)$ as in Definition 3.32, we define a \mathbb{G}_m -action on the de Rham complex

$$\mathcal{A}^*(X, O(\mathbb{B}_\rho)) = \mathcal{A}^*(X, \mathbb{R}) \otimes_{\mathbb{R}} O(\mathbb{B}_\rho)$$

by taking the \diamond -action of \mathbb{G}_m on $\mathcal{A}^*(X, \mathbb{R})$, acting trivially on $O(\mathbb{B}_\rho)$.

There is an essentially unique harmonic metric on $O(\mathbb{B}_\rho)$, and we decompose the connection D as $D = d^+ + \vartheta$ into antisymmetric and symmetric parts, and let $D^c := i \diamond d^+ - i \diamond \vartheta$. Now, we define the DGA $\tilde{A}(X, O(\mathbb{B}_\rho))$ on C by

$$\tilde{A}^\bullet(X, O(\mathbb{B}_\rho)) := (A^*(X, O(\mathbb{B}_\rho)) \otimes_{\mathbb{R}} O(C), uD + vD^c),$$

and we denote the differential by $\tilde{D} := uD + vD^c$. Note that the \diamond -action of \mathbb{G}_m makes this \mathbb{G}_m -equivariant over C . Thus $\tilde{A}(X, O(\mathbb{B}_\rho)) \in \text{DGA}(\mathbb{C}, R \times \mathbb{G}_m)$. The construction is now the same as in Theorem 5.14, except that we only have a \mathbb{G}_m -action, rather than an \mathcal{S} -action. \square

Observe that this theorem easily adapts to multiple basepoints, as considered in Remark 4.9.

Corollary 6.2. *In the scenario of Theorem 6.1, the homotopy groups $\varpi_n(X^{\rho, \text{Mal}}, x)$ for $n \geq 2$, and the Hopf algebra $O(\varpi_1(X^{\rho, \text{Mal}}, x))$ carry natural ind-MTS, functorial in (X, x) , and compatible with the action of ϖ_1 on ϖ_n , the Whitehead bracket and the Hurewicz maps $\varpi_n(X^{\rho, \text{Mal}}, x) \rightarrow H^n(X, O(\mathbb{B}_\rho))^\vee$.*

Moreover, there are \mathcal{S} -linear isomorphisms

$$\begin{aligned} \varpi_n(X^{\rho, \text{Mal}}, x)^\vee \otimes \mathcal{S} &\cong \pi_n(H^*(X, O(\mathbb{B}_\rho)))^\vee \otimes \mathcal{S} \\ O(\varpi_1(X^{\rho, \text{Mal}}, x)) \otimes \mathcal{S} &\cong O(R \rtimes \pi_1(H^*(X, O(\mathbb{B}_\rho)))) \otimes \mathcal{S} \end{aligned}$$

of quasi-MTS. The associated graded map from the weight filtration is just the pullback of the standard isomorphism $\text{gr}_W \varpi_(X^{\rho, \text{Mal}}) \cong \pi_*(H^*(X, O(\mathbb{B}_\rho)))$.*

Proof. Theorem 6.1 provides the data required by Theorems 4.20 and 4.21 to construct \mathcal{S} -split ind-MTS on homotopy groups. \square

6.1. Unitary actions. Although we only have a mixed twistor structure (rather than a mixed Hodge structure) on general Malcev homotopy types, $\varpi_1(X, x)^{\text{red}}$ has a discrete unitary action, as in Lemma 5.7. We will extend this to a discrete unitary action on the mixed twistor structure. On some invariants, this action will become algebraic, and then we have a mixed Hodge structure as in Lemma 1.36.

For the remainder of this section, assume that R is any quotient of $\varpi_1(X, x)^{\text{red}}$ to which the action of the discrete group $(S^1)^\delta$ descends, but does not necessarily act algebraically, and let $\rho : \pi_1(X, x) \rightarrow R$ be the associated representation.

Proposition 6.3. *The mixed twistor structure $(X_{\text{MTS}}^{\rho, \text{Mal}}, x)$ of Theorem 6.1 is equipped with a $(S^1)^\delta$ -action, satisfying the properties of Lemma 1.36 (except algebraicity of the action). Moreover, there is a $(S^1)^\delta$ -action on $\text{gr}(X_{\text{MTS}}^{\rho, \text{Mal}}, 0)$, such that the $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant splitting*

$$\mathbb{A}^1 \times \text{gr}(X_{\text{MTS}}^{\rho, \text{Mal}}, 0) \times \text{SL}_2 \cong (X_{\text{MTS}}^{\rho, \text{Mal}}, x) \times_{\mathbb{C}^*, \text{row}_1}^{\mathbf{R}} \text{SL}_2$$

of Theorem 6.1 is also $(S^1)^\delta$ -equivariant.

Proof. Since $(S^1)^\delta$ acts on R , it acts on $O(\mathbb{B}_\rho)$, and we denote this action by $v \mapsto t \otimes v$, for $t \in (S^1)^\delta$. We may now adapt the proof of Theorem 5.14, defining the $(S^1)^\delta$ -action on $\mathcal{A}^*(X, \mathbb{R}) \otimes_{\mathbb{R}} O(\mathbb{B}_\rho)$ by setting $t \boxtimes (a \otimes v) := (t \diamond a) \otimes (t^2 \otimes v)$ for $t \in (S^1)^\delta$. \square

Remark 6.4. Note that taking $R = (\pi_1(X, x))_{\mathbb{R}}^{\text{red}}$ satisfies the conditions of the Proposition. Taking the fibre over $\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ of the \mathcal{S} -splitting from Theorem 6.1 gives the formality result of [KPT1], namely $X^{\rho, \text{Mal}} \cong X_{\mathbb{T}, (1, i)}^{\rho, \text{Mal}}$, since $-id + d^c = -2i\bar{\partial}$. Now, $(-i, 1)$ is not a stable point for the S -action, but has stabiliser $1 \times \mathbb{G}_{m, \mathbb{C}} \subset S_{\mathbb{C}}$. In [KPT1], it is effectively shown that this action of $\mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^*$ lifts to a discrete action on $X_{\mathbb{T}, (1, i)}^{\rho, \text{Mal}}$. From our algebraic \mathbb{G}_m -action and discrete S^1 -action on $X_{\mathbb{T}}^{\rho, \text{Mal}}$, we may recover the restriction of this action to $S^1 \subset \mathbb{C}^*$, with t^2 acting as the composition of $t \in \mathbb{G}_m(\mathbb{C})$ and $t \in S^1$.

Another type of Hodge structure defined on $X^{\rho, \text{Mal}}$ was the real Hodge structure (i.e. S -action) of [Pri1]. This corresponded to taking the fibre of the splitting over $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, giving an isomorphism $X^{\rho, \text{Mal}} \cong \text{gr} X_{\text{MTS}}^{\rho, \text{Mal}}$, and then considering the S -action on the latter. However, that Hodge structure was not in general compatible with the Hodge filtration.

Now, Proposition 6.3 implies that the mixed twistor structures on homotopy groups given in Theorem 4.20 have discrete S^1 -actions. By Lemma 1.36, we know that this will give a mixed Hodge structure whenever the S^1 -action is algebraic.

6.1.1. Evaluation maps. For a group Γ , let $\mathbb{S}(\Gamma)$ denote the category of Γ -representations in simplicial sets.

Definition 6.5. Given $X \in \mathbb{S}(R(A))$, define $\mathbf{C}^\bullet(X, \mathbb{O}(R) \otimes A) \in \text{cAlg}(R)$ by

$$\mathbf{C}^n(X, \mathbb{O}(R) \otimes A) := \text{Hom}_{R(A)}(X_n, A \otimes O(R)).$$

Lemma 6.6. *Given a real algebra A , the functor $s\text{Aff}_A(R) \rightarrow \mathbb{S}(R(A))$ given by $Y \mapsto Y(A)$ is right Quillen, with left adjoint $X \mapsto \text{Spec } \mathbf{C}^\bullet(X, \mathbb{O}(R) \otimes A)$.*

Proof. This is essentially the same as [Pri3, Lemma 3.52], which takes the case $A = \mathbb{R}$. \square

Recall from [GJ, Lemma VI.4.6] that there is a right Quillen equivalence $\text{holim}_{\rightarrow R(A)} \mathbb{S}(R(A)) \rightarrow \mathbb{S} \downarrow BR(A)$, with left adjoint given by the covering system functor $X \mapsto \tilde{X}$.

Definition 6.7. Given $f : X \rightarrow BR(A)$, define

$$C^\bullet(X, O(\mathbb{B}_f)) := C^\bullet(\tilde{X}, \mathbb{O}(R) \otimes A).$$

Lemma 6.8. *Given a real algebra A , the functor $s\text{Aff}_A(R) \rightarrow \mathbb{S} \downarrow BR(A)$ given by $Y \mapsto \text{holim}_{\rightarrow R(A)} Y(A)$ is right Quillen, with left adjoint*

$$(X \xrightarrow{f} BR(A)) \mapsto \text{Spec } C^\bullet(X, O(\mathbb{B}_f)).$$

Proof. The functor $s\text{Aff}(R) \rightarrow \mathbb{S}(R(A))$ given by $Y \mapsto Y(A)$ is right Quillen, with left adjoint as in Lemma 6.6. Composing this right Quillen functor with $\text{holim}_{\rightarrow R(A)}$ gives the right Quillen functor required. \square

6.1.2. *Continuity.*

Definition 6.9. Let $(S^1)^{\text{cts}}$ be the real affine scheme given by setting $O((S^1)^{\text{cts}})$ to be the ring of real-valued continuous functions on the circle.

Lemma 6.10. *There is a group homomorphism*

$$\sqrt{h} : \pi_1(X, x) \rightarrow R((S^1)^{\text{cts}}),$$

invariant with respect to the $(S^1)^\delta$ -action given by combining the actions on R and $(S^1)^{\text{cts}}$, such that $1^ \sqrt{h} = \rho : \pi_1(X, x) \rightarrow R(\mathbb{R})$, for $1 : \text{Spec } \mathbb{R} \rightarrow (S^1)^{\text{cts}}$.*

Proof. This is just the unitary action from Lemma 5.7, given on connections by $\sqrt{h}(t)(d^+, \vartheta) = (d^+, t \diamond \vartheta)$, for $t \in (S^1)$. By [Sim3, Theorem 7], the map $(S^1) \times \pi_1(X, x) \rightarrow R(\mathbb{R})$ is continuous, which is precisely the property we need. \square

Informally, this gives a continuity property of the discrete S^1 -action, and we now wish to show a similar continuity property for the $(S^1)^\delta$ -action on the mixed twistor structure $(X^{\rho, \text{Mal}}, x)_{\text{MTS}}$ of Proposition 6.3. Recalling that $X_{\mathbb{T}} = X_{\text{MTS}} \times_{\mathbb{A}^1}^{\mathbf{R}} \{1\}$, we want a continuous map

$$(X, x) \times S^1 \rightarrow \mathbf{R} \text{holim}_{\rightarrow R} (X^{\rho, \text{Mal}}, x)_{\mathbb{T}}$$

over C^* .

The following is essentially [Pri1, §3.3.2]:

Proposition 6.11. *For the $(S^1)^\delta$ -actions on $\text{gr} X_{\text{MTS}}^{\rho, \text{Mal}}$ of Proposition 6.3 and on $(S^1)^{\text{cts}}$, there is a $(S^1)^\delta$ -invariant map*

$$h \in \text{Hom}_{\text{Ho}(\mathbb{S}_0 \downarrow BR((S^1)^{\text{cts}}))}(\text{Sing}(X, x), \mathbf{R} \text{holim}_{\rightarrow R((S^1)^{\text{cts}})} (X, x)_{\mathbb{T}}^{\rho, \text{Mal}}((S^1)^{\text{cts}})_{C^*}),$$

extending the map $h : X \rightarrow BR((S^1)^{\text{cts}})$ corresponding to the group homomorphism $h : \pi_1(X, x) \rightarrow R((S^1)^{\text{cts}})$ given by $h(t) = \sqrt{h}(t^2)$, for \sqrt{h} as in Lemma 6.10 and $t \in S^1$. Here, $(X^{\rho, \text{Mal}}, x)_{\mathbb{T}}((S^1)^{\text{cts}})_{C^} := \text{Hom}_{C^*}((S^1)^{\text{cts}}, (X^{\rho, \text{Mal}}, x)_{\mathbb{T}})$.*

Moreover, for $1 : \text{Spec } \mathbb{R} \rightarrow (S^1)^{\text{cts}}$, the map

$$1^* h : \text{Sing}(X, x) \rightarrow (\mathbf{R} \text{holim}_{\rightarrow R(\mathbb{R})} (X^{\rho, \text{Mal}}, x)_{\mathbb{T}}((S^1)^{\text{cts}})_{C^*}) \times_{BR((S^1)^{\text{cts}})} BR(\mathbb{R})$$

in $\text{Ho}(\mathbb{S}_0 \downarrow BR(\mathbb{R}))$ is just the canonical map

$$\text{Sing}(X, x) \rightarrow \mathbf{R} \text{holim}_{\rightarrow R(\mathbb{R})} (X^{\rho, \text{Mal}}(\mathbb{R}), x).$$

Proof. By Lemma 6.8, this is equivalent to giving a $(S^1)^\delta$ -equivariant morphism

$$\mathrm{Spec} \mathbf{C}^\bullet(\mathrm{Sing}(X), O(\mathbb{B}_h)) \rightarrow (X^{\rho, \mathrm{Mal}}, x)_{\mathbb{T}} \times_{C^*}^{\mathbf{R}} (S^1)^{\mathrm{cts}}$$

in $\mathrm{Ho}((R \times (S^1)^{\mathrm{cts}}) \downarrow s\mathrm{Aff}_{(S^1)^{\mathrm{cts}}}(R))$, noting that for the trivial map $f : \{x\} \rightarrow BR((S^1)^{\mathrm{cts}})$, we have $\mathbf{C}^\bullet(\{x\}, O(\mathbb{B}_f)) = O(R) \otimes O((S^1)^{\mathrm{cts}})$, so $x \hookrightarrow X$ gives a map $R \times (S^1)^{\mathrm{cts}} \rightarrow \mathrm{Spec} \mathbf{C}^\bullet(\mathrm{Sing}(X), O(\mathbb{B}_h))$.

Now, the description of the S^1 -action in Lemma 6.10 shows that the local system $O(\mathbb{B}_h)$ on X has a resolution given by

$$(\mathcal{A}^*(X, O(\mathbb{B}_\rho)) \otimes_{\mathbb{R}} O((S^1)^{\mathrm{cts}}), d^+ + t^{-2} \diamond \vartheta),$$

for t the complex co-ordinate on S^1 , so $\mathbf{C}^\bullet(\mathrm{Sing}(X), O(\mathbb{B}_h)) \xrightarrow{x^*} O(R) \otimes O((S^1)^{\mathrm{cts}})$ is quasi-isomorphic to $E^\bullet \xrightarrow{x^*} O(R) \otimes O((S^1)^{\mathrm{cts}})$, where

$$E^\bullet := D(A^*(X, O(\mathbb{B}_\rho)) \otimes_{\mathbb{R}} O((S^1)^{\mathrm{cts}}), d^+ + t^{-2} \diamond \vartheta),$$

for D the denormalisation functor.

Now, $O(S^1)$ is the quotient of $O(S)$ given by $\mathbb{R}[u, v]/(u^2 + v^2 - 1)$, where $t = u + iv$, and then

$$uD + vD^c = t \diamond d^+ + \bar{t} \diamond \vartheta = t \diamond (d^+ + t^{-2} \diamond \vartheta).$$

Thus $t \diamond$ gives a $(S^1)^\delta$ -equivariant quasi-isomorphism from $R \times (S^1)^{\mathrm{cts}} \xrightarrow{x} \mathrm{Spec} E^\bullet$ to $(X_{\mathbb{T}}, x)^{\rho, \mathrm{Mal}} \times_{C^*} (S^1)^{\mathrm{cts}}$, as required. \square

Corollary 6.12. *For all n , the map $\pi_n(X, x) \times S^1 \rightarrow \varpi_n(X^{\rho, \mathrm{Mal}}, x)_{\mathbb{T}}$, given by composing the map $\pi_n(X, x) \rightarrow \varpi_n(X^{\rho, \mathrm{Mal}}, x)$ with the $(S^1)^\delta$ -action on $(X^{\rho, \mathrm{Mal}}, x)_{\mathbb{T}}$, is continuous.*

Proof. Proposition 6.11 gives a $(S^1)^\delta$ -invariant map

$$\pi_n(h) : \pi_n(X, x) \rightarrow \pi_n(\mathbf{R} \mathop{\mathrm{holim}}_{R((S^1)^{\mathrm{cts}})} (X^{\rho, \mathrm{Mal}}, x)_{\mathbb{T}}((S^1)^{\mathrm{cts}})_{C^*}).$$

It therefore suffices to prove that

$$\pi_n(\mathbf{R} \mathop{\mathrm{holim}}_{R((S^1)^{\mathrm{cts}})} (X^{\rho, \mathrm{Mal}}, x)_{\mathbb{T}}((S^1)^{\mathrm{cts}})_{C^*}) = \varpi_n(X^{\rho, \mathrm{Mal}}, x)_{\mathbb{T}}((S^1)^{\mathrm{cts}})_{C^*}.$$

Observe that the morphism $S \rightarrow C^*$ factors through $\mathrm{row}_1 : \mathrm{SL}_2 \rightarrow C^*$, via the map $S \rightarrow \mathrm{SL}_2$ given by the S -action on the identity matrix. This gives us a factorisation of $(S^1)^{\mathrm{cts}} \rightarrow C^*$ through SL_2 , using the maps $(S^1)^{\mathrm{cts}} \rightarrow S^1 \subset S$. It thus gives a morphism $(S^1)^{\mathrm{cts}} \rightarrow \mathrm{Spec} \mathbf{R}O(C^*)$, so the SL_2 -splitting of Theorem 6.1 gives an equivalence

$$(X^{\rho, \mathrm{Mal}}, x)_{\mathbb{T}} \times_{C^*}^{\mathbf{R}} (S^1)^{\mathrm{cts}} \simeq (\underline{\mathrm{gr}} X_{\mathrm{MTS}}^{\rho, \mathrm{Mal}}, 0) \times (S^1)^{\mathrm{cts}}.$$

Similarly, we may pull back the grouplike MTS $G(X, x)_{\mathrm{MTS}}^{\rho, \mathrm{Mal}}$ from Proposition 4.19 to a dg pro-algebraic group over $(S^1)^{\mathrm{cts}}$, and the SL_2 -splitting then gives us an isomorphism

$$\varpi_n(X, x)_{\mathbb{T}}^{\rho, \mathrm{Mal}} \times_{C^*} (S^1)^{\mathrm{cts}} \cong \varpi_n(\underline{\mathrm{gr}} X_{\mathrm{MTS}}^{\rho, \mathrm{Mal}}, 0) \times (S^1)^{\mathrm{cts}},$$

compatible with the equivalence above.

Thus it remains only to show that

$$\pi_n(\mathbf{R} \mathop{\mathrm{holim}}_{R((S^1)^{\mathrm{cts}})} (\underline{\mathrm{gr}} X_{\mathrm{MTS}}^{\rho, \mathrm{Mal}}, 0)((S^1)^{\mathrm{cts}})) = \varpi_n(\underline{\mathrm{gr}} X_{\mathrm{MTS}}^{\rho, \mathrm{Mal}}, 0)((S^1)^{\mathrm{cts}}).$$

Now, write $\underline{\mathrm{gr}} X_{\mathrm{MTS}}^{\rho, \mathrm{Mal}} \simeq \bar{W}N\mathfrak{g}$ under the equivalences of Theorem 3.29, for $\mathfrak{g} \in s\hat{\mathcal{N}}(R)$. By [Pri3, Lemma 3.53], the left-hand side becomes $\pi_n(\bar{W}(R \times \exp(\mathfrak{g}))((S^1)^{\mathrm{cts}}))$, which is just $\pi_{n-1}((R \times \exp(\mathfrak{g}))((S^1)^{\mathrm{cts}}))$, giving $(R \times \exp(\pi_0 \mathfrak{g}))((S^1)^{\mathrm{cts}})$ for $n = 1$, and $(\pi_{n-1} \mathfrak{g})((S^1)^{\mathrm{cts}})$ for $n \geq 2$. Meanwhile, the right-hand side is $(R \times \exp(H_0 N \mathfrak{g}))((S^1)^{\mathrm{cts}})$ for $n = 1$, and $(H_{n-1} N \mathfrak{g})((S^1)^{\mathrm{cts}})$ for $n \geq 2$. Thus the required isomorphism follows from the Dold-Kan correspondence. \square

Hence (for R any quotient of $(\varpi_1(X, x))_{\mathbb{R}}^{\text{red}}$ to which the $(S^1)^\delta$ -action descends), we have:

Corollary 6.13. *If the group $\varpi_n(X, x)^{\rho, \text{Mal}}$ is finite-dimensional and spanned by the image of $\pi_n(X, x)$, then the former carries a natural \mathcal{S} -split mixed Hodge structure, which extends the mixed twistor structure of Corollary 6.2. This is functorial in (X, x) and compatible with the action of ϖ_1 on ϖ_n , the Whitehead bracket, the R -action, and the Hurewicz maps $\varpi_n(X, x)^{\rho, \text{Mal}} \rightarrow \mathbb{H}^n(X, O(\mathbb{B}_\rho))^\vee$.*

Proof. The splittings of Theorem 4.21 and Proposition 6.3 combine with Corollary 6.12 to show that the map

$$\pi_n(X, x) \times S^1 \rightarrow \varpi_n(\underline{\text{gr}}X_{\text{MTS}}^{\rho, \text{Mal}}, 0)$$

is continuous. Since the splitting also gives an isomorphism $\varpi_n(\underline{\text{gr}}X_{\text{MTS}}^{\rho, \text{Mal}}, 0) \cong \varpi_n(X, x)^{\rho, \text{Mal}}$, we deduce that $\pi_n(X, x)$ spans $\varpi_n(\underline{\text{gr}}X_{\text{MTS}}^{\rho, \text{Mal}}, 0)$, so the S^1 action on $\varpi_n(\underline{\text{gr}}X_{\text{MTS}}^{\rho, \text{Mal}}, 0)$ is continuous.

Since any finite-dimensional continuous S^1 -action is algebraic, this gives us an algebraic S^1 -action on $\varpi_n(\underline{\text{gr}}X_{\text{MTS}}^{\rho, \text{Mal}}, 0)$. Retracing our steps through the splitting isomorphisms, this implies that the S^1 -action on $\varpi_n(X, x)^{\rho, \text{Mal}}$ is algebraic. As in Lemma 1.36, this gives an algebraic $\mathbb{G}_m \times S$ -action on $\text{row}_1^* \varpi_n(X_{\text{MTS}}^{\rho, \text{Mal}}, x)$, so we have a mixed Hodge structure. That this is \mathcal{S} -split follows from Proposition 6.3, since the \mathcal{S} -splitting of the MTS in Corollary 6.2 is S^1 -equivariant. \square

Remark 6.14. Observe that if $\pi_1(X, x)$ is algebraically good with respect to R and the homotopy groups $\pi_n(X, x)$ have finite rank for all $n \geq 2$, with the local system $\pi_n(X, -) \otimes_{\mathbb{Z}} \mathbb{R}$ an extension of R -representations, then Theorem 3.17 implies that $\varpi_n(X^{\rho, \text{Mal}}, x) \cong \pi_n(X, x) \otimes \mathbb{R}$, ensuring that the hypotheses of Corollary 6.13 are satisfied.

7. VARIATIONS OF MIXED HODGE AND MIXED TWISTOR STRUCTURES

Fix a compact Kähler manifold X .

Definition 7.1. Define the sheaf $\tilde{\mathcal{A}}^\bullet(X)$ of DGAs on $X \times C$ by

$$\tilde{\mathcal{A}}^\bullet = (\mathcal{A}^* \otimes_{\mathbb{R}} O(C), ud + vd^c),$$

for co-ordinates u, v as in Remark 1.3. We denote the differential by $\tilde{d} := ud + vd^c$. Note that $\Gamma(X, \tilde{\mathcal{A}}^\bullet) = \tilde{A}^\bullet(X)$, as given in Definition 2.1.

Definition 7.2. Define a real \mathcal{C}^∞ family of mixed Hodge (resp. mixed twistor) structures \mathcal{E} on X to be of a finite locally free S -equivariant (resp. \mathbb{G}_m -equivariant) $j^{-1}\tilde{\mathcal{A}}_X^0$ -sheaf on $X \times C^*$ equipped with a finite increasing filtration $W_i \mathcal{E}$ by locally free S -equivariant (resp. \mathbb{G}_m -equivariant) subbundles such that for all $x \in X$, the pullback of \mathcal{E} to x corresponds under Proposition 1.41 to a mixed Hodge structure (resp. corresponds under Corollary 1.49 to a mixed twistor structure).

Lemma 7.3. *A (real) variation of mixed Hodge structures (in the sense of [SZ]) on X is equivalent to a real \mathcal{C}^∞ family of mixed twistor structures \mathcal{E} on X , equipped with a flat S -equivariant \tilde{d} -connection*

$$\tilde{D} : \mathcal{E} \rightarrow \mathcal{E} \otimes_{j^{-1}\tilde{\mathcal{A}}_X^0} j^{-1}\tilde{\mathcal{A}}_X^1,$$

compatible with the filtration W .

Proof. Given a real VMHS \mathbb{V} , we obtain a \mathcal{C}^∞ family $\mathcal{E} := \xi(\mathbb{V} \otimes \mathcal{A}^0, \mathbb{F})$ of mixed Hodge structures (in the notation of Corollary 1.9), and the connection $D : \mathbb{V} \otimes \mathcal{A}^0 \rightarrow \mathbb{V} \otimes \mathcal{A}^1$ gives $\tilde{D} = \xi(D, \mathbb{F})$. S -equivariance of \tilde{D} is equivalent to the condition

$$D : F^p(\mathbb{V} \otimes \mathcal{A}^0 \otimes \mathbb{C}) \rightarrow F^p(\mathbb{V} \otimes \mathcal{A}^0 \otimes \mathbb{C}) \otimes_{\mathcal{A}_X^0 \otimes \mathbb{C}} \mathcal{A}^{01} \oplus F^{p-1}(\mathbb{V} \otimes \mathcal{A}^0 \otimes \mathbb{C}) \otimes_{\mathcal{A}_X^0 \otimes \mathbb{C}} \mathcal{A}^{10},$$

corresponding to a Hodge filtration on $\mathbb{V} \otimes \mathcal{O}_X$, with $D : F^p(\mathbb{V} \otimes \mathcal{O}_X) \rightarrow F^{p-1}(\mathbb{V} \otimes \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_X$. \square

Definition 7.4. Adapting [Sim2, §1] from complex to real structures, we define a (real) variation of mixed twistor structures (or VMTS) on X to consist of a real \mathcal{C}^∞ family of mixed twistor structures \mathcal{E} on X , equipped with a flat \mathbb{G}_m -equivariant \tilde{d} -connection

$$D : \mathcal{E} \rightarrow \mathcal{E} \otimes_{j^{-1}\tilde{\mathcal{A}}^0} j^{-1}\tilde{\mathcal{A}}^1,$$

compatible with the filtration W .

Definition 7.5. Given an ind-MHS (resp. ind-MTS) structure on a Hopf algebra $O(\Pi)$, define an MHS (resp. MTS) representation of G to consist of a MHS (resp. MTS) V , together with a morphism

$$V \rightarrow V \otimes O(\Pi)$$

of ind-MHS (resp. ind-MTS), co-associative with respect to the Hopf algebra comultiplication.

Fix a representation $\rho : \pi_1(X, x) \rightarrow R$ as in Theorem 5.14.

Theorem 7.6. For $\varrho : \pi_1(X, x) \rightarrow {}^{\text{VHS}}\varpi_1(X, x)$ (resp. $\varrho : \pi_1(X, x) \rightarrow \varpi_1(X, x)^{\text{red}}$) the category of MHS (resp. MTS) representations of $\varpi_1(X^{\rho, \text{Mal}}, x)$ is equivalent to the category of real variations of mixed Hodge structure (resp. variations of mixed twistor structure) on X . Under this equivalence, the forgetful functor to real MHS (resp. MTS) sends a real VMHS (resp. VMTS) \mathbb{V} to \mathbb{V}_x .

For R any quotient of ${}^{\text{VHS}}\varpi_1(X, x)$ (resp. $\varpi_1(X, x)^{\text{red}}$) and $\rho : \pi_1(X, x) \rightarrow R$, MHS (resp. MTS) representations of $\varpi_1(X^{\rho, \text{Mal}}, x)$ correspond to real VMHS (resp. VMTS) \mathbb{V} whose underlying local systems are extensions of R -representations.

Proof. We will prove this for VMHS. The proof for VMTS is almost identical, replacing S with \mathbb{G}_m , and Proposition 1.41 with Proposition 1.49.

Given an MHS representation $\psi : \varpi_1(X^{\rho, \text{Mal}}, x) \rightarrow \text{GL}(V)$ for an MHS V , let S_0V be the maximal semisimple subrepresentation of V , and define the increasing filtration S_iV inductively by the property that $(S_iV/S_{i-1}V)$ is the maximal semisimple subrepresentation of $V/(S_{i-1}V)$. Then let $\text{GL}^S(V) \leq \text{GL}(V)$ consist of automorphisms respecting the filtration S . Then ψ induces a morphism $R \rightarrow \prod_i \text{GL}(\text{gr}_i^S V) = \text{GL}^S(V)^{\text{red}}$, and we set $G = \text{GL}^S(V) \times_{\text{GL}^S(V)^{\text{red}}} R$. The Hopf algebra $O(G)$ then inherits an ind-MHS structure from V , and $U := \ker(G \rightarrow R)$ is the matrix group $I + S_{-1}\text{End}(V)$.

The \mathcal{S} -splitting of $\varpi_1(X^{\rho, \text{Mal}}, x)$ gives a section $R \times \text{Spec } \mathcal{S} \rightarrow \varpi_1(X^{\rho, \text{Mal}}, x)$ compatible with the ind-MHS, which combines with ψ to give a section $\sigma_G : R \times \text{Spec } \mathcal{S} \rightarrow G$. As in §4.4, this gives rise to $\gamma_G \in \gamma^0(S_{-1}\text{End}(V) \hat{\otimes} \mathcal{S}(-1))$ with $\sigma_G + N\sigma_G \epsilon = \text{ad}_{1+\gamma_G \epsilon} \circ \sigma_G$. If we set

$$V' := \ker(\text{id} \otimes N - \gamma_G : V \otimes \mathcal{S} \rightarrow V \otimes \mathcal{S}(-1)),$$

then it follows that V' is a real R -representation, with $V' \rightarrow V' \otimes O(R)$ a morphism of quasi-MHS. Since γ_G is nilpotent, $V \otimes \mathcal{S} \cong V' \otimes \mathcal{S}$ and $\text{gr}^S V = \text{gr}^S V'$. Since $O(\varpi_1(X^{\rho, \text{Mal}}, x))$ is of non-negative weights, with $O(\varpi_1(X^{\rho, \text{Mal}}, x)^{\text{red}}) = O(R)$ of weight 0, this also implies $\text{gr}^W V \cong \text{gr}^W V'$. Thus V' is a MHS, and V' is an MHS representation of R .

Proposition 1.41 then associates to V' a locally free $\mathcal{O}_{\mathbb{A}^1} \otimes \mathcal{O}_{C^*}$ -module \mathbb{E}' on $X \times \mathbb{A}^1 \times C^*$, equipped with a $\mathbb{G}_m \times S$ -action on $\mathbb{E}' \otimes \mathcal{A}_X^0$, compatible with the $O(\mathbb{A}^1) \otimes O(\text{SL}_2)$ -multiplication. The fibre at $(1, 1) \in \mathbb{A}^1 \times C^*$ is the local system \mathbb{V}' associated to the

R -representation V' . If $D' = \text{id} \otimes d : \mathbb{E}' \otimes \mathcal{A}_X^0 \rightarrow \mathbb{E}' \otimes \mathcal{A}_X^1$ is the associated connection, then by Proposition 5.12 the element $(s, \lambda) \in \mathbb{G}_m(R) \times \mathcal{S}(\mathbb{R})$ sends D' to $\frac{\lambda}{\bar{\lambda}} \otimes D'$.

Proposition 1.41 then gives a MHS on the \mathcal{C}^∞ -family $\mathcal{V}' := \mathbb{V}' \otimes \mathcal{A}_X^0$, with $\mathbb{E}' \otimes \mathcal{A}_X^0 = \xi(\mathcal{V}', \text{MHS})$, and the connection $D' : \mathcal{V}' \rightarrow \mathcal{V}' \otimes_{\mathcal{A}_X^0} \mathcal{A}_X^1$ preserves W , with

$$\begin{aligned} \partial : F^p \bar{F}^q(\mathcal{V}' \otimes \mathbb{C}) &\rightarrow \mathcal{A}_X^{10} \otimes F^p \bar{F}^q(\mathcal{V}' \otimes \mathbb{C}), \\ \theta : F^p \bar{F}^q(\mathcal{V}' \otimes \mathbb{C}) &\rightarrow \mathcal{A}_X^{10} \otimes F^{p-1} \bar{F}^{q+1}(\mathcal{V}' \otimes \mathbb{C}). \end{aligned}$$

This implies that the filtration F descends to $\mathbb{V}' \otimes \mathcal{O}_X = \ker(\bar{\partial} + \bar{\theta})$, that the connection D' satisfies Griffiths transversality, and that $\text{gr}^W \mathbb{V}'$ is a VHS. Thus \mathbb{V}' is a semisimple VMHS on X .

We next put a quasi-MHS on the DG Lie algebra

$$A^\bullet(X, S_{-1} \text{End}(\mathbb{V}')),$$

with weight filtration and Hodge filtration given by

$$\begin{aligned} W_n A^\bullet(X, S_{-1} \text{End}(\mathbb{V}')) &= \sum_{i+j=n} \tau^{\leq i} A^\bullet(X, W_j S_{-1} \text{End}(\mathbb{V}')) \\ F^p A^m(X, S_{-1} \text{End}(\mathbb{V}') \otimes \mathbb{C}) &= \sum_{i+j=p} A^{i, m-i}(X, F^j S_{-1} \text{End}(\mathbb{V}') \otimes \mathbb{C}). \end{aligned}$$

Now, the derivation

$$\alpha_G = \text{id} \otimes N - [\gamma_G, -] : S_{-1} \text{End}(V) \otimes \mathcal{S} \rightarrow S_{-1} \text{End}(V) \otimes \mathcal{S}(-1)$$

corresponds under the isomorphism $V \otimes \mathcal{S} \cong V' \otimes \mathcal{S}$ to the derivation

$$\text{id} \otimes N : S_{-1} \text{End}(V') \otimes \mathcal{S} \rightarrow S_{-1} \text{End}(V') \otimes \mathcal{S}(-1),$$

so the morphism of Deligne groupoids from Proposition 4.26 is

$$\mathfrak{Dcl}(W_0 \gamma^0(A^\bullet(X, S_{-1} \text{End}(\mathbb{V}')) \otimes (\mathcal{S} \xrightarrow{N} \mathcal{S}(-1)))) \rightarrow \mathfrak{Dcl}(W_0 \gamma^0(S_{-1} \text{End}(V') \otimes (\mathcal{S} \xrightarrow{N} \mathcal{S}(-1)))).$$

Objects of the first groupoid are elements

$$(\omega, \eta) \in \gamma^0 A^1(X, W_{-1} S_{-1} \text{End}(\mathbb{V}') \otimes \mathcal{S}) \times \gamma^0 A^0(X, W_0 S_{-1} \text{End}(\mathbb{V}') \otimes \mathcal{S})$$

satisfying $[D', \omega] + \omega^2 = 0$, $[D' + \omega, \eta] + N\omega = 0$. This is equivalent to giving a $(d + N)$ -connection

$$D = (\text{id} \otimes d \otimes \text{id}) + (\text{id} \otimes \text{id} \otimes N) + \omega + \eta : \mathbb{V}' \otimes \mathcal{A}^0 \otimes \mathcal{S} \rightarrow \mathbb{V}' \otimes \mathcal{A}^1 \otimes \mathcal{S} \oplus \mathbb{V}' \otimes \mathcal{A}^0 \otimes \mathcal{S}(-1)$$

with the composite

$$D^2 : \mathbb{V}' \otimes \mathcal{A}^0 \otimes \mathcal{S} \rightarrow \mathbb{V}' \otimes \mathcal{A}^2 \otimes \mathcal{S} \oplus \mathbb{V}' \otimes \mathcal{A}^1 \otimes \mathcal{S}(-1)$$

vanishing. If we let $\mathbb{V} = \ker D$, then this gives $\mathbb{V} \otimes \mathcal{A}^0 \otimes \mathcal{S} \cong \mathbb{V}' \otimes \mathcal{A}^0 \otimes \mathcal{S}$, with $\text{gr}^W \mathbb{V} = \text{gr}^W \mathbb{V}'$, so it follows that \mathbb{V} is a VMHS.

A morphism in the second groupoid from $x^*(\omega, \eta)$ to γ_G is

$$g \in \text{id} + W_0 \gamma^0(S_{-1} \text{End}(V') \otimes \mathcal{S})$$

with the property that $g D_x g^{-1} = \alpha_G + \gamma_g$. Since $V = \ker(\alpha + \gamma_G : V' \otimes \mathcal{S} \rightarrow V' \otimes \mathcal{S}(-1))$, this means that g is an isomorphism $\mathbb{V}_x \rightarrow V$ of MHS.

Thus the MHS representation V gives rise to a VMHS \mathbb{V} equipped with an isomorphism $\mathbb{V}_x \cong V$ of MHS.

Conversely, given a VMHS \mathbb{V} with $\mathbb{V}_x = V$, let \mathbb{V}' be its semisimplification. Since \mathbb{V}' is a semisimple VMHS, the corresponding R -representation on $V' = \mathbb{V}'_x$ is an MHS representation, giving $\sigma : R \times \text{Spec } \mathcal{S} \rightarrow \text{GL}(V')$. We may then adapt Proposition 1.26 to get an isomorphism $\mathbb{V}' \otimes \mathcal{A}_X^0 \otimes \mathcal{S} \cong \mathbb{V}' \otimes \mathcal{A}_X^0 \otimes \mathcal{S}$ of \mathcal{C}^∞ -families of quasi-MHS, since \mathcal{A}_X^0 is flabby. We may therefore consider the difference

$$D - D' : \mathbb{V}' \otimes \mathcal{A}_X^0 \otimes \mathcal{S} \rightarrow \mathbb{V}' \otimes \mathcal{A}^1 \otimes \mathcal{S} \oplus \mathbb{V}' \otimes \mathcal{A}^0 \otimes \mathcal{S}(-1)$$

between the $(d + N)$ -connections associated to \mathbb{V} and vv' . We may now reverse the argument above to show that this gives an object of the Deligne groupoid, and hence an MHS representation $\varpi_1(X^{\rho, \text{Mal}}, x) \rightarrow \text{GL}(V)$. \square

For ρ as in Theorem 7.6, we now have the following.

Corollary 7.7. *There is a canonical algebra $\mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})$ in ind-VMHS (resp. ind-VMTS) on $X \times X$, with $\mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})_{x,x} = O(\varpi_1(X^{\rho, \text{Mal}}, x))$. This has a comultiplication*

$$\text{pr}_{13}^{-1} \mathbb{O}(\varpi_1 X^{\rho, \text{Mal}}) \rightarrow \text{pr}_{12}^{-1} \mathbb{O}(\varpi_1 X^{\rho, \text{Mal}}) \otimes \text{pr}_{23}^{-1} \mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})$$

on $X \times X \times X$, a co-identity $\Delta^{-1} \mathbb{O}(\varpi_1 X^{\rho, \text{Mal}}) \rightarrow \mathbb{R}$ on X (where $\Delta(x) = (x, x)$) and a co-inverse $\tau^{-1} \mathbb{O}(\varpi_1 X^{\rho, \text{Mal}}) \rightarrow \mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})$ (where $\tau(x, y) = (y, x)$), all of which are morphisms of algebras in ind-VMHS (resp. ind-VMTS).

There are canonical ind-VMHS (resp. ind-VMTS) $\Pi^n(X^{\rho, \text{Mal}})$ on X for all $n \geq 2$, with $\Pi^n(X^{\rho, \text{Mal}})_x = \varpi_n(X^{\rho, \text{Mal}}, x)^\vee$.

Proof. The left and right actions of $\varpi_1(X^{\rho, \text{Mal}}, x)$ on itself make $O(\varpi_1(X^{\rho, \text{Mal}}, x))$ into an ind-MHS (resp. ind-MTS) representation of $\varpi_1(X^{\rho, \text{Mal}}, x)^2$, so it corresponds under Theorem 7.6 to an ind-VMHS (resp. ind-VMTS) $\mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})$ with the required properties. Theorem 5.16 makes $\varpi_n(X^{\rho, \text{Mal}}, x)^\vee$ into an ind-MHS-representation of $\varpi_1(X^{\rho, \text{Mal}}, x)$, giving $\Pi^n(X^{\rho, \text{Mal}})$. \square

Note that for any VMHS (resp. VMTS) \mathbb{V} , this means that we have a canonical morphism $\text{pr}_2^{-1} \mathbb{V} \rightarrow \text{pr}_1^{-1} \mathbb{V} \otimes \mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})$ of ind-VMHS (resp. ind-VMTS) on $X \times X$, for ρ as in Theorem 7.6.

