Shifted symplectic and Poisson structures

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Building blocks for Derived AG

- Classical AG built from affine schemes, i.e (commutative rings)^{op}
- Derived AG from $(CDG^{\leqslant 0}A_{\mathbb{Q}})^{op}$:
  - cochain complex $A^0 \xleftarrow{\delta} A^{-1} \xleftarrow{\delta} A^{-2} \xleftarrow{\delta} \ldots$,
  - graded-commutative multiplication $A^i \otimes A^j \to A^{i+j}$, δ a derivation, $1 \in A_0$.
  - Quasi-isomorphisms $f\colon A \to B$ with $\mathrm{H}^*(A) \cong \mathrm{H}^*(B)$.

Think of these as functions on a derived scheme.

Example: derived critical locus

- Y a smooth affine variety, $f : Y \to \mathbb{A}^1$ a function.
- Functions $\mathscr{O}_X$ on $X := \mathrm{DCrit}(f)$ are

$$\mathscr{O}_Y \xleftarrow{df} \mathscr{T}_Y \xleftarrow{df} \Lambda^2 \mathscr{T}_Y \xleftarrow{df} \dots$$

- $\mathrm{H}^0 \mathscr{O}_X$ recovers critical locus, but X has more structure.
- $\mathrm{DCrit}(0) = \operatorname{Spec} \Lambda^* \mathscr{T}_Y$.

de Rham complexes I

- $A = (A, \delta)$ a CDGA over R.
- Kähler differentials $\Omega^1_{A/R}$ (a complex). (Universal among derivations of A.)
- Exterior powers $\Omega^p_{A/R}$.
- de Rham differential $d \colon \Omega^p_{A/R} \to \Omega^{p+1}_{A/R}$.
- Take product total complex for de Rham complex

de Rham complexes II

$$\mathrm{DR}(A/R) := (\prod_p \Omega^p_{A/R}[-p], d \pm \delta),$$

- Hodge filtration $F^p\mathrm{DR}(A) = \prod \Omega_A^{\geqslant p}$.
- Closed form $\omega \in F^p\mathrm{DR}(A)^i$ consists of $(\omega_p, \omega_{p+1}, \ldots)$,

$$\omega_n \in (\Omega^n_{A/R})^{i-n},$$
$$d\omega_n = \delta\omega_{n+1}.$$

Derived de Rham cohomology

- Always quasi-isos $\tilde{A} \xrightarrow{\sim} A$ for $\tilde{A}$ cofibrant ($\approx$ quasi-free).
  (e.g. $k[x]/x^2 \xleftarrow{\sim} (k[x,y], \delta y = x^2)$)
- [In $\mathcal{C}^\infty$ setting, cofibrant $\approx \mathcal{C}^\infty(\mathbb{R}^{m|n})$.]
- Derived de Rham
  $\mathbf{L}\mathrm{DR}(A/R) := \mathrm{DR}(\tilde{A}/R)$, for $\tilde{A} \to A$ a cofibrant (or even quasi-smooth) res'n.
- $\mathbf{L}F^p\mathrm{DR}(A/R)$, $\mathbf{L}\Omega^p(A)$ similarly.

n-shifted pre-symplectic structures

- $\omega \in \mathrm{Z}^{n+2}\mathbf{L}F^2\mathrm{DR}(A/R)$ [PTVV].
- Explicitly, $\omega = \sum_{p \geqslant 2} \omega_p$, with

$$\delta\omega_2 = 0, \quad d\omega_p = \delta\omega_{p+1}.$$

- Morphisms given by chain homotopies.
- Sheafify for global definition (e.g. via hypercovers).
- Symplectic if non-degenerate:

$$\omega_2^\sharp \colon (\mathbf{L}\Omega^1_X)^\vee \xrightarrow{\sim} \mathbf{L}\Omega^1_X[n].$$

