The pro-unipotent radical of the pro-algebraic fundamental group of a compact Kähler manifold

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Abstract

The aim of this paper is to study the pro-algebraic fundamental group of a compact Kähler manifold. Following work by Simpson, the structure of this group's pro-reductive quotient is already well understood. We show that Hodge-theoretic methods can also be used to establish that the pro-unipotent radical is quadratically presented. This generalises both Deligne et al.'s result on the de Rham fundamental group, and Goldman and Millson's result on deforming representations of Kähler groups, and can be regarded as a consequence of formality of the schematic homotopy type. New examples are given of groups which cannot arise as Kähler groups.

Contents

Introduction

For X a compact connected Kähler manifold, consider the real pro-algebraic completion $\varpi_1(X, x)$ of the fundamental group $\pi_1(X, x)$. In [Sim92], Simpson defined a Hodge structure on the complex pro-algebraic fundamental group $\varpi_1(X, x)_{\mathbb{C}}$, in the form of a discrete $\mathbb{C}^*$-action. The Levi decomposition for pro-algebraic groups allows us to write

$$\varpi_1(X, x) \cong R_u(\varpi_1(X, x)) \rtimes \varpi_1^{\mathrm{red}}(X, x),$$

where $R_u(\varpi_1(X,x))$ is the pro-unipotent radical of $\varpi_1(X,x)$ and $\varpi_1^{\mathrm{red}}(X,x)$ is the reductive quotient of $\varpi_1(X,x)$. This decomposition is unique up to conjugation by $R_u(\varpi_1(X,x))$. By studying the Hodge structure on $\varpi_1^{\mathrm{red}}(X,x)$, Simpson established restrictions on its possible group structures.

The purpose of this paper is to use Hodge theory to show that $R_u(\varpi_1(X,x))$ is quadratically presented as a pro-unipotent group, in the sense that its Lie algebra can be defined by equations of bracket length two. This generalises both Goldman and Millson's result on deforming reductive representations of the fundamental group ([GM88]), and Deligne et al.'s result on the de Rham fundamental group $\pi_1(X,x) \otimes \mathbb{R}$ ([DGMS75]). The idea behind this paper is that in both of these cases, we are taking a reductive representation

$$\rho_0 : \pi_1(X,x) \to G,$$

and considering deformations

$$\rho : \pi_1(X,x) \to U \rtimes G$$

of ρ_0, for U unipotent.

Effectively, [GM88] considers only $U = \exp(\mathrm{Lie}(G) \otimes \mathfrak{m}_A)$, for $\mathfrak{m}_A$ a maximal ideal of an Artinian local $\mathbb{R}$-algebra, while [DGMS75] considers only the case when $G = 1$. Since taking $U = R_u(\varpi_1(X,x))$ pro-represents this functor when $G = \varpi_1^{\mathrm{red}}(X,x)$, the quadratic presentation for U gives quadratic presentations both for the hull of [GM88] and for the Lie algebra of [DGMS75]. This also generalises Hain's results ([**?**]) on relative Malcev completions of variations of Hodge structure, since here we are taking relative Malcev completions of arbitrary reductive representations.

Section 1 summarises standard definitions and properties of pro-algebraic groups which are used throughout the rest of the paper.

Section 2 develops a theory of deformations over nilpotent Lie algebras with G-actions. This can be thought of as a generalisation of the theory introduced in [Pri04], which corresponds to the case $G = 1$. The essential philosophy is that all the concepts for deformations over Artinian rings, developed by Schlessinger in [Sch68] and later authors, can be translated to this context.

Section 3 introduces the notion of twisted differential graded algebras, which are analogous to the DGAs used in [DGMS75] to characterise the real homotopy type. They are equivalent to the G-equivariant DGAs used in [KPT08] to study the schematic homotopy type.

Section 4 contains various technical lemmas about pro-algebraic groups.