Remark 7.8. Using Remarks 4.18 and 5.17, we can adapt Theorem 7.6 to any MHS/MTS representation V of the groupoid $\varpi_1(X^{\rho, \text{Mal}}; T)$ with several basepoints (i.e. require that $V(x) \rightarrow O(\varpi_1(X^{\rho, \text{Mal}}; x, y)) \otimes V(y)$ be a morphism of ind-MHS/MTS). This gives a VMHS/VMTS \mathbb{V} , with canonical isomorphisms $\mathbb{V}_x \cong V(x)$ of MHS/MTS for all $x \in T$.

Corollary 7.7 then adapts to multiple basepoints, since there is a natural representation of $\varpi_1(X^{\rho, \text{Mal}}; T) \times \varpi_1(X^{\rho, \text{Mal}}; T)$ given by $(x, y) \mapsto O(\varpi_1(X^{\rho, \text{Mal}}; x, y))$. This gives a canonical Hopf algebra $\mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})$ in ind-VMHS/VMTS on $X \times X$, with $\mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})_{x,y} = O(\varpi_1(X^{\rho, \text{Mal}}; x, y))$ for all $x, y \in T$. Since this construction is functorial for sets of basepoints, we deduce that this is the VMHS /VMTS $\mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})$ of Corollary 7.7 (which is therefore independent of the basepoint x). This generalises [Hai4, Corollary 13.11] (which takes $R = \text{GL}(\mathbb{V}_x)$ for a VHS \mathbb{V}).

Likewise, the representation $x \mapsto \varpi_n(X^{\rho, \text{Mal}}, x)^\vee$ of $\varpi_1(X^{\rho, \text{Mal}}; T)$ gives an ind-VMHS/VMTS $\Pi^n(X^{\rho, \text{Mal}})$ (independent of x) on X with $\Pi^n(X^{\rho, \text{Mal}})_x = \varpi_n(X^{\rho, \text{Mal}}, x)^\vee$, for all $x \in X$.

Remark 7.9. [Ara] introduces a quotient $\varpi_1(X, x)_k^{\text{alg}} \rightarrow \pi_1(X, x)_k^{\text{hodge}}$ over any field $k \subset \mathbb{R}$, characterised by the property that representations of $\pi_1(X, x)_k^{\text{hodge}}$ correspond to local systems underlying k -VMHS on X .

Over any field $k \subset \mathbb{R}$, there is a pro-algebraic group MT_k over k , whose representations correspond to mixed Hodge structures over k . If $\rho : \varpi_1(X, x)_k^{\text{alg}} \rightarrow {}^{\text{VHS}}\varpi_1(X, x)_k$ is the largest quotient of the k -pro-algebraic completion with the property that the surjection $\varpi_1(X, x)_k^{\text{alg}} \rightarrow \varpi_1(X, x)_k^{\text{alg}} \otimes_k \mathbb{R}$ factors through ${}^{\text{VHS}}\varpi_1(X, x)_k$, then Theorem 5.16 and Remark 5.18 give an algebraic action of MT_k on $\varpi_1(X, x)_k^{\rho, \text{Mal}}$, with representations of $\varpi_1(X, x)_k^{\rho, \text{Mal}} \rtimes \text{MT}_k$ being representations of $\varpi_1(X, x)_k^{\rho, \text{Mal}}$ in k -MHS. Theorem 7.6 implies that these are precisely k -VMHS on X , so [Ara, Lemma 2.8] implies that $\varpi_1(X, x)_k^{\rho, \text{Mal}} = \pi_1(X, x)_k^{\text{hodge}}$.

For any quotient $\rho' : {}^{\text{VHS}}\varpi_1(X, x)_k \rightarrow R$ (in particular if R is the image of the monodromy representation of a k -VHS), Theorem 7.6 then implies that $\varpi_1(X, x)^{\rho', \text{Mal}}$ is a quotient of $\pi_1(X, x)_k^{\text{hodge}}$, proving the second part of [Ara, Conjecture 5.5].

Note that this also implies that if \mathbb{V} is a local system on X whose semisimplification \mathbb{V}^{ss} underlies a VHS, then \mathbb{V} underlies a VMHS (which need not be compatible with the VHS on \mathbb{V}^{ss}).

Example 7.10. One application of the ind-VMHS on $\mathbb{O}(\varpi_1 X^{\rho, \text{Mal}})$ from Corollary 7.7 is to look at deformations of the representation associated to a VHS \mathbb{V} . Explicitly, \mathbb{V} gives representations $\rho_x : {}^{\text{VHS}}\varpi_1(X, x) \rightarrow \text{GL}(\mathbb{V}_x)$ for all $x \in X$, and for any Artinian local \mathbb{R} -algebra A with residue field \mathbb{R} , we consider the formal scheme F_{ρ_x} given by

$$F_{\rho_x}(A) = \text{Hom}(\pi_1(X, x), \text{GL}(\mathbb{V}_x \otimes A)) \times_{\text{Hom}(\pi_1(X, x), \text{GL}(\mathbb{V}_x))} \{\rho_x\}.$$

Now, $\text{GL}(\mathbb{V}_x \otimes A) = \text{GL}(\mathbb{V}_x) \ltimes \exp(\mathfrak{gl}(\mathbb{V}_x) \otimes \mathfrak{m}(A))$, where $\mathfrak{m}(A)$ is the maximal ideal of A . If $R(x)$ is the image of ρ_x , and $\rho'_x : {}^{\text{VHS}}\varpi_1(X, x) \rightarrow R(x)$ is the induced morphism, then

$$F_{\rho_x}(A) = \text{Hom}(\varpi_1(X, x)^{\rho'_x, \text{Mal}}, R(x) \ltimes \exp(\mathfrak{gl}(\mathbb{V}_x) \otimes \mathfrak{m}(A)))_{\rho_x}.$$

Thus F_{ρ_x} is a formal subscheme contained in the germ at 0 of $\mathcal{O}(\varpi_1(X, x)^{\rho'_x, \text{Mal}}) \otimes \mathfrak{gl}(\mathbb{V}_x)$, defined by the conditions

$$f(a \cdot b) = f(a) \star (\text{ad}_{\rho'_x(a)}(f(b)))$$

for $a, b \in \varpi_1(X, x)^{\rho'_x, \text{Mal}}$, where \star is the Campbell–Baker–Hausdorff product $a \star b = \log(\exp(a) \cdot \exp(b))$.

Those same conditions define a family $\mathbb{F}(\rho)$ on X of formal subschemes contained in $(\Delta^{-1}\mathbb{O}(\varpi_1 X^{\rho', \text{Mal}})) \otimes \mathfrak{gl}(\mathbb{V})$, with $\mathbb{F}(\rho)_x = F_{\rho_x}$. If $\mathbb{F} = \text{Spf } \mathcal{B}$, the VMHS on $\mathbb{O}(\varpi_1 X^{\rho', \text{Mal}})$ and \mathbb{V} then give \mathcal{B} the natural structure of a (pro-Artinian algebra in) pro-VMHS. This generalises [ES] to real representations, and also adapts easily to S -equivariant representations in more general groups than GL_n . Likewise, if we took \mathbb{V} to be any variation of twistor structures, the same argument would make \mathcal{B} a pro-VMTS.

7.1. Enriching VMTS. Say we have some quotient R of $\varpi_1(X, x)^{\text{red}}$ to which the action of the discrete group $(S^1)^\delta$ descends, but does not necessarily act algebraically, and let $\rho : \pi_1(X, x) \rightarrow R$ be the associated representation. Corollary 6.2 puts an ind-MTS on the Hopf algebra $\mathcal{O}(\varpi_1(X, x)^{\rho, \text{Mal}})$, and Proposition 6.3 puts a $(S^1)^\delta$ action on $\xi(\mathcal{O}(\varpi_1(X, x)^{\rho, \text{Mal}}), \text{MTS})$, satisfying the conditions of Lemma 1.36.

Now take an MHS V , and assume that we have an MTS representation $\varpi_1(X, x)^{\rho, \text{Mal}} \rightarrow \text{GL}(V)$, with the additional property that the corresponding morphism

$$\xi(V, \text{MHS}) \rightarrow \xi(V, \text{MHS}) \otimes \xi(\mathcal{O}(\varpi_1(X, x)^{\rho, \text{Mal}}), \text{MTS})$$

of ind-MTS is equivariant for the $(S^1)^\delta$ -action.

Now, $\text{gr}_n^W V$ is an MTS representation of $\text{gr}_0^W \varpi_1(X, x)^{\rho, \text{Mal}} = R$, giving a $(S^1)^\delta$ -equivariant map

$$\text{gr}_n^W V \rightarrow \text{gr}_n^W V \otimes \mathcal{O}(R).$$

If \mathbb{V} is the local system associated to V , then this is equivalent to giving a compatible system of isomorphisms $\text{gr}_n^W \mathbb{V} \cong t \otimes \text{gr}_n^W \mathbb{V}$ for $t \in S^1$. Therefore Proposition 5.12 implies that $\text{gr}^W V$ is a representation of ${}^{\text{VHS}}\varpi_1(X, x)$. Letting R' be the largest common quotient of R and ${}^{\text{VHS}}\varpi_1(X, x)$, this means that $\text{gr}^W V$ is an R' -representation, so V is a representation of $\varpi_1(X, x)^{\rho', \text{Mal}}$, for $\rho' : \pi_1(X, x) \rightarrow R'$.

Then we have a S^1 -equivariant morphism

$$\xi(V, \text{MHS}) \rightarrow \xi(V, \text{MHS}) \otimes \xi(\mathcal{O}(\varpi_1(X, x)^{\rho', \text{Mal}}), \text{MHS})$$

of ind-MTS, noting that S^1 now acts algebraically on both sides (using Corollary 4.20), so Lemma 1.36 implies that this is a morphism of ind-MHS, and therefore that V is an MHS representation of $\varpi_1(X, x)^{\rho', \text{Mal}}$. Theorem 7.6 then implies that this amounts to \mathbb{V} being a VMHS on X .

Combining this argument with Corollary 6.12 immediately gives:

Proposition 7.11. *Under the conditions of Corollary 6.13, the local system associated to the $\pi_1(X, x)$ -representation $\varpi_n(X, x)^{\rho, \text{Mal}}$ naturally underlies a VMHS, which is independent of the basepoint x .*

8. MONODROMY AT THE ARCHIMEDEAN PLACE

Remark 4.22 shows that the mixed Hodge (resp. mixed twistor) structure on $G(X, x_0)^{R, \text{Mal}}$ can be recovered from a nilpotent monodromy operator $\beta : O(R \times \exp(\mathfrak{g})) \rightarrow O(R \times \exp(\mathfrak{g})) \otimes \mathcal{S}(-1)$, where $\mathfrak{g} = \bar{G}(\mathbb{H}^*(X, O(\mathbb{B}_\rho)))$. In this section, we show how to calculate the monodromy operator in terms of standard operations on the de Rham complex.

Definition 8.1. If there is an algebraic action of S^1 on the reductive pro-algebraic group R , set $S' := S$. Otherwise, set $S' := \mathbb{G}_m$. These two cases will correspond to mixed Hodge and mixed twistor structures, respectively.

We now show how to recover β explicitly from the formality quasi-isomorphism of Theorem 5.14. By Corollary 4.12, β can be regarded as an element of

$$W_{-1} \mathbb{E}xt_{\mathbb{H}^*(X, O(\mathbb{B}_\rho))}^0(\mathbb{L}_{\mathbb{H}^*(X, O(\mathbb{B}_\rho))}^\bullet, (\mathbb{H}^*(X, O(\mathbb{B}_\rho)) \rightarrow O(R)) \otimes O(\text{SL}_2)(-1))^{R \times S'}.$$

Definition 8.2. Recall that we set $\tilde{D} = uD + vD^c$, and define $\tilde{D}^c := xD + yD^c$, for co-ordinates $\begin{pmatrix} u & v \\ x & y \end{pmatrix}$ on SL_2 . Note that \tilde{D}^c is of type $(0, 0)$ with respect to the S -action, while \tilde{D} is of type $(1, 1)$.

As in the proof of Theorem 5.14, Corollary 2.9 adapts to give $R \times S'$ -equivariant quasi-isomorphisms

$$\mathbb{H}^*(X, O(\mathbb{B}_\rho)) \otimes O(\text{SL}_2) \xleftarrow{p} Z_{\tilde{D}^c} \xrightarrow{i} \text{row}_1^* \tilde{A}^\bullet(X, O(\mathbb{B}_\rho))$$

of DGAs, where $Z_{\tilde{D}^c} := \ker(\tilde{D}^c) \cap \text{row}_1^* \tilde{A}^\bullet$ (so has differential \tilde{D}). These are moreover compatible with the augmentation maps to $O(\mathbb{B}_\rho)_{x_0} \otimes O(\text{SL}_2) = O(R) \otimes O(\text{SL}_2)$.

Definition 8.3. For simplicity of exposition, we denote these objects by $\mathcal{H}^*, \mathcal{Z}^\bullet, \mathcal{A}^\bullet$, so the quasi-isomorphisms become

$$\mathcal{H}^* \otimes O(\text{SL}_2) \xleftarrow{p} \mathcal{Z}^\bullet \xrightarrow{i} \mathcal{A}^\bullet.$$

We also set $\mathcal{O} := O(R)$, $\underline{\mathcal{H}}^* := \mathcal{H}^* \otimes O(\text{SL}_2)$ and $\underline{\mathcal{O}} := \mathcal{O} \otimes O(\text{SL}_2)$.

This gives the following $R \times S'$ -equivariant quasi-isomorphisms of Hom-complexes:

$$\begin{array}{ccc} \mathbf{R}Hom_{\mathcal{A}}(\mathbb{L}_{\mathcal{A}/O(C)}, \mathcal{A}(-1) \xrightarrow{x_0^*} \underline{\mathcal{O}}(-1)) & \xrightarrow{i^*} & \mathbf{R}Hom_{\mathcal{Z}}(\mathbb{L}_{\mathcal{Z}/O(C)}, \mathcal{A}(-1) \xrightarrow{x_0^*} \underline{\mathcal{O}}(-1)) \\ & \nearrow i_* & \\ \mathbf{R}Hom_{\mathcal{Z}}(\mathbb{L}_{\mathcal{Z}/O(C)}, \mathcal{Z}(-1) \rightarrow \underline{\mathcal{O}}(-1)) & \xrightarrow{p^*} & \mathbf{R}Hom_{\mathcal{Z}}(\mathbb{L}_{\mathcal{Z}/O(C)}, \underline{\mathcal{H}}(-1) \rightarrow \underline{\mathcal{O}}(-1)) \\ & \nearrow p^* & \\ \mathbf{R}Hom_{\underline{\mathcal{H}}}(\mathbb{L}_{\underline{\mathcal{H}}/O(C)}, \underline{\mathcal{H}}(-1) \rightarrow \underline{\mathcal{O}}(-1)). & & \end{array}$$

(Note that, since $\mathcal{H}^0 = \mathbb{R}$ and $\mathcal{Z}^0 = O(\text{SL}_2)$, in both cases the augmentation maps to \mathcal{O} are independent of the basepoint x_0 .) The final expression simplifies, as

$$\mathbb{L}_{(\mathcal{H} \otimes O(\text{SL}_2))/O(C)} \cong (\mathbb{L}_{\mathcal{H}/\mathbb{R}} \otimes O(\text{SL}_2)) \oplus (\mathcal{H} \otimes \Omega(\text{SL}_2/C)).$$

The derivation $N : O(\mathrm{SL}_2) \rightarrow O(\mathrm{SL}_2)(-1)$ has kernel $O(C)$, so yields an $O(C)$ derivation $\mathcal{A} \rightarrow \mathcal{A}(-1)$, and hence an element

$$(N, 0) \in \mathrm{Hom}_{\mathcal{A}, R \times S'}(\mathbb{L}_{\mathcal{A}}, \mathcal{A}(-1) \xrightarrow{x_0^*} \mathcal{O}(-1))^0 \quad \text{with} \quad d(N, 0) = (0, N \circ x_0^*).$$

The chain of quasi-isomorphisms then yields a homotopy-equivalent element f in the final space, and we may choose the homotopies to annihilate $O(\mathrm{SL}_2)(-1) = \Omega(\mathrm{SL}_2/C) \subset \mathbb{L}_{\mathcal{Z}}$, giving

$$\beta \in \mathbf{R}\mathrm{Hom}_{\mathcal{H}, R \times S'}(\mathbb{L}_{\mathcal{H}}, \mathcal{H}(-1) \otimes O(\mathrm{SL}_2) \rightarrow \mathcal{O}(-1))^0 \quad \text{with} \quad d\beta = 0,$$

noting that $N \circ x_0^* = 0$ on $\mathcal{H} \subset \mathcal{H} \otimes O(\mathrm{SL}_2)$, and that f restricted to $\mathcal{H} \otimes \Omega(\mathrm{SL}_2/C)$ is just the identification $\mathcal{H} \otimes \Omega(\mathrm{SL}_2/C) \cong \mathcal{H}(-1) \otimes O(\mathrm{SL}_2)$.

8.1. Reformulation via E_∞ derivations.

Definition 8.4. Given a commutative DG algebra B without unit, define $E(B)$ to be the real graded Lie coalgebra $\mathrm{CoLie}(B[1])$ freely cogenerated by $B[1]$. Explicitly, $\mathrm{CoLie}(V) = \bigoplus_{n \geq 1} \mathrm{CoLie}^n(V)$, where $\mathrm{CoLie}^n(V)$ is the quotient of $V^{\otimes n}$ by the elements

$$\mathrm{sh}_{pq}(v_1 \otimes \dots \otimes v_n) := \sum_{\sigma \in \mathrm{Sh}(p, q)} \pm v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)},$$

for $p, q > 0$ with $p + q = n$. Here, $\mathrm{Sh}(p, q)$ is the set of (p, q) shuffle permutations, and \pm is the Koszul sign.

$E(B)$ is equipped with a differential $d_{E(B)}$ defined on cogenerators $B[1]$ by

$$(q_B + d_B) : (\bigwedge^2(B[1]) \oplus B[1])[-1] \rightarrow B[1],$$

where $q_B : \mathrm{Symm}^2 B \rightarrow B$ is the product on B . Since $d_{E(B)}^2 = 0$, this turns $E(B)$ into a differential graded Lie coalgebra.

Freely cogenerated differential graded Lie coalgebras are known as strong homotopy commutative algebras (SHCAs). A choice of cogenerators V for and SHCA E is then known as an E_∞ or C_∞ algebra. For more details, and analogies with L_∞ algebras associated to DGLAs, see [Kon]. Note that when B is concentrated in strictly positive degrees, $E(B)$ is dual to the dg Lie algebra $G(B \oplus \mathbb{R})$ of Definition 3.23.

Definition 8.5. The functor E has a left adjoint $O(\bar{W}_+)$, given by $O(\bar{W}_+(C)) := \bigoplus_{n > 0} \mathrm{Symm}^n(C[-1])$, with differential as in Definition 3.23. In particular, if $C = \mathfrak{g}^\vee$, for $\mathfrak{g} \in \mathrm{dg}\hat{\mathcal{N}}$, then $\mathbb{R} \oplus O(\bar{W}_+C) = O(\bar{W}\mathfrak{g})$.

For any dg Lie coalgebra C , we therefore define $O(\bar{W}C)$ to be the unital dg algebra $\mathbb{R} \oplus O(\bar{W}_+C)$.

Now, the crucial property of this construction is that $O(\bar{W}_+E(B))$ is a cofibrant replacement for B in the category of non-unital dg algebras (as follows for instance from the proof of [Pri5, Theorem 4.55], interchanging the rôles of Lie and commutative algebras). Therefore for any dg algebra B over A , $O(\bar{W}E(B)) \otimes_{O(\bar{W}E(A))} A$ is a cofibrant replacement for B over A , so

$$\mathbb{L}_{B/A} \simeq \ker(\Omega(O(\bar{W}E(B))) \otimes_{O(\bar{W}E(B))} B \rightarrow \Omega(O(\bar{W}E(A))) \otimes_{O(\bar{W}E(A))} B).$$

Thus

$$\mathbf{R}\mathrm{Hom}_{\mathcal{Z}}(\mathbb{L}_{\mathcal{Z}}, B) \simeq \mathrm{Der}(O(\bar{W}E(\mathcal{Z})) \otimes_{O(\bar{W}E(\mathbb{R}))} \mathbb{R}, B),$$

the complex of derivations over \mathbb{R} . This in turn is isomorphic to the complex

$$\mathrm{Der}_{E(\mathbb{R})}(E(\mathcal{Z}), E(B))$$

of dg Lie coalgebra derivations. The remainder of this section is devoted to constructing explicit homotopy inverses for the equivalences above, thereby deriving the element

$$\beta = (\alpha, \gamma) \in \mathcal{D}er(E(\mathcal{H}), E(\mathcal{H}) \otimes O(\mathrm{SL}_2))^0 \times \mathcal{D}er(E(\mathcal{H}), E(\mathbb{R}) \otimes O(\mathrm{SL}_2))^{-1}$$

required by Remark 4.22, noting that the second term can be rewritten to give $\gamma \in G(\mathcal{H})^0$.

8.2. Kähler identities. By [Sim3, §1], we have first-order Kähler identities

$$D^* = -[\Lambda, D^c], \quad (D^c)^* = [\Lambda, D]$$

(noting that our operator D^c differs from Simpson's by a factor of $-i$), with Laplacian

$$\Delta = [D, D^*] = [D^c, (D^c)^*] = -D\Lambda D^c + D^c\Lambda D + DD^c\Lambda + \Lambda DD^c.$$

Since $uy - vx = 1$, we also have

$$\Delta = -\tilde{D}\Lambda\tilde{D}^c + \tilde{D}^c\Lambda\tilde{D} + \tilde{D}\tilde{D}^c\Lambda + \Lambda\tilde{D}\tilde{D}^c.$$

Definition 8.6. Define a semilinear involution $*$ on $O(\mathrm{SL}_2) \otimes \mathbb{C}$ by $u^* = y, v^* = -x$. This corresponds to the map $A \mapsto (A^\dagger)^{-1}$ on $\mathrm{SL}_2(\mathbb{C})$. The corresponding involution on S is given by $\lambda^* = \bar{\lambda}^{-1}$, for $\lambda \in S(\mathbb{R}) \cong \mathbb{C}^*$.

The calculations above combine to give:

Lemma 8.7.

$$\tilde{D}^* = -[\Lambda, \tilde{D}^c] \quad \tilde{D}^{c*} := [\Lambda, \tilde{D}].$$

Note that this implies that $\tilde{D}\tilde{D}^{c*} + \tilde{D}^{c*}\tilde{D} = 0$. Also note that Green's operator G commutes with \tilde{D} and \tilde{D}^c as well as with Λ , and hence with \tilde{D}^* and \tilde{D}^{c*} .

The working above yields the following.

Lemma 8.8.

$$\Delta = [\tilde{D}, \tilde{D}^*] = [\tilde{D}^c, \tilde{D}^{c*}] = -\tilde{D}\Lambda\tilde{D}^c + \tilde{D}^c\Lambda\tilde{D} + \tilde{D}\tilde{D}^c\Lambda + \Lambda\tilde{D}\tilde{D}^c.$$

8.3. Monodromy calculation. Given any operation f on \mathcal{A} or \mathcal{Z} , we will simply denote the associated dg Lie coalgebra derivation on $E(\mathcal{A})$ or $E(\mathcal{Z})$ by f , so $d_{E(\mathcal{A})} = \tilde{D} + q = d_{E(\mathcal{Z})}$.

Note that the complex $\mathcal{D}er(C, C)$ of coderivations of a dg Lie coalgebra C has the natural structure of a DGLA, with bracket $[f, g] = f \circ g - (-1)^{\deg f \deg g} g \circ f$. When $C = E(B)$, this DGLA is moreover pro-nilpotent, since $E(B) = \bigoplus_{n \geq 1} \mathrm{CoLie}^n(B[1])$, so

$$\mathcal{D}er(E(B), E(B)) = \varprojlim_n \mathcal{D}er\left(\bigoplus_{1 \leq m \leq n} \mathrm{CoLie}^m(B[1]), \bigoplus_{1 \leq m \leq n} \mathrm{CoLie}^m(B[1])\right).$$

Since $[\tilde{D}^c, \tilde{D}^*] = 0$ and

$$\mathrm{id} = \mathrm{pr}_{\mathcal{H}} + G\Delta = \mathrm{pr}_{\mathcal{H}} + G(\tilde{D}^c\tilde{D}^{c*} + \tilde{D}^{c*}\tilde{D}^c),$$

it follows that $\mathrm{Im}(\tilde{D}^{c*})$ is a subcomplex of $\mathcal{Z} = \ker(\tilde{D}^c)$, and $\mathcal{Z} = \mathcal{H} \otimes O(\mathrm{SL}_2) \oplus \mathrm{Im}(\tilde{D}^{c*})$.

Definition 8.9. Decompose $\mathrm{Im}(\tilde{D}^{c*})$ as $B \oplus C$, where $B = \ker(\tilde{D}) \cap \mathrm{Im}(\tilde{D}^{c*})$, and C is its orthogonal complement. Since $i : \mathcal{Z} \rightarrow \mathcal{A}$ is a quasi-isomorphism, $\tilde{D} : C \rightarrow B$ is an isomorphism, and we may define $h_i : \mathcal{A} \rightarrow \mathcal{A}[-1]$ by $h_i(z + b + c) = \tilde{D}^{-1}b \in C$, for $z \in \mathcal{Z}, b \in B, c \in C$. Thus $h_i^2 = 0$, and $\mathrm{id} = \mathrm{pr}_{\mathcal{Z}} + \tilde{D}h_i + h_i\tilde{D}$. Explicitly,

$$h_i := G\tilde{D}^* \circ (1 - \mathrm{pr}_{\mathcal{Z}}) = G^2\tilde{D}^*\tilde{D}^c\tilde{D}^c = G^2\tilde{D}^c\tilde{D}^c\tilde{D}^*,$$

where G is Green's operator and $\mathrm{pr}_{\mathcal{Z}}$ is orthogonal projection onto \mathcal{Z} . Since $\tilde{D}\tilde{D}^c = DD^c$, we can also rewrite this as $G^2D^*D^{c*}\tilde{D}^c$.

Lemma 8.10. *Given a derivation $f \in \text{Der}(E(\mathcal{Z}), E(\mathcal{A}))^0$ with $[q, f] + [\tilde{D}, f] = 0$, let*

$$\gamma^i(f) := \sum_{n \geq 0} (-1)^{n+1} h_i \circ [q, h_i]^n \circ (f + h_i \circ [q, f]).$$

Then $\gamma^i(f) \in \text{Der}(E(\mathcal{Z}), E(\mathcal{A}))^{-1}$, and

$$f + [d_E, \gamma^i(f)] = \text{pr}_{\mathcal{Z}} \circ \left(\sum_{n \geq 0} (-1)^n \circ [q, h_i]^n \circ (f + h_i \circ [q, f]) \right),$$

so lies in $\text{Der}(E(\mathcal{Z}), E(\mathcal{Z}))^0$.

Proof. First, observe that h_i is 0 on \mathcal{Z} , so $g \circ h_i = 0$ for all $g \in \text{Der}(E(\mathcal{Z}), E(\mathcal{A}))$, and therefore $h_i \circ g = [h_i, g]$ is a derivation. If we write $\text{ad}_q(g) = [q, g]$, then $\text{ad}_q(h_i \circ e) = [q, h_i] \circ e$, for any $e \in \text{Der}(E(\mathcal{Z}), E(\mathcal{A}))^0$ with $[q, e] = 0$. Then

$$\text{ad}_q(h_i \circ \text{ad}_q(h_i \circ e)) = [q, h_i \circ [q, h_i] \circ e] = [q, h_i] \circ [q, h_i] \circ e + h_i \circ \frac{1}{2} [[q, q], h_i] \circ e,$$

which is just $[q, h_i]^2 \circ e$, since $q^2 = 0$ (which amounts to saying that the multiplication on \mathcal{A} is associative), so $\text{ad}_q^2 = 0$.

Now,

$$\text{ad}_q(h_i \circ f) = [q, h_i \circ f] = [q, h_i] \circ f - h_i \circ [q, f],$$

and this lies in $\ker(\text{ad}_q)$. Proceeding inductively, we get

$$\gamma^i(f) := \sum_{n \geq 0} (-1)^{n+1} (\text{ad}_{h_i} \text{ad}_q)^n \text{ad}_{h_i} f,$$

which is clearly a derivation. Note that the sum is locally finite because the n th term maps $\text{CoLie}^m(\mathcal{Z})$ to $\text{CoLie}^{m-n}(\mathcal{A})$.

Now, let $y := \sum_{n \geq 0} (-1)^n (\text{ad}_q \text{ad}_{h_i})^n f$, so $\gamma^i(f) = -[h_i, y] = -h_i \circ y$. Set $f' := f + [d_E, \gamma^i(f)]$; we wish to show that $[\tilde{D}, h_i \circ f'] + h_i \circ [\tilde{D}, f'] = 0$. Note that $f + [q, \gamma^i(f)] = y$, so

$$f' = f - [q, h_i \circ y] - [\tilde{D}, h_i \circ y] = y - [\tilde{D}, h_i] \circ y - h_i \circ [\tilde{D}, y].$$

Since $\text{pr}_{\mathcal{Z}} = (\text{id} - [\tilde{D}, h_i])$, it only remains to show that $h_i \circ [\tilde{D}, y] = 0$, or equivalently that $h_i \circ [\tilde{D}, f'] = 0$.

Now, $0 = [d_E, f'] = [\tilde{D}, f'] + [q, f']$. Since $[q, \mathcal{Z}] \subset \mathcal{Z}$, this means that

$$h_i \circ [\tilde{D}, f'] = -h_i \circ [q, h_i \circ [\tilde{D}, f']].$$

Since $h_i \circ \text{ad}_q$ maps $\text{CoLie}^n(\mathcal{A}[1])$ to $\text{CoLie}^{n-1}(\mathcal{A}[1])$, this means that $h_i \circ [\tilde{D}, f'] = 0$, since $h_i \circ [\tilde{D}, f'] = (-h_i \circ \text{ad}_q)^n \circ (h_i \circ [\tilde{D}, f'])$ for all n , and this is 0 on $\text{CoLie}^n(\mathcal{A}[1])$. \square

Definition 8.11. On the complex \mathcal{Z} , define $h_p := G\tilde{D}^*$, noting that this is also isomorphic to $G\tilde{D}^c\Lambda$ here.

Lemma 8.12. *Given a derivation $f \in \text{Der}(E(\mathcal{Z}), E(\mathcal{H}))^0$ with $[q, f] + [\tilde{D}, f] = 0$, let*

$$\gamma^p(f) := \sum_{n \geq 0} (-1)^{n+1} (f + [q, f] \circ h_p) \circ [q, h_p]^n \circ h_p.$$

Then $\gamma^p(f) \in \text{Der}(E(\mathcal{Z}), E(\mathcal{H}))^{-1}$, and

$$f + [d_E, \gamma^p(f)] = \left(\sum_{n \geq 0} (-1)^n (f + [q, f] \circ h_p) \circ [q, h_p]^n \right) \circ \text{pr}_{\mathcal{H}},$$

where $\text{pr}_{\mathcal{H}}$ is orthogonal projection onto harmonic forms. Thus $f + [d_E, \gamma^p(f)]$ lies in $\text{Der}(E(\mathcal{H}), E(\mathcal{H}))^0$.

Proof. The proof of Lemma 8.10 carries over, since the section of $p : \mathcal{Z} \rightarrow \underline{\mathcal{H}}$ given by harmonic forms corresponds to a decomposition $\mathcal{Z} = \underline{\mathcal{H}} \oplus \text{Im}(\tilde{D}^c)$. Then h_p makes p into a deformation retract, as $[h_p, \tilde{D}] = \text{pr}_{\underline{\mathcal{H}}}$ on \mathcal{Z} . \square

Theorem 8.13. *For $\mathfrak{g} = G(\mathbb{H}^*(X, O(\mathbb{B}_\rho)))$, the monodromy operator*

$$\beta : O(R \rtimes \exp(\mathfrak{g})) \rightarrow O(R \rtimes \exp(\mathfrak{g})) \otimes O(\text{SL}_2)(-1)$$

at infinity, corresponding to the MHS (or MTS) on the homotopy type $(X, x_0)^{\rho, \text{Mal}}$ is given $\beta = \alpha + \text{ad}_{\gamma_{x_0}}$, where $\alpha : \mathfrak{g}^\vee \rightarrow \mathfrak{g}^\vee \otimes O(\text{SL}_2)(-1)$ is

$$\begin{aligned} \alpha &= \sum_{b>0, a \geq 0} (-1)^{a+b+1} \text{pr}_{\underline{\mathcal{H}}} \circ ([q, G^2 D^* D^{c*}] \circ \tilde{D}^c)^b \circ (\tilde{D} \circ [q, G\Lambda]) \circ (\tilde{D}^c \circ [q, G\Lambda])^a \circ s \\ &\quad + \sum_{b>0, a>0} (-1)^{a+b} [q, \text{pr}_{\underline{\mathcal{H}}}] \circ ([q, G^2 D^* D^{c*}] \circ \tilde{D}^c)^b \circ (\tilde{D} \circ G\Lambda) \circ (\tilde{D}^c \circ [q, G\Lambda])^a \circ s, \end{aligned}$$

for $s : \mathcal{H} \rightarrow \mathcal{A}$ the inclusion of harmonic forms. Meanwhile, $\gamma_{x_0} \in \mathfrak{g} \hat{\otimes} O(\text{SL}_2)(-1)$ is

$$\begin{aligned} \gamma_{x_0} \circ s &= \sum_{a \geq 0, b \geq 0} (-1)^{a+b} x_0^* \circ h_i \circ ([q, G^2 D^* D^{c*}] \circ \tilde{D}^c)^b \circ (\tilde{D} \circ [q, G\Lambda]) \circ (\tilde{D}^c \circ [q, G\Lambda])^a \circ s \\ &\quad + \sum_{a>0, b>0} (-1)^{a+b} x_0^* \circ ([q, G^2 D^* D^{c*}] \circ \tilde{D}^c)^b \circ (\tilde{D} \circ G\Lambda) \circ (\tilde{D}^c \circ [q, G\Lambda])^a \circ s. \end{aligned}$$

Proof. The derivation $N : \mathcal{A} \rightarrow \mathcal{A}(-1)$ yields a coderivation $N \in \text{Der}(E(\mathcal{Z}), E(\mathcal{A}))^0$ with $[q, f] = [\tilde{D}, f] = 0$. Lemma 8.10 then gives $\gamma^i(N) \in \text{Der}(E(\mathcal{Z}), E(\mathcal{A}))^{-1}$ with $N + [d_E, \gamma^i(N)] \in \text{Der}(E(\mathcal{Z}), E(\mathcal{Z}))^0$. Therefore, in the cone complex $\text{Der}(E(\mathcal{Z}), E(\mathcal{A})) \xrightarrow{x^*} \text{Der}(E(\mathcal{Z}), E(\underline{cQ}))$, the derivation N is homotopic to

$$(N + [d_E, \gamma^i(N)], \gamma^i(N)_{x_0}) \in \text{Der}(E(\mathcal{Z}), E(\mathcal{Z}))^0 \oplus \text{Der}(E(\mathcal{Z}), E(\underline{cQ}))^{-1}.$$

Explicitly,

$$\begin{aligned} \gamma^i(N) &= \sum_{n \geq 0} (-1)^{n+1} h_i \circ [q, h_i]^n \circ N \\ N + [d_E, \gamma^i(N)] &= \text{pr}_{\mathcal{Z}} \circ \left(\sum_{n \geq 0} (-1)^n \circ [q, h_i]^n \circ N \right). \end{aligned}$$

Setting $f := N + [d_E, \gamma^i(N)]$, we next apply Lemma 8.12 to the pair $(p \circ f, \gamma^i(N)_{x_0})$. If $s : \underline{\mathcal{H}} \rightarrow \mathcal{Z}$ denotes the inclusion of harmonic forms, we obtain

$$\begin{aligned} \alpha \circ s &= p \circ f + [d_E, \gamma^p(p \circ f)], \\ \gamma_{x_0} \circ s &= \gamma^i(N)_{x_0} + \gamma^p(p \circ f)_{x_0} + [d_E, \gamma^p(\gamma^i(N)_{x_0}) + \gamma^p(p \circ f)_{x_0}]. \end{aligned}$$

Now,

$$\begin{aligned} \alpha &= \sum_{m \geq 0} (-1)^m (p \circ f + [q, p \circ f] \circ h_p) \circ [q, h_p]^m \circ s \\ &= \sum_{m \geq 0, n \geq 0} (-1)^{m+n} \text{pr}_{\underline{\mathcal{H}}} \circ [q, h_i]^n \circ N \circ [q, h_p]^m \circ s \\ &\quad + \sum_{m \geq 0, n \geq 0} (-1)^{m+n} [q, \text{pr}_{\underline{\mathcal{H}}}] \circ [q, h_i]^n \circ N \circ h_p \circ [q, h_p]^m \circ s \end{aligned}$$

since $p \circ \text{pr}_{\mathcal{Z}} = \text{pr}_{\underline{\mathcal{H}}}$, $[q, N] = 0$ and $\text{ad}_q^2 = 0$.

Now, $N \circ g = [N, g] + g \circ N$, but N is 0 on $\mathcal{H} \subset \underline{\mathcal{H}}$, while $[N, s] = 0$ (since s is SL_2 -linear). Since $h_i = G^2 D^* D^{c*} \tilde{D}^c$, $h_p = \tilde{D}^c G\Lambda$ and $[q, \tilde{D}^c] = 0$, we get $[q, h_i] = [q, G^2 D^* D^{c*}] \circ \tilde{D}^c$ and $[q, h_p] = \tilde{D}^c \circ [q, G\Lambda]$. In particular, this implies that $[q, h_i] \circ \tilde{D}^c = 0$ and that $[N, [q, h_p]] = \tilde{D} \circ [q, G\Lambda]$, since $[N, G\Lambda] = [N, q] = 0$.

Thus

$$\begin{aligned} \alpha &= \sum_{n,a,c \geq 0} (-1)^{n+a+c+1} \text{pr}_{\underline{\mathcal{H}}} \circ [q, h_i]^n \circ [q, h_p]^c \circ (\tilde{D} \circ [q, G\Lambda]) \circ [q, h_p]^a \circ s \\ &+ \sum_{m,n \geq 0} (-1)^{m+n} [q, \text{pr}_{\underline{\mathcal{H}}}] \circ [q, h_i]^n \circ (\tilde{D} \circ G\Lambda) \circ [q, h_p]^m \circ s \\ &+ \sum_{n,a,c \geq 0} (-1)^{m+n} [q, \text{pr}_{\underline{\mathcal{H}}}] \circ [q, h_i]^n \circ \tilde{D}^c \circ G\Lambda \circ [q, h_p]^c \circ (\tilde{D} \circ [q, G\Lambda]) \circ [q, h_p]^a \circ s. \end{aligned}$$

When $n = 0$, all terms are 0, since $\text{pr}_{\underline{\mathcal{H}}} \circ \tilde{D} = \text{pr}_{\underline{\mathcal{H}}} \circ \tilde{D}^c = 0$, and $[q, h_p] = \tilde{D}^c \circ [q, G\Lambda]$. For $n \neq 0$, the first sum is 0 whenever $c \neq 0$, and the final sum is always 0 (since $[q, h_i] \circ \tilde{D}^c = 0$). If $m = 0$, the second sum is also 0, as $\tilde{D} \circ G\Lambda$ equals $G\tilde{D}^{c*}$ on $\ker(\tilde{D})$, so is 0 on \mathcal{H} . Therefore (writing $b = n$), we get

$$\begin{aligned} \alpha &= \sum_{b>0, a \geq 0} (-1)^{a+b+1} \text{pr}_{\underline{\mathcal{H}}} \circ [q, h_i]^b \circ (\tilde{D} \circ [q, G\Lambda]) \circ [q, h_p]^a \circ s \\ &+ \sum_{b>0, a > 0} (-1)^{a+b} [q, \text{pr}_{\underline{\mathcal{H}}}] \circ [q, h_i]^b \circ (\tilde{D} \circ G\Lambda) \circ [q, h_p]^a \circ s, \end{aligned}$$

and substituting for $[q, h_i]$ and $[q, h_p]$ gives the required expression.