Examples

- Symplectic structure on smooth variety is 0-shifted.
- Derived critical locus is (-1)-shifted symplectic.
- Stack $B\mathrm{GL}_n$ is 2-shifted symplectic.
- Mapping stack $\mathrm{map}(X, B\mathrm{GL}_n)$ is $(2-d)$-shifted symplectic for $d = \dim X$ whenever $\Omega_X^d \cong \mathscr{O}_X$ [PTVV].

n-shifted polyvectors

- $\widehat{\mathrm{Pol}}(A/R, n)$: multiderivations

  $$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathrm{CoSymm}_A(\mathbf{L}\Omega^1_{A/R}[n+1]), A),$$

  $$(\approx \prod_i \mathrm{Symm}^i_A(T_{A/R}[n+1])).$$
- Filtration $F^p\widehat{\mathrm{Pol}}(A/R, n)$:

  $$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathrm{CoSymm}^{\geqslant p}_A(\mathbf{L}\Omega^1_{A/R}[n+1]), A).$$
- Commutative product $F^p \cdot F^q \subset F^{p+q}$.
- Schouten–Nijenhuis Lie bracket on $\widehat{\mathrm{Pol}}(A/R, n)[n+1]$; $[F^p, F^q] \subset F^{p+q-1}$.

n-Poisson structures I

- $\pi = \sum_{p \geqslant 2} \pi_p \in F^2\widehat{\mathrm{Pol}}(A/R, n)^{n+2}$,
$$\delta\pi + \frac{1}{2}[\pi, \pi] = 0 \text{ (Maurer–Cartan).}$$
- Explicitly, higher Jacobi identities:
$$\pi_p \colon \ \mathrm{CoSymm}^p_A(\mathbf{L}\Omega^1_{A/R}[n+1]) \to A[n+2]$$
$$\delta\pi_p = -\frac{1}{2}\sum_{i+j=p+1}[\pi_i, \pi_j].$$

n-Poisson structures II

- Morphisms from Thom–Sullivan homotopies

$$A \rightleftarrows A \otimes \Omega^\bullet(\Delta^1) \leftleftarrows A \otimes \Omega^\bullet(\Delta^2) \cdots$$

give space $\mathcal{P}(A/R)$ of n-Poisson structures. (Gauge equivalences suffice for $\pi_0\mathcal{P}(A/R)$.)

- Non-degenerate if

$$\pi_2^\sharp \colon \mathbf{L}\Omega^1_A[n] \xrightarrow{\sim} (\mathbf{L}\Omega^1_A)^\vee.$$

n-Poisson algebras

- P_{n+1}-algebra is CDGA with compatible Lie bracket of degree $-n$.
- $\pi \in \mathcal{P}(A)$ gives L_∞-structure on $A[n]$.
- n-Poisson structures $\Longleftrightarrow$ homotopy P_{n+1}-algebra structures on A [Melani].

What about stacks (like BGL_n)?

- Don't have smooth functoriality.
- So resolve by more exotic objects.

- Example
  - Y acted on by G, Lie algebra $\mathfrak{g}$.
  - Chart for $[Y/G]$ will be $[Y/\mathfrak{g}]$.
  - $O([Y/\mathfrak{g}])$ Chevalley–Eilenberg complex

$$O(Y) \xrightarrow{\partial} O(Y) \otimes \mathfrak{g}^\vee \xrightarrow{\partial} O(Y) \otimes \Lambda^2 \mathfrak{g}^\vee \xrightarrow{\partial} \dots$$

  - *bigraded* CDGA when $O(Y)$ a CDGA.
  - Tweak definitions for Poisson structures.

Symplectic versus Poisson

- Classical case: 2-form ω is symplectic iff inverse π is Poisson.
- Standard proof uses Darboux theorem (cotangent bundle).
- Shifted Darboux theorems [B-BBBJ], [BG] give local comparison for shifted structures.

A more direct approach

- ω_2 homotopy inverse to π_2.
- Higher components??
- Look to generalise

$$\pi^\sharp \circ \omega^\sharp \circ \pi^\sharp = \pi^\sharp.$$

- Then globalise (via hypercovers).