In Section 5, the twisted DGA arising from C^∞-sections is defined. It is shown that this can be used to recover $R_u(\varpi_1(X,x))$.

Section 6 uses Hodge theory to prove that, for a compact Kähler manifold, this twisted DGA is formal, i.e. quasi-isomorphic to its cohomology DGA. This can be thought of as equivalent to formality of the real schematic homotopy type. In consequence, $R_u(\varpi_1(X,x))$ is quadratically presented — a direct analogue of the demonstration in [DGMS75] that formality of the real homotopy type forces the de Rham fundamental group to be quadratically presented. This implies that if $\pi_1(X,x)$ is of the form $\Delta \rtimes \Lambda$, with Λ acting reductively on $\Delta \otimes \mathbb{R}$, then $\Delta \otimes \mathbb{R}$ must be quadratically presented.

I would like to thank Bertrand Toën for suggesting the connection between this work and the schematic homotopy type.

1 Review of pro-algebraic groups

In this section, we recall some definitions and standard properties of pro-algebraic groups, most of which can be found in [DMOS82] II§2 and [Sim92]. Fix a field k of characteristic zero.

Definition 1.1. An algebraic group over k is defined to be an affine group scheme G of finite type over k. These all arise as Zariski-closed subgroups of general linear groups $\mathrm{GL}_n(k)$. A pro-algebraic group is a filtered inverse limit of algebraic groups, or equivalently an arbitrary affine group scheme over k.

Definition 1.2. Given a pro-algebraic group G, let $O(G)$ denote global sections of the structure sheaf of G, so that $G = \mathrm{Spec}\, O(G)$. This is a sum of finite-dimensional $G \times G$-representations, the actions corresponding to right and left translation. The group structure on G corresponds to a comultiplication $\Delta : O(G) \to O(G) \otimes O(G)$, coidentity $\varepsilon : O(G) \to k$, and coinverse $S : O(G) \to O(G)$, satisfying coassociativity, coidentity and coinverse axioms.

Lemma 1.3. *If G is a pro-algebraic group, and we regard $O(G)$ as a sum of finite-dimensional G-representations via the left action, then for any finite-dimensional G-representation V,*

$$V \cong (V \otimes O(G))^G := \{a \in V \otimes O(G) \,|\, (g \otimes \mathrm{id})a = (\mathrm{id} \otimes g)a, \, \forall g \in G\},$$

with the G-action on the latter coming from the right action on $O(G)$.

Proof. This follows immediately from [DMOS82] II Proposition 2.2, which states that G-representations correspond to $O(G)$-comodules. Under this correspondence, $v \in V$ is associated to the function $g \mapsto g \cdot v$. $\square$

Definition 1.4. An algebraic group G is said to be unipotent if the coproduct $\Delta : O(G) \to O(G) \otimes O(G)$ is counipotent. Unipotent algebraic groups all arise as Zariski-closed subgroups of the groups of upper triangular matrices:

$$\begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix}.$$

A pro-algebraic group is said to be pro-unipotent if it is an inverse limit of unipotent algebraic groups. This is equivalent to saying that Δ is ind-counipotent.

Lemma 1.5. *There is a one-to-one correspondence between unipotent algebraic groups over k, and finite-dimensional nilpotent Lie algebras over k.*

Proof. The Lie algebra $\mathfrak{u}$ of any unipotent algebraic group U is necessarily finite-dimensional and nilpotent. Conversely, if $\mathfrak{u}$ is any finite-dimensional nilpotent Lie algebra, we define a unipotent algebraic group $\exp(\mathfrak{u})$ by

$$\exp(\mathfrak{u})(A) := \exp(\mathfrak{u} \otimes A).$$

Here, for any Lie algebra $\mathfrak{g}$, the group $\exp(\mathfrak{g})$ has underlying set $\mathfrak{g}$ and multiplication given by the Campbell-Baker-Hausdorff formula

$$g \cdot h := g + h + \frac{1}{2}[g,h] + \dots,$$

which in this case is a finite sum, by nilpotence, so the group $\exp(\mathfrak{u})$ is indeed algebraic. To see that these functors are inverse is most easily done by considering groups of upper triangular matrices. $\qquad\square$