Next, we look at γ_{x_0} . First, note that $\Lambda|_{\mathcal{Z}^1} = 0$, so $h_p|_{\mathcal{Z}^1} = 0$, and therefore h_p (and hence $\gamma^p(p \circ f)$) restricted to $\text{CoLie}^n(\mathcal{Z}^1)$ is 0, so $x_0^* \circ \gamma^p(p \circ f) = 0$. Thus

$$\begin{aligned} \gamma_{x_0} \circ s &= \gamma^i(N)_{x_0} + [d_E, \gamma^p(\gamma^i(N)_{x_0})] \\ &= \sum_{m \geq 0} (-1)^m (\gamma^i(N)_{x_0} + [q, \gamma^i(N)_{x_0}] \circ h_p) \circ [q, h_p]^m \circ s \\ &= \sum_{m,n \geq 0} (-1)^{m+n+1} x_0^* \circ h_i \circ [q, h_i]^n \circ N \circ [q, h_p]^m \circ s \\ &+ \sum_{m,n \geq 0} (-1)^{m+n+1} [q, x_0^* \circ h_i \circ [q, h_i]^n \circ N] \circ h_p \circ [q, h_p]^m \circ s. \end{aligned}$$

On restricting to $\mathcal{H} \subset \underline{\mathcal{H}}$, we may replace $N \circ g$ with $[N, g]$ (using the same reasoning as for α). Now, $[q, h_i]^{n+1} \circ h_p = 0$, and on expanding out $\tilde{D}^c \circ [N, [q, h_p]^m]$, all terms but one vanish, giving

$$\begin{aligned} \gamma_{x_0} \circ s &= \sum_{m>0, n \geq 0} (-1)^{m+n+1} x_0^* \circ h_i \circ [q, h_i]^n \circ (\tilde{D} \circ [q, G\Lambda]) \circ [q, h_p]^{m-1} \circ s \\ &+ \sum_{m>0, n \geq 0} (-1)^{m+n+1} x_0^* \circ [q, h_i]^{n+1} \circ (\tilde{D} \circ G\Lambda) \circ [q, h_p]^m \circ s, \end{aligned}$$

which expands out to give the required expression. \square

Remark 8.14. This implies that the MHS $O(\varpi_1(X, x_0)^{\rho, \text{Mal}})$ is just the kernel of

$$\beta \otimes \text{id} + \text{ad}_{\gamma_{x_0}} \otimes \text{id} + \text{id} \otimes N : O(R \times \exp(\mathbf{H}_0 \mathfrak{g})) \otimes \mathcal{S} \rightarrow O(R \times \exp(\mathbf{H}_0 \mathfrak{g})) \otimes \mathcal{S}(-1),$$

where β, γ_{x_0} here denote the restrictions of β, γ_{x_0} in Theorem 8.13 to $\text{Spec } \mathcal{S} = \begin{pmatrix} 1 & 0 \\ \mathbb{A}^1 & 1 \end{pmatrix} \subset \text{SL}_2$.

Likewise, $(\varpi_n(X, x_0)^{\rho, \text{Mal}})^\vee$ is the kernel of

$$\beta \otimes \text{id} + \text{ad}_{\gamma_{x_0}} \otimes \text{id} + \text{id} \otimes N : (\mathbf{H}_{n-1} \mathfrak{g})^\vee \otimes \mathcal{S} \rightarrow (\mathbf{H}_{n-1} \mathfrak{g})^\vee \otimes \mathcal{S}(-1)$$

Examples 8.15. Since q maps $\text{CoLie}^n(\mathcal{H})$ to $\text{CoLie}^{n-1}(\mathcal{H})$, we need only look at the truncations of the sums in Theorem 8.13 to calculate the MHS or MTS on $G(X, x_0)^{R, \text{Mal}} / [\mathbf{R}_u G(X, x_0)^{R, \text{Mal}}]_m$, where $[K]_1 = K$ and $[K]_{n+1} = [K, [K]_m]$.

- (1) Since all terms involve q , this means that $G(X, x_0)^{R, \text{Mal}} / [\mathbb{R}_u G(X, x_0)^{R, \text{Mal}}]_2 \simeq R \times \mathbb{H}^{>0}(X, O(\mathbb{B}_\rho))^\vee[1]$, the equivalence respecting the MHS (or MTS). This just corresponds to the quasi-isomorphism $s : \mathbb{H}^*(X, O(\mathbb{B}_\rho)) \rightarrow A^\bullet(X, O(\mathbb{B}_\rho))$ of cochain complexes, since the ring structure on $A^\bullet(X, O(\mathbb{B}_\rho))$ is not needed to recover $G(X, x_0)^{R, \text{Mal}} / [\mathbb{R}_u G(X, x_0)^{R, \text{Mal}}]_2$.
- (2) The first non-trivial case is $G(X, x_0)^{R, \text{Mal}} / [\mathbb{R}_u G(X, x_0)^{R, \text{Mal}}]_3$. The only contributions to β here come from terms of degree 1 in q . Thus α vanishes on this quotient, which means that the obstruction to splitting the MHS is a unipotent inner automorphism.

The element γ_{x_0} becomes

$$\begin{aligned} x_0^* \circ h_i \circ \tilde{D} \circ [q, G\Lambda] \circ s &= x_0^* \circ G^2 D^* D^{c*} \tilde{D}^c \tilde{D} \circ [q, G\Lambda] \circ s \\ &= x_0^* \circ G^2 D^* D^{c*} D^c D \circ [q, G\Lambda] \circ s, \end{aligned}$$

which we can rewrite as $x_0^* \circ \text{pr}_{\text{Im}(D^* D^{c*})} \circ [q, G\Lambda] \circ s$, where $\text{pr}_{\text{Im}(D^* D^{c*})}$ is orthogonal projection onto $\text{Im}(D^* D^{c*})$. Explicitly, $\gamma_{x_0} \in ([\mathfrak{g}]_2 / [\mathfrak{g}]_3) \hat{\otimes} O(\text{SL}_2)$ corresponds to the morphism $\wedge^2 \mathcal{H}^1 \rightarrow O(R) \otimes O(\text{SL}_2)$ given by

$$v \otimes w \mapsto (\text{pr}_{\text{Im}(D^* D^{c*})} G\Lambda(s(v) \wedge s(w)))_{x_0},$$

since $\Lambda|_{\mathcal{H}^1} = 0$.

Since $[\mathfrak{g}]_2 / [\mathfrak{g}]_3$ lies in the centre of $\mathfrak{g} / [\mathfrak{g}]_3$, this means that $\text{ad}_{\gamma_{x_0}}$ acts trivially on $\mathbb{R}_u(G(X, x_0)^{R, \text{Mal}}) / [\mathbb{R}_u G(X, x_0)^{R, \text{Mal}}]_3$, so $G(X, x_0)^{R, \text{Mal}} / [\mathbb{R}_u G(X, x_0)^{R, \text{Mal}}]_3$ is an extension

$$1 \rightarrow \mathbb{R}_u(G(X, x_0)^{R, \text{Mal}}) / [\mathbb{R}_u G(X, x_0)^{R, \text{Mal}}]_3 \rightarrow G(X, x_0)^{R, \text{Mal}} / [\mathbb{R}_u G(X, x_0)^{R, \text{Mal}}]_3 \rightarrow R \rightarrow 1$$

of split MHS. Thus γ_{x_0} is the obstruction to any Levi decomposition respecting the MHS, and allowing x_0 to vary gives us the associated VMHS on X .

In particular, taking $R = 1$, the MHS on $G(X, x_0)^{1, \text{Mal}} / [G(X, x_0)^{1, \text{Mal}}]_3$ is split, and specialising further to the case when X is simply connected,

$$(\pi_3(X, x_0) \otimes \mathbb{R})^\vee \cong \mathbb{H}^3(X, \mathbb{R}) \oplus \ker(\text{Symm}^2 \mathbb{H}^2(X, \mathbb{R})) \xrightarrow{\cup} \mathbb{H}^4(X, \mathbb{R})$$

is an isomorphism of real MHS. This shows that the phenomena in [CCM] are entirely due to the lattice $\pi_3(X, x_0)$ in $\pi_3(X, x_0) \otimes \mathbb{R}$.

- (3) The first case in which α is non-trivial is $\mathbb{R}_u(G(X, x_0)^{R, \text{Mal}}) / [\mathbb{R}_u G(X, x_0)^{R, \text{Mal}}]_4$. We then have

$$\begin{aligned} \alpha &= \text{pr}_{\mathcal{H}} \circ [q, G^2 D^* D^{c*}] \circ D^c D \circ [q, G\Lambda] \circ s \\ &= \text{pr}_{\mathcal{H}} \circ q \circ \text{pr}_{\text{Im}(D^* D^{c*})} \circ [q, G\Lambda] \circ s, \end{aligned}$$

and this determines the MHS on $G(X, x_0)^{R, \text{Mal}}$ up to pro-unipotent inner automorphism. In particular, if X is simply connected, this determines the MHS on $\pi_4(X, x_0) \otimes \mathbb{R}$ as follows.

Let $V := \text{CoLie}^3(\mathbb{H}^2(X, \mathbb{R})[1])[-2]$, i.e. the quotient of $\mathbb{H}^2(X, \mathbb{R})^{\otimes 3}$ by the subspace generated by $a \otimes b \otimes c - a \otimes c \otimes b + c \otimes a \otimes b$ and $a \otimes b \otimes c - b \otimes a \otimes c + b \otimes c \otimes a$, then set K to be the kernel of the map $q : V \rightarrow \mathbb{H}^4(X, \mathbb{R}) \otimes \mathbb{H}^2(X, \mathbb{R})$ given by $q(a \otimes b \otimes c) = (a \cup b) \otimes c - (b \cup c) \otimes a$. If we let $C := \text{coker}(\text{Symm}^2 \mathbb{H}^2(X, \mathbb{R}) \xrightarrow{\cup} \mathbb{H}^4(X, \mathbb{R}))$ and $L := \ker(\mathbb{H}^2(X, \mathbb{R}) \otimes \mathbb{H}^3(X, \mathbb{R}) \xrightarrow{\cup} \mathbb{H}^5(X, \mathbb{R}))$, then

$$\text{gr}^W(\pi_4(X) \otimes \mathbb{R})^\vee \cong C \oplus L \oplus K.$$

The MHS is then determined by $\alpha : K \rightarrow C(-1)$, corresponding to the restriction to K of the map $\alpha' : V \rightarrow C(-1)$ given by setting $\alpha'(a \otimes b \otimes c)$ to be

$$\begin{aligned} & \operatorname{pr}_{\mathcal{H}}(\operatorname{pr}_I((G\Lambda\tilde{a}) \wedge \tilde{b}) \wedge \tilde{c}) - \operatorname{pr}_{\mathcal{H}}((\operatorname{pr}_I G\Lambda\tilde{a}) \wedge (\tilde{b} \wedge \tilde{c})) \\ & - \operatorname{pr}_{\mathcal{H}}(\operatorname{pr}_I(\tilde{a} \wedge (G\Lambda\tilde{b})) \wedge \tilde{c}) - \operatorname{pr}_{\mathcal{H}}(\tilde{a} \wedge \operatorname{pr}_I((G\Lambda\tilde{b}) \wedge \tilde{c})) \\ & - \operatorname{pr}_{\mathcal{H}}(\tilde{a} \wedge \tilde{b} \wedge (\operatorname{pr}_I G\Lambda\tilde{c})) + \operatorname{pr}_{\mathcal{H}}(\tilde{a} \wedge \operatorname{pr}_I(\tilde{b} \wedge (G\Lambda\tilde{c}))) \end{aligned}$$

where $\tilde{a} := sa$, for s the identification of cohomology with harmonic forms, while pr_I and $\operatorname{pr}_{\mathcal{H}}$ are orthogonal projection onto $\operatorname{Im}(d^*d^{c*})$ and harmonic forms, respectively.

Explicitly, the MHS $(\pi_4(X) \otimes \mathbb{R})^\vee$ is then given by the subspace

$$(c - x\alpha(k), l, k) \subset (C \oplus L \oplus K) \otimes \mathcal{S},$$

for $c \in C$, $l \in L$ and $k \in K$, with \mathcal{S} the quasi-MHS of Lemma 1.19.

9. SIMPLICIAL AND SINGULAR VARIETIES

In this section, we will show how the techniques of cohomological descent allow us to extend real mixed Hodge and twistor structures to all proper complex varieties. By [SD, Remark 4.1.10], the method of [Gro1, §9] shows that a surjective proper morphism of topological spaces is universally of effective cohomological descent.

Lemma 9.1. *If $f : X \rightarrow Y$ is a map of compactly generated Hausdorff topological spaces inducing an equivalence on fundamental groupoids, such that $R^i f_* \mathbb{V} = 0$ for all local systems \mathbb{V} on X and all $i > 0$, then f is a weak equivalence.*

Proof. Without loss of generality, we may assume that X and Y are path-connected. If $\tilde{X} \xrightarrow{\pi} X, \tilde{Y} \xrightarrow{\pi'} Y$ are the universal covering spaces of X, Y , then it will suffice to show that $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is a weak equivalence, since the fundamental groups are isomorphic.

As \tilde{X}, \tilde{Y} are simply connected, it suffices to show that $R^i \tilde{f}_* \mathbb{Z} = 0$ for all $i > 0$. By the Leray-Serre spectral sequence, $R^i \pi_* \mathbb{Z} = 0$ for all $i > 0$, and similarly for Y . The result now follows from the observation that $\pi_* \mathbb{Z}$ is a local system on X . \square

Proposition 9.2. *If $a : X_\bullet \rightarrow X$ is a morphism (of simplicial topological spaces) of effective cohomological descent, then $|a| : |X_\bullet| \rightarrow X$ is a weak equivalence, where $|X_\bullet|$ is the geometric realisation of X_\bullet .*

Proof. We begin by showing that the fundamental groupoids are equivalent. Since $H^0(|X_\bullet|, \mathbb{Z}) \cong H^0(X, \mathbb{Z})$, we know that $\pi_0 |X_\bullet| \cong \pi_0 X$, so we may assume that $|X_\bullet|$ and X are both connected.

Now the fundamental groupoid of $|X_\bullet|$ is isomorphic to the fundamental groupoid of the simplicial set $\operatorname{diag} \operatorname{Sing}(X_\bullet)$ (the diagonal of the bisimplicial set given by the singular simplicial sets of the X_n). For any group G , the groupoid of G -torsors on $|X_\bullet|$ is thus equivalent to the groupoid of pairs (T, ω) , where T is a G -torsor on X_0 , and the descent datum $\omega : \partial_0^{-1} T \rightarrow \partial_1^{-1} T$ is a morphism of G -torsors satisfying

$$\partial_2^{-1} \omega \circ \partial_0^{-1} \omega = \partial_1^{-1} \omega, \quad \sigma_0^{-1} \omega = 1.$$

Since a is effective, this groupoid is equivalent to the groupoid of G -torsors on X , so the fundamental groups are isomorphic.

Given a local system \mathbb{V} on $|X_\bullet|$, there is a corresponding $\operatorname{GL}(V)$ -torsor T , which therefore descends to X . Since $\mathbb{V} = T \times^{\operatorname{GL}(V)} V$ and $T = |a|^{-1} |a|_* T$, we can deduce that $\mathbb{V} = |a|^{-1} |a|_* \mathbb{V}$, so $R^i |a|_* \mathbb{V} = 0$ for all $i > 0$, as a is of effective cohomological descent. Thus $|a|$ satisfies the conditions of Lemma 9.1, so is a weak equivalence. \square

Corollary 9.3. *Given a proper complex variety X , there exists a smooth proper simplicial variety X_\bullet , unique up to homotopy, and a map $a : X_\bullet \rightarrow X$, such that $|X_\bullet| \rightarrow X$ is a weak equivalence.*

In fact, we may take each X_n to be projective, and these resolutions are unique up to simplicial homotopy.

Proof. Apply [Del2, 6.2.8, 6.4.4 and §8.2]. □

9.1. Semisimple local systems. From now on, X_\bullet will be a fixed simplicial proper complex variety (*a fortiori*, this allows us to consider any proper complex variety).

In this section, we will define the real holomorphic S^1 -action on a suitable quotient of the real reductive pro-algebraic fundamental group $\varpi_1(|X_\bullet|, x)^{\text{red}}$.

Recall that a local system on a simplicial diagram X_\bullet of topological spaces is equivalent to the category of pairs (\mathbb{V}, α) , where \mathbb{V} is a local system on X_0 , and $\alpha : \partial_0^{-1}\mathbb{V} \rightarrow \partial_1^{-1}\mathbb{V}$ is an isomorphism of local systems satisfying

$$\partial_2^{-1}\alpha \circ \partial_0^{-1}\alpha = \partial_1^{-1}\alpha, \quad \sigma_0^{-1}\alpha = 1.$$

Definition 9.4. Given a simplicial diagram X_\bullet of smooth proper varieties and a point $x \in X_0$, define the fundamental group $\varpi_1(|X_\bullet|, x)^{\text{norm}}$ to be the quotient of $\varpi_1(|X_\bullet|, x)$ by the normal subgroup generated by the image of $\text{R}_u\varpi_1(X_0, x)$. We call its representations normally semisimple local systems on $|X_\bullet|$ — these correspond to local systems \mathbb{W} (on the connected component of $|X|$ containing x) for which $a_0^{-1}\mathbb{W}$ is semisimple, for $a_0 : X_0 \rightarrow |X_\bullet|$.

Then define $\varpi_1(|X_\bullet|, x)^{\text{norm,red}}$ to be the reductive quotient of $\varpi_1(|X_\bullet|, x)^{\text{norm}}$. Its representations are semisimple and normally semisimple local systems on the connected component of $|X|$ containing x .

Lemma 9.5. *If $f : X_\bullet \rightarrow Y_\bullet$ is a homotopy equivalence of simplicial smooth proper varieties, then $\varpi_1(|X_\bullet|, x)^{\text{norm}} \simeq \varpi_1(|Y_\bullet|, fx)^{\text{norm}}$.*

Proof. Without loss of generality, we may assume that the matching maps

$$X_n \rightarrow Y_n \times_{\text{Hom}_{\mathbb{S}}(\partial\Delta^n, Y)} \text{Hom}_{\mathbb{S}}(\partial\Delta^n, X)$$

of f are faithfully flat and proper for all $n \geq 0$ (since morphisms of this form generate all homotopy equivalences), and that $|X|$ is connected. Here, \mathbb{S} is the category of simplicial sets and $\partial\Delta^n$ is the boundary of Δ^n , with the convention that $\partial\Delta^0 = \emptyset$.

Topological and algebraic effective descent then imply that f^{-1} induces an equivalence on the categories of local systems, and that f^* induces an equivalence on the categories of quasi-coherent sheaves, and hence on the categories of Higgs bundles. Since representations of $\varpi_1(|X_\bullet|, x)^{\text{norm}}$ correspond to objects in the category of Higgs bundles on X_\bullet , this completes the proof. □

Definition 9.6. If $X_\bullet \rightarrow X$ is any resolution as in Corollary 9.3, with $x_0 \in X_0$ mapping to $x \in X$, we denote the corresponding pro-algebraic group by $\varpi_1(X, x)^{\text{norm}} := \varpi_1(|X_\bullet|, x_0)^{\text{norm}}$, noting that this is independent of the choice of x_0 , since $|X_\bullet| \rightarrow X$ is a weak equivalence.

Proposition 9.7. *If X is a proper complex variety with a smooth proper resolution $a : X_\bullet \rightarrow X$, then normally semisimple local systems on X_\bullet correspond to local systems on X which become semisimple on pulling back to the normalisation $\pi : X^{\text{norm}} \rightarrow X$ of X .*

Proof. First observe that $\varpi_1(|X_\bullet|, x_0)^{\text{norm}} = \varpi_1(X, x_0) / \langle a_0 \text{R}_u(\varpi_1(X_0, x_0)) \rangle$. Lemma 9.5 ensures that $\varpi_1(|X_\bullet|, x_0)^{\text{norm}}$ is independent of the choice of resolution X_\bullet of X , so can be defined as $\varpi_1(X, x_0) / \langle f \text{R}_u(\varpi_1(Y, y)) \rangle$ for any smooth projective variety Y and proper faithfully flat $f : Y \rightarrow X$, with $fy = x$.

Now, since X^{norm} is normal, we may make use of an observation on pp. 9–10 of [ABC⁺] (due to M. Ramachandran). X^{norm} admits a proper faithfully flat morphism g from a smooth variety Y with connected fibres over X^{norm} . If $\tilde{x} \in X^{\text{norm}}$ is a point above $x \in X$, and $y \in Y$ is a point above \tilde{x} , then this implies the morphism $\pi_1 g: \pi_1(Y, y) \rightarrow \pi_1(X^{\text{norm}}, \tilde{x})$ is surjective (from the long exact sequence of homotopy), and therefore $g(R_u \varpi_1(Y, y)) = R_u \varpi_1(X^{\text{norm}}, \tilde{x})$.

Taking $f: Y \rightarrow X$ to be the composition $Y \xrightarrow{g} X^{\text{norm}} \xrightarrow{\pi} X$, we see that $f R_u \varpi_1(Y, y) = \pi(R_u \varpi_1(X^{\text{norm}}, \tilde{x}))$. This shows that $\varpi_1(X, x)^{\text{norm}} = \varpi_1(X, x) / \langle \pi(R_u \varpi_1(X^{\text{norm}}, \tilde{x})) \rangle$, as required. \square

Proposition 9.8. *If X_\bullet is a simplicial diagram of compact Kähler manifolds, then there is a discrete action of the circle group S^1 on $\varpi_1(|X_\bullet|, x)^{\text{norm}}$, such that the composition $S^1 \times \pi_1(X_\bullet, x) \rightarrow \varpi_1(|X_\bullet|, x)^{\text{norm}}$ is continuous. We denote this last map by $\sqrt{h}: \pi_1(|X_\bullet|, x) \rightarrow \varpi_1(|X_\bullet|, x)^{\text{norm}}((S^1)^{\text{cts}})$.*

This also holds if we replace X_\bullet with any proper complex variety X .

Proof. The key observation is that the S^1 -action defined in [Sim3] is functorial in X , and that semisimplicity is preserved by pullbacks between compact Kähler manifolds (since Higgs bundles pull back to Higgs bundles), so there is a canonical isomorphism $t(\partial_i^{-1} \mathbb{V}) \cong \partial_i^{-1}(t\mathbb{V})$ for $t \in S^1$; thus it makes sense for us to define

$$t(\mathbb{V}, \alpha) := (t\mathbb{V}, t(\alpha)),$$

whenever \mathbb{V} is semisimple on X_0 .

If \mathcal{C} is the category of finite-dimensional real local systems on X_\bullet , this defines a S^1 -action on the full subcategory $\mathcal{C}' \subset \mathcal{C}$ consisting of those local systems \mathbb{V} on X_\bullet whose restrictions to X_0 (or equivalently to all X_n) are semisimple. Now, the category of $\varpi_1(|X_\bullet|, x)^{\text{norm}}$ -representations is equivalent to \mathcal{C}' (assuming, without loss of generality, that $|X_\bullet|$ is connected). By Tannakian duality, this defines a S^1 -action on $\varpi_1(|X_\bullet|, x)^{\text{norm}}$.

Since X_0, X_1 are smooth and proper, the actions of S^1 on their reductive pro-algebraic fundamental groupoids are continuous by Lemma 6.10, corresponding to maps

$$\pi_1(X_i; T) \rightarrow \varpi_1(X_i; T)^{\text{red}}((S^1)^{\text{cts}}).$$

The morphisms $\varpi_1(X_i; a_i^{-1}(x)) \rightarrow \varpi_1(|X_\bullet|, x)$ (coming from $a_i: X_i \rightarrow |X_\bullet|$) then give us maps

$$\pi_1(X_i; a_i^{-1}(x)) \rightarrow \varpi_1(|X_\bullet|, x)^{\text{norm, red}}((S^1)^{\text{cts}}),$$

compatible with the simplicial operations on X_\bullet . Since

$$\pi_1(X_1; a_1^{-1}(x)) \begin{array}{c} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{array} \pi_1(X_0; a_0^{-1}(x)) \rightarrow \pi_1(|X_\bullet|, x)$$

is a coequaliser diagram in the category of groupoids, this gives us a map

$$\pi_1(|X_\bullet|, x) \rightarrow \varpi_1(|X_\bullet|, x)^{\text{norm, red}}((S^1)^{\text{cts}}).$$

For the final part, replace a proper complex variety with a simplicial smooth proper resolution, as in Corollary 9.3. \square

9.2. The Malcev homotopy type. Now fix a simplicial diagram X_\bullet of compact Kähler manifolds, and take a full and essentially surjective representation $\rho: \varpi_1(|X_\bullet|, x)^{\text{norm, red}} \rightarrow R$. As in Definition 3.32, this gives rise to an R -torsor \mathbb{B}_ρ on X .

Definition 9.9. Define the cosimplicial DGAs

$$A^\bullet(X_\bullet, O(\mathbb{B}_\rho)), H^*(X_\bullet, O(\mathbb{B}_\rho)) \in cDGA\text{lg}(R)$$

by $n \mapsto A^\bullet(X_n, O(\mathbb{B}_\rho))$ and $n \mapsto H^*(X_n, O(\mathbb{B}_\rho))$.

Definition 9.10. Given a point $x_0 \in X_0$, define $x_0^* : A^\bullet(X_\bullet, O(\mathbb{B}_\rho)) \rightarrow O(R)$ to be given in cosimplicial degree n by $((\sigma_0)^n x_0)^* : A^\bullet(X_n, O(\mathbb{B}_\rho)) \rightarrow O(\mathbb{B}_\rho)_{(\sigma_0)^n x_0} \cong O(R)$.

Lemma 9.11. *The relative Malcev homotopy type $|X_\bullet|^{\rho, \text{Mal}}$ is represented by the morphism*

$$(\text{Th}(A^\bullet(X_\bullet, O(\mathbb{B}_\rho))) \xrightarrow{x_0^*} O(R)) \in \text{Ho}(D\text{GAlg}(R)),$$

where $\text{Th} : cD\text{GAlg}(R) \rightarrow D\text{GAlg}(R)$ is the Thom-Sullivan functor (Definition 3.28) mapping cosimplicial DG algebras to DG algebras.

Proof. This is true for any simplicial diagram of manifolds, and follows by combining Propositions 3.29 and 3.35. \square

9.3. Mixed Hodge structures. Retaining the hypothesis that X_\bullet is a simplicial proper complex variety, observe that a representation of $\varpi_1(|X_\bullet|, x)^{\text{norm, red}}$ corresponds to a semisimple representation of X_\bullet whose pullbacks to each X_n are all semisimple. This follows because the morphisms $X_n \rightarrow X_0$ of compact Kähler manifolds all preserve semisimplicity under pullback, as observed in Proposition 9.8.

Theorem 9.12. *If R is any quotient of $\varpi_1(|X_\bullet|, x)^{\text{norm, red}}$ (resp. any quotient to which the $(S^1)^\delta$ -action of Proposition 9.8 descends and acts algebraically), then there is an algebraic mixed twistor structure (resp. mixed Hodge structure) $(|X_\bullet|, x)_{\text{MTS}}^{\rho, \text{Mal}}$ (resp. $(|X_\bullet|, x)_{\text{MHS}}^{\rho, \text{Mal}}$) on the relative Malcev homotopy type $(|X_\bullet|, x)^{\rho, \text{Mal}}$, where ρ denotes the quotient map.*

There is also a \mathbb{G}_m -equivariant (resp. S -equivariant) splitting

$$\mathbb{A}^1 \times (\underline{\text{gr}}(|X_\bullet|^{\rho, \text{Mal}}, 0)_{\text{MTS}}) \times \text{SL}_2 \simeq (|X_\bullet|, x)_{\text{MTS}}^{\rho, \text{Mal}} \times_{C^*, \text{row}_1}^{\mathbf{R}} \text{SL}_2$$

(resp.

$$\mathbb{A}^1 \times (\underline{\text{gr}}(|X_\bullet|^{\rho, \text{Mal}}, 0)_{\text{MHS}}) \times \text{SL}_2 \simeq (|X_\bullet|, x)_{\text{MHS}}^{\rho, \text{Mal}} \times_{C^*, \text{row}_1}^{\mathbf{R}} \text{SL}_2)$$

on pulling back along $\text{row}_1 : \text{SL}_2 \rightarrow C^*$, whose pullback over $0 \in \mathbb{A}^1$ is given by the opposedness isomorphism.

Proof. We define the cosimplicial DGA $\tilde{A}(X_\bullet, O(\mathbb{B}_\rho))$ on C by $n \mapsto \tilde{A}^\bullet(X_n, O(\mathbb{B}_\rho))$, observing that functoriality (similarly to Proposition 5.20) ensures that the simplicial and DGA structures are compatible. This has an augmentation $x^* : \tilde{A}(X_\bullet, O(\mathbb{B}_\rho)) \rightarrow O(R) \otimes O(C)$ given in level n by $((\sigma_0)^n x)^*$.

We then define the mixed twistor structure by

$$|X_\bullet|_{\text{MHS}}^{\rho, \text{Mal}} := (\text{Spec Th } \xi(\tilde{A}(X_\bullet, O(\mathbb{B}_\rho)), \tau_{\tilde{A}})) \times_C C^* \in dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(\mathbb{G}_m \times R \rtimes S),$$

with

$$\underline{\text{gr}}|X_\bullet|_{\text{MHS}}^{\rho, \text{Mal}} = \text{Spec}(\text{Th } H^*(X_\bullet, O(\mathbb{B}_\rho))) \in dg_{\mathbb{Z}}\text{Aff}(R \rtimes S);$$

the definitions of $|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}$ are similar, replacing S with \mathbb{G}_m .

For any DGA B , we may regard B as a cosimplicial DGA (with constant cosimplicial structure), and then $\text{Th}(B) = B$. In particular, $\text{Th}(O(R)) = O(R)$, so we have a basepoint $\text{Spec Th}(x^*) : \mathbb{A}^1 \times R \times C^* \rightarrow |X_\bullet|_{\text{MHS}}^{\rho, \text{Mal}}$, giving

$$(|X_\bullet|, x)_{\text{MHS}}^{\rho, \text{Mal}} \in dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathbb{G}_m \times S),$$

and similarly for $|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}$.

The proof of Theorem 5.14 now carries over. For a singular variety X , apply Proposition 9.2 to substitute a simplicial smooth proper variety X_\bullet . \square

Corollary 9.13. *In the scenario of Theorem 9.12, the homotopy groups $\varpi_n(|X_\bullet|^{\rho, \text{Mal}}, x)$ for $n \geq 2$, and the Hopf algebra $O(\varpi_1(|X_\bullet|^{\rho, \text{Mal}}, x))$ carry natural ind-MTS (resp. ind-MHS), functorial in (X_\bullet, x) , and compatible with the action of ϖ_1 on ϖ_n , the Whitehead bracket and the Hurewicz maps $\varpi_n(|X_\bullet|^{\rho, \text{Mal}}, x) \rightarrow H^n(|X_\bullet|, O(\mathbb{B}_\rho))^\vee$.*

Moreover, there are \mathcal{S} -linear isomorphisms

$$\begin{aligned} \varpi_n(|X_\bullet|^\rho, \text{Mal}, x)^\vee \otimes \mathcal{S} &\cong \pi_n(\text{Th H}^*(X_\bullet, O(\mathbb{B}_\rho)))^\vee \otimes \mathcal{S} \\ O(\varpi_1(|X_\bullet|^\rho, \text{Mal}, x)) \otimes \mathcal{S} &\cong O(R \times \pi_1(\text{Th H}^*(X_\bullet, O(\mathbb{B}_\rho))) \otimes \mathcal{S} \end{aligned}$$

of quasi-MTS (resp. quasi-MHS). The associated graded map from the weight filtration is just the pullback of the standard isomorphism $\text{gr}_W \varpi_* (|X_\bullet|^\rho, \text{Mal}) \cong \pi_*(\text{Th H}^*(X_\bullet, O(\mathbb{B}_\rho)))$.

Here, $\pi_*(B)$ are the homotopy groups $\text{H}_{*-1} \bar{G}(B)$ associated to the $R \times S$ -equivariant DGA $\text{H}^*(X, O(\mathbb{B}_\rho))$ (as constructed in Definition 3.23), with the induced real twistor (resp. Hodge) structure.

Furthermore, $W_0 O(\varpi_1(|X_\bullet|^\rho, \text{Mal}, x)) = O(\varpi_1(|X_\bullet|^\rho, \text{Mal}, x)^{\text{norm}})$.

Proof. This is essentially the same as Corollary 5.16. Note that we may simplify the calculation of $\pi_*(\text{Th H}^*(X_\bullet, O(\mathbb{B}_\rho)))$ by observing that $\pi_*(C^\bullet) = \pi_* \text{Spec}(DC^\bullet)$, where D denotes cosimplicial denormalisation, so $\pi_*(\text{Th H}^*(X_\bullet, O(\mathbb{B}_\rho))) = \pi_* \text{Spec}(\text{diag } DH^*(X_\bullet, O(\mathbb{B}_\rho)))$.

For the final statement, note that representations of $\text{gr}_0^W \varpi_1(|X_\bullet|^\rho, \text{Mal}, x) := \text{Spec } W_0 O(\varpi_1(|X_\bullet|^\rho, \text{Mal}, x))$ correspond to representations of $\varpi_1(|X_\bullet|^\rho, \text{Mal}, x)$ which annihilate the image of $W_{-1} \varpi_1(X_n^{\rho, \text{Mal}}, x)$ for all n . Since X_n is smooth and projective, we just have $W_{-1} \varpi_1(X_n^{\rho, \text{Mal}}, x) = R_{\text{u}} \varpi_1(X_n^{\rho, \text{Mal}}, x)$, so these are precisely the normally semisimple representations. \square

Corollary 9.14. *If $\pi_1(|X_\bullet|, x)$ is algebraically good with respect to R and the homotopy groups $\pi_n(|X_\bullet|, x)$ have finite rank for all $n \geq 2$, with $\pi_n(|X_\bullet|, x) \otimes_{\mathbb{Z}} \mathbb{R}$ an extension of R -representations, then Corollary 9.13 gives mixed twistor (resp. mixed Hodge) structures on $\pi_n(|X_\bullet|, x) \otimes \mathbb{R}$ for all $n \geq 2$, by Theorem 3.17.*

Proposition 9.15. *When $R = 1$, the mixed Hodge structures of Corollary 9.12 agree with those defined in [Hai2, Theorem 6.3.1].*

Proof. Proposition 5.6 adapts to simplicial varieties, showing that our algebraic mixed Hodge structure on the simplicial variety recovers the mixed Hodge complex of [Hai2, Theorem 5.6.4], by applying the Thom-Sullivan functor to pass from cosimplicial to DG algebras.

Since the reduced bar construction is just our functor \bar{G} , it follows from Theorem 3.29 that our characterisation of homotopy groups (Definition 3.7) is the same as that given in [Hai2], so our construction of Hodge structures on homotopy groups is essentially the same as [Hai2, Theorem 4.2.1]. \square

9.4. Enriching twistor structures. For the remainder of this section, assume that R is any quotient of $(\pi_1(|X_\bullet|, x))_{\mathbb{R}}^{\text{red, norm}}$ to which the $(S^1)^\delta$ -action descends, but does not necessarily act algebraically.

Proposition 9.16. *There is a natural $(S^1)^\delta$ -action on $\text{gr}|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}$, giving a $(S^1)^\delta$ -invariant map*

$$h \in \text{Hom}_{\text{Ho}(\mathbb{S} \downarrow BR((S^1)^{\text{cts}}))}(\text{Sing}(|X_\bullet|, x), \mathbf{R} \text{holim}_{R((S^1)^{\text{cts}})} (|X_\bullet|, x)_{\mathbb{T}}^{\rho, \text{Mal}}((S^1)^{\text{cts}})_{C^*}),$$

where $(|X_\bullet|, x)_{\mathbb{T}}^{\rho, \text{Mal}}((S^1)^{\text{cts}})_{C^*} := \text{Hom}_{C^*}((S^1)^{\text{cts}}, (|X_\bullet|^\rho, \text{Mal}, x)_{\mathbb{T}})$.

Moreover, for $1 : \text{Spec } \mathbb{R} \rightarrow (S^1)^{\text{cts}}$, the map

$$1^* h : \text{Sing}(|X_\bullet|, x) \rightarrow \mathbf{R} \text{holim}_{R(\mathbb{R})} (|X_\bullet|^\rho, \text{Mal}, x)_{\mathbb{T}}((S^1)^{\text{cts}})_{C^*} \times_{BR((S^1)^{\text{cts}})} BR(\mathbb{R})$$

in $\text{Ho}(\mathbb{S} \downarrow BR(\mathbb{R}))$ is just the canonical map

$$\text{Sing}(|X_\bullet|, x) \rightarrow \mathbf{R} \text{holim}_{R(\mathbb{R})} (|X_\bullet|^\rho, \text{Mal}, x)_{\mathbb{T}}(\mathbb{R}).$$

Proof. We first note that Proposition 6.3 adapts by functoriality to give a $(S^1)^\delta$ -action on the mixed twistor structure $|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}$ of Theorem 9.12. It also gives a $(S^1)^\delta$ -action on $\underline{\text{gr}}|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}$, for which the $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant splitting

$$\mathbb{A}^1 \times \underline{\text{gr}}|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}} \times \text{SL}_2 \cong (|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}, x) \times_{C^*, \text{row}_1}^{\mathbf{R}} \text{SL}_2$$

of Theorem 9.12 is also $(S^1)^\delta$ -equivariant.

The proof of Proposition 6.11 also adapts by functoriality, with h above extending the map $h : (|X_\bullet|, x) \rightarrow BR((S^1)^{\text{cts}})$ corresponding to the group homomorphism $h : \pi_1(|X_\bullet|, x) \rightarrow R((S^1)^{\text{cts}})$ given by $h(t) = \sqrt{h}(t^2)$, for \sqrt{h} as in Proposition 9.8. \square

Thus (for R any quotient of $\varpi_1(|X_\bullet|, x)^{\text{red, norm}}$ to which the $(S^1)^\delta$ -action descends), we have:

Corollary 9.17. *If the group $\varpi_n(|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}, x)$ is finite-dimensional and spanned by the image of $\pi_n(|X_\bullet|, x)$, then the former carries a natural mixed Hodge structure, which splits on tensoring with \mathcal{S} and extends the mixed twistor structure of Corollary 9.13. This is functorial in X_\bullet and compatible with the action of ϖ_1 on ϖ_n , the Whitehead bracket, the R -action, and the Hurewicz maps $\varpi_n(|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}, x) \rightarrow \text{H}^n(|X_\bullet|, O(\mathbb{B}_\rho))^\vee$.*

Proof. We first note that Corollary 6.12 adapts to show that for all n , the homotopy class of maps $\pi_n(|X_\bullet|, x) \times S^1 \rightarrow \varpi_n(|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}, x)_\mathbb{T}$, given by composing the maps $\pi_n(|X_\bullet|, x) \rightarrow \varpi_n(|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}, x)$ with the $(S^1)^\delta$ -action on $(|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}, x)_\mathbb{T}$, are analytic. The proof of Corollary 6.13 then carries over to this context. \square

Remark 9.18. Observe that if $\pi_1(|X_\bullet|, x)$ is algebraically good with respect to R and the homotopy groups $\pi_n(|X_\bullet|, x)$ have finite rank for all $n \geq 2$, with $\pi_n(|X_\bullet|, x) \otimes_{\mathbb{Z}} \mathbb{R}$ an extension of R -representations, then Theorem 3.17 implies that $\varpi_n(|X_\bullet|_{\text{MTS}}^{\rho, \text{Mal}}, x) \cong \pi_n(|X_\bullet|, x) \otimes \mathbb{R}$, ensuring that the hypotheses of Corollary 6.13 are satisfied.

10. ALGEBRAIC MHS/MTS FOR QUASI-PROJECTIVE VARIETIES I

Fix a smooth compact Kähler manifold X , a divisor D locally of normal crossings, and set $Y := X - D$. Let $j : Y \rightarrow X$ be the inclusion morphism.

Definition 10.1. Denote the sheaf of real \mathcal{C}^∞ n -forms on X by \mathcal{A}_X^n , and let \mathcal{A}_X^\bullet be the resulting complex (the real sheaf de Rham complex on X).

Let $\mathcal{A}_X^\bullet[D] \subset j_*\mathcal{A}_Y$ be the sheaf of dg \mathcal{A}_X^\bullet -subalgebras locally generated by $\{\log r_i, d \log r_i, d^c \log r_i\}_{1 \leq i \leq m}$, where D is given in local co-ordinates by $D = \bigcup_{i=1}^m \{z_i = 0\}$, and $r_i = |z_i|$.

Let $\mathcal{A}_X^\bullet\langle D \rangle \subset j_*\mathcal{A}_Y \otimes \mathbb{C}$ be the sheaf of dg $\mathcal{A}_X^\bullet \otimes \mathbb{C}$ -subalgebras locally generated by $\{d \log z_i\}_{1 \leq i \leq m}$.

Note that $d^c \log r_i = d \arg z_i$.