The canonical tangent vector

- Tangent space $T_\pi\mathcal{P}$ of Poisson structures at π (Poisson cohomology):

$$\alpha \in F^2\widehat{\mathrm{Pol}}(A/R, n)^{n+2},$$
$$\delta\alpha + [\pi, \alpha] = 0$$

(i.e. $\pi + \alpha\epsilon$ Poisson for $\epsilon^2 = 0$).

- Differentiating $\mathbb{G}_m$-action gives

$$\sigma(\pi) := \sum_{p \geqslant 2} (p-1)\pi_p \in T_\pi\mathcal{P}.$$

Compatibility (the key)

- Contraction $\mu(-, \pi)$ from de Rham to Poisson cohomology (cf. [K-S M]).
- "Derivative" ν.
- Relates de Rham / Schouten–Nijenhuis:

$$[\pi, \mu(\omega, \pi)] = \mu(d\omega, \pi) + \nu(\omega, \pi; \frac{1}{2}[\pi, \pi]).$$

- $\mu(\omega, \pi) \in T_\pi \mathcal{P}$ (Poisson cohomology class) for ω pre-symplectic, π Poisson.

In detail

When $\phi = adf_1 \wedge \ldots \wedge df_p$,

$$\mu(\phi, \pi) = a[\pi, f_1] \ldots [\pi, f_p,],$$
$$\nu(\phi, \pi; b) = \sum_i \pm a[\pi, f_1] \ldots [b, f_i] \ldots [\pi, f_p].$$

- Thus $\mu(\omega_2, \pi_2)^\sharp = \pi_2^\sharp \circ \omega_2^\sharp \circ \pi_2^\sharp$.
- $\mu(-, \pi)$ a qu-iso for π non-degenerate, so $\exists! \omega$ with $\mu(\omega, \pi) \simeq \sigma(\pi)$.

The equivalence

Weak equivalences of ∞-groupoids:

$$\begin{array}{c}\{\text{Symplectic structures } \omega\} \\ \uparrow \\ \{\text{Compatible pairs } \mu(\omega,\pi) \simeq \sigma(\pi)\} \\ \downarrow \\ \{\text{Non-degenerate Poisson structures } \pi\}\end{array}$$

Governing DGLAs

- Symplectic: $F^2\mathbf{L}\mathrm{DR}(A/R)[n+1]$ (abelian).
- Poisson: $F^2\widehat{\mathrm{Pol}}(A/R, n)[n+1]$.
- $T\mathcal{P}$: $F^2\widehat{\mathrm{Pol}}(A/R, n)[n+1][\epsilon]$.
- Compatible pairs a homotopy limit.
- Equivalence via obstruction theory:

Obstructions

$\mathrm{gr}_F^p \to F^2\widehat{\mathrm{Pol}}(A)/F^{p+1} \to F^2\widehat{\mathrm{Pol}}(A)/F^p$
central extension of DGLAs.

- Maurer–Cartan obstruction map $\mathcal{P}(A/R, n)/F^p \to \underline{\mathrm{MC}}(\mathrm{gr}_F^p[n+2])$, fibre $\mathcal{P}(A/R, n)/F^{p+1}$.
- Similar obstructions for symplectic structures, compatible pairs.
- Graded pieces equivalent via powers of $(\mathbf{L}\Omega^1_X)^\vee \simeq \mathbf{L}\Omega^1_X[n]$.

Idea of proof

- Comparison stepwise: seek $(\pi_2, \ldots, \pi_p)$ compatible with $(\omega_2, \ldots, \omega_p)$.
- Equivalences of obstruction spaces show that if $(\omega_2, \ldots, \omega_p)$ compatible with $(\pi_2, \ldots, \pi_p)$, then lifts $(\omega_2, \ldots, \omega_p, \omega_{p+1})$ correspond to lifts $(\pi_2, \ldots, \pi_p, \pi_{p+1})$.
- Étale functoriality (or tweak for Artin stacks) gives global equivalence.