Definition 1.6. The pro-unipotent radical $R_u(G)$ of a pro-algebraic group G is defined to be the maximal pro-unipotent normal closed subgroup of G. A pro-algebraic group G is said to be reductive if $R_u(G) = 1$, and for an arbitrary pro-algebraic group G, the reductive quotient of G is defined by $G^{\text{red}} := G/R_u(G)$.

Theorem 1.7 (Levi decomposition)**.** *For any pro-algebraic group G, there is a decomposition*

$$G \cong R_u(G) \rtimes G^{\text{red}},$$

unique up to conjugation by the pro-unipotent radical $R_u(G)$.

Proof. This is the Levi decomposition for pro-algebraic groups in characteristic zero, proved in [HM69], which states that for every pro-algebraic group G, the surjection $G \to G^{\text{red}}$ has a section, unique up to conjugation by $R_u(G)$, inducing an isomorphism $G \cong R_u(G) \rtimes G^{\text{red}}$. $\qquad\square$

Theorem 1.8 (Tannakian duality)**.** *A pro-algebraic group G over k can be recovered from its (tensor) category of finite-dimensional k-representations. Representations of G^{red} correspond to the subcategory of semisimple representations.*

Proof. The first part is [DMOS82] Theorem II.2.11. The second part is just the observation that, in characteristic zero, "reductive" and "linearly reductive" are equivalent. $\qquad\square$

Definition 1.9. Given a discrete group Γ, define the pro-algebraic completion Γ^{alg} of Γ to be the pro-algebraic group G universal among group homomorphisms $\Gamma \to G(k)$. In other words Γ^{alg} pro-represents the functor which sends an algebraic group G to the set of representations $\Gamma \to G(k)$. Under Tannakian duality, the finite-dimensional linear k-representations of Γ^{alg} are just the finite-dimensional linear k-representations of Γ.

The reductive quotient of Γ^{alg} is denoted Γ^{red}, and is universal among Zariski-dense group homomorphisms $\Gamma \to G(k)$, with G reductive. In other words Γ^{red} pro-represents the functor which sends an algebraic group G to the set of reductive representations $\Gamma \to G(k)$. Under Tannakian duality, the finite-dimensional linear k-representations of Γ^{red} are just the semisimple finite-dimensional linear k-representations of Γ.

The pro-unipotent (or Malcev) completion $\Gamma \otimes k$ of Γ is universal among group homomorphisms $\Gamma \to G(k)$, with G pro-unipotent. In other words $\Gamma \otimes k$ pro-represents the functor which sends an algebraic group G to the set of unipotent representations $\Gamma \to G(k)$. Under Tannakian duality, the finite-dimensional linear k-representations of $\Gamma \otimes k$ are just the unipotent finite-dimensional linear k-representations of Γ.

Definition 1.10. Given a representation $\rho : \Gamma \to G(k)$, for some pro-algebraic group G, the Malcev completion of Γ relative to ρ is defined in [**?**] to represent the functor which sends a pro-unipotent extension $H \to G$ to the set of representations

$$\Gamma \to H(k)$$

lifting ρ. Thus $\Gamma \otimes k$ is the Malcev completion of Γ relative to the trivial representation, and Γ^{alg} is the Malcev completion of Γ relative to the canonical representation $\Gamma \to \Gamma^{\mathrm{red}}$.

2 Functors on nilpotent Lie algebras with G-actions

This section extends the ideas of [Sch68] to a slightly different context.