Definition 10.2. Construct increasing filtrations on $\mathcal{A}_X^\bullet\langle D \rangle$ and $\mathcal{A}_X^\bullet[D]$ by setting

$$\begin{aligned} J_0\mathcal{A}_X^\bullet[D] &= \mathcal{A}_X^\bullet, \\ J_0\mathcal{A}_X^\bullet\langle D \rangle &= \mathcal{A}_X^\bullet \otimes \mathbb{C}, \end{aligned}$$

then forming $J_r\mathcal{A}_X^\bullet\langle D \rangle \subset \mathcal{A}_X^\bullet\langle D \rangle$ and $J_r\mathcal{A}_X^\bullet[D] \subset \mathcal{A}_X^\bullet[D]$ inductively by the local expressions

$$\begin{aligned} J_r\mathcal{A}_X^\bullet\langle D \rangle &= \sum_i J_{r-1}\mathcal{A}_X^\bullet\langle D \rangle d \log z_i, \\ J_r\mathcal{A}_X^\bullet[D] &= \sum_i J_{r-1}\mathcal{A}_X^\bullet[D] \log r_i + \sum_i J_{r-1}\mathcal{A}_X^\bullet[D] d \log r_i + \sum_i J_{r-1}\mathcal{A}_X^\bullet[D] d^c \log r_i, \end{aligned}$$

for local co-ordinates as above.

Given any cochain complex V , we denote the good truncation filtration by $\tau_n V := \tau^{\leq n} V$.

Lemma 10.3. *The maps*

$$\begin{aligned} (\mathcal{A}_X^\bullet \langle D \rangle, J) &\leftarrow (\mathcal{A}_X^\bullet \langle D \rangle, \tau) \rightarrow (j_* \mathcal{A}_Y^\bullet \otimes \mathbb{C}, \tau) \\ (\mathcal{A}_X^\bullet \llbracket D \rrbracket, J) &\leftarrow (\mathcal{A}_X^\bullet \llbracket D \rrbracket, \tau) \rightarrow (j_* \mathcal{A}_Y^\bullet, \tau) \end{aligned}$$

are filtered quasi-isomorphisms of complexes of sheaves on X .

Proof. This is essentially the same as [Del1] Prop 3.1.8, noting that the inclusion $\mathcal{A}_X^\bullet \langle D \rangle \hookrightarrow \mathcal{A}_X^\bullet \llbracket D \rrbracket \otimes \mathbb{C}$ is a filtered quasi-isomorphism, because $\mathcal{A}_X^\bullet \llbracket D \rrbracket \otimes \mathbb{C}$ is locally freely generated over $\mathcal{A}_X^\bullet \langle D \rangle$ by the elements $\log r_i$ and $d \log r_i$. \square

An immediate consequence of this lemma is that for all $m \geq 0$, the flabby complex $\mathrm{gr}_m^J \mathcal{A}_X \llbracket D \rrbracket$ is quasi-isomorphic to $\mathbf{R}^m j_* \mathbb{R}$.

Definition 10.4. For any real local system \mathbb{V} on X , define

$$\begin{aligned} \mathcal{A}_X^\bullet(\mathbb{V}) &:= \mathcal{A}_X^\bullet \otimes_{\mathbb{R}} \mathbb{V}, & \mathcal{A}_X^\bullet(\mathbb{V}) \langle D \rangle &:= \mathcal{A}_X^\bullet \langle D \rangle \otimes_{\mathbb{R}} \mathbb{V}, & \mathcal{A}_X^\bullet(\mathbb{V}) \llbracket D \rrbracket &:= \mathcal{A}_X^\bullet \llbracket D \rrbracket \otimes_{\mathbb{R}} \mathbb{V}. \\ A^\bullet(X, \mathbb{V}) &:= \Gamma(X, \mathcal{A}_X^\bullet(\mathbb{V})), & A^\bullet(X, \mathbb{V}) \langle D \rangle &:= \Gamma(X, \mathcal{A}_X^\bullet(\mathbb{V}) \langle D \rangle), \\ & & A^\bullet(X, \mathbb{V}) \llbracket D \rrbracket &:= \Gamma(X, \mathcal{A}_X^\bullet(\mathbb{V}) \llbracket D \rrbracket). \end{aligned}$$

These inherit filtrations, given by

$$\begin{aligned} J_r A^\bullet(X, \mathbb{V}) \langle D \rangle &:= \Gamma(X, J_r \mathcal{A}_X^\bullet \langle D \rangle \otimes \mathbb{V}), \\ J_r A^\bullet(X, \mathbb{V}) \llbracket D \rrbracket &:= \Gamma(X, J_r \mathcal{A}_X^\bullet \llbracket D \rrbracket \otimes \mathbb{V}). \end{aligned}$$

Note that Lemma 10.3 implies that for all $m \geq 0$, the flabby complex $\mathrm{gr}_m^J \mathcal{A}_X(\mathbb{V}) \llbracket D \rrbracket$ (resp. $\mathrm{gr}_m^J \mathcal{A}_X^\bullet(\mathbb{V}) \langle D \rangle$) is quasi-isomorphic to $\mathbf{R}^m j_*(j^{-1} \mathbb{V}) \cong \mathbb{V} \otimes \mathbf{R}^m j_* \mathbb{R}$ (resp. $\mathbf{R}^m j_*(j^{-1} \mathbb{V}) \otimes \mathbb{C}$).

Remark 10.5. The filtration J essentially corresponds to the weight filtration W of [Del1, 3.1.5]. However, the true weight filtration on cohomology, and hence on homotopy types, is given by the décalage $\mathrm{Dec} J$ (as in [Del1, Theorem 3.2.5] or [Mor]). Since $\mathrm{Dec} J$ gives the correct notion of weights, not only for mixed Hodge structures but also for Frobenius eigenvalues in the ℓ -adic case of [Pri6], we reserve the terminology “weight filtration” for $W := \mathrm{Dec} J$.

10.0.1. *Decreasing Hodge and twistor filtrations.* We now introduce refinements of the constructions from §1 in order to deal with the non-abelian analogue of decreasing filtrations $F^0 \supset F^1 \supset \dots$

Definition 10.6. Mat_n is the algebraic monoid of $n \times n$ -matrices. Thus $\mathrm{Mat}_1 \cong \mathbb{A}^1$, so acts on \mathbb{A}^1 by multiplication. Note that the inclusion $\mathbb{G}_m \hookrightarrow \mathrm{Mat}_1$ identifies Mat_1 -representations with non-negatively weighted \mathbb{G}_m -representations.

Let $\bar{S} := (\mathrm{Mat}_1 \times S^1)/(-1, -1)$, giving a real algebraic monoid whose subgroup of units is S , via the isomorphism $S \cong (\mathbb{G}_m \times S^1)/(-1, -1)$. There is thus a morphism $\bar{S} \rightarrow S^1$ given by $(m, u) \mapsto u^2$, extending the isomorphism $S/\mathbb{G}_m \cong S^1$.

Note that \bar{S} -representations correspond via the morphism $S \rightarrow \bar{S}$ to real Hodge structures of non-negative weights. In the co-ordinates of Remark 1.3,

$$\bar{S} = \mathrm{Spec} \mathbb{R}[u, v, \frac{u^2 - v^2}{u^2 + v^2}, \frac{2uv}{u^2 + v^2}].$$

The following adapts Definition 4.4 to non-positive weights, replacing \mathbb{G}_m and S with Mat_1 and \bar{S} respectively.

Definition 10.7. A non-positively weighted algebraic mixed Hodge structure $(X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}}$ on a pointed Malcev homotopy type $(X, x)^{R, \mathrm{Mal}}$ consists of the following data:

- (1) an algebraic action of S^1 on R ,
- (2) an object

$$(X, x)_{\text{MHS}}^{R, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\text{Mat}_1 \times S)),$$

where S acts on R via the S^1 -action, using the canonical isomorphism $S^1 \cong S/\mathbb{G}_m$,

- (3) an object

$$\underline{\text{gr}}(X, x)_{\text{MHS}}^{R, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*(\bar{S})),$$

- (4) an isomorphism $(X, x)^{R, \text{Mal}} \cong (X, x)_{\text{MHS}}^{R, \text{Mal}} \times_{(\mathbb{A}^1 \times C^*), (1,1)}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)$,
- (5) an isomorphism (called the opposedness isomorphism)

$$\theta^\sharp(\underline{\text{gr}}(X, x)_{\text{MHS}}^{R, \text{Mal}}) \times C^* \cong (X, x)_{\text{MHS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^*}(R)_*(\text{Mat}_1 \times S)),$$

for the canonical map $\theta : \text{Mat}_1 \times S \rightarrow \bar{S}$ given by combining the inclusion $\text{Mat}_1 \hookrightarrow \bar{S}$ with the inclusion $S \hookrightarrow \bar{S}$.

Definition 10.8. An non-positively weighted algebraic mixed twistor structure $(X, x)_{\text{MTS}}^{R, \text{Mal}}$ on a pointed Malcev homotopy type $(X, x)^{R, \text{Mal}}$ consists of the following data:

- (1) an object

$$(X, x)_{\text{MTS}}^{R, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\text{Mat}_1 \times \mathbb{G}_m)),$$

- (2) an object $\underline{\text{gr}}(X, x)_{\text{MTS}}^{R, \text{Mal}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*(\text{Mat}_1))$,
- (3) an isomorphism $(X, x)^{R, \text{Mal}} \cong (X, x)_{\text{MTS}}^{R, \text{Mal}} \times_{(\mathbb{A}^1 \times C^*), (1,1)}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)$,
- (4) an isomorphism (called the opposedness isomorphism)

$$\theta^\sharp(\underline{\text{gr}}(X, x)_{\text{MTS}}^{R, \text{Mal}}) \times C^* \cong (X, x)_{\text{MTS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^*}(R)_*(\text{Mat}_1 \times \mathbb{G}_m)),$$

for the canonical map $\theta : \text{Mat}_1 \times \mathbb{G}_m \rightarrow \text{Mat}_1$ given by combining the identity on Mat_1 with the inclusion $\mathbb{G}_m \hookrightarrow \text{Mat}_1$.

10.1. The Hodge and twistor filtrations. We begin by generalising some constructions from §7.

Definition 10.9. Given a semisimple real local system \mathbb{V} on X , define the sheaf $\tilde{\mathcal{A}}_X^\bullet(\mathbb{V})[[D]]$ of cochain complexes on $X_{\text{an}} \times C_{\text{Zar}}$ by

$$\tilde{\mathcal{A}}_X^\bullet(\mathbb{V})[[D]] = (\mathcal{A}_X^*(\mathbb{V})[[D]] \otimes_{\mathbb{R}} \mathcal{O}_C, uD + vD^c),$$

for co-ordinates u, v as in Remark 1.3. We denote the differential by $\tilde{D} := uD + vD^c$.

Define the quasi-coherent sheaf $\tilde{A}^\bullet(X, \mathbb{V})[[D]]$ of cochain complexes on C by $\tilde{A}^\bullet(X, \mathbb{V})[[D]] := \text{pr}_{C^*}(\tilde{\mathcal{A}}_X^\bullet(\mathbb{V})[[D]])$.

Definition 10.10. Note that the \diamond action on \mathcal{A} from Definition 2.2 gives an action of $\mathbb{G}_m \subset S$ on $\tilde{\mathcal{A}}_X^\bullet(\mathbb{V})[[D]]$ over C . If we have a semisimple local system \mathbb{V} , equipped with a discrete (resp. algebraic) action of S^1 on $\mathcal{A}_X^0(\mathbb{V})$, recall that the proof of Proposition 6.3 (resp. Theorem 5.14) gives a discrete $S(\mathbb{R}) = \mathbb{C}^\times$ -action (resp. an algebraic S -action) \boxtimes on $\tilde{\mathcal{A}}_X^\bullet(\mathbb{V})$, and note that this extends naturally to $\tilde{\mathcal{A}}_X^\bullet(\mathbb{V})[[D]]$.

Definition 10.11. Given a Zariski-dense representation $\rho : \pi_1(X, jy) \rightarrow R(\mathbb{R})$, for R a pro-reductive pro-algebraic group, define an algebraic twistor filtration on the relative Malcev homotopy type $(Y, y)^{R, \text{Mal}}$ by

$$(Y, y)_{\mathbb{T}}^{R, \text{Mal}} := (R \times C^* \xrightarrow{\text{Spec}(jy)^*} \mathbf{Spec}_{C^*} \tilde{A}^\bullet(X, \mathbb{O}(R))[[D]]|_{C^*}),$$

in $\text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^*}(R)_*(\mathbb{G}_m))$, where $\mathbb{O}(R)$ is the local system of Proposition 3.35, which is necessarily a sum of finite-dimensional semisimple local systems, and $\mathbb{G}_m \subset S$ acts via the \boxtimes action above.

A Zariski-dense representation $\rho: \pi_1(X, jy) \rightarrow R(\mathbb{R})$ is equivalent to a morphism $\varpi_1(X, jy)^{\text{red}} \twoheadrightarrow R$ of pro-algebraic groups, where $\varpi_1(X, jy)^{\text{red}}$ is the reductive quotient of the real pro-algebraic fundamental group $\varpi_1(X, jy)$. [Sim3] effectively gives a discrete S^1 -action on $\varpi_1(X, jy)^{\text{red}}$, corresponding (as in Lemma 5.7) to the \otimes action on semisimple local systems from Lemma 5.7. This S^1 -action thus descends to R if and only if $\mathbb{O}(R)$ satisfies the conditions of Definition 10.10. Moreover, the S^1 -action is algebraic on R if and only if $\mathbb{O}(R)$ becomes a weight 0 variation of Hodge structures under the \otimes action, by Proposition 5.12.

Definition 10.12. Take a Zariski-dense representation $\rho: \pi_1(X, jy) \rightarrow R(\mathbb{R})$, for R a pro-reductive pro-algebraic group to which the S^1 -action on $\varpi_1(X, jy)^{\text{red}}$ descends and acts algebraically. Then define an algebraic Hodge filtration on the relative Malcev homotopy type $(Y, y)^{R, \text{Mal}}$ by

$$(Y, y)_{\mathbb{F}}^{R, \text{Mal}} := (R \times C^* \xrightarrow{\text{Spec}(jy)^*} \mathbf{Spec}_{C^*} \tilde{A}^\bullet(X, \mathbb{O}(R)) \llbracket D \rrbracket_{C^*}),$$

in $\text{Ho}(\text{dg}_{\mathbb{Z}} \text{Aff}_{C^*}(R)_*(S))$, where the S -action is given by the \boxtimes action of Definition 10.10.

If the S^1 action descends to R but is not algebraic, we still have the following:

Proposition 10.13. *The algebraic twistor filtration $(Y, y)_{\mathbb{T}}^{R, \text{Mal}}$ of Definition 10.11 is equipped with an $(S^1)^\delta$ -action (i.e. a discrete S^1 -action) with the properties that*

- (1) *the S^1 -action and \mathbb{G}_m -actions commute,*
- (2) *the projection $(Y, y)_{\mathbb{T}}^{R, \text{Mal}} \rightarrow C^*$ is S^1 -equivariant, and*
- (3) *$-1 \in S^1$ acts as $-1 \in \mathbb{G}_m$.*

Proof. This is the same as the proof of Proposition 6.3. The action comes from Definition 10.10, with $t \in (S^1)^\delta$ acting on $\mathcal{A}_X^*(\mathbb{O}(R)) \llbracket D \rrbracket$ by $t \boxtimes (a \otimes v) = (t \diamond a) \otimes (t^2 \otimes v)$. \square

10.2. Higher direct images and residues.

Definition 10.14. Let $D^m \subset X$ denote the union of all m -fold intersections of local components of the divisor $D \subset X$, and set $D^{(m)}$ to be its normalisation. Write $\nu_m: D^{(m)} \rightarrow X$ for the composition of the normalisation map with the embedding of D^m , and set $C^{(m)} := \nu_m^{-1} D^{m+1}$.

As in [Tim2, 1.2], observe that $D^m - D^{m+1}$ is a smooth quasi-projective variety, isomorphic to $D^{(m)} - C^{(m)}$. Moreover, $D^{(m)}$ is a smooth projective variety, with $C^{(m)}$ a normal crossings divisor.

Definition 10.15. Recall from [Del1] Definition 2.1.13 that for $n \in \mathbb{Z}$, $\mathbb{Z}(n)$ is the lattice $(2\pi i)^n \mathbb{Z}$, equipped with the pure Hodge structure of type $(-n, -n)$. Given an abelian group A , write $A(n) := A \otimes_{\mathbb{Z}} \mathbb{Z}(n)$.

Definition 10.16. On $D^{(m)}$, define ε^m by the property that $\varepsilon^m(m)$ is the integral local system of orientations of D^m in X . Thus ε^n is the local system $\varepsilon_{\mathbb{Z}}^n$ defined in [Del1, 3.1.4].

Lemma 10.17. $\mathbf{R}^m j_* \mathbb{Z} \cong \nu_{m*} \varepsilon^m$.

Proof. This is [Del1, Proposition 3.1.9]. \square

Lemma 10.18. *For any local system \mathbb{V} on X , there is a canonical quasi-isomorphism*

$$\text{Res}_m: \text{gr}_m^J \mathcal{A}_X^\bullet(\mathbb{V}) \langle D \rangle \rightarrow \nu_{m*} \mathcal{A}_{D^{(m)}}^\bullet(\mathbb{V} \otimes_{\mathbb{R}} \varepsilon_{\mathbb{C}}^m) [-m]$$

of cochain complexes on X .

Proof. We follow the construction of [Del1, 3.1.5.1]. In a neighbourhood where D is given locally by $\bigcup_i \{z_i = 0\}$, with $\omega \in \mathcal{A}_X^\bullet(\mathbb{V})$, we set

$$\mathrm{Res}_m(\omega \wedge d \log z_1 \wedge \dots \wedge d \log z_m) := \omega|_{D^{(m)}} \otimes \epsilon(z_1, \dots, z_m),$$

where $\epsilon(z_1, \dots, z_m)$ denotes the orientation of the components $\{z_1 = 0\}, \dots, \{z_m = 0\}$.

That Res_m is a quasi-isomorphism follows immediately from Lemmas 10.3 and 10.17. \square

10.3. Opposedness. Fix a Zariski-dense representation $\rho: \pi_1(X, jy) \rightarrow R(\mathbb{R})$, for R a pro-reductive pro-algebraic group.

Proposition 10.19. *If the S^1 -action on $\varpi_1(X, jy)^{\mathrm{red}}$ descends to an algebraic action on R , then for the algebraic Hodge filtration $(Y, y)_{\mathbb{F}}^{R, \mathrm{Mal}}$ of Definition 10.12, the $R \rtimes S$ -equivariant cohomology sheaf*

$$\mathcal{H}^a(\mathrm{gr}_b^J \mathcal{O}(Y, y)_{\mathbb{F}}^{R, \mathrm{Mal}})$$

on C^* defines a pure ind-Hodge structure of weight $a + b$, corresponding to the \boxtimes S -action on

$$\mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b).$$

Proof. We need to show that $\mathrm{H}^a(\mathrm{gr}_b^J \tilde{A}^\bullet(X, \mathbb{O}(R))[[D]])|_{C^*}$ corresponds to a pure ind-Hodge structure of weight $a + b$, or equivalently to a sum of vector bundles of slope $a + b$. We are therefore led to study the complex $\mathrm{gr}_b^J \tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R))[[D]]|_{C^*}$ on $X \times C^*$, since

$$\mathrm{H}^a(\mathrm{gr}_b^J \tilde{A}^\bullet(X, \mathbb{O}(R))[[D]])|_{C^*} = \mathrm{H}^a(X, \mathrm{gr}_b^J \tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R))[[D]])|_{C^*}.$$

In a neighbourhood where D is given locally by $\bigcup_i \{z_i = 0\}$, $\mathrm{gr}^J \tilde{\mathcal{A}}_X^\bullet[[D]]$ is the $\tilde{\mathcal{A}}_X^\bullet$ -algebra generated by the classes $[\log |z_i|]$, $[d \log |z_i|]$ and $[d^c \log |z_i|]$ in gr_1^J . Let $\tilde{C}^* \rightarrow C^*$ be the étale covering of Definition 1.12. Now, $\tilde{d} = ud + vd^c = (u + iv)\partial + (u - iv)\bar{\partial}$, so $\mathrm{gr}^J \tilde{\mathcal{A}}_X^\bullet[[D]]|_{\tilde{C}^*}$ is the $\mathrm{gr}^J \tilde{\mathcal{A}}_X^\bullet|_{\tilde{C}^*}$ -algebra generated by $[\log |z_i|]$, $\tilde{d}[\log |z_i|]$, $[d \log z_i]$.

Since $\tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R))[[D]] = \tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{A}_X^\bullet} \mathcal{A}_X^\bullet[[D]]$, we have an S -equivariant quasi-isomorphism

$$\tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{A}_X^\bullet} \mathrm{gr}^J \mathcal{A}_X^\bullet \langle D \rangle|_{\tilde{C}^*} \hookrightarrow \mathrm{gr}^J \tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R))[[D]]|_{\tilde{C}^*},$$

as the right-hand side is generated over the left by $[\log |z_i|]$, $\tilde{d}[\log |z_i|]$.

Now, Lemma 10.18 gives a quasi-isomorphism

$$\mathrm{Res}_b: \tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{A}_X^\bullet} \mathrm{gr}_b^J \mathcal{A}_X^\bullet \langle \mathbb{V} \rangle \langle D \rangle \rightarrow \nu_{b*} \tilde{\mathcal{A}}_X^\bullet \otimes_{\mathcal{A}_X^\bullet} \mathcal{A}_{D^{(b)}}^\bullet ((\mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon_{\mathbb{C}}^b)[-b]),$$

and the right-hand side is just

$$\nu_{b*} \tilde{\mathcal{A}}_{D^{(b)}}^\bullet(\mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon_{\mathbb{C}}^b)[-b].$$

Therefore

$$\mathrm{gr}_b^J \tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R))[[D]]|_{\tilde{C}^*} \simeq \nu_{b*} \tilde{\mathcal{A}}_{D^{(b)}}^\bullet(\mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon_{\mathbb{C}}^b)[-b]|_{\tilde{C}^*},$$

and in particular Res_b defines an isomorphism

$$\mathrm{H}^a(\mathrm{gr}_b^J \tilde{A}^\bullet(X, \mathbb{O}(R))[[D]]|_{\tilde{C}^*}) \cong \mathrm{H}^{a-b}(\tilde{A}^\bullet(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon_{\mathbb{C}}^b)|_{\tilde{C}^*}).$$

As in §1.1.2, we have an étale pushout $C^* = \tilde{C}^* \cup_{S_{\mathbb{C}}} S$ of affine schemes, so to give an isomorphism $\mathcal{F} \rightarrow \mathcal{G}$ of quasi-coherent sheaves on C^* is the same as giving an isomorphism $f: \mathcal{F}|_{\tilde{C}^*} \rightarrow \mathcal{G}|_{\tilde{C}^*}$ such that $f|_{S_{C_x}}$ is real, in the sense that $f = \bar{f}$ on $S_{\mathbb{C}}$. Since $\tilde{d} \log |z_i| = (u + iv)d \log z_i + (u - iv)d \log \bar{z}_i$ is a boundary, we deduce that $[i(u - iv)^{-1}d \log z_i] \sim [-i(u + iv)^{-1}d \log \bar{z}_i]$, so

$$\overline{(u - iv)^b \mathrm{Res}_b} = (u - iv)^b \mathrm{Res}_b,$$

making use of the fact that ε^b already contains a factor of i^b (coming from $\mathbb{Z}(-b)$).

Therefore $(u - iv)^b \mathrm{Res}_b$ gives an isomorphism

$$\mathrm{H}^a(\mathrm{gr}_b^J \tilde{A}^\bullet(X, \mathbb{O}(R))[[D]]|_{C^*}) \cong \mathrm{H}^{a-b}(\tilde{A}^\bullet(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b)|_{C^*}).$$

Now, $d \log z_i$ is of type $(1, 0)$, while ε^b is of type (b, b) and $(u - iv)$ is of type $(0, -1)$, so it follows that $(u - iv)^b \text{Res}_b$ is of type $(0, 0)$, i.e. S -equivariant.

As in Theorem 5.14, inclusion of harmonic forms gives an S -equivariant isomorphism

$$H^{a-b}(\tilde{A}^\bullet(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b))|_{C^*} \cong H^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b) \otimes \mathcal{O}_{C^*},$$

which is a pure twistor structure of weight $(a - b) + 2b = a + b$. Therefore

$$\mathcal{H}^a(\text{gr}_b^J \tilde{\mathcal{A}}_X^\bullet(\mathbb{O}(R))|_{C^*}) \cong H^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b) \otimes \mathcal{O}_{C^*}$$

is pure of weight $a + b$, as required. \square

Proposition 10.20. *For the algebraic twistor filtration $(Y, y)_{\mathbb{T}}^{R, \text{Mal}}$ of Definition 10.11, the $R \times \mathbb{G}_m$ -equivariant cohomology sheaf*

$$\mathcal{H}^a(\text{gr}_b^J \mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}})$$

on C^* defines a pure ind-twistor structure of weight $a + b$, corresponding to the canonical \mathbb{G}_m -action on

$$H^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b).$$

Proof. The proof of Proposition 10.19 carries over, replacing S -equivariance with \mathbb{G}_m -equivariance, and Theorem 5.14 with Theorem 6.1. \square

Proposition 10.21. *If the S^1 -action on $\varpi_1(X, jy)^{\text{red}}$ descends to R , then the associated discrete S^1 -action of Proposition 10.13 on $\mathcal{H}^a(\text{gr}_b^J \mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}})$ corresponds to the \boxtimes action of $S^1 \subset S$ (see Definition 10.10) on*

$$H^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b).$$

Proof. The proof of Proposition 10.19 carries over, replacing S -equivariance with discrete S -equivariance. \square

Theorem 10.22. *There is a canonical non-positively weighted mixed twistor structure $(Y, y)_{\text{MTS}}^{R, \text{Mal}}$ on $(Y, y)^{R, \text{Mal}}$, in the sense of Definition 10.8.*

Proof. On $\mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}} = \tilde{A}^\bullet(X, \mathbb{O}(R))|_{C^*}$, we define the filtration $\text{Dec } J$ by

$$(\text{Dec } J)_r(\mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}})^n = \{a \in J_{r-n}(\mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}})^n : \tilde{D}a \in J_{r-n-1}(\mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}})^{n+1}\}.$$

For the Rees algebra construction ξ of Lemma 1.7, we then set $\mathcal{O}(Y, y)_{\text{MTS}}^{R, \text{Mal}} \in DG_{\mathbb{Z}} \text{Alg}_{\mathbb{A}^1 \times C^*}(R)_*(\text{Mat}_1 \times \mathbb{G}_m)$ to be

$$\mathcal{O}(Y, y)_{\text{MTS}}^{R, \text{Mal}} := \xi(\mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}}, \text{Dec } J),$$

noting that this is flat and that $(Y, y)_{\text{MTS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 1} \text{Spec } \mathbb{R} = (Y, y)_{\mathbb{T}}^{R, \text{Mal}}$, so

$$(Y, y)_{\text{MTS}}^{R, \text{Mal}} \times_{(\mathbb{A}^1 \times C^*), (1, 1)} \text{Spec } \mathbb{R} \simeq (Y, y)^{R, \text{Mal}}.$$

We define $\underline{\text{gr}}(Y, y)_{\text{MTS}}^{R, \text{Mal}} \in dg_{\mathbb{Z}} \text{Aff}(R)_*(\text{Mat}_1)$ by

$$\underline{\text{gr}}(Y, y)_{\text{MTS}}^{R, \text{Mal}} = \text{Spec} \left(\bigoplus_{a, b} H^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b)[-a], d_1 \right),$$

where $d_1 : H^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b) \rightarrow H^{a-b+2}(D^{(b-1)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^{b-1})$ is the differential in the E_1 sheet of the spectral sequence associated to the filtration J . Combining Lemmas 10.17 and 10.18, it follows that this is the same as the differential $H^{a-b}(X, \mathbf{R}^b j_* j^{-1} \mathbb{O}(R)) \rightarrow H^{a-b+2}(X, \mathbf{R}^{b-1} j_* j^{-1} \mathbb{O}(R))$ in the E_2 sheet of the Leray spectral sequence for $j : Y \rightarrow X$. The augmentation $\bigoplus_{a, b} H^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b) \rightarrow \mathcal{O}(R)$ is just defined to be the unique ring homomorphism $H^0(X, \mathbb{O}(R)) = \mathbb{R} \rightarrow \mathcal{O}(R)$.

In order to show that this defines a mixed twistor structure, it only remains to establish opposedness. Since $(Y, y)_{\text{MTS}}^{R, \text{Mal}}$ is flat,

$$(Y, y)_{\text{MTS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R} \simeq (Y, y)_{\text{MTS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 0} \text{Spec } \mathbb{R},$$

and properties of Rees modules mean that this is just given by

$$\text{Spec } C^*(\text{gr}^{\text{Dec } J} \mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}}) \in \text{dg}_{\mathbb{Z}} \text{Aff}_{C^*}(R)_*(\text{Mat}_1 \times \mathbb{G}_m),$$

where the Mat_1 -action assigns $\text{gr}_n^{\text{Dec } J}$ the weight n .

By [Del1, Proposition 1.3.4], décalage has the formal property that the canonical map

$$\text{gr}_n^{\text{Dec } J} \mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}} \rightarrow \left(\bigoplus_a \mathcal{H}^a(\text{gr}_{n-a}^J \mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}})[-a], d_1 \right)$$

is a quasi-isomorphism. Since the right-hand side is just

$$\left(\bigoplus_a \text{H}^{2a-n}(D^{(n-a)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^{n-a})[-a], d_1 \right) \otimes \mathcal{O}_{C^*}$$

by Proposition 10.20, we have a quasi-isomorphism

$$\underline{\text{gr}}(Y, y)_{\text{MTS}}^{R, \text{Mal}} \times C^* \simeq (Y, y)_{\text{MTS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R}.$$

That this is $(\text{Mat}_1 \times \mathbb{G}_m)$ -equivariant follows because $\text{H}^{2a-n}(D^{(n-a)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^{n-a})$ is of weight $2a - n + 2(n - a) = n$ for the \mathbb{G}_m -action, and of weight n for the Mat_1 -action, being $\text{gr}_n^{\text{Dec } J}$. \square

Theorem 10.23. *If the local system on X associated to any R -representation underlies a polarisable variation of Hodge structure, then there is a canonical non-positively weighted mixed Hodge structure $(Y, y)_{\text{MHS}}^{R, \text{Mal}}$ on $(Y, y)^{R, \text{Mal}}$, in the sense of Definition 10.7.*

Proof. We adapt the proof of Theorem 10.22, replacing Proposition 10.20 with Proposition 10.19. The first condition is equivalent to saying that the S^1 -action descends to R and is algebraic, by Proposition 5.12. We therefore set

$$\mathcal{O}(Y, y)_{\text{MHS}}^{R, \text{Mal}} := \xi(\mathcal{O}(Y, y)_{\mathbb{F}}^{R, \text{Mal}}, \text{Dec } J),$$

for $(Y, y)_{\mathbb{F}}^{R, \text{Mal}}$ as in Definition 10.12, and let

$$\underline{\text{gr}}(Y, y)_{\text{MTS}}^{R, \text{Mal}} = \text{Spec} \left(\bigoplus_{a, b} \text{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b)[-a], d_1 \right),$$

which is now in $\text{dg}_{\mathbb{Z}} \text{Aff}(R)_*(\bar{S})$, since $\mathbb{O}(R)$ is a sum of weight 0 VHS, making $\text{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b)$ a weight $a - b + 2b = a + b$ Hodge structure, and hence an \bar{S} -representation. \square

Proposition 10.24. *If the discrete S^1 -action on $\varpi_1(X, jy)^{\text{red}}$ descends to R , then there are natural $(S^1)^\delta$ -actions on $(Y, y)_{\text{MTS}}^{R, \text{Mal}}$ and $\underline{\text{gr}}(Y, y)_{\text{MTS}}^{R, \text{Mal}}$, compatible with the opposedness isomorphism, and with $-1 \in S^1$ acting as $-1 \in \mathbb{G}_m$.*

Proof. This is a direct consequence of Proposition 10.13 and Proposition 10.21, since the Rees module construction transfers the discrete S^1 -action. \square

10.4. Singular and simplicial varieties.

Proposition 10.25. *If Y is any separated complex scheme of finite type, there exists a simplicial smooth proper complex variety X_\bullet , a simplicial divisor $D_\bullet \subset X_\bullet$ with normal crossings, and a map $(X_\bullet - D_\bullet) \rightarrow Y$ such that $|X_\bullet - D_\bullet| \rightarrow Y$ is a weak equivalence, where $|Z_\bullet|$ is the geometric realisation of the simplicial space $Z_\bullet(\mathbb{C})$.*

Proof. The results in [Del2, §8.2] and [SD, Propositions 5.1.7 and 5.3.4], adapted as in Corollary 9.3, give the equivalence required. \square

Now, let X_\bullet be a simplicial smooth proper complex variety, and $D_\bullet \subset X_\bullet$ a simplicial divisor with normal crossings. Set $Y_\bullet = X_\bullet - D_\bullet$, assume that $|Y_\bullet|$ is connected, and pick a point $y \in |Y_\bullet|$. Let $j : |Y_\bullet| \rightarrow |X_\bullet|$ be the natural inclusion map. We will look at representations of the fundamental group $\varpi_1(|X_\bullet|, jy)^{\text{norm,red}}$ from Definition 9.4.

Using Proposition 10.25, the following gives mixed twistor or mixed Hodge structures on relative Malcev homotopy types of arbitrary complex varieties.

Theorem 10.26. *If R is any quotient of $\varpi_1(|X_\bullet|, jy)^{\text{norm,red}}$ (resp. any quotient to which the $(S^1)^\delta$ -action of Proposition 9.8 descends and acts algebraically), then there is an algebraic mixed twistor structure (resp. mixed Hodge structure) $(|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}}$ (resp. $(|Y_\bullet|, y)_{\text{MHS}}^{R, \text{Mal}}$) on the relative Malcev homotopy type $(|Y_\bullet|, y)^{R, \text{Mal}}$.*

There is also a canonical \mathbb{G}_m -equivariant (resp. S -equivariant) splitting

$$\mathbb{A}^1 \times (\underline{\text{gr}}(|Y_\bullet|^{R, \text{Mal}}, 0)_{\text{MTS}}) \times \text{SL}_2 \simeq (|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}} \times_{C^*, \text{row}_1}^{\mathbf{R}} \text{SL}_2$$

(resp.

$$\mathbb{A}^1 \times (\underline{\text{gr}}(|Y_\bullet|^{R, \text{Mal}}, 0)_{\text{MHS}}) \times \text{SL}_2 \simeq (|Y_\bullet|, y)_{\text{MHS}}^{R, \text{Mal}} \times_{C^*, \text{row}_1}^{\mathbf{R}} \text{SL}_2)$$

on pulling back along $\text{row}_1 : \text{SL}_2 \rightarrow C^*$, whose pullback over $0 \in \mathbb{A}^1$ is given by the opposedness isomorphism.

Proof. We adapt the proof of Theorem 9.12. Define the cosimplicial DGA $\tilde{A}(X_\bullet, \mathbb{O}(R))[[D_\bullet]]$ on C by $n \mapsto \tilde{A}^\bullet(X_n, \mathbb{O}(R))[[D_n]]$, observing that functoriality ensures that the cosimplicial and DGA structures are compatible. This has an augmentation $(jy)^* : \tilde{A}(X_\bullet, \mathbb{O}(R))[[D_\bullet]] \rightarrow \mathbb{O}(R) \otimes \mathbb{O}(C)$ given in level n by $((\sigma_0)^n x)^*$, and inherits a filtration J from the DGAs $\tilde{A}^\bullet(X_n, \mathbb{O}(R))[[D_n]]$.

We then define the mixed Hodge structure to be the object of $dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(\text{Mat}_1 \times R \rtimes S)$ given by

$$|Y_\bullet|_{\text{MHS}}^{R, \text{Mal}} := (\text{Spec Th } \xi(\tilde{A}(X_\bullet, \mathbb{O}(R))[[D_\bullet]], \text{Dec Th}(J))) \times_C C^*,$$

for Th the Thom–Sullivan construction of Definition 3.28. $|Y_\bullet|_{\text{MTS}}^{R, \text{Mal}}$ is defined similarly, replacing S with \mathbb{G}_m . The graded object is given by

$$\underline{\text{gr}}|Y_\bullet|_{\text{MHS}}^{R, \text{Mal}} = \text{Spec Th} \left(\bigoplus_{a,b} \mathbb{H}^{a-b}(D_\bullet^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{Z}} \varepsilon^b)[-a], d_1 \right)$$

in $dg_{\mathbb{Z}}\text{Aff}(R \rtimes \bar{S})$, with $\underline{\text{gr}}|Y_\bullet|_{\text{MTS}}^{R, \text{Mal}}$ given by replacing S with \mathbb{G}_m .

For any DGA B , we may regard B as a cosimplicial DGA (with constant cosimplicial structure), and then $\text{Th}(B) = B$. In particular, $\text{Th}(\mathbb{O}(R)) = \mathbb{O}(R)$, so we have a basepoint $\text{Spec Th}((jy)^*) : \mathbb{A}^1 \times R \times C^* \rightarrow |Y_\bullet|_{\text{MHS}}^{R, \text{Mal}}$, giving

$$(|Y_\bullet|, y)_{\text{MHS}}^{R, \text{Mal}} \in dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\text{Mat}_1 \times S),$$

and similarly for $|Y_\bullet|_{\text{MTS}}^{R, \text{Mal}}$.

The proofs of Theorems 10.23 and 10.22 now carry over for the remaining statements. \square

11. ALGEBRAIC MHS/MTS FOR QUASI-PROJECTIVE VARIETIES II — NON-TRIVIAL MONODROMY

In this section, we assume that X is a smooth projective complex variety, with $Y = X - D$ (for D still a divisor locally of normal crossings). The hypothesis in Theorems 10.22 and 10.23 that R be a quotient of $\varpi_1(X, jy)$ is unnecessarily strong, and corresponds to allowing only those semisimple local systems on Y with trivial monodromy around the divisor. By [Moc1], every semisimple local system on Y carries an essentially unique tame imaginary pluriharmonic metric, so it is conceivable that Theorem 10.22 could hold for any reductive quotient R of $\varpi_1(Y, y)$.

However, Simpson's discrete S^1 -action on $\varpi_1(X, jy)^{\text{red}}$ does not extend to the whole of $\varpi_1(Y, y)^{\text{red}}$, but only to a quotient ${}^\nu\varpi_1(Y, y)^{\text{red}}$. This is because given a tame pure imaginary Higgs form θ and $\lambda \in S^1$, the Higgs form $\lambda\theta$ is only pure imaginary if either $\lambda = \pm 1$ or θ is nilpotent. The group ${}^\nu\varpi_1(Y, y)^{\text{red}}$ is characterised by the property that its representations are semisimple local systems whose associated Higgs form has nilpotent residues. This is equivalent to saying that ${}^\nu\varpi_1(Y, y)^{\text{red}}$ -representations are semisimple local systems on Y for which the monodromy around any component of D has unitary eigenvalues. Thus the greatest generality in which Proposition 10.24 could possibly hold is for any S^1 -equivariant quotient R of ${}^\nu\varpi_1(Y, y)^{\text{red}}$.

Denote the maximal quotient of ${}^\nu\varpi_1(Y, y)^{\text{red}}$ on which the S^1 -action is algebraic by ${}^{\text{VHS}}\varpi_1(Y, y)$. Arguing as in Proposition 5.12, representations of ${}^{\text{VHS}}\varpi_1(Y, y)$ correspond to real local systems underlying variations of Hodge structure on Y , and representations of ${}^{\text{VHS}}\varpi_1(Y, y) \rtimes S^1$ correspond to weight 0 real VHS. The greatest generality in which Theorem 10.23 could hold is for any S^1 -equivariant quotient R of ${}^{\text{VHS}}\varpi_1(Y, y)^{\text{red}}$.

Definition 11.1. Given a semisimple real local system \mathbb{V} on Y , use Mochizuki's tame imaginary pluriharmonic metric to decompose the associated connection $D : \mathcal{A}_Y^0(\mathbb{V}) \rightarrow \mathcal{A}_Y^1(\mathbb{V})$ as $D = d^+ + \vartheta$ into antisymmetric and symmetric parts, and let $D^c := i \diamond d^+ - i \diamond \vartheta$. Also write $D' = \partial + \bar{\theta}$ and $D'' = \bar{\partial} + \theta$. Note that these definitions are independent of the choice of pluriharmonic metric, since the metric is unique up to global automorphisms $\Gamma(X, \text{Aut}(\mathbb{V}))$.

11.1. Constructing mixed Hodge structures. We now outline a strategy for adapting Theorem 10.23 to more general R .

Proposition 11.2. *Let R be a quotient of ${}^{\text{VHS}}\varpi_1(Y, y)$ to which the S^1 -action descends, and assume we have the following data.*

- For each weight 0 real VHS \mathbb{V} on Y corresponding to an $R \rtimes S^1$ -representation, an S -equivariant \mathbb{R} -linear graded subsheaf

$$\mathcal{T}^*(\mathbb{V}) \subset j_*\mathcal{A}_Y^*(\mathbb{V}) \otimes \mathbb{C},$$

on X , closed under the operations D and D^c . This must be functorial in \mathbb{V} , with

- $\mathcal{T}^*(\mathbb{V} \oplus \mathbb{V}') = \mathcal{T}^*(\mathbb{V}) \oplus \mathcal{T}^*(\mathbb{V}')$,
- the image of $\mathcal{T}^*(\mathbb{V}) \otimes \mathcal{T}^*(\mathbb{V}') \xrightarrow{\wedge} j_*\mathcal{A}_Y^*(\mathbb{V} \otimes \mathbb{V}') \otimes \mathbb{C}$ contained in $\mathcal{T}^*(\mathbb{V} \otimes \mathbb{V}')$, and
- $1 \in \mathcal{T}^*(\mathbb{R})$.

- An increasing non-negative S -equivariant filtration J of $\mathcal{T}^*(\mathbb{V})$ with $J_r \mathcal{T}^n(\mathbb{V}) = \mathcal{T}^n(\mathbb{V})$ for all $n \leq r$, compatible with the tensor structures, and closed under the operations D and D^c .

Set $F^p \mathcal{T}^\bullet(\mathbb{V}) := \mathcal{T}^\bullet(\mathbb{V}) \cap F^p \mathcal{A}^\bullet(Y, \mathbb{V})_{\mathbb{C}}$, where the Hodge filtration F is defined in the usual way in terms of the S -action, and assume that

- (1) The map $\mathcal{T}^\bullet(\mathbb{V}) \rightarrow j_*\mathcal{A}_Y^\bullet(\mathbb{V})_{\mathbb{C}}$ is a quasi-isomorphism of sheaves on X for all \mathbb{V} .
- (2) For all $i \neq r$, the sheaf $\mathcal{H}^i(\text{gr}_r^J \mathcal{T}^\bullet(\mathbb{V}))$ on X is 0.
- (3) For all a, b and p , the map

$$\mathbb{H}^{a+b}(X, F^p \text{gr}_b^J \mathcal{T}^\bullet(\mathbb{V})) \rightarrow \mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{V})_{\mathbb{C}}$$

is injective, giving a Hodge filtration $F^p \mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{V})_{\mathbb{C}}$ which defines a pure Hodge structure of weight $a + 2b$ on $\mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{V})$.

Then there is a non-negatively weighted mixed Hodge structure $(Y, y)_{\text{MHS}}^{R, \text{Mal}}$, with

$$\underline{\text{gr}}(Y, y)_{\text{MHS}}^{R, \text{Mal}} \simeq \text{Spec} \left(\bigoplus_{a, b} \mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a - b], d_2 \right),$$

where $H^a(X, \mathbf{R}^b j_* \mathbb{O}(R))$ naturally becomes a pure Hodge structure of weight $a + 2b$, and $d_2: H^a(X, \mathbf{R}^b j_* \mathbb{O}(R)) \rightarrow H^{a+2}(X, \mathbf{R}^{b-1} j_* \mathbb{O}(R))$ is the differential from the E_2 sheet of the Leray spectral sequence for j .

Proof. We proceed along similar lines to [Mor]. To construct the Hodge filtration, we first define $\tilde{\mathcal{T}}^\bullet(\mathbb{V}) \subset j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{V})_{\mathbb{C}}$ to be given by the differential \tilde{D} on the graded sheaf $\mathcal{T}^\bullet(\mathbb{V}) \otimes \mathcal{O}(C)$, then let $\mathcal{E}_{\mathbb{F}}(\mathbb{O}(R))$ be the homotopy fibre product

$$(\tilde{\mathcal{T}}^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{O}(C) \otimes \mathbb{C}} \mathcal{O}(\tilde{C}^*)) \times_{(j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{O}(C)} \mathcal{O}(S) \otimes \mathbb{C})}^h (j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{O}(C)} \mathcal{O}(S))$$

in the category of $R \times S$ -equivariant DGAs on $X \times C_{\mathbb{Z}\text{ar}}^*$, quasi-coherent over C^* . Here, we are extending \mathcal{T}^\bullet to ind-VHS by setting $\mathcal{T}^\bullet(\varinjlim_{\alpha} \mathbb{V}_{\alpha}) := \varinjlim_{\alpha} \mathcal{T}^\bullet(\mathbb{V}_{\alpha})$, and similarly for $\tilde{\mathcal{T}}^\bullet$.

Explicitly, a homotopy fibre product $C \times_D^h F$ is defined by replacing $C \rightarrow D$ with a quasi-isomorphic surjection $C' \rightarrow D$, then setting $C \times_D^h F := C' \times_D F$. Equivalently, we could replace $F \rightarrow D$ with a surjection. That such surjections exist and give well-defined homotopy fibre products up to quasi-isomorphism follows from the observation in Proposition 3.46 that the homotopy category of quasi-coherent DGAs on a quasi-affine scheme can be realised as the homotopy category of a right proper model category.

Observe that for co-ordinates u, v on C as in Remark 1.3,

$$\tilde{\mathcal{T}}^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{O}(C) \otimes \mathbb{C}} \mathcal{O}(\tilde{C}^*) \cong \left(\bigoplus_{p \in \mathbb{Z}} F^p \mathcal{T}^\bullet(\mathbb{O}(R))(u + iv)^{-p} \right) [(u - iv), (u - iv)^{-1}],$$

while $(j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{O}(C)} \mathcal{O}(S)) \cong j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)) \otimes \mathcal{O}(S)$ (with the same reasoning as Lemma 2.4).

Note that $\tilde{C}^* \times_C \tilde{C}^* \cong \tilde{C}^* \sqcup S_C$, so $\mathcal{E}_{\mathbb{F}}(\mathbb{O}(R))|_{\tilde{C}^*}$ is

$$\begin{aligned} & [\tilde{\mathcal{T}}^\bullet(\mathbb{O}(R))|_{\tilde{C}^*} \oplus \tilde{\mathcal{T}}^\bullet(\mathbb{O}(R))|_{S_C}] \times_{[j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))|_{S_C} \oplus j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))|_{S_C}]}^h [j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))|_{S_C}] \\ & \simeq [\tilde{\mathcal{T}}^\bullet(\mathbb{O}(R))|_{\tilde{C}^*} \oplus j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))|_{S_C}] \times_{[j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))|_{S_C} \oplus j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))|_{S_C}]}^h [j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))|_{S_C}] \\ & \simeq \tilde{\mathcal{T}}^\bullet(\mathbb{O}(R))|_{\tilde{C}^*}. \end{aligned}$$

Similarly, $\mathcal{E}_{\mathbb{F}}(\mathbb{O}(R))|_S \simeq j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{O}(C)} \mathcal{O}(S)$.

If we let $\mathbf{C}^\bullet(X, -)$ denote either the cosimplicial Čech or Godement resolution on X , then the Thom–Sullivan functor Th of Definition 3.28 gives us a functor $\text{Th} \circ \mathbf{C}^\bullet(X, -)$ from sheaves of DG algebras on X to DG algebras. We denote this by $\mathbf{R}\Gamma(X, -)$, since it gives a canonical choice for derived global sections. We then define the Hodge filtration by

$$\mathcal{O}(Y, y)_{\mathbb{F}}^{R, \text{Mal}} := \mathbf{R}\Gamma(X, \mathcal{E}_{\mathbb{F}}(\mathbb{O}(R)))$$

as an object of $\text{Ho}(DG_{\mathbb{Z}}\text{Alg}_{C^*}(R)_*(S))$. Note that condition (1) above ensures that the pullback of $(Y, y)_{\mathbb{F}}^{R, \text{Mal}}$ over $1 \in C^*$ is quasi-isomorphic to $\text{Spec } \mathbf{R}\Gamma(X, j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)))$. Since the map

$$A^\bullet(Y, \mathbb{O}(R)) \rightarrow \mathbf{R}\Gamma(X, j_* \tilde{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)))$$

is a quasi-isomorphism, this means that $(Y, y)_{\mathbb{F}}^{R, \text{Mal}}$ indeed defines an algebraic Hodge filtration on $(Y, y)_{\mathbb{F}}^{R, \text{Mal}}$.

To define the mixed Hodge structure, we first note that condition (2) above implies that

$$(\tilde{\mathcal{T}}^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{O}(C)} \mathcal{O}(S), \tau) \rightarrow (\tilde{\mathcal{T}}^\bullet(\mathbb{O}(R)) \otimes_{\mathcal{O}(C)} \mathcal{O}(S), J)$$

is a filtered quasi-isomorphism of complexes, where τ denotes the good truncation filtration. We then define $\mathcal{O}(Y, y)_{\text{MHS}}^{R, \text{Mal}}$ to be the homotopy limit of the diagram

$$\begin{array}{ccc} \xi(\mathbf{R}\Gamma(X, \tilde{\mathcal{F}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}}), \text{Dec } \mathbf{R}\Gamma(J)) & \longrightarrow & \xi(\mathbf{R}\Gamma(X, \tilde{\mathcal{F}}^\bullet(\mathbb{O}(R))|_{S_C}), \text{Dec } \mathbf{R}\Gamma(J)) \\ \xi(\mathbf{R}\Gamma(X, \tilde{\mathcal{F}}^\bullet(\mathbb{O}(R))|_{S_C}), \text{Dec } \mathbf{R}\Gamma(\tau)) & \longrightarrow & \xi(\mathbf{R}\Gamma(X, j_* \mathcal{A}_Y^\bullet(\mathbb{O}(R))|_{S_C}), \text{Dec } \mathbf{R}\Gamma(\tau)) \\ \xi(\mathbf{R}\Gamma(X, j_* \mathcal{A}_Y^\bullet(\mathbb{O}(R))|_S), \text{Dec } \mathbf{R}\Gamma(\tau)) & & \end{array}$$

which can be expressed as an iterated homotopy fibre product of the form $E_1 \times_{E_2}^h E_3 \times_{E_4}^h E_5$. Here, ξ denotes the Rees algebra construction as in Lemma 1.7. The basepoint $iy \in X$ gives an augmentation of this DG algebra, so we have defined an object of $\text{Ho}(DG_{\mathbb{Z}} \text{Alg}_{\mathbb{A}^1 \times C^*}(R)_*(\text{Mat}_1 \times S))$.

Conditions (2) and (1) above ensure that the second and third maps in the diagram above are both quasi-isomorphisms, with the second map becoming an isomorphism on pulling back along $1 \in \mathbb{A}^1$ (corresponding to forgetting the filtrations). The latter observation means that we do indeed have

$$(Y, y)_{\text{MHS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 1}^{\mathbf{R}} \text{Spec } \mathbb{R} \simeq (Y, y)_{\mathbb{F}}^{R, \text{Mal}}.$$

Setting $\text{gr}(Y, y)_{\text{MHS}}^{R, \text{Mal}}$ as in the statement above, it only remains to establish opposedness.

Now, the pullback of $\xi(M, W)$ along $0 \in \mathbb{A}^1$ is just $\text{gr}^W M$. Moreover, [Del1, Proposition 1.3.4] shows that for any filtered complex (M, J) , the map

$$\text{gr}^{\text{Dec } J} M \rightarrow \left(\bigoplus_{a,b} \mathbb{H}^a(\text{gr}_b^J M)[-a], d_1^J \right)$$

is a quasi-isomorphism, where d_1^J is the differential in the E_1 sheet of the spectral sequence associated to J . Thus the structure sheaf \mathcal{G} of $(Y, y)_{\text{MHS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R}$ is the homotopy limit of the diagram

$$\begin{array}{ccc} \left(\bigoplus_{a,b} \mathbb{H}^a(X, \text{gr}_b^J \tilde{\mathcal{F}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}})[-a], d_1^J \right) & \longrightarrow & \left(\bigoplus_{a,b} \mathbb{H}^a(X, \text{gr}_b^J \tilde{\mathcal{F}}^\bullet(\mathbb{O}(R))|_{S_C})[-a], d_1^J \right) \\ \left(\bigoplus_{a,b} \mathbb{H}^a(X, \mathcal{H}^b \tilde{\mathcal{F}}^\bullet(\mathbb{O}(R))|_{S_C})[-a], d_2 \right) & \longrightarrow & \left(\bigoplus_{a,b} \mathbb{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))|_{S_C})[-a], d_2 \right) \\ \left(\bigoplus_{a,b} \mathbb{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))|_S)[-a], d_2 \right) & & \end{array}$$

where d_2 denotes the differential on the E_2 sheet of the spectral sequence associated to a bigraded complex.

The second and third maps in the diagram above are isomorphisms, so we can write \mathcal{G} as the homotopy fibre product of

$$\begin{array}{ccc} \left(\bigoplus_{a,b} \mathbb{H}^{a+b}(X, \text{gr}_b^J \tilde{\mathcal{F}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}})[-a-b], d_1^J \right) & \longrightarrow & \left(\bigoplus_{a,b} \mathbb{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))|_{S_C})[-a-b], d_2 \right) \\ \left(\bigoplus_{a,b} \mathbb{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))|_S)[-a-b], d_2 \right) & & \end{array}$$

By condition (3) above, $\mathbb{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R)))$ has the structure of an S -representation of weight $a + 2b$ — denote this by E^{ab} , and set $E := (\bigoplus_{a,b} E^{ab}, d_2)$. Then we can apply Lemma 1.41 to rewrite \mathcal{G} as

$$\left(\bigoplus_{p \in \mathbb{Z}} F^p(E \otimes \mathbb{C})(u + iv)^{-p}[(u - iv), (u - iv)^{-1}] \times_{E \otimes O(S_C)}^h E \otimes O(S) \right).$$

Since $(\bigoplus_{p \in \mathbb{Z}} F^p(E \otimes \mathbb{C})(u + iv)^{-p}[(u - iv), (u - iv)^{-1}] \cong E \otimes O(\widetilde{C^*}))$, this is just

$$E \otimes (O(\widetilde{C^*}) \times_{O(S_C)}^h O(S)) \simeq E \otimes \mathcal{O}(C^*),$$

as required. \square

11.2. Constructing mixed twistor structures. Proposition 11.2 does not easily adapt to mixed twistor structures, since an S -equivariant morphism $M \rightarrow N$ of quasi-coherent sheaves on S is an isomorphism if and only if the fibres $M_1 \rightarrow N_1$ are isomorphisms of vector spaces, but the same is not true of a \mathbb{G}_m -equivariant morphism of quasi-coherent sheaves on S . Our solution is to introduce holomorphic properties, the key idea being that for t the co-ordinate on S^1 , the connection $t \otimes D: \mathcal{A}_Y^0(\mathbb{V}) \otimes O(S^1) \rightarrow \mathcal{A}_Y^1(\mathbb{V}) \otimes O(S^1)$ does not define a local system of $O(S^1)$ -modules, essentially because iterated integration takes us outside $O(S^1)$. However, as observed in [Sim2, end of §3], $t \otimes D$ defines a holomorphic family of local systems on X , parametrised by $S^1(\mathbb{C}) = \mathbb{C}^\times$.

Definition 11.3. Given a smooth complex affine variety Z , define $O(Z)^{\text{hol}}$ to be the ring of holomorphic functions $f: Z(\mathbb{C}) \rightarrow \mathbb{C}$. Given a smooth real affine variety Z , define $O(Z)^{\text{hol}}$ to be the ring of $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant holomorphic functions $f: Z(\mathbb{C}) \rightarrow \mathbb{C}$.

In particular, $O(S^1)^{\text{hol}}$ is the ring of functions $f: \mathbb{C}^\times \rightarrow \mathbb{C}$ for which $\overline{f(z)} = f(\bar{z}^{-1})$, or equivalently convergent Laurent series $\sum_{n \in \mathbb{Z}} a_n t^n$ for which $\bar{a}_n = a_{-n}$.

Definition 11.4. Given a smooth complex variety Z , define $\mathcal{A}_Y^0 \mathcal{O}_Z^{\text{hol}}$ to be the sheaf on $Y \times Z(\mathbb{C})$ consisting of smooth complex functions which are holomorphic along Z . Write $\mathcal{A}_Y^\bullet \mathcal{O}_Z^{\text{hol}} := \mathcal{A}_Y^\bullet \otimes_{\mathcal{A}_Y^0} \mathcal{A}_Y^0 \mathcal{O}_Z^{\text{hol}}$, and, given a local system \mathbb{V} on Y , set $\mathcal{A}_Y^\bullet \mathcal{O}_Z^{\text{hol}}(\mathbb{V}) := \mathcal{A}_Y^\bullet(\mathbb{V}) \otimes_{\mathcal{A}_Y^0} \mathcal{A}_Y^0 \mathcal{O}_Z^{\text{hol}}$.

Given a smooth real variety Z , define $\mathcal{A}_Y^0 \mathcal{O}_Z^{\text{hol}}$ to be the $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant sheaf $\mathcal{A}_Y^0 \mathcal{O}_{Z_{\mathbb{C}}}^{\text{hol}}$ on $Y \times Z(\mathbb{C})$, where the non-trivial element $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$ acts by $\sigma(f)(y, z) = \overline{f(y, \sigma z)}$.

Definition 11.5. Define $P := C^*/\mathbb{G}_m$ and $\tilde{P} := \tilde{C}^*/\mathbb{G}_m$. As in Definition 1.32, we have $S^1 = S/\mathbb{G}_m$, and hence a canonical inclusion $S^1 \hookrightarrow P$ (given by cutting out the divisor $\{(u : v) : u^2 + v^2 = 0\}$). For co-ordinates u, v on C as in Remark 1.3, fix co-ordinates $t = \frac{u+iv}{u-iv}$ on \tilde{P} , and $a = \frac{u^2-v^2}{u^2+v^2}$, $b = \frac{2uv}{u^2+v^2}$ on S^1 (so $a^2 + b^2 = 1$).

Thus $P \cong \mathbb{P}_{\mathbb{R}}^1$ and $\tilde{P} \cong \mathbb{A}_{\mathbb{C}}^1$, the latter isomorphism using the co-ordinate t . The canonical map $\tilde{P} \rightarrow P$ is given by $t \mapsto (1+t : i-it)$, and the map $S_{\mathbb{C}}^1 \rightarrow \tilde{P}$ by $(a, b) \mapsto a+ib$.

Also note that the étale pushout $C^* = \tilde{C}^* \cup_{S_{\mathbb{C}}^1} S$ corresponds to an étale pushout

$$P = \tilde{P} \cup_{S_{\mathbb{C}}^1} S^1,$$

where $S_{\mathbb{C}}^1 \cong \mathbb{G}_{m, \mathbb{C}}$ is given by the subscheme $t \neq 0$ in $\mathbb{A}_{\mathbb{C}}^1$. Note that the $\text{Gal}(\mathbb{C}/\mathbb{R})$ -action on $\mathbb{C}[t, t^{-1}]$ given by the real form S^1 is determined by the condition that the non-trivial element $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$ maps t to t^{-1} .

Definition 11.6. Define $\check{\mathcal{A}}_Y^\bullet(\mathbb{V})$ to be the sheaf $\bigoplus_{n \geq 0} \mathcal{A}_Y^n(\mathbb{V}) \mathcal{O}_P^{\text{hol}}(n)$ of graded algebras on $Y \times P(\mathbb{C})$, equipped with the differential $uD + vD^c$, where $u, v \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the weight 1 generators $u, v \in O(C)$.

Definition 11.7. Given a polarised scheme $(Z, \mathcal{O}_Z(1))$ (where Z need not be projective), and a sheaf \mathcal{F} of \mathcal{O}_Z -modules, define $\underline{\Gamma}(Z, \mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(Z, \mathcal{F}(n))$. This is regarded as a \mathbb{G}_m -representation, with $\Gamma(Z, \mathcal{F}(n))$ of weight n .

Lemma 11.8. *The \mathbb{G}_m -equivariant sheaf $\check{\mathcal{A}}_Y^\bullet(\mathbb{V})$ of $O(C)$ -complexes on Y (from Definition 7.1) is given by*

$$\check{\mathcal{A}}_Y^\bullet(\mathbb{V}) \cong \underline{\Gamma}(P(\mathbb{C}), \mathcal{A}_Y^\bullet(\mathbb{V}))^{\text{Gal}(\mathbb{C}/\mathbb{R})}.$$

Proof. We first consider $\Gamma(P(\mathbb{C}), \check{\mathcal{A}}_Y^0(\mathbb{V}))$. This is the sheaf on Y which sends any open subset $U \subset Y$ to the ring of consisting of those smooth functions $f: U \times \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{C}$

which are holomorphic along $\mathbb{P}^1(\mathbb{C})$. Thus for any $y \in U$, $f(y, -)$ is a global holomorphic function on $\mathbb{P}^1(\mathbb{C})$, so is constant. Therefore $\Gamma(P(\mathbb{C}), \check{\mathcal{A}}_Y^0(\mathbb{V})) = \mathcal{A}_Y^0 \otimes \mathbb{C}$.

For general n , a similar argument using finite-dimensionality of $\Gamma(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n)^{\text{hol}})$ shows that

$$\Gamma(P(\mathbb{C}), \check{\mathcal{A}}_Y^0(\mathbb{V})(n)) \cong \mathcal{A}_Y^0 \otimes \Gamma(P(\mathbb{C}), \mathcal{O}_P(n)^{\text{hol}}).$$

Now by construction of P , we have $\underline{\Gamma}(P(\mathbb{C}), \mathcal{O}_P^{\text{hol}}) \cong O(C) \otimes \mathbb{C}$ with the grading corresponding to the \mathbb{G}_m -action. Thus

$$\underline{\Gamma}(P(\mathbb{C}), \check{\mathcal{A}}_Y^*(\mathbb{V})(n))^{\text{Gal}(\mathbb{C}/\mathbb{R})} \cong \check{\mathcal{A}}_Y^*(\mathbb{V}).$$

Since the differential in both cases is given by $uD + vD^c$, this establishes the isomorphism of complexes. \square

Definition 11.9. On the schemes S^1 and \tilde{P} , define the sheaf $\mathcal{O}(1)$ by pulling back $\mathcal{O}_P(1)$ from P . Thus the corresponding module $A(1)$ on $\text{Spec } A$ is given by

$$A(1) = A(u, v)/(t(u - iv) - (u + iv)).$$

Hence $\mathcal{O}_{\tilde{P}}(1) = \mathcal{O}_{\tilde{P}}(u - iv)$ and $\mathcal{O}_{S^1}(1) \otimes \mathbb{C} = \mathcal{O}_{S^1} \otimes \mathbb{C}(u - iv)$ are trivial line bundles, but $\mathcal{O}_{S^1}(1) = \mathcal{O}_{S^1}(u, v)/(au + bv - u, bu - av - v)$.

Proposition 11.10. *Let R be a quotient of $\varpi_1(Y, y)^{\text{red}}$, and assume that we have the following data.*

- For each finite rank local real system \mathbb{V} on Y corresponding to an R -representation, a flat graded $(\mathcal{A}_X^0 \otimes \mathbb{C})$ -submodule

$$\mathcal{T}^*(\mathbb{V}) \subset j_*\check{\mathcal{A}}_Y^*(\mathbb{V}) \otimes \mathbb{C},$$

closed under the operations D and D^c . This must be functorial in \mathbb{V} , with

- $\mathcal{T}^*(\mathbb{V} \oplus \mathbb{V}') = \mathcal{T}^*(\mathbb{V}) \oplus \mathcal{T}^*(\mathbb{V}')$,
- the image of $\mathcal{T}^*(\mathbb{V}) \otimes \mathcal{T}^*(\mathbb{V}') \xrightarrow{\wedge} j_*\check{\mathcal{A}}_Y^*(\mathbb{V} \otimes \mathbb{V}') \otimes \mathbb{C}$ contained in $\mathcal{T}^*(\mathbb{V} \otimes \mathbb{V}')$, and
- $1 \in \mathcal{T}^*(\mathbb{R})$.

- An increasing non-negative filtration J of $\mathcal{T}^*(\mathbb{V})$ with $J_r \mathcal{T}^*(\mathbb{V}) = \mathcal{T}^*(\mathbb{V})$ for all $n \leq r$, compatible with the tensor structure, and closed under the operations D and D^c .

Set $\check{\mathcal{T}}^\bullet(\mathbb{V}) \subset j_*\check{\mathcal{A}}_Y^\bullet(\mathbb{V})$ to be the complex on $X \times P(\mathbb{C})$ whose underlying sheaf is $\bigoplus_{n \geq 0} \mathcal{T}^n(\mathbb{V}) \otimes_{\mathcal{A}_X^0 \otimes \mathbb{C}} \mathcal{A}_X^0 \mathcal{O}_P^{\text{hol}}(n)$, and assume that

- (1) For $S^1(\mathbb{C}) \subset P(\mathbb{C})$, the map $\check{\mathcal{T}}^\bullet(\mathbb{V})|_{S^1(\mathbb{C})} \rightarrow j_*\check{\mathcal{A}}_Y^\bullet(\mathbb{V})|_{S^1(\mathbb{C})}$ is a quasi-isomorphism of sheaves of $\mathcal{O}_{S^1}^{\text{hol}}$ -modules on $X \times S^1(\mathbb{C})$ for all \mathbb{V} .
- (2) For all $i \neq r$, the sheaf $\mathcal{H}^i(\text{gr}_r^J \check{\mathcal{T}}^\bullet(\mathbb{V})|_S)$ of $\mathcal{O}_{S^1}^{\text{hol}}$ -modules on $X \times S^1(\mathbb{C})$ is 0.
- (3) For all $a, b \geq 0$, the $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant sheaf

$$\ker(\mathbb{H}^a(X, \text{gr}_b^J \check{\mathcal{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})} \oplus \overline{\sigma^* \mathbb{H}^a(X, \text{gr}_b^J \check{\mathcal{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})}} \rightarrow \mathbb{H}^a(X, \mathcal{H}^b(j_*\check{\mathcal{A}}_Y^\bullet(\mathbb{V}))|_{S^1(\mathbb{C})}))$$

is a finite locally free $\mathcal{O}_{\tilde{P}}^{\text{hol}}$ -module of slope $a + 2b$.

Then there is a non-negatively weighted mixed twistor structure $(Y, y)_{\text{MTS}}^{R, \text{Mal}}$, with

$$\underline{\text{gr}}(Y, y)_{\text{MTS}}^{R, \text{Mal}} \simeq \text{Spec} \left(\bigoplus_{a, b} \mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a - b], d_2 \right),$$

where $\mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R))$ is assigned the weight $a + 2b$, and $d_2: \mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R)) \rightarrow \mathbb{H}^{a+2}(X, \mathbf{R}^{b-1} j_* \mathbb{O}(R))$ is the differential from the E_2 sheet of the Leray spectral sequence for j .

Proof. Define $\mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}}$ to be the homotopy fibre product

$$\mathbf{R}\Gamma(X, \underline{\Gamma}(\tilde{P}(\mathbb{C}), \check{\mathcal{T}}^\bullet(\mathcal{O}(R)))) \times_{\mathbf{R}\Gamma(X, j_* \underline{\Gamma}(S^1(\mathbb{C}), \check{\mathcal{A}}_Y^\bullet(\mathcal{O}(R)))}^h \mathbf{R}\Gamma(X, j_* \underline{\Gamma}(S^1(\mathbb{C}), \check{\mathcal{A}}_Y^\bullet(\mathcal{O}(R))))^{\text{Gal}(\mathbb{C}/\mathbb{R})}$$

as an object of $\text{Ho}(DG_{\mathbb{Z}}\text{Alg}_{C^*}(R)_*(\mathbb{G}_m))$, and let $\mathcal{O}(Y, y)_{\text{MTS}}^{R, \text{Mal}}$ be the homotopy limit of the diagram

$$\begin{array}{c} \xi(\mathbf{R}\Gamma(X, \underline{\Gamma}(\tilde{P}(\mathbb{C}), \check{\mathcal{T}}^\bullet(\mathcal{O}(R)))) , \text{Dec } \mathbf{R}\Gamma(J)) \\ \downarrow \\ \xi(\mathbf{R}\Gamma(X, \underline{\Gamma}(S^1(\mathbb{C}), \check{\mathcal{T}}^\bullet(\mathcal{O}(R)))) , \text{Dec } \mathbf{R}\Gamma(J)) \\ \uparrow \\ \xi(\mathbf{R}\Gamma(X, \underline{\Gamma}(S^1(\mathbb{C}), \check{\mathcal{T}}^\bullet(\mathcal{O}(R)))) , \text{Dec } \mathbf{R}\Gamma(\tau)) \\ \downarrow \\ \xi(\mathbf{R}\Gamma(X, j_* \underline{\Gamma}(S^1(\mathbb{C}), \check{\mathcal{A}}_Y^\bullet(\mathcal{O}(R)))) , \text{Dec } \mathbf{R}\Gamma(\tau)) \\ \uparrow \\ \xi(\mathbf{R}\Gamma(X, j_* \underline{\Gamma}(S^1(\mathbb{C}), \check{\mathcal{A}}_Y^\bullet(\mathcal{O}(R))))^{\text{Gal}(\mathbb{C}/\mathbb{R})} , \text{Dec } \mathbf{R}\Gamma(\tau)) \end{array}$$

as an object of $\text{Ho}(DG_{\mathbb{Z}}\text{Alg}_{\mathbb{A}^1 \times C^*}(R)_*(\text{Mat}_1 \times \mathbb{G}_m))$. Here, we are extending \mathcal{T}^\bullet to ind-local systems by setting $\mathcal{T}^\bullet(\varinjlim_{\alpha} \mathbb{V}_{\alpha}) := \varinjlim_{\alpha} \mathcal{T}^\bullet(\mathbb{V}_{\alpha})$, and similarly for $\check{\mathcal{T}}^\bullet$.

Given a $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant sheaf \mathcal{F} of $\mathcal{O}_P^{\text{hol}}$ -modules on $X \times P(\mathbb{C})$, the group cohomology complex gives a $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant cosimplicial sheaf $\mathbf{C}^\bullet(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathcal{F})$ on $X \times P(\mathbb{C})$ — this is a resolution of \mathcal{F} , with $\mathbf{C}^0(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathcal{F}) = \mathcal{F} \oplus \sigma^* \mathcal{F}$. Applying the Thom–Whitney functor Th , this means that

$$\text{Th } \mathbf{C}^\bullet(\text{Gal}(\mathbb{C}/\mathbb{R}), j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{V}))$$

is a $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant $\mathcal{O}_P^{\text{hol}}$ -DGA on $X \times P(\mathbb{C})$, equipped with a surjection to $j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{V}) \oplus \sigma^* j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{V})$.

This allows us to consider the $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant sheaf $\mathcal{B}_{\mathbb{T}}^\bullet$ of $\mathcal{O}_P^{\text{hol}}$ -DGAs on $P(\mathbb{C})$ given by the fibre product of

$$\begin{array}{ccc} (\check{\mathcal{T}}^\bullet(\mathcal{O}(R)))|_{\tilde{P}(\mathbb{C})} \oplus \sigma^* \overline{\check{\mathcal{T}}^\bullet(\mathcal{O}(R))}|_{\tilde{P}(\mathbb{C})} & \longrightarrow & (j_* \check{\mathcal{A}}_Y^\bullet(\mathcal{O}(R)) \oplus \sigma^* \overline{j_* \check{\mathcal{A}}_Y^\bullet(\mathcal{O}(R))})|_{S^1(\mathbb{C})} \\ & \searrow & \\ \text{Th } \mathbf{C}^\bullet(\text{Gal}(\mathbb{C}/\mathbb{R}), j_* \check{\mathcal{A}}_Y^\bullet(\mathcal{O}(R)))|_{S^1(\mathbb{C})} & & \end{array}$$

Note that since the second map is surjective, this fibre product is in fact a homotopy fibre product. In particular,

$$\mathcal{O}(Y, y)_{\mathbb{T}}^{R, \text{Mal}} \simeq \mathbf{R}\Gamma(X, \underline{\Gamma}(P(\mathbb{C}), \mathcal{B}_{\mathbb{T}}^\bullet)^{\text{Gal}(\mathbb{C}/\mathbb{R})})|_{C^*}.$$

Now, $\underline{\Gamma}(P(\mathbb{C}), -)$ gives a functor from Zariski sheaves to $\mathcal{O}_P^{\text{hol}}$ -modules to $\mathcal{O}(C)$ -modules, and we consider the functor $\underline{\Gamma}(P(\mathbb{C}), -)|_{C^*}$ to quasi-coherent sheaves on C^* . There is a right derived functor $\mathbf{R}\underline{\Gamma}(P(\mathbb{C}), -)$; by [Ser], the map

$$\underline{\Gamma}(P(\mathbb{C}), \mathcal{F})|_{C^*} \rightarrow \mathbf{R}\underline{\Gamma}(P(\mathbb{C}), \mathcal{F})|_{C^*}$$

is a quasi-isomorphism for all coherent $\mathcal{O}_P^{\text{hol}}$ -modules \mathcal{F} . Given a morphism $f : Z \rightarrow P_{\mathbb{C}}$ of polarised varieties, with Z affine, and a quasi-coherent Zariski sheaf \mathcal{F} of $\mathcal{O}_Z^{\text{hol}}$ -modules on Z , note that

$$\mathbf{R}\underline{\Gamma}(P(\mathbb{C}), f_* \mathcal{F}) \simeq \mathbf{R}\underline{\Gamma}(P(\mathbb{C}), \mathbf{R}f_* \mathcal{F}) \simeq \mathbf{R}\underline{\Gamma}(Z(\mathbb{C}), \mathcal{F}) \simeq \underline{\Gamma}(Z(\mathbb{C}), \mathcal{F}).$$

There are convergent spectral sequences

$$\mathbb{H}^a(P(\mathbb{C}), \mathcal{H}^b(\mathcal{B}_{\mathbb{T}}^\bullet)(n)) \implies \mathbb{H}^{a+b}(P(\mathbb{C}), \mathcal{B}_{\mathbb{T}}^\bullet(n))$$

for all n , and Condition (3) above ensures that $\mathcal{H}^b(\mathcal{B}_\mathbb{T}^\bullet)$ is a direct sum of coherent sheaves. Since $\mathbf{H}^i \mathbf{R}\Gamma(P(\mathbb{C}), \mathcal{B}_\mathbb{T}^\bullet) = \bigoplus_{n \in \mathbb{Z}} \mathbf{H}^i(P(\mathbb{C}), \mathcal{B}_\mathbb{T}^\bullet(n))$, this means that the map

$$\Gamma(P(\mathbb{C}), \mathcal{B}_\mathbb{T}^\bullet)|_{C^*} \rightarrow \mathbf{R}\Gamma(P(\mathbb{C}), \mathcal{B}_\mathbb{T}^\bullet)|_{C^*}$$

is a quasi-isomorphism. Combining these observations shows that

$$\mathcal{O}(Y, y)_\mathbb{T}^{R, \text{Mal}} \simeq \mathbf{R}\Gamma(X, \mathbf{R}\Gamma(P(\mathbb{C}), \mathcal{B}_\mathbb{T}^\bullet))^{\text{Gal}(\mathbb{C}/\mathbb{R})}|_{C^*}.$$

In particular,

$$\mathcal{O}(Y, y)_\mathbb{T}^{R, \text{Mal}} \otimes_{\mathcal{O}_{C^*}}^{\mathbf{L}} \mathcal{O}(\mathbb{G}_m) \rightarrow \mathbf{R}\Gamma(X, \Gamma(\text{Spec } \mathbb{C}, \mathcal{B}_\mathbb{T}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^1(1:0)}}^{\text{hol}} \mathbb{C})^{\text{Gal}(\mathbb{C}/\mathbb{R})})$$

is a quasi-isomorphism, and note that right-hand side is just

$$\mathbf{R}\Gamma(X, (\mathcal{B}_\mathbb{T}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^1(1:0)}}^{\text{hol}} \mathbb{C})^{\text{Gal}(\mathbb{C}/\mathbb{R})}) \otimes \mathcal{O}(\mathbb{G}_m),$$

which is the homotopy fibre

$$\mathbf{R}\Gamma(X, [\mathcal{F}^\bullet(\mathbb{O}(R)) \times_{j_* \mathcal{A}_Y^\bullet(\mathbb{O}(R)) \otimes \mathbb{C}}^h j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))]),$$

and hence quasi-isomorphic to $\mathbf{R}\Gamma(X, j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)))$ by condition (1) above. This proves that

$$(Y, y)_\mathbb{T}^{R, \text{Mal}} \times_{C^*, 1}^{\mathbf{R}} \text{Spec } \mathbb{R} \simeq (Y, y)^{R, \text{Mal}},$$

so $(Y, y)_\mathbb{T}^{R, \text{Mal}}$ is indeed a twistor filtration on $(Y, y)^{R, \text{Mal}}$.

The proof that $\mathcal{O}(Y, y)_\mathbb{T}^{R, \text{Mal}} \simeq \mathcal{O}(Y, y)_{\text{MTS}}^{R, \text{Mal}} \otimes_{\mathcal{O}_{\mathbb{A}^1, 1}}^{\mathbf{L}} \text{Spec } \mathbb{R}$ follows along exactly the same lines as in Proposition 11.2, so it only remains to establish opposedness.

Arguing as in the proof of Proposition 11.2, we see that the structure sheaf \mathcal{G} of $\text{gr}(Y, y)_{\text{MTS}}^{R, \text{Mal}} \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec } \mathbb{R}$ is the homotopy fibre product of the diagram

$$\begin{array}{c} (\bigoplus_{a,b} \Gamma(\tilde{P}(\mathbb{C}), \mathbb{H}^{a+b}(X, \text{gr}_b^J \check{\mathcal{F}}^\bullet(\mathbb{O}(R))))[-a-b], d_1^J) \\ \downarrow \\ (\bigoplus_{a,b} \Gamma(S^1(\mathbb{C}), \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)))))[-a-b], d_2) \\ \uparrow \\ (\bigoplus_{a,b} \Gamma(S^1(\mathbb{C}), \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))))^{\text{Gal}(\mathbb{C}/\mathbb{R})})[-a-b], d_2), \end{array}$$

as a $(\text{Mat}_1 \times R \times \mathbb{G}_m)$ -equivariant sheaf of DGAs over C^* .

Set $\text{gr}\mathcal{B}_{\text{MHS}}^{a,b}$ to be the sheaf on $P(\mathbb{C})$ given by the fibre product of the diagram

$$\begin{array}{c} \mathbb{H}^{a+b}(X, \text{gr}_b^J \check{\mathcal{F}}^\bullet(\mathbb{O}(R)))|_{\tilde{P}(\mathbb{C})} \oplus \overline{\sigma^* \mathbb{H}^{a+b}(X, \text{gr}_b^J \check{\mathcal{F}}^\bullet(\mathbb{O}(R)))|_{\tilde{P}(\mathbb{C})}} \\ \downarrow \\ \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)))) \oplus \overline{\sigma^* \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))))|_{S^1(\mathbb{C})}} \\ \uparrow \\ \text{Th } C^\bullet(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))))|_{S^1(\mathbb{C})}, \end{array}$$

and observe that

$$\mathcal{G} \simeq \left(\bigoplus_{a,b} \Gamma(P(\mathbb{C}), \text{gr}\mathcal{B}_{\text{MHS}}^{a,b})^{\text{Gal}(\mathbb{C}/\mathbb{R})}|_{C^*}, d_1^J \right).$$

Now, $\underline{\text{gr}}\mathcal{B}_{\text{MHS}}^{a,b}$ is just the homotopy fibre product of

$$\begin{array}{c} \mathbb{H}^{a+b}(X, \text{gr}_b^J \check{\mathcal{F}}^\bullet(\mathbb{O}(R)))|_{\hat{P}(\mathbb{C})} \oplus \overline{\sigma^* \mathbb{H}^{a+b}(X, \text{gr}_b^J \check{\mathcal{F}}^\bullet(\mathbb{O}(R)))|_{\hat{P}(\mathbb{C})}} \\ \downarrow \\ \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R)))) \oplus \overline{\sigma^* \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))))}|_{S^1(\mathbb{C})} \\ \uparrow \\ \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))))|_{S^1(\mathbb{C})}; \end{array}$$

condition (1) ensures that the first map is injective, so $\underline{\text{gr}}\mathcal{B}_{\text{MHS}}^{a,b}$ is quasi-isomorphic to the kernel of

$$\mathbb{H}^a(X, \text{gr}_b^J \check{\mathcal{F}}^\bullet(\mathbb{O}(R)))|_{\hat{P}(\mathbb{C})} \oplus \overline{\sigma^* \mathbb{H}^a(X, \text{gr}_b^J \check{\mathcal{F}}^\bullet(\mathbb{O}(R)))|_{\hat{P}(\mathbb{C})}} \rightarrow \mathbb{H}^a(X, \mathcal{H}^b(j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{O}(R))))|_{S^1(\mathbb{C})}.$$

By condition (3), this is a holomorphic vector bundle on $P(\mathbb{C})$ of slope $a + 2b$.

Now, we just observe that for any holomorphic vector bundle \mathcal{F} of slope m , the map $\Gamma(P(\mathbb{C}), \mathcal{F}(-m)) \rightarrow 1^* \mathcal{F}$, given by taking the fibre at $1 \in P(\mathbb{R})$, is an isomorphism of complex vector spaces, and that the maps

$$\Gamma(P(\mathbb{C}), \mathcal{F}(-m)) \otimes \Gamma(P(\mathbb{C}), \mathcal{O}(n)) \rightarrow \Gamma(P(\mathbb{C}), \mathcal{F}(n-m))$$

are isomorphisms for $n \geq 0$. This gives an isomorphism

$$\underline{\Gamma}(P(\mathbb{C}), \mathcal{F})|_{C^*} \cong (1^* \mathcal{F}) \otimes \mathcal{O}_{C^*}$$

over C^* , which becomes \mathbb{G}_m -equivariant if we set $1^* \mathcal{F}$ to have weight m .