Fix a field k of characteristic zero. Take a pro-algebraic group G over k, and let $\mathrm{Rep}(G)$ be the category of finite-dimensional representations of G over k. If G is reductive, then every such representation will be decomposable into irreducibles, so Hom will be an exact functor on this category. Consider the category $\widehat{\mathrm{Rep}}(G) := \mathrm{pro}(\mathrm{Rep}(G))$, whose objects are filtered inverse systems $\{V_\alpha\}_{\alpha \in I}$, with morphisms given by

$$\mathrm{Hom}_{\mathrm{pro}(\mathrm{Rep}(G))}(\{V_\alpha\}, \{W_\beta\}) = \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}_{\mathrm{Rep}(G)}(V_\alpha, W_\beta).$$

Given a set $\{V_i\}_{i \in I}$, with $V_i \in \mathrm{Rep}(G)$, we make the vector space $\prod_{i \in I} V_i$ an object of $\widehat{\mathrm{Rep}}(G)$ via the formula

$$\prod_{i \in I} V_i = \varprojlim_{J \subset I \text{ finite}} \prod_{j \in J} V_j.$$

Lemma 2.1. *If G is reductive, then every object of $\widehat{\mathrm{Rep}}(G)$ can be expressed as a product of irreducible finite-dimensional G-representations.*

Proof. Since $\mathrm{Rep}(G)$ is an Artinian category, i.e. it satisfies the descending chain condition for sub-objects, we may use [Gro95] to observe that $\widehat{\mathrm{Rep}}(G)$ is isomorphic to the category of left-exact set-valued functors on $\mathrm{Rep}(G)$.

Take $W \in \widehat{\mathrm{Rep}}(G)$, and let $\{V_s : s \in S\}$ be a set of representatives for isomorphism classes of irreducible representations in $\mathrm{Rep}(G)$. Now, $\mathrm{Hom}_{\widehat{\mathrm{Rep}}(G)}(W, V_s)$ has the natural structure of a vector space over k. Choose a basis $t_i : i \in I_s$ for this vector space, and let

$$U := \prod_{s \in S} V_s^{I_s}.$$

There is then a natural isomorphism between $\mathrm{Hom}_{\widehat{\mathrm{Rep}}(G)}(W, V_s)$ and $\mathrm{Hom}_{\widehat{\mathrm{Rep}}(G)}(U, V_s)$ for all $s \in S$, so the left-exact functors defined on $\mathrm{Rep}(G)$ by U and W must be isomorphic, and therefore $U \cong W$. $\square$

Definition 2.2. For any pro-algebraic group G, define $\mathcal{N}(G)$ to be the category whose objects are pairs $(\mathfrak{u}, \rho)$, where $\mathfrak{u}$ is a finite-dimensional nilpotent Lie algebra over k, and $\rho : G \to \mathrm{Aut}(\mathfrak{u})$ is a representation to the group of Lie algebra automorphisms of $\mathfrak{u}$. A morphism θ from $(\mathfrak{u}, \rho)$ to $(\mathfrak{u}', \rho')$ is a morphism $\theta : \mathfrak{u} \to \mathfrak{u}'$ of Lie algebras such that $\theta \circ \rho = \rho'$. Observe that $\mathcal{N}(G)$ is an Artinian category, and write $\widehat{\mathcal{N}}(G)$ for the category $\mathrm{pro}(\mathcal{N}(G))$.

Given $\mathcal{L} \in \widehat{\mathcal{N}}(G)$, let $\mathcal{N}(G)_{\mathcal{L}}$ be the category of pairs $(N \in \mathcal{N}(G), \mathcal{L} \xrightarrow{\phi} N)$, and $\widehat{\mathcal{N}}(G)_{\mathcal{L}}$ the category of pairs $(N \in \widehat{\mathcal{N}}(G), \mathcal{L} \xrightarrow{\phi} N)$. We will almost always consider the case $\mathcal{L} = 0$ (note that $\mathcal{N}(G)_0 = \mathcal{N}(G)$), although a few technical lemmas (Propositions 2.30 and 2.31 and Corollary 2.32) require the full generality of $\mathcal{N}(G)_{\mathcal{L}}$. In $\widehat{\mathcal{N}}(G)_{\mathcal{L}}$, $\mathcal{L}$ is the initial object, and 0 the final object.

Definition 2.3. For $N = \varprojlim_{\alpha} N_{\alpha} \in \widehat{\mathcal{N}}(G)_{\mathcal{L}}$, define the tangent space functor

$$
\begin{aligned}
t_{N/\mathcal{L}} : \mathrm{Rep}(G) &\to \mathrm{Vect} \\
V &\mapsto \mathrm{Hom}_{\widehat{\mathcal{N}}(G)_{\mathcal{L}}}(N, V\epsilon),
\end{aligned}
$$

where $V\epsilon \in \mathcal{N}(G)$ is the representation V regarded as an abelian Lie algebra, i.e. $[V\epsilon, V\epsilon] = 0$, with structure morphism $\mathcal{L} \xrightarrow{0} V\epsilon$. This is clearly a vector space over k, and

$$
t_{N/\mathcal{L}}(V) = \varinjlim_{\alpha} \mathrm{Hom}_{\mathrm{Rep}(G)}(N_{\alpha}/\langle [N_{\alpha}, N_{\alpha}] + \mathcal{L}\rangle, V).
$$

We define the cotangent space

$$
t^*_{N/\mathcal{L}} := N/\langle [N, N] + \mathcal{L}\rangle = \varinjlim_{\alpha} N_{\alpha}/\langle [N_{\alpha}, N_{\alpha}] + \mathcal{L}\rangle \in \widehat{\mathrm{Rep}}(G).
$$

Definition 2.4. Given V in $\widehat{\mathrm{Rep}}(G)$, denote the free pro-nilpotent Lie algebra on generators V by $L(V)$. This has a natural continuous G-action, so is in $\widehat{\mathcal{N}}(G)$. Equivalently, we may use [Gro95] to define $L(V)$ as the object of $\widehat{\mathcal{N}}(G)$ pro-representing the functor $N \mapsto \mathrm{Hom}_{\widehat{\mathrm{Rep}}(G)}(V, N)$.

Definition 2.5. Given a Lie algebra L, we define the lower central series of ideals $\Gamma_n(L)$ inductively by

$$
\Gamma_1(L) = L, \quad \Gamma_{n+1}(L) = [L, \Gamma_n(L)],
$$

and define the associated graded algebra $\mathrm{gr}L$ by $\mathrm{gr}_n L = \Gamma_n L / \Gamma_{n+1} L$

For $L \in \mathcal{N}(G)_{\mathcal{L}}$, define the ideals $\Gamma^{\mathcal{L}}_n(L)$ by

$$
\Gamma^{\mathcal{L}}_1(L) = L, \quad \Gamma^{\mathcal{L}}_2(L) = [L, L] + \langle \mathcal{L}\rangle \quad \Gamma^{\mathcal{L}}_{n+1}(L) = [L, \Gamma_n(L)],
$$

and define the associated graded algebra $\mathrm{gr}^{\mathcal{L}}L$ by $\mathrm{gr}^{\mathcal{L}}_n L = \Gamma^{\mathcal{L}}_n L / \Gamma^{\mathcal{L}}_{n+1} L$.

Definition 2.6. Given Lie algebras $N, M \in \widehat{\mathcal{N}}(G)$, let $N * M$ be the completed free Lie algebra product, i.e. the completion with respect to the commutator filtration of the free product of the pro- Lie algebras N and M. Note that $*$ is sum in $\widehat{\mathcal{N}}(G)$ — the analogue in the category $\mathrm{pro}(\mathcal{C}_{\Lambda})$ of pro-Artinian Λ-algebras is $\hat{\otimes}$.