Therefore

$$\underline{\Gamma}(P(\mathbb{C}), \underline{\text{gr}}\mathcal{B}_{\text{MHS}}^{a,b})|_{C^*}^{\text{Gal}(\mathbb{C}/\mathbb{R})} \cong \mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R)) \otimes \mathcal{O}_{C^*},$$

making use of condition (1) to show that $\mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R)) \otimes \mathbb{C}$ is the fibre of $\underline{\text{gr}}\mathcal{B}_{\text{MHS}}^{a,b}$ at $1 \in P(\mathbb{R})$. This completes the proof of opposedness. \square

Proposition 11.11. *Let R be a quotient of $\nu\varpi_1(Y, y)^{\text{red}}$ to which the discrete S^1 -action descends, assume that the conditions of Proposition 11.10 hold, and assume in addition that for all $\lambda \in \mathbb{C}^\times$, the map $\lambda \diamond: j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{V}) \rightarrow j_* \check{\mathcal{A}}_Y^\bullet(\frac{\lambda}{\lambda} \otimes \mathbb{V})$ maps $\mathcal{F}(\mathbb{V})$ isomorphically to $\mathcal{F}(\frac{\lambda}{\lambda} \otimes \mathbb{V})$. Then there are natural $(S^1)^\delta$ -actions on $(Y, y)_{\text{MTS}}^{R, \text{Mal}}$ and $\underline{\text{gr}}(Y, y)_{\text{MTS}}^{R, \text{Mal}}$, compatible with the opposedness isomorphism, and with the action of $-1 \in S^1$ coinciding with that of $-1 \in \mathbb{G}_m$.*

Proof. The proof of Proposition 10.24 carries over, substituting Proposition 11.10 for Theorem 10.22. \square

11.3. Unitary monodromy. In this section, we will consider only semisimple local systems \mathbb{V} on Y with unitary monodromy around the local components of D (i.e. semisimple monodromy with unitary eigenvalues),

Definition 11.12. For \mathbb{V} as above, let $\mathcal{M}(\mathbb{V}) \subset j_* \mathcal{A}^0(\mathbb{V}) \otimes \mathbb{C}$ consist of locally L_2 -integrable functions for the Poincaré metric, holomorphic in the sense that they lie in $\ker \bar{\partial}$, where $D = \partial + \bar{\partial} + \theta + \bar{\theta}$.

Then set

$$\mathcal{A}_X^*(\mathbb{V})\langle D \rangle := \mathcal{M}(\mathbb{V}) \otimes_{\mathcal{O}_X} \mathcal{A}_X^*(\mathbb{R})\langle D \rangle \subset j_* \mathcal{A}_Y(\mathbb{V}) \otimes \mathbb{C},$$

where \mathcal{O}_X denotes the sheaf of holomorphic functions on X .

The crucial observation which we now make is that $\mathcal{A}_X^*(\mathbb{V})\langle D \rangle$ is closed under the operations D and D^c . Closure under $\bar{\partial}$ is automatic, and closure under ∂ follows because Mochizuki's metric is tame, so $\partial: \mathcal{M}(\mathbb{V}) \rightarrow \mathcal{M}(\mathbb{V}) \otimes_{\mathcal{O}_X} \Omega_X^1\langle D \rangle$. Since \mathbb{V} has unitary monodromy around the local components of D , the Higgs form θ is holomorphic, which

ensures that $\mathcal{A}_X^*(\mathbb{V})\langle D \rangle$ is closed under both θ and $\bar{\theta}$. We can thus write $\mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle$ for the complex given by $\mathcal{A}_X^*(\mathbb{V})\langle D \rangle$ with differential D .

Lemma 11.13. *For all $m \geq 0$, there is a morphism*

$$\text{Res}_m : \mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle \rightarrow \nu_{m*} \mathcal{A}_{D^{(m)}}^\bullet(\nu_m^{-1} j_* \mathbb{V} \otimes \varepsilon^m)\langle C^{(m)} \rangle[-m],$$

compatible with both D and D^c , for $D^{(m)}, C^{(m)}$ as in Definition 10.14 .

Proof. As in [Tim2, 1.4], Res_m is given in level q by the composition

$$\begin{aligned} \mathcal{A}_X^q(\mathbb{V})\langle D \rangle &= \mathcal{M}(\mathbb{V}) \otimes_{\mathcal{O}_X} \mathcal{A}_X^q\langle D \rangle \\ &\xrightarrow{\text{id} \otimes \text{Res}_m} \mathcal{M}(\mathbb{V}) \otimes_{\mathcal{O}_X} \nu_{m*} \mathcal{A}_{D^{(m)}}^{q-m}(\varepsilon_{\mathbb{R}}^m)\langle C^{(m)} \rangle \\ &= \nu_{m*}[\varepsilon^m \otimes_{\mathbb{Z}} \nu_m^* \mathcal{M}(\mathbb{V}) \otimes_{\mathcal{O}_{D^{(m)}}} \mathcal{A}_{D^{(m)}}^{q-m}(\varepsilon^m)\langle C^{(m)} \rangle] \\ &\rightarrow \nu_{m*}[\varepsilon^m \otimes_{\mathbb{Z}} \nu_m^* \mathcal{M}(\mathbb{V}) \otimes_{\mathcal{O}_{D^{(m)}}} \mathcal{A}_{D^{(m)}}^{q-m}(\varepsilon^m)\langle C^{(m)} \rangle], \end{aligned}$$

where the final map is given by orthogonal projection. The proof of [Tim2, Lemma 1.5] then adapts to show that Res_m is compatible with both D and D^c . \square

Note that $(j_* \mathbb{V} \otimes \varepsilon^m)|_{D^m - D^{m+1}}$ inherits a pluriharmonic metric from \mathbb{V} , so is necessarily a semisimple local system on the quasi-projective variety $D^m - D^{m+1} = D^{(m)} - C^{(m)}$.

Definition 11.14. Define a filtration on $\mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle$ by

$$J_r \mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle := \ker(\text{Res}_{r+1}),$$

for $r \geq 0$. This generalises [Tim2, Definition 1.6].

Definition 11.15. Define the graded sheaf $\mathcal{L}_{(2)}^*(\mathbb{V})$ on X to consist of $j_* \mathbb{V}$ -valued L^2 distributional forms a for which ∂a and $\bar{\partial} a$ are also L^2 . Write $L_{(2)}^*(X, \mathbb{V}) := \Gamma(X, \mathcal{L}_{(2)}^*(\mathbb{V}))$.

Since θ is holomorphic, note that the operators θ and $\bar{\theta}$ are bounded, so also act on $\mathcal{L}_{(2)}^*(\mathbb{V}) \otimes \mathbb{C}$.

11.3.1. Mixed Hodge structures.

Theorem 11.16. *If R is a quotient of ${}^{\text{VHS}}\varpi_1(Y, y)$ for which the representation $\pi_1(Y, y) \rightarrow R(\mathbb{R})$ has unitary monodromy around the local components of D , then there is a canonical non-positively weighted mixed Hodge structure $(Y, y)_{\text{MHS}}^{R, \text{Mal}}$ on $(Y, y)^{R, \text{Mal}}$, in the sense of Definition 10.7. The associated split MHS is given by*

$$\underline{\text{gr}}(Y, y)_{\text{MHS}}^{R, \text{Mal}} \simeq \text{Spec} \left(\bigoplus_{a, b} \mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a - b], d_2 \right),$$

with $\mathbb{H}^a(X, \mathbf{R}^b j_ \mathbb{O}(R))$ a pure ind-Hodge structure of weight $a + 2b$.*

Proof. We apply Proposition 11.2, taking $\mathcal{S}^*(\mathbb{V}) := \mathcal{A}_X^*(\mathbb{V})\langle D \rangle$, equipped with its filtration J . The first condition to check is compatibility with tensor operations. This follows because, although a product of arbitrary L^2 functions is not L^2 , a product of meromorphic L^2 functions is so.

Next, we check that $\mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle \rightarrow j_* \mathcal{A}_Y^\bullet(\mathbb{V})_{\mathbb{C}}$ is a quasi-isomorphism, with $\text{gr}_m^J \mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle \simeq \mathbf{R}^m j_* \mathbb{V}[-m]$. [Tim2, Proposition 1.7] (which deals with unitary local systems), adapts to show that Res_m gives a quasi-isomorphism

$$\text{gr}_m^J \mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle \rightarrow J_0 \nu_{m*} \mathcal{A}_{D^{(m)}}^\bullet(\nu_m^{-1} j_* \mathbb{V} \otimes \varepsilon^m)\langle C^{(m)} \rangle[-m].$$

Since $\mathbf{R}^m j_* \mathbb{V} \cong \nu_{m*}(\nu_m^{-1} j_* \mathbb{V} \otimes \varepsilon^m)$, this means that it suffices to establish the quasi-isomorphism for $m = 0$ (replacing X with $D^{(m)}$ for the higher cases). The proof of [Tim1, Theorem D.2(a)] adapts to this generality, showing that $j_* \mathbb{V} \rightarrow J_0 \mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle$ is a quasi-isomorphism.

It only remains to show that for all a, b , the groups $\mathbb{H}^{a+b}(X, F^p \text{gr}_b^J \mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle)$ define a Hodge filtration on $\mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{V})_{\mathbb{C}}$, giving a pure Hodge structure of weight $a + 2b$. This is essentially [Tim2, Proposition 6.4]: the quasi-isomorphism induced above by Res_m is in fact a filtered quasi-isomorphism, provided we set ε^m to be of type (m, m) . By applying a twist, we can therefore reduce to the case $b = 0$ (replacing X with $D^{(b)}$ for the higher cases), so we wish to show that the groups $\mathbb{H}^a(X, F^p J_0 \mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle)$ define a Hodge filtration on $\mathbb{H}^a(X, j_* \mathbb{V})$ of weight a .

The proof of [Tim1, Proposition D.4] adapts to give this result, by identifying $\mathbb{H}^*(X, j_* \mathbb{V})$ with L^2 cohomology, which in turn is identified with the space of harmonic forms. We have a bicomplex $(\Gamma(X, \mathcal{L}_{(2)}^*(\mathbb{V}) \otimes \mathbb{C}), D', D'')$ satisfying the principle of two types, with $F^p J_0 \mathcal{A}_X^\bullet(\mathbb{V})\langle D \rangle \rightarrow F^p \mathcal{L}_{(2)}^\bullet(\mathbb{V}) \otimes \mathbb{C}$ and $j_* \mathbb{V} \rightarrow \mathcal{L}_{(2)}^\bullet(\mathbb{V})$ both being quasi-isomorphisms. \square

11.3.2. Mixed twistor structures.

Definition 11.17. Given a smooth complex variety Z , let $\mathcal{L}_{(2)}^*(\mathbb{V}) \mathcal{O}_Z^{\text{hol}}$ be the sheaf on $X \times Z(\mathbb{C})$ consisting of holomorphic families of L^2 distributions on X , parametrised by $Z(\mathbb{C})$. Explicitly, given a local co-ordinate z on $Z(\mathbb{C})$, the space $\Gamma(U \times \{|z| < R\}, \mathcal{L}_{(2)}^n(\mathbb{V}) \mathcal{O}_P^{\text{hol}})$ consists of power series

$$\sum_{m \geq 0} a_m z^m$$

with $a_m \in \Gamma(U, \mathcal{L}_{(2)}^n(\mathbb{V})) \otimes \mathbb{C}$, such that for all $K \subset U$ compact and all $r < R$, the sum

$$\sum_{m \geq 0} \|a_m\|_{2,K} r^m$$

converges, where $\| - \|_{2,K}$ denotes the L^2 norm on K .

Definition 11.18. Set $\check{L}_{(2)}^n(X, \mathbb{V})$ to be the complex of $\mathcal{O}_P^{\text{hol}}$ -modules on $P(\mathbb{C})$ given by

$$\check{L}_{(2)}^n(X, \mathbb{V}) := \Gamma(X, \mathcal{L}_{(2)}^n(\mathbb{V}) \mathcal{O}_P^{\text{hol}}(n)),$$

with differential $uD + vD^c$. Note that locally on $P(\mathbb{C})$, elements of $\check{L}_{(2)}^n(X, \mathbb{V})$ can be characterised as convergent power series with coefficients in $L_{(2)}^n(X, \mathbb{V}) \otimes \mathbb{C}$.

Theorem 11.19. *If $\pi_1(Y, y) \rightarrow R(\mathbb{R})$ is Zariski-dense, with unitary monodromy around the local components of D , then there is a canonical non-positively weighted mixed twistor structure $(Y, y)_{\text{MTS}}^{R, \text{Mal}}$ on $(Y, y)^{R, \text{Mal}}$, in the sense of Definition 10.8. The associated split MTS is given by*

$$\underline{\text{gr}}(Y, y)_{\text{MTS}}^{R, \text{Mal}} \simeq \text{Spec} \left(\bigoplus_{a,b} \mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a-b], d_2 \right),$$

with $\mathbb{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R))$ of weight $a + 2b$.

Proof. We verify the conditions of Proposition 11.10, setting

$$\mathcal{T}^*(\mathbb{V}) \subset j_* \mathcal{A}_Y(\mathbb{V}) \otimes \mathbb{C}$$

to be $\mathcal{T}^*(\mathbb{V}) =: \mathcal{A}_X^*(\mathbb{V})\langle D \rangle$, with its filtration J defined above. This gives the complex $\check{\mathcal{T}}^\bullet(\mathbb{V}) \subset j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{V})$ on $X \times P(\mathbb{C})$ whose underlying sheaf is $\bigoplus_{n \geq 0} \mathcal{T}^n(\mathbb{V}) \otimes_{\mathcal{A}_X^0} \mathcal{A}_X^0 \mathcal{O}_P^{\text{hol}}(n)$, with differential $uD + vD^c$.

This leads us to study the restriction to $S^1(\mathbb{C}) \subset P(\mathbb{C})$, where we can divide $\mathcal{T}^{pq}(\mathbb{V})$ by $(u + iv)^p (u - iv)^q$, giving

$$j_* \check{\mathcal{A}}_Y^\bullet(\mathbb{V})|_{S^1(\mathbb{C})} \cong (j_* \mathcal{A}_Y^*(\mathbb{V}) \mathcal{O}_{S^1}^{\text{hol}}, t^{-1} \otimes D),$$

where (adapting Lemma 5.7),

$$t^{-1} \circledast D := d^+ + t^{-1} \diamond \vartheta = \partial + \bar{\partial} + t^{-1}\theta + t\bar{\theta},$$

for $t \in \mathbb{C}^\times \cong S^1(\mathbb{C})$. There is a similar expression for $\check{\mathcal{T}}^\bullet(\mathbb{V})|_{S^1(\mathbb{C})}$.

Now, as observed in [Sim2, end of §3], $t^{-1} \circledast D$ defines a holomorphic family $\mathcal{K}(\mathbb{V})$ of local systems on Y , parametrised by $S^1(\mathbb{C}) = \mathbb{C}^\times$. Beware that for non-unitary points $\lambda \in \mathbb{C}^\times$, the canonical metric is not pluriharmonic on the fibre $\mathcal{K}(\mathbb{V})_\lambda$, since $\lambda^{-1}\theta + \lambda\bar{\theta}$ is not Hermitian. The proof of Theorem 11.16 (essentially [Tim2, Proposition 1.7] and [Tim1, Theorem D.2(a)]) still adapts to verify conditions (1) and (2) from Proposition 11.10, replacing \mathbb{V} with $\mathcal{K}(\mathbb{V})$, so that for instance

$$j_*\mathcal{K}(\mathbb{V}) \rightarrow J_0\check{\mathcal{T}}^\bullet(\mathbb{V})|_{S^1(\mathbb{C})}$$

is a quasi-isomorphism.

It remains to verify condition (3) from Proposition 11.10: we need to show that for all $a, b \geq 0$, the $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant sheaf

$$\ker(\mathbb{H}^a(X, \text{gr}_b^J \check{\mathcal{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})} \oplus \overline{\sigma^*\mathbb{H}^a(X, \text{gr}_b^J \check{\mathcal{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})}} \rightarrow \mathbb{H}^a(X, \mathcal{H}^b(j_*\check{\mathcal{A}}_Y^\bullet(\mathbb{V})))|_{S^1(\mathbb{C})})$$

is a finite locally free $\mathcal{O}_P^{\text{hol}}$ -module of slope $a + 2b$.

Arguing as in the proof of Theorem 11.16, we may apply a twist to reduce to the case $b = 0$ (replacing X with $D^{(b)}$ for the higher cases), so we wish to show that

$$\mathcal{E}^a := \ker(\mathbb{H}^a(X, J_0\check{\mathcal{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})} \oplus \overline{\sigma^*\mathbb{H}^a(X, J_0\check{\mathcal{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})}} \rightarrow \mathbb{H}^a(X, j_*\mathcal{K}(\mathbb{V}))|_{S^1(\mathbb{C})})$$

is a holomorphic vector bundle on $P(\mathbb{C})$ of slope a .

We do this by considering the graded sheaf $\mathcal{L}_{(2)}^*(\mathbb{V})$ of L^2 -integrable distributions from Definition 11.15, and observe that [Tim1, Proposition D.4] adapts to show that

$$j_*\mathcal{K}(\mathbb{V}) \rightarrow (\mathcal{L}_{(2)}^*(\mathbb{V})\mathcal{O}_{S^1}^{\text{hol}}, t^{-1} \circledast D)$$

is a quasi-isomorphism on $X \times S^1(\mathbb{C})$.

On restricting to $\tilde{P}(\mathbb{C}) \subset P(\mathbb{C})$, Definition 11.5 gives the co-ordinate t on $\tilde{P}(\mathbb{C})$ as $t = \frac{u+iv}{u-iv}$, and dividing $\mathcal{T}^n(\mathbb{V})$ by $(u-iv)^n$ gives an isomorphism

$$\check{\mathcal{T}}^\bullet(\mathbb{V})|_{\tilde{P}(\mathbb{C})} \cong (\mathcal{A}_X^*(\mathbb{V})\langle D \rangle \mathcal{O}_{\tilde{P}}^{\text{hol}}, tD' + D''),$$

and similarly for $j_*\check{\mathcal{A}}_Y^\bullet(\mathbb{V})|_{\tilde{P}(\mathbb{C})}$

Thus we also wish to show that

$$J_0\check{\mathcal{T}}^\bullet(\mathbb{V})|_{\tilde{P}(\mathbb{C})} \rightarrow (\mathcal{L}_{(2)}^*(\mathbb{V})\mathcal{O}_{\tilde{P}}^{\text{hol}}, tD' + D'')$$

is a quasi-isomorphism. Condition (1) from Proposition 11.10 combines with the quasi-isomorphism above to show that we have a quasi-isomorphism on $S^1(\mathbb{C}) \subset \tilde{P}(\mathbb{C})$, so cohomology of the quotient is supported on $0 \in \tilde{P}(\mathbb{C})$. Studying the fibre over this point, it thus suffices to show that

$$(J_0\mathcal{T}(\mathbb{V}), D'') \rightarrow (\mathcal{L}_{(2)}^*(\mathbb{V}) \otimes \mathbb{C}, D'')$$

is a quasi-isomorphism, which also follows by adapting [Tim1, Proposition D.4].

Combining the quasi-isomorphisms above gives an isomorphism

$$\mathcal{E}^a \cong \mathcal{H}^a(\check{L}_{(2)}^\bullet(X, \mathbb{V})),$$

and inclusion of harmonic forms $\mathcal{H}^a(X, \mathbb{V}) \hookrightarrow L_{(2)}^a(X, \mathbb{V})$ gives a map

$$\mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathcal{O}_P^{\text{hol}}(a) \rightarrow \mathcal{H}^a(\check{L}_{(2)}^\bullet(X, \mathbb{V})).$$

The Green's operator G behaves well in holomorphic families, so gives a decomposition

$$\check{L}_{(2)}^a(X, \mathbb{V}) = (\mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathcal{O}_P^{\text{hol}}(a)) \oplus \Delta \check{L}_{(2)}^a(X, \mathbb{V}),$$

making use of finite-dimensionality of $\mathcal{H}^a(X, \mathbb{V})$ to give the isomorphism $\mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathcal{O}_P^{\text{hol}}(a) \cong \ker \Delta \cap \check{L}_{(2)}^a(X, \mathbb{V})$.

Since these expressions are $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant, it suffices to work on $\tilde{P}(\mathbb{C})$. Dividing $\mathcal{F}^n(\mathbb{V})$ by $(u - iv)^n$ gives

$$\check{L}_{(2)}^{\bullet}(X, \mathbb{V})|_{\tilde{P}(\mathbb{C})} \cong (L_{(2)}^*(X, \mathbb{V}) \mathcal{O}_{\tilde{P}}^{\text{hol}}, tD' + D'').$$

Now, since $D'(D'')^* + (D'')^*D' = 0$, we can write

$$\frac{1}{2}\Delta = (tD' + D'')(D'')^* + (D'')^*(tD' + D''),$$

giving us a direct sum decomposition

$$\check{L}_{(2)}^a(X, \mathbb{V})|_{\tilde{P}(\mathbb{C})} = (\mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathcal{O}_{\tilde{P}}^{\text{hol}}) \oplus (tD' + D'')\check{L}_{(2)}^a(X, \mathbb{V})|_{\tilde{P}(\mathbb{C})} \oplus (D'')^*\check{L}_{(2)}^n(X, \mathbb{V})|_{\tilde{P}(\mathbb{C})},$$

with the principle of two types (as in [Sim3] Lemmas 2.1 and 2.2) showing that $(tD' + D'') : \text{Im}((D'')^*) \rightarrow \text{Im}(tD' + D'')$ is an isomorphism.

We have therefore shown that $\mathcal{E}^a \cong \mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathcal{O}_P^{\text{hol}}(a)$, which is indeed of slope a . \square

Proposition 11.20. *Assume that a Zariski-dense representation $\pi_1(Y, y) \rightarrow R(\mathbb{R})$ has unitary monodromy around the local components of D , and that the discrete S^1 -action on ${}^v\varpi_1(Y, y)^{\text{red}}$ descends to R . Then there are natural $(S^1)^\delta$ -actions on $(Y, y)_{\text{MTS}}^{R, \text{Mal}}$ and $\text{gr}(Y, y)_{\text{MTS}}^{R, \text{Mal}}$, compatible with the opposedness isomorphism, and with the action of $-1 \in S^1$ coinciding with that of $-1 \in \mathbb{G}_m$.*

Proof. We just observe that the construction $\mathcal{F}^*(\mathbb{V}) = \mathcal{A}_X^*(\mathbb{V})\langle D \rangle$ of Theorem 11.19 satisfies the conditions of Proposition 11.11, being closed under the \boxtimes -action of \mathbb{C}^\times . \square

11.4. Singular and simplicial varieties. Fix a smooth proper simplicial complex variety X_\bullet , and a simplicial divisor $D_\bullet \subset X_\bullet$ with normal crossings. Set $Y_\bullet := X_\bullet - D_\bullet$, with a point $y \in Y_0$, and write $j : Y_\bullet \rightarrow X_\bullet$ for the embedding. Note that Proposition 10.25 shows that for any separated complex scheme Y of finite type, there exists such a simplicial variety Y_\bullet with an augmentation $a : Y_\bullet \rightarrow Y$ for which $|Y_\bullet| \rightarrow Y$ is a weak equivalence.

Theorem 11.21. *Take $\rho : \pi_1(|Y_\bullet|, y) \rightarrow R(\mathbb{R})$ Zariski-dense with R pro-reductive, and assume that for every local system \mathbb{V} on $|Y_\bullet|$ corresponding to an R -representation, the local system $a_0^{-1}\mathbb{V}$ on Y_0 is semisimple, with unitary monodromy around the local components of D_0 . Then there is a canonical non-positively weighted mixed twistor structure $(|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}}$ on $(|Y_\bullet|, y)^{R, \text{Mal}}$, in the sense of Definition 10.8.*

The associated split MTS is given by

$$\text{gr}(|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}} \simeq \text{Spec Th} \left(\bigoplus_{p, q} \text{HP}^p(X_\bullet, a^{-1}\mathbf{R}^q j_* \mathbb{O}(R))[-p - q], d_2 \right),$$

with $\text{HP}^p(X_n, \mathbf{R}^q j_* a_n^{-1}\mathbb{O}(R))$ of weight $p + 2q$. Here, $\text{HP}^p(X_\bullet, a^{-1}\mathbb{V})$ denotes the cosimplicial vector space $n \mapsto \text{HP}^p(X_n, a_n^{-1}\mathbb{V})$, and Th is the Thom-Whitney functor of Definition 3.28.

Proof. Our first observation is that the pullback of a holomorphic pluriharmonic metric is holomorphic, so for any local system \mathbb{V} corresponding to an R -representation, the local system $a_n^{-1}\mathbb{V}$ on Y_n is semisimple for all n , with unitary monodromy around the local components of D_n . We may therefore form objects

$$(Y_n, (\sigma_0)^n y)_{\text{MTS}}^{R, \text{Mal}} \in \text{dg}_{\mathbb{Z}} \text{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\text{Mat}_1 \times \mathbb{G}_m),$$

and $\text{gr}(Y_n, (\sigma_0)^n y)_{\text{MTS}}^{R, \text{Mal}} \in \text{dg}_{\mathbb{Z}} \text{Aff}(R)_*(\text{Mat}_1)$ as in the proof of Theorem 11.19, together with opposedness quasi-isomorphisms.

These constructions are functorial, giving cosimplicial DGAs

$$\mathcal{O}(Y_\bullet, y)_{\text{MTS}}^{R, \text{Mal}} \in \text{cDG}_{\mathbb{Z}} \text{Alg}_{\mathbb{A}^1 \times C^*}(R)_*(\text{Mat}_1 \times \mathbb{G}_m),$$

and $\mathcal{O}(\underline{\text{gr}}(Y_\bullet, y)_{\text{MTS}}^{R, \text{Mal}}) \in cDG_{\mathbb{Z}}\text{Aff}(R)_*(\text{Mat}_1)$. We now apply the Thom-Whitney functor, giving an algebraic MTS with $\underline{\text{gr}}(|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}}$ as above, and

$$\mathcal{O}(|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}} := \text{Th}(\mathcal{O}(Y_\bullet, y)_{\text{MTS}}^{R, \text{Mal}}).$$

Taking the fibre over $(1, 1) \in \mathbb{A}^1 \times C^*$ gives $\text{Th}(\mathcal{O}(Y_\bullet, y)_{\text{MTS}}^{R, \text{Mal}})$, which is quasi-isomorphic to $\mathcal{O}(|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}}$, by Lemma 9.11. \square

Theorem 11.22. *Take $\rho: \pi_1(|Y_\bullet|, y) \rightarrow R(\mathbb{R})$ Zariski-dense with R pro-reductive, and assume that for every local system \mathbb{V} on $|Y_\bullet|$ corresponding to an R -representation, the local system $a_0^{-1}\mathbb{V}$ underlies a variation of Hodge structure with unitary monodromy around the local components of D_0 . Then there is a canonical non-positively weighted mixed Hodge structure $(Y, y)_{\text{MHS}}^{R, \text{Mal}}$ on $(Y, y)^{R, \text{Mal}}$, in the sense of Definition 10.7. The associated split MTS is given by*

$$\underline{\text{gr}}(Y, y)_{\text{MHS}}^{R, \text{Mal}} \simeq \text{Spec Th}\left(\bigoplus_{p, q} \text{HP}(X_\bullet, \mathbf{R}^q j_* a^{-1} \mathbb{O}(R))[-p - q], d_2\right),$$

with $\text{HP}(X_n, \mathbf{R}^q j_* a_n^{-1} \mathbb{O}(R))$ a pure ind-Hodge structure of weight $p + 2q$.

Proof. The proof of Theorem 11.21 carries over, replacing Theorem 11.19 with Theorem 11.16, and observing that variations of Hodge structure are preserved by pullback. \square

Definition 11.23. Define ${}^\nu\varpi_1(|Y_\bullet|, y)^{\text{norm}}$ to be the quotient of $\varpi_1(|Y_\bullet|, y)^{\text{norm}}$ characterised as follows. Representations of ${}^\nu\varpi_1(|Y_\bullet|, y)^{\text{norm}}$ correspond to local systems \mathbb{V} on $|Y_\bullet|$ for which $a_0^{-1}\mathbb{V}$ is a semisimple local system on Y_0 whose monodromy around local components of D_0 has unitary eigenvalues.

Proposition 11.24. *There is a discrete action of the circle group S^1 on ${}^\nu\varpi_1(|Y_\bullet|, y)^{\text{norm}}$, such that the composition $S^1 \times \pi_1(|Y_\bullet|, y) \rightarrow {}^\nu\varpi_1(|Y_\bullet|, y)^{\text{norm}}$ is continuous. We denote this last map by $\sqrt{h}: \pi_1(|Y_\bullet|, y) \rightarrow {}^\nu\varpi_1(|Y_\bullet|, y)^{\text{norm}}((S^1)^{\text{cts}})$.*

Proof. The proof of Proposition 9.8 carries over to the quasi-projective case. \square

Proposition 11.25. *Take a pro-reductive S^1 -equivariant quotient R of ${}^\nu\varpi_1(|Y_\bullet|, x)^{\text{norm}}$, and assume that for every local system \mathbb{V} on $|Y_\bullet|$ corresponding to an R -representation, the local system $a_0^{-1}\mathbb{V}$ has unitary monodromy around the local components of D_0 . Then there are natural $(S^1)^\delta$ -actions on $(|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}}$ and $\underline{\text{gr}}(|Y_\bullet|, y)_{\text{MTS}}^{R, \text{Mal}}$, compatible with the opposedness isomorphism, and with the action of $-1 \in S^1$ coinciding with that of $-1 \in \mathbb{G}_m$.*

Proof. This just follows from the observation that the S^1 -action of Proposition 11.20 is functorial, hence compatible with the construction of Theorem 11.21. \square

11.5. More general monodromy. It is natural to ask whether the hypotheses of Theorems 11.16 and 11.19 are optimal, or whether algebraic mixed Hodge and mixed twistor structures can be defined more widely. The analogous results to Theorem 11.16 for ℓ -adic pro-algebraic homotopy types in [Pri6] hold in full generality (i.e. for any Galois-equivariant quotient R of $\varpi_1(Y, y)^{\text{red}}$). However the proofs of Theorems 11.16 and 11.19 clearly do not extend to non-unitary monodromy, since if θ is not holomorphic, then $\bar{\theta}$ does not act on $\mathcal{A}_X^*(\mathbb{V})\langle D \rangle$. Thus any proof adapting those theorems would have to take some modification of $\mathcal{A}_X^*(\mathbb{V})\langle D \rangle$ closed under the operator $\bar{\theta}$.

A serious obstruction to considering non-semisimple monodromy around the divisor is that the principle of two types plays a crucial rôle in the proofs of Theorems 11.16 and 11.19, and for quasi-projective varieties this is only proved for L^2 cohomology. The map $H^*(X, j_*\mathbb{V}) \rightarrow H_{(2)}^*(X, \mathbb{V})$ is only an isomorphism either for X a curve or for semisimple monodromy, so $\mathcal{L}_{(2)}^\bullet(\mathbb{V})$ will no longer have the properties we require. There is not even any prospect of modifying the filtrations in Propositions 11.2 or 11.10 so that

$J_0H^*(Y, \mathbb{V}) := H_{(2)}^*(X, \mathbb{V})$, because L^2 cohomology does not carry a cup product *a priori* (and nor does intersection cohomology). This means that there is little prospect of applying the decomposition theorems of [Sab] and [Moc2], except possibly in the case of curves.

If the groups $H^n(X, j_*\mathbb{V})$ all carry natural MTS or MHS, then the other terms in the Leray spectral sequence should inherit MHS or MTS via the isomorphisms

$$H^n(X, \mathbf{R}^m j_*\mathbb{V}) \cong H^n(X, \mathbf{R}^m j_*\mathbb{R} \otimes (j_*\mathbb{V}^\vee)^\vee) \cong H^n(D^{(m)}, j_{m*}j_m^{-1}\nu_m^{-1}(j_*\mathbb{V}^\vee)^\vee \otimes \varepsilon^m),$$

for $j_m: (D^m - D^{m+1}) \rightarrow D^{(m)}$ the canonical open immersion. Note that $j_m^{-1}\nu_m^{-1}(j_*\mathbb{V}^\vee)^\vee$ is a local system on $D^m - D^{m+1}$ — this will hopefully inherit a tame pluriharmonic metric from \mathbb{V} by taking residues.

It is worth noting that even for non-semisimple monodromy, the weight filtration on homotopy types should just be the one associated to the Leray spectral sequence. Although the monodromy filtration is often involved in such weight calculations, [Del3] shows that for \mathbb{V} pure of weight 0 on Y , we still expect $j_*\mathbb{V}$ to be pure of weight 0 on X . It is only at generic (not closed) points of X that the monodromy filtration affects purity.

Adapting L^2 techniques to the case of non-semisimple monodromy around the divisor would have to involve some complex of Fréchet spaces to replace $L_{(2)}^\bullet(X, \mathbb{V})$, with the properties that it calculates $H^*(X, j_*\mathbb{V})$ and is still amenable to Hodge theory. When monodromy around D is trivial, a suitable complex is $A^\bullet(X, j_*\mathbb{V})$, since $j_*\mathbb{V}$ is a local system. In general, one possibility is a modification of Foth's complex $\mathcal{B}^\bullet(\mathbb{V})$ from [Fot], based on bounded forms. Another possibility might be the complex given by $\bigcap_{p \in (0, \infty)} L_{(p)}^\bullet(X, \mathbb{V})$, i.e. the complex consisting of distributions which are L^p for all $p < \infty$. Beware that these are not the same as bounded forms — p -norms are all defined, but the limit $\lim_{p \rightarrow \infty} \|f\|_p$ might be infinite (as happens for $\log|\log|z||$).

Rather than using Fréchet space techniques directly, another approach to defining the MHS or MTS we need (including for \mathbb{V} with non-semisimple monodromy) might be via Saito's mixed Hodge modules or Sabbah's mixed twistor modules. Since $H^n(X, j_*\mathbb{V}) \cong \mathrm{IH}^n(X, \mathbb{V})$ for curves X , fibring by families of curves then opens the possibility of putting MHS or MTS on $H^n(X, j_*\mathbb{V})$ for general X . Again, the main difficulty would lie in defining the cup products needed to construct DGAs.

12. CANONICAL SPLITTINGS

12.1. Splittings of mixed Hodge structures.

Definition 12.1. Define MHS to be the category of finite-dimensional mixed Hodge structures.

Write $\mathrm{row}_2: \mathrm{SL}_2 \rightarrow \mathbb{A}^2$ for projection onto the second row, so $\mathrm{row}_2^\sharp O(\mathbb{A}^2)$ is a subring of $O(\mathrm{SL}_2)$. This subring is equivariant for the S -action on SL_2 from Definition 1.15.

Definition 12.2. Define SHS (resp. $\mathrm{ind}(\mathrm{SHS})$) to be the category of pairs (V, β) , where V is a finite-dimensional S -representation (resp. an S -representation) in real vector spaces and $\beta: V \rightarrow V \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$ is S -equivariant. A morphism $(V, \beta) \rightarrow (V', \beta')$ is an S -equivariant map $f: V \rightarrow V'$ with $\beta' \circ f = (f \otimes \mathrm{id}) \circ \beta$.

Definition 12.3. Given $(V, \beta) \in \mathrm{SHS}$, observe that taking duals gives rise to a map $\beta^\vee: V^\vee \rightarrow V^\vee \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$. Then define the dual in SHS by $(V, \beta)^\vee := (V^\vee, \beta^\vee)$.

Likewise, we define the tensor product $(U, \alpha) \otimes (V, \beta) := (U \otimes V, \alpha \otimes \mathrm{id} + \mathrm{id} \otimes \beta)$.

Observe that for $(V, \beta), (V', \beta') \in \mathrm{SHS}$,

$$\mathrm{Hom}_{\mathrm{SHS}}((V, \beta), (V', \beta')) \cong \mathrm{Hom}_{\mathrm{SHS}}((\mathbb{R}, 0), (V, \beta)^\vee \otimes (V', \beta')).$$

Lemma 12.4. *A (commutative) algebra (A, δ) in $\text{ind}(\text{SHS})$ consists of an S -equivariant (commutative) algebra A , together with an S -equivariant derivation $\delta : A \rightarrow A \otimes \text{row}_2^\sharp O(\mathbb{A}^2)(-1)$.*

Proof. We need to endow $(A, \delta) \in \text{SHS}$ with a unit $(\mathbb{R}, 0) \rightarrow (A, \delta)$, which is the same as a unit $1 \in A$, and with a (commutative) associative multiplication

$$\mu: (A, \delta) \otimes (A, \delta) \rightarrow (A, \delta).$$

Substituting for \otimes , this becomes $\mu: (A \otimes A, \delta \otimes \text{id} + \text{id} \otimes \delta) \rightarrow (A, \delta)$, so μ is a (commutative) associative multiplication on A , and for $a, b \in A$, we must have $\delta(ab) = a\delta(b) + b\delta(a)$. \square

Definition 12.5. Given an S -representation V , the inclusion $\mathbb{G}_m \hookrightarrow S$ (given by $v = 0$ in the co-ordinates of Remark 1.3) gives a grading on V , which we denote by

$$V = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n V.$$

Equivalently, $\mathcal{W}_n(V \otimes \mathbb{C})$ is the sum of elements of type (p, q) for $p + q = n$.

Theorem 12.6. *The categories MHS and SHS are equivalent. This equivalence is additive, and compatible with tensor products and duals.*

Proof. Given $(V, \beta) \in \text{SHS}$ as above, define a weight filtration on V by $W_r V = \bigoplus_{i \leq r} \mathcal{W}_i V$. Since β is S -equivariant and $\text{row}_2^\sharp O(\mathbb{A}^2)(-1)$ is of strictly positive weights, we have

$$\beta : W_r V \rightarrow (W_{r-1} V) \otimes \text{row}_2^\sharp O(\mathbb{A}^2)(-1).$$

Thus β gives rise to an S -equivariant map $V \rightarrow V \otimes O(\text{SL}_2)(-1)$ for which $\beta(W_r V) \subset (W_{r-1} V) \otimes O(\text{SL}_2)(-1)$ for all r . In particular, $(W_r V, \beta|_{W_r V}) \in \text{SHS}$ for all r .

We now form $V \otimes O(\text{SL}_2)$, then look at the S -equivariant derivation $N_\beta : V \otimes O(\text{SL}_2) \rightarrow V \otimes O(\text{SL}_2)(-1)$ given by $N_\beta = \text{id} \otimes N + \beta \otimes \text{id}$. Since $\ker N = O(C)$, this map is $O(C)$ -linear; by Lemma 1.19, it corresponds under Lemma 1.9 to a real derivation

$$N_\beta : V \otimes \mathcal{S} \rightarrow V(-1) \otimes \mathcal{S}$$

such that $N_\beta \otimes_{\mathbb{R}} \mathbb{C}$ preserves Hodge filtrations F . The previous paragraph shows that $N_\beta((W_r V) \otimes \mathcal{S}) \subset (W_{r-1} V) \otimes \mathcal{S}$, with

$$\text{gr}^W N_\beta = (\text{id} \otimes N) : (\text{gr}^W V) \otimes \mathcal{S} \rightarrow (\text{gr}^W V)(-1) \otimes \mathcal{S}.$$

Therefore $M(V, \beta) := \ker(N_\beta) \subset V \otimes \mathcal{S}$ is a real vector space, equipped with an increasing filtration W , and a decreasing filtration F on $M(V, \beta) \otimes \mathbb{C}$. We need to show that $M(V, \beta)$ is a mixed Hodge structure.

Since $N : \mathcal{S} \rightarrow \mathcal{S}(-1)$ is surjective, the observation above that $\text{gr}^W N_\beta = (\text{id} \otimes N)$ implies that N_β must also be surjective (as the filtration W is bounded), so

$$0 \rightarrow M(V, \beta) \rightarrow V \otimes \mathcal{S} \xrightarrow{N_\beta} V(-1) \otimes \mathcal{S} \rightarrow 0$$

is an exact sequence; this implies that the functor M is exact.

Since $\text{gr}_r^W(V, \beta) = (\mathcal{W}_r V, 0)$, we get that $M(\text{gr}_r^W(V, \beta)) = \mathcal{W}_r V$. As M is exact, $\text{gr}_r^W M(V, \beta) = M(\text{gr}_r^W(V, \beta))$, so we have shown that $\text{gr}_r^W M(V, \beta)$ is a pure weight r Hodge structure, and hence that $M(V, \beta) \in \text{MHS}$. Thus we have an exact functor

$$M : \text{SHS} \rightarrow \text{MHS};$$

it is straightforward to check that this is compatible with tensor products and duals.

We need to check that M is an equivalence of categories. First, observe that for any S -representation V , we have an object $(V, 0) \in \text{SHS}$ with $M(V) = V$.

Write

$$\text{Ext}_{\text{SHS}}^1((U, \alpha), (V, \beta)) := \text{coker}(\beta_* - \alpha^* : \text{Hom}_S(U, V) \xrightarrow{\beta_* - \alpha^*} \text{Hom}_S(U, V \otimes O(C))).$$

This gives an exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_{\mathrm{SHS}}((U, \alpha), (V, \beta)) &\rightarrow \mathrm{Hom}_S(U, V) \xrightarrow{\beta_* - \alpha^*} \mathrm{Hom}_S(U, V \otimes O(C)) \\ &\rightarrow \mathrm{Ext}_{\mathrm{SHS}}^1((U, \alpha), (V, \beta)) \rightarrow 0. \end{aligned}$$

Note that $\mathrm{Ext}_{\mathrm{SHS}}^1((U, \alpha), (V, \beta))$ does indeed parametrise extensions of (U, α) by (V, β) : given an exact sequence

$$0 \rightarrow (V, \beta) \rightarrow (W, \gamma) \rightarrow (U, \alpha) \rightarrow 0,$$

we may choose an S -equivariant section s of $W \twoheadrightarrow U$, so $W \cong U \oplus V$. The obstruction to this being a morphism in SHS is $o(s) := s^*\gamma - \alpha \in \mathrm{Hom}_S(U, V \otimes O(C))$, and another choice of section differs from s by some $f \in \mathrm{Hom}_S(U, V)$, with $o(s+f) = o(s) + \beta_*f - \alpha^*f$.

Write $\mathbf{R}^i\Gamma_{\mathrm{SHS}}(V, \beta) := \mathrm{Ext}_{\mathrm{SHS}}^i((\mathbb{R}, 0), (V, \beta))$ for $i = 0, 1$, noting that

$$\mathrm{Ext}_{\mathrm{SHS}}^i((U, \alpha), (V, \beta)) = \mathbf{R}^i\Gamma_{\mathrm{SHS}}((V, \beta) \otimes (U, \alpha)^\vee).$$

We thus have morphisms

$$\begin{array}{ccccccc} 0 \rightarrow \Gamma_{\mathrm{SHS}}(V, \beta) & \longrightarrow & V^S & \xrightarrow{\beta} & (V \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1))^S & \rightarrow & \mathbf{R}^1\Gamma_{\mathrm{SHS}}(V, \beta) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow \Gamma_{\mathcal{H}}M(V, \beta) & \rightarrow & (V \otimes O(\mathrm{SL}_2))^S & \xrightarrow{\beta+N} & (V \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1))^S & \rightarrow & \mathbf{R}^1\Gamma_{\mathcal{H}}M(V, \beta) \rightarrow 0 \end{array}$$

of exact sequences, making use of the calculations of §1.3.1. For any short exact sequence in SHS, the morphisms $\rho^i: \mathbf{R}^i\Gamma_{\mathrm{SHS}}(V, \beta) \rightarrow \mathbf{R}^i\Gamma_{\mathcal{H}}M(V, \beta)$ are thus compatible with the long exact sequences of cohomology.

The crucial observation on which the construction hinges is that the map $\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1) \rightarrow \mathrm{coker}(N : O(\mathrm{SL}_2) \rightarrow O(\mathrm{SL}_2)(-1))$ is an isomorphism, making $\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$ a section for $O(\mathrm{SL}_2)(-1) \twoheadrightarrow \mathrm{H}^1(C^*, \mathcal{O}_{C^*})$. This implies that when $\beta = 0$, the maps ρ^i are isomorphisms. Since each object $(V, \beta) \in \mathrm{SHS}$ is an Artinian extension of S -representations, we deduce that the maps ρ^i must be isomorphisms for all such objects.

Taking $i = 1$ gives that $\mathrm{Ext}_{\mathrm{SHS}}^1((U, \alpha), (V, \beta)) \rightarrow \mathrm{Ext}_{\mathcal{H}}^1(M(U, \alpha), M(V, \beta))$ is an isomorphism; we deduce that every extension in MHS lifts uniquely to an extension in SHS, so $M : \mathrm{SHS} \rightarrow \mathrm{MHS}$ is essentially surjective. Taking $i = 0$ shows that M is full and faithful. \square

Remark 12.7. Note that the Tannakian fundamental group (in the sense of [DMOS]) of the category SHS is

$$\Pi(\mathrm{SHS}) = S \ltimes \mathrm{Fr}(\mathcal{W}_{>0}(\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)))^\vee,$$

where $\mathrm{Fr}(V)$ denotes the free pro-unipotent group generated by the pro-finite-dimensional vector space V . In other words, SHS is canonically equivalent to the category of finite-dimensional $\Pi(\mathrm{SHS})$ -representations. Likewise, $\mathrm{ind}(\mathrm{SHS})$ is equivalent to the category of all $\Pi(\mathrm{SHS})$ -representations.

The categories SHS and MHS both have vector space-valued forgetful functors. Tannakian formalism shows that the functor $\mathrm{SHS} \rightarrow \mathrm{MHS}$, together with a choice of natural isomorphism between the respective forgetful functors, gives a morphism $\Pi(\mathrm{MHS}) \rightarrow \Pi(\mathrm{SHS})$. The choice of natural isomorphism amounts to choosing a Levi decomposition for $\Pi(\mathrm{MHS})$, or equivalently a functorial isomorphism $V \cong \mathrm{gr}^W V$ of vector spaces for $V \in \mathrm{MHS}$.

A canonical choice b_0 of such an isomorphism is given by composing the embedding $b: M(V, \beta) \hookrightarrow V \otimes \mathcal{S}$ with the map $p_0: \mathcal{S} \rightarrow \mathbb{R}$ given by $x \mapsto 0$. This allows us to put a new MHS on V , with Hodge filtration $b_0(F)$ and the same weight filtration as V , so $b_0: M(V, \beta) \rightarrow (V, W, b_0(F))$ is an isomorphism of MHS. To describe this new MHS, first

observe that $\mathcal{S}(-1) \cong \Omega(\mathcal{S}/\mathbb{R}) = \mathcal{S}dx$, and that for $\beta : V \rightarrow V \otimes \Omega(\mathcal{S}/\mathbb{R})$, we get an isomorphism $\exp(-\int_0^x \beta) : V \rightarrow M(V, \beta)$, which is precisely b_0^{-1} .

Since the map $p_i : \mathcal{S} \rightarrow \mathbb{C}$ given by $x \mapsto i$ preserves F , it follows that the map

$$p_i \circ b_0^{-1} = \exp(-\int_0^i \beta) : V \rightarrow V \otimes \mathbb{C}$$

satisfies $\exp(-\int_0^i \beta)(b_0(F)) = F$, so the new MHS is

$$(V, W, b_0(F)) = (V, W, \exp(\int_0^i \beta)(F)).$$

Remark 12.8. In Proposition 1.26, it was shown that every mixed Hodge structure M admits a non-unique splitting $M \otimes \mathcal{S} \cong (\text{gr}^W M) \otimes \mathcal{S}$, compatible with the filtrations. Theorem 12.6 is a refinement of that result, showing that such a splitting can be chosen canonically, by requiring that the image of $\text{gr}^W M$ under the derivation $(\text{id}_M \otimes N : M \otimes O(\text{SL}_2) \rightarrow M \otimes O(\text{SL}_2)(-1))$ lies in $\text{row}_2^{\sharp} O(\mathbb{A}^2)(-1)$. This is because β is just the restriction of $\text{id}_M \otimes N$ to $V := \text{gr}^W M$.

This raises the question of which F -preserving maps $\beta : V \rightarrow V \otimes \Omega(\mathcal{S}/\mathbb{R})$ correspond to maps $V \rightarrow V \otimes \text{row}_2^{\sharp} O(\mathbb{A}^2)(-1)$ (rather than just $V \rightarrow V \otimes O(\text{SL}_2)(-1)$). Using the explicit description from the proof of Lemma 1.19, we see that this amounts to the restriction that

$$\beta(V_{\mathbb{C}}^{p,q}) \subset \sum_{a \geq 0, b \geq 0} V_{\mathbb{C}}^{p-a-1, q-b-1} (x-i)^a (x+i)^b dx.$$

Remark 12.9. In [Del4], Deligne established a characterisation of real MHS in terms of S -representations equipped with additional structure.

For any $\lambda \in \mathbb{C}$, we have a map $p_{\lambda} : \mathcal{S} \rightarrow \mathbb{C}$ given by $x \mapsto \lambda$, and $b_{\lambda}^{-1} := (p_{\lambda} \circ b)^{-1} = \exp(-\int_{\lambda}^x \beta) : V \rightarrow M(V, \beta)$. Comparing the filtrations $b_0(F)$ and $b_0(\bar{F})$ on V , we are led to consider

$$d := b_{-i} \circ b_i^{-1} = \exp(\int_{-i}^i \beta).$$

This maps V to V , and has the properties that $\bar{d} = d^{-1}$ and

$$(d - \text{id})(V_{\mathbb{C}}^{p,q}) \subset \bigoplus_{r < p, s < q} V_{\mathbb{C}}^{r,s}.$$

This is precisely the data of an \mathfrak{M} -representation in the sense of [Del4, Proposition 2.1], so corresponds to a MHS. Explicitly, we first find the unique operator $d^{1/2}$ with $d := (d^{1/2})^2$ satisfying the properties above, then define the mixed Hodge structure $M(V, d)$ to have underlying vector space V , with the same weight filtration, and with $F^p M(V, d) := d^{1/2}(F^p V)$.

For our choice of d as above, we then have an isomorphism

$$a := d^{1/2} \circ b_i = d^{-1/2} \circ b_{-i} : M(V, \beta) \rightarrow V$$

of vector spaces. Since $b_i(F^p M(V, \beta)) = F^p V$, this means that $a(F^p M(V, \beta)) = F^p M(V, d)$, so a is an isomorphism of MHS.

We have therefore shown directly how our category SHS is equivalent to Deligne's category of \mathfrak{M} -representations by sending the pair (V, β) to $(V, \exp(\int_{-i}^i \beta))$. This also gives a canonical isomorphism $\mathfrak{M} \cong \Pi(\text{SHS})$, once we specify the associated isomorphism $a \circ b_0^{-1} : V \rightarrow V$ on fibre functors. The Archimedean monodromy operator β thus provides a more canonical generator for the Lie algebra of $\text{R}_u \mathfrak{M}$ than is given by the operator d of [Del4] — providing such a generator was also the goal of the Hodge correlator \mathbb{G} of [Gon].

For an explicit quasi-inverse functor from \mathfrak{M} -representations to SHS, take a pair (V, d) . Since d is unipotent, $\delta := \log d: V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ is well-defined, and decomposes into types as $\delta = \sum_{p,q < 0} \delta^{p,q}$. We now just set

$$\begin{aligned} \beta &:= \sum_{a \geq 0, b \geq 0} \frac{\delta^{-a-1, -b-1} (x-i)^a (x+i)^b dx}{\int_{-i}^i (x-i)^a (x+i)^b dx}, \\ &= \sum_{a \geq 0, b \geq 0} \frac{(-1)^a (a+b+1)! \delta^{-a-1, -b-1} (x-i)^a (x+i)^b dx}{(2i)^{a+b+1} a! b!}. \end{aligned}$$

This equivalence $\mathfrak{M} \simeq \text{SHS}$ can be understood in terms of identifying the generating elements of [Del4, Construction 1.6] with explicit elements of $(\mathcal{W}_{>0} \text{row}_2^{\sharp}(O(\mathbb{A}^2)(-1)))^{\vee} \otimes \mathbb{C}$. Explicitly, Deligne's generating set $\{\delta^{-a-1, -b-1}\}$ is the dual basis to

$$\left\{ \frac{(-1)^a (a+b+1)! (x-iy)^a (x+iy)^b}{(2i)^{a+b+1} a! b!} \right\} \subset (\mathcal{W}_{>0} \text{row}_2^{\sharp}(O(\mathbb{A}^2)(-1))) \otimes \mathbb{C}.$$

12.2. Splittings of mixed twistor structures. The following lemma ensures that a mixed twistor structure can be regarded as an Artinian extension of \mathbb{G}_m -representations.

Lemma 12.10. *If \mathcal{E} and \mathcal{F} are pure twistor structures of weights m and n respectively, then*

$$\text{Hom}_{\text{MTS}}(\mathcal{E}, \mathcal{F}) \cong \begin{cases} \text{Hom}_{\mathbb{R}}(\mathcal{E}_1, \mathcal{F}_1) & m = n \\ 0 & m \neq n. \end{cases}$$

Proof. By hypothesis, $\mathcal{E} = \text{gr}_m^W \mathcal{E}$ and $\mathcal{F} = \text{gr}_n^W \mathcal{F}$. Thus we may assume that $\mathcal{E} = \mathcal{O}(m)$ and $\mathcal{F} = \mathcal{O}(n)$. Since homomorphisms must respect the weight filtration, we have

$$\text{Hom}_{\text{MTS}}(\mathcal{O}(m), \mathcal{O}(n)) = \text{Hom}_{\mathbb{P}^1}(\mathcal{O}(m), W_m \mathcal{O}(n)),$$

which is 0 unless $m \geq n$. When $m \geq n$, we have $W_m \mathcal{O}(n) = \mathcal{O}(n)$, so

$$\text{Hom}_{\text{MTS}}(\mathcal{O}(m), \mathcal{O}(n)) = \Gamma(\mathbb{P}^1, \mathcal{O}(n-m)),$$

which is 0 for $m > n$ and \mathbb{R} for $n = m$, as required. \square

Definition 12.11. Define STS to be the category of pairs (V, β) , where V is an \mathbb{G}_m -representation in real vector spaces and $\beta: V \rightarrow V \otimes \text{row}_2^{\sharp} O(\mathbb{A}^2)(-1)$ is \mathbb{G}_m -equivariant. A morphism $(V, \beta) \rightarrow (V', \beta')$ is a \mathbb{G}_m -equivariant map $f: V \rightarrow V'$ with $\beta' \circ f = (f \otimes \text{id}) \circ \beta$.

Note that the only difference between Definitions 12.2 and 12.11 is that the latter replaces S with \mathbb{G}_m throughout.

Definition 12.12. Given $(V, \beta) \in \text{STS}$, observe that taking duals gives rise to a map $\beta^{\vee}: V^{\vee} \rightarrow V^{\vee} \otimes \text{row}_2^{\sharp} O(\mathbb{A}^2)(-1)$. Then define the dual in STS by $(V, \beta)^{\vee} := (V^{\vee}, \beta^{\vee})$.

Likewise, we define the tensor product by $(U, \alpha) \otimes (V, \beta) := (U \otimes V, \alpha \otimes \text{id} + \text{id} \otimes \beta)$.

Observe that for $(V, \beta), (V', \beta') \in \text{STS}$,

$$\text{Hom}_{\text{STS}}((V, \beta), (V', \beta')) \cong \text{Hom}_{\text{STS}}((\mathbb{R}, 0), (V, \beta)^{\vee} \otimes (V', \beta')).$$

Theorem 12.13. *The categories MTS and STS are equivalent. This equivalence is additive, and compatible with tensor products and duals.*

Proof. As in the proof of Theorem 12.6, every object $(V, \beta) \in \text{STS}$ inherits a weight filtration W from V , and β gives rise to a \mathbb{G}_m -equivariant map

$$N_{\beta}: V \otimes O(\text{SL}_2) \rightarrow V \otimes O(\text{SL}_2)(-1)$$

respecting the weight filtration on V , with $\text{gr}^W N_{\beta} = (\text{id} \otimes N)$.

For the projection $\text{row}_1 : \text{SL}_2 \rightarrow C^*$ of Definition 1.15, we then get a \mathbb{G}_m -equivariant map

$$\text{row}_{1*}N_\beta : \text{row}_{1*}(V \otimes \mathcal{O}_{\text{SL}_2}) \rightarrow \text{row}_{1*}(V \otimes \mathcal{O}_{\text{SL}_2}(-1));$$

Then $\ker(\text{row}_{1*}N_\beta)$ is a \mathbb{G}_m -equivariant vector bundle on C^* . Using the isomorphism $C \cong \mathbb{A}^2$ of Remark 1.3 and the projection $\pi : (\mathbb{A}^2 - \{0\}) \rightarrow \mathbb{P}^1$, this corresponds to a vector bundle $M(V, \beta) := (\pi_* \ker(\text{row}_{1*}N_\beta))^{\mathbb{G}_m}$ on \mathbb{P}^1 .

Now, $M(V, \beta)$ inherits a weight filtration W from V , and surjectivity of N_β implies that

$$0 \rightarrow \ker(\text{row}_{1*}N_\beta) \rightarrow \text{row}_{1*}(V \otimes \mathcal{O}_{\text{SL}_2}) \rightarrow \text{row}_{1*}(V \otimes \mathcal{O}_{\text{SL}_2}(-1)) \rightarrow 0$$

is an exact sequence, so M is an exact functor. In particular, this gives $\text{gr}_n^W M(V, \beta) = M(\mathcal{W}_n V, 0)$, which is just the vector bundle on \mathbb{P}^1 corresponding to the \mathbb{G}_m -equivariant vector bundle $(\mathcal{W}_n V) \otimes \mathcal{O}_{C^*}$ on C^* . Since $\mathcal{W}_n V$ has weight n for the \mathbb{G}_m -action, this means that $\text{gr}_n^W M(V, \beta)$ has slope n , so we have defined an exact functor

$$M : \text{STS} \rightarrow \text{MTS},$$

which is clearly compatible with tensor products and duals.

If we define $\Gamma_{\text{STS}}(V, \beta) := \ker(\beta : V \rightarrow V \otimes \text{row}_2^\sharp \mathcal{O}(\mathbb{A}^2)(-1))^{\mathbb{G}_m}$ and $\mathbf{R}^1 \Gamma_{\text{STS}}(V, \beta) := (\text{coker } \beta)^{\mathbb{G}_m}$, then the proof of Theorem 12.6 gives us morphisms

$$\rho^i : \mathbf{R}^i \Gamma_{\text{STS}}(V, \beta) \rightarrow W_0 H^i(\mathbb{P}^1, M(V, \beta))$$

for $i = 0, 1$. These are automatically isomorphisms when $\beta = 0$, and the long exact sequences of cohomology then give that ρ^i is an isomorphism for all (V, β) . We therefore have isomorphisms

$$\text{Ext}_{\text{STS}}^i((U, \alpha), (V, \beta)) \rightarrow W_0 \text{Ext}_{\mathbb{P}^1}^i(M(U, \alpha), M(V, \beta)),$$

and arguing as in Theorem 12.6, this shows that M is an equivalence of categories, using Lemma 12.10 in the pure case. \square

Remark 12.14. Note that the Tannakian fundamental group (in the sense of [DMOS]) of the category STS is

$$\Pi(\text{STS}) = \mathbb{G}_m \times \text{Fr}(\text{row}_2^\sharp \mathcal{O}(\mathbb{A}^2)(-1))^\vee,$$

where $\text{Fr}(V)$ denotes the free pro-unipotent group generated by the pro-finite-dimensional vector space V .

The functor $\text{STS} \rightarrow \text{MTS}$ then gives a morphism $\Pi(\text{MTS}) \rightarrow \Pi(\text{STS})$, but this is not unique, since it depends on a choice of natural isomorphism between the fibre functors (at $1 \in C^*$) on MTS and on STS. This amounts to choosing a Levi decomposition for $\Pi(\text{MTS})$, or equivalently a functorial isomorphism $\mathcal{E}_1 \cong \text{gr}^W \mathcal{E}_1$ of vector spaces for $\mathcal{E} \in \text{MHS}$. A canonical choice of such an isomorphism is to take the fibre at $I \in \text{SL}_2$.

We can think of Theorem 12.13 as an analogue of [Del4] for real mixed twistor structures, in that for any MTS \mathcal{E} , it gives a canonical splitting of the weight filtration on \mathcal{E}_1 , together with unique additional data required to recover \mathcal{E} .

13. SL_2 SPLITTINGS OF NON-ABELIAN MTS/MHS AND STRICTIFICATION

13.1. Simplicial structures.

Definition 13.1. Let $s\text{Cat}$ be the category of simplicially enriched categories, which we will refer to as simplicial categories. Explicitly, an object $\mathcal{C} \in s\text{Cat}$ consists of a class $\text{Ob } \mathcal{C}$ of objects, together with simplicial sets $\underline{\text{Hom}}_{\mathcal{C}}(x, y)$ for all $x, y \in \text{Ob } \mathcal{C}$, equipped with an associative composition law and identities.

Lemma 13.2. *For a reductive pro-algebraic monoid M and an M -representation A in DG algebras, there is a cofibrantly generated model structure on $DG_{\mathbb{Z}}\text{Alg}_A(M)$, in which fibrations are surjections, and weak equivalences are quasi-isomorphisms.*

Proof. When M is a group, this is Lemma 3.39, but the same proof carries over to the monoid case. \square

Definition 13.3. Given $B \in DG_{\mathbb{Z}}\text{Alg}_A(M)$ define $B^{\Delta^n} := B \otimes_{\mathbb{Q}} \Omega(|\Delta^n|)$, for $\Omega(|\Delta^n|)$ as in Definition 3.28. Make $DG_{\mathbb{Z}}\text{Alg}_A(M)$ into a simplicial category by setting $\underline{\text{Hom}}(B, B')$ to be the simplicial set

$$\underline{\text{Hom}}_{DG_{\mathbb{Z}}\text{Alg}_A(M)}(B, C)_n := \text{Hom}_{DG_{\mathbb{Z}}\text{Alg}_A(M)}(B, C^{\Delta^n}).$$

Beware that $DG_{\mathbb{Z}}\text{Alg}_A(M)$ does not then satisfy the axioms of a simplicial model category from [GJ, Ch. II], because $\underline{\text{Hom}}(-, B) : DG_{\mathbb{Z}}\text{Alg}_A(M)^{\text{opp}} \rightarrow \mathbb{S}$ does not have a left adjoint. However, $DG_{\mathbb{Z}}\text{Alg}_A(M)$ is a simplicial model category in the weaker sense of [Qui].

Now, as in [Hov, §5], for any pair X, Y of objects in a model category \mathcal{C} , there is a derived function complex $\mathbf{RMap}_{\mathcal{C}}(X, Y) \in \mathbb{S}$, defined up to weak equivalence. One construction is to take a cofibrant replacement \tilde{X} for X and a fibrant resolution \hat{Y}_{\bullet} for Y in the Reedy category of simplicial diagrams in \mathcal{C} , then to set

$$\mathbf{RMap}_{\mathcal{C}}(X, Y)_n := \text{Hom}_{\mathcal{C}}(\tilde{X}, \hat{Y}_n).$$

In fact, Dwyer and Kan showed in [DK] that $\mathbf{RMap}_{\mathcal{C}}$ is completely determined by the weak equivalences in \mathcal{C} . In particular, $\pi_0 \mathbf{RMap}_{\mathcal{C}}(X, Y) = \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y)$, where $\text{Ho}(\mathcal{C})$ is the homotopy category of \mathcal{C} , given by formally inverting weak equivalences.

To see that $C^{\Delta^{\bullet}}$ is a Reedy fibrant simplicial resolution of C in $DG_{\mathbb{Z}}\text{Alg}_A(M)$, note that the matching object $M_n C^{\Delta^{\bullet}}$ is given by

$$C \otimes M_n \Omega(|\Delta^{\bullet}|) = C \otimes \Omega(|\Delta^n|) / (t_0 \cdots t_n, \sum_i t_0 \cdots t_{i-1} (dt_i) t_{i+1} \cdots t_n),$$

so the matching map $C^{\Delta^n} \rightarrow M_n C^{\Delta^{\bullet}}$ is a fibration (i.e. surjective).

Therefore for $\tilde{B} \rightarrow B$ a cofibrant replacement,

$$\mathbf{RMap}_{DG_{\mathbb{Z}}\text{Alg}_A(M)}(B, C) \simeq \underline{\text{Hom}}_{DG_{\mathbb{Z}}\text{Alg}_A(M)}(\tilde{B}, C).$$

Definition 13.4. Given an object $D \in DG_{\mathbb{Z}}\text{Alg}_A(M)$, make the comma category $DG_{\mathbb{Z}}\text{Alg}_A(M) \downarrow D$ into a simplicial category by setting

$$\underline{\text{Hom}}_{DG_{\mathbb{Z}}\text{Alg}_A(M) \downarrow D}(B, C)_n := \text{Hom}_{DG_{\mathbb{Z}}\text{Alg}_A(M)}(B, C^{\Delta^n} \times_{D^{\Delta^n}} D).$$

Now, $C \rightarrow C^{\Delta^{\bullet}} \times_{D^{\Delta^{\bullet}}} D$ is a Reedy fibrant resolution of C in $DG_{\mathbb{Z}}\text{Alg}_A(M) \downarrow D$ for every fibration $C \rightarrow D$. Thus for $\tilde{B} \rightarrow B$ a cofibrant replacement and $C \rightarrow \hat{C}$ a fibrant replacement,

$$\mathbf{RMap}_{DG_{\mathbb{Z}}\text{Alg}_A(M) \downarrow D}(B, C) \simeq \underline{\text{Hom}}_{DG_{\mathbb{Z}}\text{Alg}_A(M) \downarrow D}(\tilde{B}, \hat{C}).$$

Definition 13.5. Given a simplicial category \mathcal{C} , recall from [Ber] that the category $\pi_0 \mathcal{C}$ is defined to have the same objects as \mathcal{C} , with morphisms

$$\text{Hom}_{\pi_0 \mathcal{C}}(x, y) = \pi_0 \underline{\text{Hom}}_{\mathcal{C}}(x, y).$$

A morphism in $\underline{\text{Hom}}_{\mathcal{C}}(x, y)_0$ is said to be a homotopy equivalence if its image in $\pi_0 \mathcal{C}$ is an isomorphism.

If the objects of a simplicial category \mathcal{C} are the fibrant cofibrant objects of a model category \mathcal{M} , with $\underline{\text{Hom}}_{\mathcal{C}} = \mathbf{RMap}_{\mathcal{M}}$, then observe that homotopy equivalences in \mathcal{C} are precisely weak equivalences in \mathcal{M} .

13.2. Functors parametrising Hodge and twistor structures. Recall from Definition 1.22 that we write $\mathbf{RO}(C^*)$ for the DG algebra $O(\mathrm{SL}_2) \xrightarrow{N\epsilon} O(\mathrm{SL}_2)(-1)\epsilon$, with ϵ of degree 1. By Proposition 3.46, this induces an equivalence

$$\mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{A \otimes \mathbf{RO}(C^*)}(R') \downarrow B \otimes \mathbf{RO}(C^*)) \rightarrow \mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{\mathrm{Spec} A \times C^*}(R') \downarrow B \otimes \mathcal{O}_{C^*})$$

for any R' -representation B in A -algebras.

Definition 13.6. For $A \in \mathrm{Alg}(\mathrm{Mat}_1)$, define $\mathcal{PT}(A)_*$ (resp. $\mathcal{PH}(A)_*$) to be the full simplicial subcategory of the category

$$\begin{aligned} & DG_{\mathbb{Z}}\mathrm{Alg}_{A \otimes \mathbf{RO}(C^*)}(\mathrm{Mat}_1 \times R \times \mathbb{G}_m) \downarrow A \otimes O(R) \otimes \mathbf{RO}(C^*) \\ & \text{(resp. } DG_{\mathbb{Z}}\mathrm{Alg}_{A \otimes \mathbf{RO}(C^*)}(\mathrm{Mat}_1 \times R \rtimes S) \downarrow A \otimes O(R) \otimes \mathbf{RO}(C^*)) \end{aligned}$$

on fibrant cofibrant objects. These define functors

$$\mathcal{PT}_*, \mathcal{PH}_* : DG_{\mathbb{Z}}\mathrm{Alg}(\mathrm{Mat}_1) \rightarrow s\mathrm{Cat}.$$

Remark 13.7. Since $\mathcal{PT}(A)_*$ and $\mathcal{PH}(A)_*$ are defined in terms of derived function complexes, it follows that a morphism in any of these categories is a homotopy equivalence (in the sense of Definition 13.5) if and only if it is weak equivalence in the associated model category, i.e. a quasi-isomorphism.

Remark 13.8. Let $\mathbb{R}[t] \in \mathrm{Alg}(\mathrm{Mat}_1)$ be given by setting t to be of weight 1. After applying Proposition 3.46 and taking fibrant cofibrant replacements, observe that a pointed algebraic non-abelian mixed twistor structure consists of

$$O(\underline{\mathrm{gr}}X_{\mathrm{MTS}}) \in DG_{\mathbb{Z}}\mathrm{Alg}(R \times \mathrm{Mat}_1) \downarrow O(R),$$

together with an object $O(X_{\mathrm{MTS}}) \in \mathcal{PT}_*(\mathbb{R}[t])$ and a weak equivalence

$$O(X_{\mathrm{MTS}}) \otimes_{\mathbb{R}[t]} \mathbb{R} \rightarrow O(\underline{\mathrm{gr}}X_{\mathrm{MTS}})$$

in $\mathcal{PT}_*(\mathbb{R})$.

Likewise, a pointed algebraic non-abelian mixed Hodge structure consists of

$$O(\underline{\mathrm{gr}}X_{\mathrm{MHS}}) \in DG_{\mathbb{Z}}\mathrm{Alg}(R \rtimes \bar{S}) \downarrow O(R),$$

together with an object $O(X_{\mathrm{MHS}}) \in \mathcal{PH}_*(\mathbb{R}[t])$, and a weak equivalence

$$O(X_{\mathrm{MHS}}) \otimes_{\mathbb{R}[t]} \mathbb{R} \rightarrow O(\underline{\mathrm{gr}}X_{\mathrm{MHS}})$$

in $\mathcal{PH}_*(\mathbb{R})$.

13.3. Deformations.

13.3.1. *Quasi-presmoothness.* The following is [Pri7, Definition 2.22]:

Definition 13.9. Say that a morphism $F : \mathcal{A} \rightarrow \mathcal{B}$ in $s\mathrm{Cat}$ is a 2-fibration if

- (F1) for any objects a_1 and a_2 in \mathcal{A} , the map $\underline{\mathrm{Hom}}_{\mathcal{A}}(a_1, a_2) \rightarrow \underline{\mathrm{Hom}}_{\mathcal{B}}(Fa_1, Fa_2)$ is a fibration of simplicial sets;
- (F2) for any objects $a_1 \in \mathcal{A}$, $b \in \mathcal{B}$, and any homotopy equivalence $e : Fa_1 \rightarrow b$ in \mathcal{B} , there is an object $a_2 \in \mathcal{A}$, a homotopy equivalence $d : a_1 \rightarrow a_2$ in \mathcal{A} and an isomorphism $\theta : Fa_2 \rightarrow b$ such that $\theta \circ Fd = e$.

The following are adapted from [Pri7]:

Definition 13.10. Say that a functor $\mathcal{D} : \mathrm{Alg}(\mathrm{Mat}_1) \rightarrow s\mathrm{Cat}$ is formally 2-quasi-presmooth if for all square-zero extensions $A \rightarrow B$, the map

$$\mathcal{D}(A) \rightarrow \mathcal{D}(B)$$

is a 2-fibration.

Say that \mathcal{D} is formally 2-quasi-presmooth if $\mathcal{D} \rightarrow \bullet$ is so.

Proposition 13.11. *The functors $\mathcal{PT}_*, \mathcal{PH}_* : \text{Alg}(\text{Mat}_1) \rightarrow s\text{Cat}$ are formally 2-quasi-presmooth.*

Proof. Apart from the augmentation maps, this is essentially the same as [Pri7, Proposition 3.14], which proves the corresponding statements for the functor on algebras given by sending A to the simplicial category of cofibrant DG $(T \otimes A)$ -algebras, for T cofibrant. The same proof carries over, the only change being to take $\text{Mat}_1 \times R \times \mathbb{G}_m$ -invariants (resp. $\text{Mat}_1 \times R \times S$ -invariants) of the André-Quillen cohomology groups. We now sketch the argument.

Let \mathcal{P} be \mathcal{PT}_* (resp. \mathcal{PH}_*), and write S' for \mathbb{G}_m (resp. S). Fix a square-zero extension $A \rightarrow B$ in $\text{Alg}(\text{Mat}_1)$. Thus an object $C \in \mathcal{P}(B)$ is a $\text{Mat}_1 \times R \times S'$ -equivariant diagram $B \otimes \mathbf{RO}(C^*) \rightarrow C \rightarrow B \otimes O(R) \otimes \mathbf{RO}(C^*)$, with the first map a cofibration and the second a fibration. Since C is cofibrant, the underlying graded algebra is smooth over $B \otimes \mathbf{RO}(C^*)$, so lifts essentially uniquely to give a smooth morphism $A^* \otimes \mathbf{RO}(C^*)^* \rightarrow \tilde{C}^*$ of graded algebras, with $\tilde{C}^* \otimes_A B \cong C^*$. As $A \otimes O(R) \otimes \mathbf{RO}(C^*) \rightarrow B \otimes O(R) \otimes \mathbf{RO}(C^*)$ is square-zero, smoothness of \tilde{C}^* gives us a lift $\tilde{p} : \tilde{C}^* \rightarrow A^* \otimes O(R) \otimes \mathbf{RO}(C^*)^*$. Since $\text{Mat}_1 \times R \times S'$ is reductive, these maps can all be chosen equivariantly.

Now, choose some equivariant A -linear derivation δ on \tilde{C} lifting d_C . The obstruction to lifting $c \in \mathcal{P}(B)$ to $\mathcal{P}(A)$ up to isomorphism is then the class

$$\begin{aligned} [(\delta^2, p \circ \delta - d \circ p)] &\in \text{H}^2 \text{HOM}_C(\Omega(C/(B \otimes \mathbf{RO}(C^*))), I \otimes_B C \xrightarrow{p} I \otimes O(R) \otimes \mathbf{RO}(C^*)) \\ &= \text{Ext}_C^2(\mathbb{L}_{\bullet}^{C/(B \otimes \mathbf{RO}(C^*))}, I \otimes_B C \xrightarrow{p} I \otimes O(R) \otimes \mathbf{RO}(C^*)). \end{aligned}$$

This is because any other choice of (δ, \tilde{p}) amounts to adding the boundary of an element in $\text{HOM}_C^1(\Omega(C/(B \otimes \mathbf{RO}(C^*))), I \otimes_B C \xrightarrow{p} I \otimes O(R) \otimes \mathbf{RO}(C^*))$.

The key observation now is that the cotangent complex is an invariant of the quasi-isomorphism class, so C lifts to $\mathcal{P}(A)$ up to isomorphism if and only if all quasi-isomorphic objects also lift. The treatment of morphisms is similar. Although augmentations are not addressed in [Pri7, Proposition 3.14], the same proof adapts. It is important to note that the André-Quillen characterisation of obstructions to lifting morphisms does not require the target to be cofibrant. \square

13.3.2. Strictification.

Proposition 13.12. *Let $\mathcal{P} : \text{Alg}(\text{Mat}_1) \rightarrow s\text{Cat}$ be one of the functors \mathcal{PT}_* or \mathcal{PH}_* . Given an object E in $\mathcal{P}(\mathbb{R})$, an object P in $\mathcal{P}(\mathbb{R}[t])$, and a quasi-isomorphism*

$$f : P/tP \rightarrow E$$

in $\mathcal{P}(\mathbb{R})$, there is an object $M \in \mathcal{P}(\mathbb{R}[t])$, a quasi-isomorphism $g : P \rightarrow M$, and an isomorphism $\theta : M/tM \rightarrow E$ such that $\theta \circ \bar{g} = f$.

Proof. If we replace $\mathbb{R}[t]$ with $\mathbb{R}[t]/t^r$, then the statement holds immediately from Proposition 13.11 and the definition of formal 2-quasi-presmoothness, since the extension $\mathbb{R}[t]/t^r \rightarrow \mathbb{R}$ is nilpotent. Proceeding inductively, we get a system of objects $M_r \in \mathcal{P}(\mathbb{R}[t]/t^r)$, quasi-isomorphisms $g_r : P/t^r P \rightarrow M_r$ and isomorphisms $\phi_r : M_r/t^{r-1} M_r \rightarrow M_{r-1}$ with $M_0 = E$, $g_0 = f$ and $\phi_r \circ \bar{g}_r = g_{r-1}$.

We may therefore set M to be the inverse limit of the system

$$\dots \xrightarrow{\phi_{r+1}} M_r \xrightarrow{\phi_r} M_{r-1} \xrightarrow{\phi_{r-1}} \dots \xrightarrow{\phi_1} M_0 = E$$

in the category of Mat_1 -representations. Explicitly, this says that the maps

$$\mathcal{W}_n M \rightarrow \varprojlim_r \mathcal{W}_n M / (t^r \mathcal{W}_{n-r} M)$$

are isomorphisms for all n . In particular, beware that the forgetful functor from Mat_1 -representations to vector spaces does not preserve inverse limits.

Let $\mathcal{M}(A)$ be one of the model categories

$$\begin{aligned} & DG_{\mathbb{Z}}\text{Alg}_{A \otimes \mathbf{RO}(C^*)}(\text{Mat}_1 \times R \times \mathbb{G}_m) \downarrow A \otimes O(R) \otimes \mathbf{RO}(C^*) \\ \text{or } & DG_{\mathbb{Z}}\text{Alg}_{A \otimes \mathbf{RO}(C^*)}(\text{Mat}_1 \times R \times S) \downarrow A \otimes O(R) \otimes \mathbf{RO}(C^*), \end{aligned}$$

so $\mathcal{P}(A)$ is the full simplicial subcategory on fibrant cofibrant objects. The maps g_r give a morphism $g : P \rightarrow M$ in $\mathcal{M}(\mathbb{R}[t])$ and the maps ϕ_r give an isomorphism $\theta : M/tM \rightarrow E$ in $\mathcal{P}(\mathbb{R})$. We need to show that M is fibrant and cofibrant (so $M \in \mathcal{P}(\mathbb{R}[t])$) and that g is a quasi-isomorphism. Fibrancy is immediate, since the deformation of a surjection is a surjection.

Given an object $A \in \mathcal{M}(\mathbb{R}[t])$, the Mat_1 -action gives a weight decomposition $A = \bigoplus_{n \geq 0} \mathcal{W}_n A$, and

$$A = \varprojlim_n^{\mathcal{M}(\mathbb{R}[t])} A/\mathcal{W}_{\geq n} A.$$

Moreover, if $A \rightarrow B$ is a quasi-isomorphism, then so is $A/\mathcal{W}_{\geq n} A \rightarrow B/\mathcal{W}_{\geq n} B$ for all n . In order to show that M is cofibrant, take a trivial fibration $A \rightarrow B$ in $\mathcal{M}(\mathbb{R}[t])$ (i.e. a surjective quasi-isomorphism) and a map $M \rightarrow B$. Then $A/\mathcal{W}_{\geq n} A \rightarrow B/\mathcal{W}_{\geq n} B$ is a trivial fibration in $\mathcal{M}(\mathbb{R}[t])$, and in fact in $\mathcal{M}(\mathbb{R}[t]/t^n)$. Since $M_n \cong M/t^n M$ is cofibrant in $\mathcal{M}(\mathbb{R}[t]/t^n)$, the map $M \rightarrow B$ lifts to a map $M \rightarrow (A/\mathcal{W}_{\geq n} A) \times_{B/\mathcal{W}_{\geq n} B} B$. We now proceed inductively, noting that

$$(A/\mathcal{W}_{\geq n+1} A) \times_{(B/\mathcal{W}_{\geq n+1} B)} B \rightarrow (A/\mathcal{W}_{\geq n} A) \times_{(B/\mathcal{W}_{\geq n} B)} B$$

is a trivial fibration in $\mathcal{M}(\mathbb{R}[t]/t^{n+1})$. This gives us a compatible system of lifts $M \rightarrow (A/\mathcal{W}_{\geq n} A) \times_{(B/\mathcal{W}_{\geq n} B)} B$, and hence

$$M \rightarrow \varprojlim_n [(A/\mathcal{W}_{\geq n} A) \times_{(B/\mathcal{W}_{\geq n} B)} B] = A.$$

Therefore M is cofibrant.

To show that g is a quasi-isomorphism, observe that for $A \in \mathcal{M}(\mathbb{R}[t])$, the map $\mathcal{W}_n A \rightarrow \mathcal{W}_n(A/t^r A)$ is an isomorphism for $n < r$. Since g_r is a quasi-isomorphism for all r , this means that g induces quasi-isomorphisms $\mathcal{W}_n P \rightarrow \mathcal{W}_n M$ for all n , so g is a quasi-isomorphism. \square

Definition 13.13. Given an R -equivariant $O(R)$ -augmented DGA \mathcal{M} in the category of ind-MTS (resp. ind-MHS) of non-negative weights, define the associated non-positively weighted algebraic mixed twistor (resp. mixed Hodge) structure $\mathbf{Spec} \zeta(\mathcal{M})$ as follows. Under Lemma 1.49 (resp. Lemma 1.41), the Rees module construction gives a flat $\text{Mat}_1 \times R \times \mathbb{G}_m$ -equivariant (resp. $\text{Mat}_1 \times R \rtimes S$ -equivariant) quasi-coherent $\mathcal{O}_{\mathbb{A}^1} \otimes O(R) \otimes \mathcal{O}_C$ -augmented algebra $\xi(\mathcal{M})$ on $\mathbb{A}^1 \times C$ associated to \mathcal{M} . We therefore define $\mathbf{Spec} \zeta(\mathcal{M}) := \mathbf{Spec}_{\mathbb{A}^1 \times C^*} \xi(\mathcal{M})|_{\mathbb{A}^1 \times C^*}$.

Now, $\text{gr}^W \mathcal{M}$ is an $O(R)$ -augmented DGA in the category of Mat_1 -representations (resp. \bar{S} -representations), so we may set $\text{gr} \mathbf{Spec} \zeta(\mathcal{M}) := \text{Spec} \text{gr}^W \mathcal{M}$. Since $\xi(\mathcal{M})$ is flat,

$$(\mathbf{Spec} \zeta(\mathcal{M})) \times_{\mathbb{A}^1, 0}^{\mathbf{R}} \text{Spec} \mathbb{R} \simeq (\mathbf{Spec} \zeta(\mathcal{M})) \times_{\mathbb{A}^1, 0} \text{Spec} \mathbb{R},$$

so Lemma 1.49 (resp. Lemma 1.41) gives the required opposedness isomorphism.

Theorem 13.14. *For every non-positively weighted algebraic mixed twistor (resp. mixed Hodge) structure $(X, x)_{\text{MTS}}^{R, \text{Mal}}$ (resp. $(X, x)_{\text{MHS}}^{R, \text{Mal}}$) on a pointed Malcev homotopy type $(X, x)^{R, \text{Mal}}$, there exists an R -equivariant $O(R)$ -augmented DGA \mathcal{M} in the category of ind-MTS (resp. ind-MHS) with $(X, x)_{\text{MTS}}^{R, \text{Mal}}$ (resp. $(X, x)_{\text{MHS}}^{R, \text{Mal}}$) quasi-isomorphic in the category of algebraic mixed twistor (resp. mixed Hodge) structures to $\mathbf{Spec} \zeta(\mathcal{M})$, for ζ as above.*

Proof. Making use of Remark 13.8, choose a fibrant cofibrant replacement E for $O(\underline{\mathrm{gr}}(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}})$ (resp. $O(\underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}})$) in the category $DG_{\mathbb{Z}}\mathrm{Alg}(R)_*(\mathrm{Mat}_1)$ (resp. $DG_{\mathbb{Z}}\mathrm{Alg}(R)_*(\bar{S})$), and a fibrant cofibrant replacement P for

$$\begin{aligned} & \Gamma(C^*, \mathcal{O}((X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}}) \otimes_{\mathcal{O}_{C^*}} \mathbf{R}\mathcal{O}_{C^*}) \\ & \text{(resp. } \Gamma(C^*, \mathcal{O}((X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}}) \otimes_{\mathcal{O}_{C^*}} \mathbf{R}\mathcal{O}_{C^*}) \end{aligned}$$

in the category

$$\begin{aligned} & DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{R}[t] \otimes \mathbf{RO}(C^*)}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m) \\ & \text{(resp. } DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{R}[t] \otimes \mathbf{RO}(C^*)}(R)_*(\mathrm{Mat}_1 \times S)). \end{aligned}$$

Since P is cofibrant, it is flat, so the data of an algebraic mixed twistor (resp. mixed Hodge) structure give a quasi-isomorphism

$$f : P/tP \rightarrow E \otimes \mathbf{RO}(C^*)$$

in

$$\begin{aligned} & DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbf{RO}(C^*)}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m) \\ & \text{(resp. } DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbf{RO}(C^*)}(R)_*(\mathrm{Mat}_1 \times S)), \end{aligned}$$

so we may apply Proposition 13.12 to obtain a fibrant cofibrant object

$$\begin{aligned} & M \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{R}[t] \otimes \mathbf{RO}(C^*)}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m) \\ & \text{(resp. } M \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{R}[t] \otimes \mathbf{RO}(C^*)}(R)_*(\mathrm{Mat}_1 \times S)) \end{aligned}$$

with an isomorphism $M/tM \cong E \otimes \mathbf{RO}(C^*)$, and a quasi-isomorphism $g : P \rightarrow M$ lifting f .

Since M is cofibrant, it is flat as an $\mathbf{RO}(C^*)$ -module. For the canonical map $\mathrm{row}_1^* : \mathbf{RO}(C^*) \rightarrow O(\mathrm{SL}_2)$, this implies that we have a short exact sequence

$$0 \rightarrow \mathrm{row}_1^* M(-1)\epsilon \rightarrow M \rightarrow \mathrm{row}_1^* M \rightarrow 0,$$

and the section $O(\mathrm{SL}_2) \rightarrow \mathbf{RO}(C^*)$ of graded rings (not respecting differentials) gives a canonical splitting of the short exact sequence for the underlying graded objects. Thus we may write $M^* = \mathrm{row}_1^* M \oplus \mathrm{row}_1^* M(-1)\epsilon$, and decompose the differential d_M as $d_M := \delta_M + N_M\epsilon$, where $\delta_M = \mathrm{row}_1^* d_M$.

Now, since $M/tM = E \otimes \mathbf{RO}(C^*)$, we know that

$$N_M : \mathrm{row}_{1*} \mathrm{row}_1^*(M/tM) \rightarrow \mathrm{row}_{1*} \mathrm{row}_1^*(M/tM)(-1)$$

is a surjection of sheaves on C^* . Since $M = \varprojlim_r M/t^r M$ in the Mat_1 -equivariant category and M is flat, this means that N_M is also surjective. We therefore set

$$K := \ker(N_M : \mathrm{row}_{1*} \mathrm{row}_1^* M \rightarrow \mathrm{row}_{1*} \mathrm{row}_1^* M(-1));$$

as $\ker(N : \mathrm{row}_{1*} O(\mathrm{SL}_2) \rightarrow \mathrm{row}_{1*} O(\mathrm{SL}_2)(-1)) = \mathcal{O}_{C^*}$, we have

$$\begin{aligned} & K \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{A}^1 \times C^*}(\mathrm{Mat}_1 \times R \times \mathbb{G}_m) \downarrow O(\mathbb{A}^1 \times R) \otimes \mathcal{O}_{C^*} \\ & \text{(resp. } K \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{A}^1 \times C^*}(\mathrm{Mat}_1 \times R \times S) \downarrow O(\mathbb{A}^1 \times R) \otimes \mathcal{O}_{C^*}), \end{aligned}$$

with

$$M = \Gamma(C^*, K \otimes_{\mathcal{O}_{C^*}} \mathbf{R}\mathcal{O}_{C^*}),$$

for $\mathbf{R}\mathcal{O}_{C^*}$ as in Definition 1.22.

Since M is flat over $\mathbf{RO}(C^*) \otimes O(\mathbb{A}^1)$, it follows that K is flat over $C^* \times \mathbb{A}^1$. Moreover, for $0 \in \mathbb{A}^1$, we have $0^* K = K/tK$, so

$$\begin{aligned} 0^* K &= \ker(N_M : \mathrm{row}_{1*} \mathrm{row}_1^*(M/tM) \rightarrow \mathrm{row}_{1*} \mathrm{row}_1^*(M/tM)(-1)) \\ &= E \otimes \ker(N : \mathrm{row}_{1*} O(\mathrm{SL}_2) \rightarrow \mathrm{row}_{1*} O(\mathrm{SL}_2)(-1)) \\ &= E \otimes \mathcal{O}_{C^*}. \end{aligned}$$

Thus K satisfies the opposedness condition, so by Lemma 1.49 (resp. Lemma 1.41) it corresponds to an ind-MTS (resp. ind-MHS) on the R -equivariant $O(R)$ -augmented DGA algebra $(1, 1)^*K$ given by pulling back along $(1, 1): \text{Spec } \mathbb{R} \rightarrow \mathbb{A}^1 \times C$. Letting this ind-MTS (resp. ind-MHS) be \mathcal{M} completes the proof. \square

13.3.3. Homotopy fibres. In Proposition 13.12, it is natural to ask how unique the model M is. We cannot expect it to be unique up to isomorphism, but only up to quasi-isomorphism. As we will see in Corollary 13.18, that quasi-isomorphism is unique up to homotopy, which in turn is unique up to 2-homotopy, and so on.

Definition 13.15. Recall from [Ber] Theorem 1.1 that a morphism $F: \mathcal{C} \rightarrow \mathcal{D}$ in $s\text{Cat}$ is said to be a weak equivalence (a.k.a. an ∞ -equivalence) whenever

- (W1) for any objects a_1 and a_2 in \mathcal{C} , the map $\underline{\text{Hom}}_{\mathcal{C}}(a_1, a_2) \rightarrow \underline{\text{Hom}}_{\mathcal{D}}(Fa_1, Fa_2)$ is a weak equivalence of simplicial sets;
- (W2) the induced functor $\pi_0 F: \pi_0 \mathcal{C} \rightarrow \pi_0 \mathcal{D}$ is an equivalence of categories.

A morphism $F: \mathcal{C} \rightarrow \mathcal{D}$ in $s\text{Cat}$ is said to be a fibration whenever

- (F1) for any objects a_1 and a_2 in \mathcal{C} , the map $\underline{\text{Hom}}_{\mathcal{C}}(a_1, a_2) \rightarrow \underline{\text{Hom}}_{\mathcal{D}}(Fa_1, Fa_2)$ is a fibration of simplicial sets;
- (F2) for any objects $a_1 \in \mathcal{C}$, $b \in \mathcal{D}$, and homotopy equivalence $e: Fa_1 \rightarrow b$ in \mathcal{D} , there is an object $a_2 \in \mathcal{C}$ and a homotopy equivalence $d: a_1 \rightarrow a_2$ in \mathcal{C} such that $Fd = e$.

Definition 13.16. Given functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$ between categories, define the 2-fibre product $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ as follows. Objects of $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ are triples (a, θ, c) , for $a \in \mathcal{A}$, $c \in \mathcal{C}$ and $\theta: Fa \rightarrow Gc$ an isomorphism in \mathcal{B} . A morphism in $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ from (a, θ, c) to (a', θ', c') is a pair (f, g) , where $f: a \rightarrow a'$ is a morphism in \mathcal{A} and $g: c \rightarrow c'$ a morphism in \mathcal{C} , satisfying the condition that

$$Gg \circ \theta = \theta' \circ Ff.$$

Remark 13.17. This definition has the property that $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ is a model for the 2-fibre product in the 2-category of categories. However, we will always use the notation $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ to mean the specific model of Definition 13.16, and not merely any equivalent category.

Also note that

$$\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C} = (\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{B}) \times_{\mathcal{B}} \mathcal{C},$$

and that a morphism $F: \mathcal{A} \rightarrow \mathcal{B}$ in $s\text{Cat}$ is a 2-fibration in the sense of Definition 13.9 if and only if $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{B} \rightarrow \mathcal{B}$ is a fibration in the sense of Definition 13.15.

Corollary 13.18. *Let $\mathcal{P}: \text{Alg}(\text{Mat}_1) \rightarrow s\text{Cat}$ be one of the functors \mathcal{PT}_* or \mathcal{PH}_* . Given an object E in $\mathcal{P}(\mathbb{R})$, the simplicial categories given by the homotopy fibre*

$$\mathcal{P}(\mathbb{R}[t]) \times_{\mathcal{P}(\mathbb{R})}^h \{E\}$$

and the 2-fibre

$$\mathcal{P}(\mathbb{R}[t]) \times_{\mathcal{P}(\mathbb{R})}^{(2)} \{E\}$$

are weakly equivalent.

Proof. By Proposition 13.11, $\mathcal{P}(\mathbb{R}[t]/t^r) \rightarrow \mathcal{P}(\mathbb{R})$ is a 2-fibration in $s\text{Cat}$. Moreover, the proof of Proposition 13.12 shows that the map

$$\begin{aligned} \mathcal{P}(\mathbb{R}[t]) &\rightarrow \varprojlim_r^{(2)} \mathcal{P}(\mathbb{R}[t]/t^r) \\ &= \varprojlim_r [\mathcal{P}(\mathbb{R}[t]/t^r) \times_{\mathcal{P}(\mathbb{R}[t]/t^{r-1})}^{(2)} \mathcal{P}(\mathbb{R}[t]/t^{r-1}) \times_{\mathcal{P}(\mathbb{R}[t]/t^{r-2})}^{(2)} \cdots \times_{\mathcal{P}(\mathbb{R})} \mathcal{P}(\mathbb{R})] \end{aligned}$$

to the inverse 2-limit is an equivalence, so $\mathcal{P}(\mathbb{R}[t]) \rightarrow \mathcal{P}(\mathbb{R})$ is also a 2-fibration.

Therefore $\mathcal{P}(\mathbb{R}[t]) \times_{\mathcal{P}(\mathbb{R})}^{(2)} \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$ is a fibration in the sense of Definition 13.15, so

$$\begin{aligned} \mathcal{P}(\mathbb{R}[t]) \times_{\mathcal{P}(\mathbb{R})}^h \{E\} &\simeq \mathcal{P}(\mathbb{R}[t]) \times_{\mathcal{P}(\mathbb{R})}^{(2)} \mathcal{P}(\mathbb{R}) \times_{\mathcal{P}(\mathbb{R})} \{E\} \\ &= \mathcal{P}(\mathbb{R}[t]) \times_{\mathcal{P}(\mathbb{R})}^{(2)} \{E\} \\ &= \mathcal{P}(\mathbb{R}[t]) \times_{\mathcal{P}(\mathbb{R})}^{(2)} \{E\}, \end{aligned}$$

as required. \square

13.3.4. SL_2 -splittings.

Corollary 13.19. *Every non-positively weighted algebraic mixed twistor (resp. mixed Hodge) structure $(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}}$ (resp. $(X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}}$) on a pointed Malcev homotopy type $(X, x)^{R, \mathrm{Mal}}$ admits a canonical SL_2 -splitting in the sense of Definition 4.10.*

Proof. By Theorem 13.14, we have an R -equivariant $O(R)$ -augmented DGA \mathcal{M} in the category of ind-MTS (resp. ind-MHS) of non-negative weights, with $(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}}$ (resp. $(X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}}$) quasi-isomorphic in the category of algebraic mixed twistor (resp. mixed Hodge) structures to $\mathbf{Spec} \zeta(\mathcal{M})$.

By Theorem 12.13 (resp. Theorem 12.6) and Lemma 12.4, there is a unique $R \times \mathbb{G}_m$ -equivariant (resp. $R \rtimes S$ -equivariant) derivation $\beta: \mathrm{gr}^W \mathcal{M} \rightarrow (\mathrm{gr}^W \mathcal{M}) \otimes \mathrm{row}_2^{\sharp} O(\mathbb{A}^2)(-1)$, with the corresponding object

$$O(\mathbb{A}^1) \otimes (\mathrm{gr}^W \mathcal{M}) \otimes O(\mathrm{SL}_2) \xrightarrow{\beta + \mathrm{id} \otimes N} O(\mathbb{A}^1) \otimes (\mathrm{gr}^W \mathcal{M}, W) \otimes O(\mathrm{SL}_2)(-1)$$

isomorphic to the object M from the proof of Theorem 13.14 (with $\mathrm{gr}^W \mathcal{M}$ canonically isomorphic to E).

In particular, it gives a $\mathbb{G}_m \times R \times \mathbb{G}_m$ -equivariant (resp. $\mathbb{G}_m \times R \rtimes S$ -equivariant) isomorphism

$$\mathrm{row}_1^* \zeta(\mathcal{M}) \cong O(\mathbb{A}^1) \otimes (\mathrm{gr}^W \mathcal{M}) \otimes O(\mathrm{SL}_2).$$

Since $\mathbf{Spec} \mathbb{A}^1 \times C^* \mathcal{M}$ is by construction quasi-isomorphic to $(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}}$ (resp. $(X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}}$), with $\mathbf{Spec} \mathrm{gr}^W \mathcal{M}$ quasi-isomorphic to $\underline{\mathrm{gr}}(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}}$ (resp. $\underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}}$), this gives us a quasi-isomorphism

$$\begin{aligned} \mathrm{row}_1^* \underline{\mathrm{gr}}(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}} &\rightarrow \mathbb{A}^1 \times \mathbf{Spec}(\mathrm{gr}^W \mathcal{M}) \times \mathrm{SL}_2 \\ (\text{resp. } \mathrm{row}_1^* \underline{\mathrm{gr}}(X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}} &\rightarrow \mathbb{A}^1 \times \mathbf{Spec}(\mathrm{gr}^W \mathcal{M}) \times \mathrm{SL}_2.) \end{aligned}$$

\square

Corollary 13.20. *If a pointed Malcev homotopy type $(X, x)^{R, \mathrm{Mal}}$ admits a non-positively weighted mixed twistor structure $(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}}$, then there is a canonical family*

$$\mathbb{A}^1 \times (X, x)^{R, \mathrm{Mal}} \simeq \mathbb{A}^1 \times \underline{\mathrm{gr}}(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}}$$

of quasi-isomorphisms over \mathbb{A}^1 .

Proof. Take the fibre of the SL_2 -splitting

$$\mathrm{row}_1^* \underline{\mathrm{gr}}(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}} \simeq \mathbb{A}^1 \times \underline{\mathrm{gr}}(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}} \times \mathrm{SL}_2$$

over $(1, 1) \in \mathbb{A}^1 \times C^*$. The fibre of $\mathrm{SL}_2 \rightarrow C^*$ over 1 is $\begin{pmatrix} 1 & 0 \\ \mathbb{A}^1 & 0 \end{pmatrix}$, giving the family of quasi-isomorphisms. \square

13.3.5. Homotopy groups.

Corollary 13.21. *Given a non-positively weighted algebraic mixed twistor (resp. mixed Hodge) structure $(X, x)_{\text{MTS}}^{R, \text{Mal}}$ (resp. $(X, x)_{\text{MHS}}^{R, \text{Mal}}$) on a pointed Malcev homotopy type $(X, x)^{R, \text{Mal}}$, there are natural ind-MTS (resp. ind-MHS) on the the duals $(\varpi_n(X, x)^{\rho, \text{Mal}})^\vee$ of the relative Malcev homotopy groups for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(X, x)^{\rho, \text{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n , with the Whitehead bracket and with the Hurewicz maps $\varpi_n(X^{\rho, \text{Mal}}) \rightarrow \mathbb{H}^n(X, \mathbb{O}(R))^\vee$ ($n \geq 2$) and $\mathbf{R}_u \varpi_1(X^{\rho, \text{Mal}}) \rightarrow \mathbb{H}^1(X, \mathbb{O}(R))^\vee$, for $\mathbb{O}(R)$ as in Proposition 3.35.

Proof. By Corollary 13.19, $(X, x)_{\text{MTS}}^{R, \text{Mal}}$ (resp. $(X, x)_{\text{MHS}}^{R, \text{Mal}}$) admits an SL_2 -splitting. Therefore the conditions of Theorem 4.20 are satisfied, giving the required result. \square

Note that Theorems 12.13 and 12.6 now show that the various homotopy groups have associated objects in STS or SHS, giving canonical SL_2 -splittings. These splittings will automatically be the same as those constructed in Theorem 4.21 from the splitting on the homotopy type. Explicitly, they give canonical isomorphisms

$$(\varpi_n(X, x)^{R, \text{Mal}})^\vee \otimes \mathcal{S} \cong (\text{gr}^W \varpi_n(X, x)^{R, \text{Mal}})^\vee \otimes \mathcal{S}$$

compatible with weight filtrations and with twistor or Hodge filtrations, and similarly for $O(\varpi_1(X, x)^{\rho, \text{Mal}})$.

13.4. Quasi-projective varieties. Fix a smooth projective complex variety X , a divisor D locally of normal crossings, and set $Y := X - D$. Let $j : Y \rightarrow X$ be the inclusion morphism. Take a Zariski-dense representation $\rho : \pi_1(Y, y) \rightarrow R(\mathbb{R})$, for R a reductive pro-algebraic group, with ρ having unitary monodromy around local components of D .

Definition 13.22. Define a functor G from DG algebras to pro-finite-dimensional chain Lie algebras as follows. First, write $\sigma A^\vee[1]$ for the brutal truncation (in non-negative degrees) of $A^\vee[1]$, and set

$$G(A) = \text{Lie}(\sigma A^\vee[1]),$$

the free pro-finite-dimensional pro-nilpotent graded Lie algebra, with differential defined on generators by $d_A + \Delta$, with $\Delta : A^\vee \rightarrow (A \otimes A)^\vee$ here being the coproduct on A^\vee .

Given a DGA A with $A^0 = \mathbb{R}$, define

$$\pi_n(A) := \mathbb{H}_{n-1}G(A).$$

Corollary 13.23. *There are natural ind-MTS on the the duals $(\varpi_n(Y, y)^{\rho, \text{Mal}})^\vee$ of the relative Malcev homotopy groups for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(Y, y)^{\rho, \text{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n , with the Whitehead bracket and with the Hurewicz maps $\varpi_n(Y^{\rho, \text{Mal}}) \rightarrow \mathbb{H}^n(Y, \mathbb{O}(R))^\vee$ ($n \geq 2$) and $\mathbf{R}_u \varpi_1(Y^{\rho, \text{Mal}}) \rightarrow \mathbb{H}^1(Y, \mathbb{O}(R))^\vee$.

Moreover, there are canonical \mathcal{S} -linear isomorphisms

$$\varpi_n(Y^{\rho, \text{Mal}}, y)^\vee \otimes \mathcal{S} \cong \pi_n\left(\bigoplus_{a,b} \mathbb{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2\right)^\vee \otimes \mathcal{S}$$

$$O(\varpi_1(Y^{\rho, \text{Mal}}, y)) \otimes \mathcal{S} \cong O(R \rtimes \pi_1\left(\bigoplus_{a,b} \mathbb{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2\right)) \otimes \mathcal{S}$$

compatible with weight and twistor filtrations.

Proof. This just combines Theorem 11.19 (or Theorem 10.22 for a simpler proof whenever ρ has trivial monodromy around the divisor) with Corollary 4.20. The splitting comes from Corollary 13.19, making use of the isomorphism

$$\underline{\text{gr}} \varpi_n(Y, y)_{\text{MTS}}^{R, \text{Mal}} = \text{gr}^W \varpi_n(Y, y)^{R, \text{Mal}}.$$

induced by the exact functor gr^W on MTS. \square

Corollary 13.24. *If the local system on X associated to any R -representation underlies a polarisable variation of Hodge structure, then there are natural ind-MHS on the the duals $(\varpi_n(Y, y)^{\rho, \text{Mal}})^\vee$ of the relative Malcev homotopy groups for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(Y, y)^{\rho, \text{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n , with the Whitehead bracket and with the Hurewicz maps $\varpi_n(Y^{\rho, \text{Mal}}) \rightarrow H^n(Y, \mathbb{O}(R))^\vee$ ($n \geq 2$) and $R_{\text{u}}\varpi_1(Y^{\rho, \text{Mal}}) \rightarrow H^1(Y, \mathbb{O}(R))^\vee$.

Moreover, there are canonical \mathcal{S} -linear isomorphisms

$$\begin{aligned} \varpi_n(Y^{\rho, \text{Mal}}, y)^\vee \otimes \mathcal{S} &\cong \pi_n\left(\bigoplus_{a,b} H^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2\right)^\vee \otimes \mathcal{S} \\ O(\varpi_1(Y^{\rho, \text{Mal}}, y)) \otimes \mathcal{S} &\cong O(R \rtimes \pi_1\left(\bigoplus_{a,b} H^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2\right)) \otimes \mathcal{S} \end{aligned}$$

compatible with weight and Hodge filtrations.

Proof. This just combines Theorem 11.16 (or Theorem 10.23 for a simpler proof whenever ρ has trivial monodromy around the divisor) with Corollary 4.20, together with the splitting of Corollary 13.19. \square

Proposition 13.25. *If the $(S^1)^\delta$ -action on ${}^\nu\varpi_1(Y, y)^{\text{red}}$ descends to R , then for all n , the map $\pi_n(Y, y) \times S^1 \rightarrow \varpi_n(Y^{\rho, \text{Mal}}, y)_{\mathbb{T}}$, given by composing the map $\pi_n(Y, y) \rightarrow \varpi_n(Y^{\rho, \text{Mal}}, y)$ with the $(S^1)^\delta$ -action on $(Y^{\rho, \text{Mal}}, y)_{\mathbb{T}}$ from Proposition 11.20, is continuous.*

Proof. The proof of Proposition 6.12 carries over to this generality. \square

Corollary 13.26. *Assume that the $(S^1)^\delta$ -action on ${}^\nu\varpi_1(Y, y)^{\text{red}}$ descends to R , and that the group $\varpi_n(Y, y)^{\rho, \text{Mal}}$ is finite-dimensional and spanned by the image of $\pi_n(Y, y)$. Then $\varpi_n(Y, y)^{\rho, \text{Mal}}$ carries a natural \mathcal{S} -split mixed Hodge structure, which extends the mixed twistor structure of Corollary 13.23.*

Proof. The proof of Corollary 6.13 adapts directly. \square

Remark 13.27. If we are willing to discard the Hodge or twistor structures, then Corollary 13.20 gives a family

$$\mathbb{A}^1 \times (Y^{\rho, \text{Mal}}, y) \simeq \mathbb{A}^1 \times \text{Spec}\left(\bigoplus_{a,b} H^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2\right)$$

of quasi-isomorphisms, and this copy of \mathbb{A}^1 corresponds to $\text{Spec } \mathcal{S}$.

If we pull back along the morphism $\mathcal{S} \rightarrow \mathbb{C}$ given by $x \mapsto i$, the resulting complex quasi-isomorphism will preserve the Hodge filtration F (in the MHS case), but not \bar{F} . This splitting is denoted by b_i in Remark 12.9, and comparison with [Del4, Remark 1.3] shows that this is Deligne's functor a_F .

Proposition 5.6 adapts to show that when $R = 1$, the mixed Hodge structure in Corollary 13.24 is the same as that of [Mor, Theorem 9.1]. Since a_F was the splitting employed in [Mor], we deduce that when $R = 1$, the complex quasi-isomorphism at $i \in \mathbb{A}^1$ (or equivalently at $(\frac{1}{i} \ 0) \in \text{SL}_2$) is precisely the quasi-isomorphism of [Mor, Corollary 9.7].

Whenever the discrete S^1 -action on $\varpi_n(Y, y)_{\text{MTS}}^{R, \text{Mal}}$ (from Proposition 11.20) is algebraic, it defines an algebraic mixed Hodge structure on $\varpi_n(Y, y)^{R, \text{Mal}}$. In the projective case ($D = \emptyset$), [KPT1] constructed a discrete \mathbb{C}^\times -action on $\varpi_n(X, x)_{\mathbb{C}}$; via Remark 6.4, the comments above show that whenever the \mathbb{C}^\times -action is algebraic, it corresponds to the complex I^{pq} decomposition of the mixed Hodge structure, with $\lambda \in \mathbb{C}^\times$ acting on I^{pq} as multiplication by λ^p .

13.4.1. *Deformations of representations.* For $Y = X - D$ as above, and some real algebraic group G , take a reductive representation $\rho: \pi_1(Y, y) \rightarrow G(\mathbb{R})$, with ρ having unitary monodromy around local components of D . Write \mathfrak{g} for the Lie algebra of G , and let $\text{ad}\mathbb{B}_\rho$ be the local system of Lie algebras on Y corresponding to the adjoint representation $\text{ad}\rho: \pi_1(Y, y) \rightarrow \text{Aut}(\mathfrak{g})$.

Proposition 13.28. *The formal neighbourhood \mathfrak{Def}_ρ of ρ in the moduli stack $[\text{Hom}(\pi_1(Y, y), G)/G]$ of representations is given by the formal stack $[(Z, 0)/\exp(\text{H}^0(Y, \text{ad}\mathbb{B}_\rho))]$, where $(Z, 0)$ is the formal germ at 0 of the affine scheme Z given by*

$$\{(\omega, \eta) \in \text{H}^1(X, j_*\text{ad}\mathbb{B}_\rho) \oplus \text{H}^0(X, \mathbf{R}^1 j_*\text{ad}\mathbb{B}_\rho) : d_2\eta + \frac{1}{2}[\omega, \omega] = 0, [\omega, \eta] = 0, [\eta, \eta] = 0\}.$$

The formal neighbourhood \mathfrak{R}_ρ of ρ in the rigidified moduli space $\text{Hom}(\pi_1(Y, y), G)$ of framed representations is given by the formal scheme

$$(Z, 0) \times_{\exp(\text{H}^0(Y, \text{ad}\mathbb{B}_\rho))} \exp(\mathfrak{g}),$$

where $\exp(\text{H}^0(Y, \text{ad}\mathbb{B}_\rho)) \subset \exp(\mathfrak{g})$ acts on $(Z, 0)$ via the adjoint action.

Proof. Let R be the Zariski closure of ρ . This satisfies the conditions of Corollary 13.23, so we have an \mathcal{S} -linear isomorphism

$$O(\varpi_1(Y^{\rho, \text{Mal}}, y)) \otimes \mathcal{S} \cong O(R \times \pi_1(\bigoplus_{a,b} \text{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2)) \otimes \mathcal{S}$$

of Hopf algebras.

Pulling back along any real homomorphism $\mathcal{S} \rightarrow \mathbb{R}$ (such as $x \mapsto 0$) gives an isomorphism

$$\varpi_1(Y^{\rho, \text{Mal}}, y) \cong O(R \times \pi_1(\bigoplus_{a,b} \text{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2)).$$

We now proceed as in [Pri2, Remarks 6.6]. Given a real Artinian local ring $A = \mathbb{R} \oplus \mathfrak{m}(A)$, observe that

$$G(A) \times_{G(\mathbb{R})} R(\mathbb{R}) \cong \exp(\mathfrak{g} \otimes \mathfrak{m}(A)) \times R(\mathbb{R}).$$

Since $\exp(\mathfrak{g} \otimes \mathfrak{m}(A))$ underlies a unipotent algebraic group, deformations of ρ correspond to algebraic group homomorphisms

$$\varpi_1(Y^{\rho, \text{Mal}}, y) \rightarrow \exp(\mathfrak{g} \otimes \mathfrak{m}(A)) \times R$$

over R .

Infinitesimal inner automorphisms are given by conjugation by $\exp(\mathfrak{g} \otimes \mathfrak{m}(A))$, and so [Pri3, Proposition 3.15] gives $\mathfrak{Def}_\rho(A)$ isomorphic to

$$[\text{Hom}_R(\pi_1(\bigoplus_{a,b} \text{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2), \exp(\mathfrak{g} \otimes \mathfrak{m}(A)))/\exp(\mathfrak{g} \otimes \mathfrak{m}(A))^R],$$

which is isomorphic to the groupoid of A -valued points of $[(Z, 0)/\exp(\text{H}^0(Y, \text{ad}\mathbb{B}_\rho))]$.

The rigidified formal scheme \mathfrak{R}_ρ is the groupoid fibre of $\mathfrak{Def}_\rho(A) \rightarrow B\exp(\mathfrak{g} \otimes \mathfrak{m}(A))$, which is just the set of A -valued points of $(Z, 0) \times_{\exp(\text{H}^0(Y, \text{ad}\mathbb{B}_\rho))} \exp(\mathfrak{g})$, as in Proposition 3.26. \square

Remarks 13.29. The mixed twistor structure on $\varpi_1(Y^{\rho, \text{Mal}}, y)$ induces a weight filtration on the pro-Artinian ring representing \mathfrak{R}_ρ . Since the isomorphisms of Corollary 13.23 respect the weight filtration, the isomorphisms of Proposition 13.28 also do so. Explicitly, the ring $O(Z)$ has a weight filtration determined by setting $\text{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))$ to be of weight $a + b$, so generators of $O(Z)$ have weights -1 and -2 . The weight filtration on the rest of the space is then characterised by the conditions that \mathfrak{g} and $\text{H}^0(Y, \text{ad}\mathbb{B}_\rho)$ both be of weight 0.

Another interesting filtration is the pre-weight filtration J of Proposition 11.10. The constructions transfer this to a filtration on $\varpi_1(Y^{\rho, \text{Mal}}, y)$, and the \mathcal{S} -splittings (and hence Proposition 13.28) also respect J . The filtration J is determined by setting $H^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))$ to be of weight b , so generators of $O(Z)$ have weights 0 and -1 . We can then define $J_0 Z := \text{Spec } O(Z)/J_{-1}O(Z)$, and obtain descriptions of $J_0 \mathfrak{D}ef_\rho \subset \mathfrak{D}ef_\rho$ and $J_0 \mathfrak{R}_\rho \subset \mathfrak{R}_\rho$ by replacing Z with $J_0 Z$. These functors can be characterised as consisting of deformations for which the conjugacy classes of monodromy around the divisors remain unchanged — these are the functors studied in [Fot].

13.4.2. *Simplicial and singular varieties.* As in §11.4, let X_\bullet be a simplicial smooth proper complex variety, and $D_\bullet \subset X_\bullet$ a simplicial divisor with normal crossings. Set $Y_\bullet = X_\bullet - D_\bullet$, assume that $|Y_\bullet|$ is connected, and pick a point $y \in |Y_\bullet|$. Let $j : |Y_\bullet| \rightarrow |X_\bullet|$ be the natural inclusion map.

Take $\rho : \pi_1(|Y_\bullet|, y) \rightarrow R(\mathbb{R})$ Zariski-dense, and assume that for every local system \mathbb{V} on $|Y_\bullet|$ corresponding to an R -representation, the local system $a_0^{-1}\mathbb{V}$ on Y_0 is semisimple, with unitary monodromy around the local components of D_0 .

Corollary 13.30. *There are natural ind-MTS on the the duals $(\varpi_n(|Y_\bullet|, y)^{\rho, \text{Mal}})^\vee$ of the relative Malcev homotopy groups for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(|Y_\bullet|, y)^{\rho, \text{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n , with the Whitehead bracket and with the Hurewicz maps $\varpi_n(|Y_\bullet|, y)^{\rho, \text{Mal}} \rightarrow H^n(|Y_\bullet|, \mathbb{O}(R))^\vee$ ($n \geq 2$) and $R_u \varpi_1(|Y_\bullet|, y)^{\rho, \text{Mal}} \rightarrow H^1(|Y_\bullet|, \mathbb{O}(R))^\vee$.

Moreover, there are canonical \mathcal{S} -linear isomorphisms

$$\varpi_n(|Y_\bullet|, y)^{\rho, \text{Mal}} \otimes \mathcal{S} \cong \pi_n(\text{Th}(\bigoplus_{p,q} H^{p-q}(X_\bullet, \mathbf{R}^q j_* a^{-1} \mathbb{O}(R))[-p], d_1))^\vee \otimes \mathcal{S}$$

$$O(\varpi_1(|Y_\bullet|, y)^{\rho, \text{Mal}}) \otimes \mathcal{S} \cong O(R \ltimes \pi_1(\text{Th}(\bigoplus_{p,q} H^{p-q}(X_\bullet, \mathbf{R}^q j_* a^{-1} \mathbb{O}(R))[-p], d_1))) \otimes \mathcal{S}$$

compatible with weight and twistor filtrations.

If $a_0^{-1}\mathbb{V}$ underlies a polarisable variation of Hodge structure on Y_0 for all \mathbb{V} as above, then the ind-MTS above all become ind-MHS, with the \mathcal{S} -linear isomorphisms above compatible with Hodge filtrations.

Proof. The proofs of Corollaries 13.23 and 13.24 carry over, substituting Theorems 11.21 and 11.22 for Theorems 11.19 and 11.16. \square

Corollary 13.31. *Assume that the $(S^1)^\delta$ -action on ${}^\nu \varpi_1(Y_0, y)^{\text{red}}$ descends to R , and that the group $\varpi_n(|Y_\bullet|, y)^{\rho, \text{Mal}}$ is finite-dimensional and spanned by the image of $\pi_n(|Y_\bullet|, y)$. Then $\varpi_n(|Y_\bullet|, y)^{\rho, \text{Mal}}$ carries a natural \mathcal{S} -split mixed Hodge structure, which extends the mixed twistor structure of Corollary 13.30.*

Proof. This is essentially the same as Corollary 13.26, replacing Proposition 11.19 with Proposition 11.25. \square

Remark 13.32. When $R = 1$, Proposition 9.15 adapts to show that the mixed Hodge structure of Corollary 13.30 agrees with that of [Hai2, Theorem 6.3.1].

13.4.3. *Projective varieties.* In Theorems 5.14 and 6.1, explicit SL_2 splittings were given for the mixed Hodge and mixed twistor structures on a connected compact Kähler manifold X . Since any MHS or MTS has many possible SL_2 -splittings, it is natural to ask whether the explicit splittings are the same as the canonical splittings of Corollary 13.19. Apparently miraculously, the answer is yes:

Theorem 13.33. *The quasi-isomorphisms*

$$\begin{aligned} \text{row}_1^*(X, x)_{\text{MTS}}^{R, \text{Mal}} &\simeq \mathbb{A}^1 \times \text{Spec}(\underline{\text{gr}}(X, x)_{\text{MTS}}^{R, \text{Mal}}) \times \text{SL}_2 \\ \text{and } \text{row}_1^*(X, x)_{\text{MHS}}^{R, \text{Mal}} &\simeq \mathbb{A}^1 \times \text{Spec}(\underline{\text{gr}}(X, x)_{\text{MHS}}^{R, \text{Mal}}) \times \text{SL}_2 \end{aligned}$$

of Corollary 13.19 are homotopic to the corresponding quasi-isomorphisms of Theorems 5.14 and 6.1.

Proof. Given a MTS or MHS V , an SL_2 -splitting $\mathrm{row}_1^*\xi(V) \cong (\mathrm{gr}^W V) \otimes O(\mathrm{SL}_2)$ gives rise to a derivation $\beta: \mathrm{gr}^W V \rightarrow \mathrm{gr}^W V \otimes \Omega(\mathrm{SL}_2/C^*)$, given by differentiation with respect to $\mathrm{row}_1^*\xi(V)$. Since $\Omega(\mathrm{SL}_2/C^*) \cong O(\mathrm{SL}_2)(-1)$, this SL_2 -splitting corresponds to the canonical SL_2 -splitting of Theorem 12.13 or 12.6 if and only if $\beta(\mathrm{gr}^W V) \subset \mathrm{gr}^W V \otimes \mathrm{row}_2^\# O(C)(-1)$.

Now, the formality quasi-isomorphisms of Theorems 5.14 and 6.1 allow us to transfer the derivation $N: \mathrm{row}_1^*\mathcal{O}((X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}}) \rightarrow \mathrm{row}_1^*\mathcal{O}((X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}})(-1)$ to an N -linear derivation (determined up to homotopy)

$$N_\beta: E \otimes O(\mathrm{SL}_2) \rightarrow E \otimes O(\mathrm{SL}_2)(-1),$$

for any fibrant cofibrant replacement E for $O(\mathrm{gr}(X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}})$, and similarly for $\mathcal{O}((X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}})$. Moreover, $\mathcal{O}((X, x)_{\mathrm{MTS}}^{R, \mathrm{Mal}})$ (resp. $\mathcal{O}((X, x)_{\mathrm{MHS}}^{R, \mathrm{Mal}})$) is then quasi-isomorphic to the cone

$$\mathrm{row}_{1*}(E \otimes O(\mathrm{SL}_2) \xrightarrow{N_\beta} E \otimes O(\mathrm{SL}_2)(-1)).$$

If we write $N_\beta = \mathrm{id} \otimes N + \beta$, for $\beta: E \rightarrow E \otimes O(\mathrm{SL}_2)(-1)$, then the key observation to make is that the formality quasi-isomorphism coincides with the canonical quasi-isomorphism of Corollary 13.19 if and only if for some choice of β in the homotopy class, we have

$$\beta(E) \subset E \otimes \mathrm{row}_2^\# O(\mathbb{A}^2)(-1) \subset E \otimes O(\mathrm{SL}_2)(-1).$$

Now, Remark 4.22 characterises the homotopy class of derivations β in terms of minimal models, with $[\beta] = [\alpha + \gamma_x]$, where γ_x characterises the basepoint, and α determines the unpointed structure. In Theorem 8.13, the operators α and γ_x are computed explicitly in terms of standard operations on the de Rham complex.

For co-ordinates $\begin{pmatrix} u & v \\ x & y \end{pmatrix}$ on SL_2 , it thus suffices to show that α and γ_x are polynomials in x and y . The explicit computation expresses these operators as expressions in $\tilde{D} = uD + vD^c$, $\tilde{D}^c = xD + yD^c$ and $h_i = G^2 D^* D^{c*} \tilde{D}^c$, where G is the Green's operator. However, each occurrence of \tilde{D} is immediately preceded by either \tilde{D}^c or by h_i . Since

$$\tilde{D}^c \tilde{D} = (xD + yD^c)(uD + vD^c) = (uy - vx)D^c D = D^c D,$$

we deduce that α and γ_x are indeed polynomials in x and y , so the formality quasi-isomorphisms of Theorems 5.14 and 6.1 are just the canonical splittings of Corollary 13.19. \square

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