Given Lie algebras $N, M \in \widehat{\mathcal{N}}(G)_{\mathcal{L}}$, define the free fibre product $N *_{\mathcal{L}} M$ similarly. Equivalently, we can use [Gro95] to define this element of $\widehat{\mathcal{N}}(G)$, since all finite colimits must exist in a pro-Artinian category.

Definition 2.7. Define the free Lie algebra $L_{\mathcal{L}}(V) := \mathcal{L} * L(V)$

We will consider only those functors F on $\mathcal{N}(G)_{\mathcal{L}}$ which satisfy

(H0) $F(0) = \bullet$, the one-point set.

We adapt the following definitions and results from [Sch68] (with identical proofs):

Definition 2.8. For $p : N \to M$ in $\widehat{\mathcal{N}}(G)_{\mathcal{L}}$ surjective, p is a semi-small extension if $[N, \ker p] = (0)$. If the pro-G-representation $\ker p$ is an absolutely irreducible G-representation, then we say that p is a small extension. Note that any surjection in $\mathcal{N}(G)_{\mathcal{L}}$ can be be factorised as a composition of small extensions.

For $F : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Set}$, define $\hat{F} : \widehat{\mathcal{N}}(G)_{\mathcal{L}} \to \mathrm{Set}$ by

$$\hat{F}(\varprojlim_{\alpha} L_{\alpha}) = \varprojlim_{\alpha} F(L_{\alpha}).$$

Note that $\hat{F}(L) \xrightarrow{\sim} \mathrm{Hom}(h_L, F)$, where

$$
\begin{aligned}
h_L : \mathcal{N}(G)_{\mathcal{L}} &\to \mathrm{Set}; \\
N &\mapsto \mathrm{Hom}(L, N).
\end{aligned}
$$

Definition 2.9. We will say a functor $F : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Set}$ is pro-representable if it is isomorphic to h_L, for some $L \in \widehat{\mathcal{N}}(G)_{\mathcal{L}}$. By the above remark, this isomorphism is determined by an element $\xi \in \hat{F}(L)$. We say the pro-couple (L, ξ) pro-represents F.

Remark 2.10. This definition is not the strict analogue of that appearing in [Sch68], which had additional hypotheses on the finiteness of tangent spaces. This terminology coincides with that of [Gro95].

Definition 2.11. A natural transformation $\phi : F \to E$ in $[\mathcal{N}(G)_{\mathcal{L}}, \mathrm{Set}]$ is called:

1. unramified if $\phi : F(V\epsilon) \to E(V\epsilon)$ is injective for all irreducible G-representations V.

2. smooth if for every surjection $M \twoheadrightarrow N$ in $\mathcal{N}(G)_{\mathcal{L}}$, the canonical map $F(M) \to E(M) \times_{E(N)} F(N)$ is surjective.

3. étale if it is smooth and unramified.

Definition 2.12. $F : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Set}$ is smooth if and only if $F \to \bullet$ is smooth.

Lemma 2.13. *A morphism $f : M \to N$ in $\widehat{\mathcal{N}}_{\mathcal{L}}(G)$ is a surjection or an isomorphism if and only if the associated graded morphism $\mathrm{gr}^{\mathcal{L}} M \to \mathrm{gr}^{\mathcal{L}} N$ is so. Therefore $f : M \to N$ in $\widehat{\mathcal{N}}(G)_{\mathcal{L}}$ is surjective if and only if the induced map $t^*_{M/\mathcal{L}} \to t^*_{N/\mathcal{L}}$ is surjective, and an endomorphism $f : M \to M$ in $\widehat{\mathcal{N}}(G)_{\mathcal{L}}$ is an automorphism whenever the induced map $t^*_{M/\mathcal{L}} \to t^*_{M/\mathcal{L}}$ is the identity.*

Definition 2.14. A map $p : M \to N$ in $\widehat{\mathcal{N}}(G)_{\mathcal{L}}$ is essential if for all morphisms $q : M' \to M$, q is surjective whenever pq is.

From now on, we will assume that G is reductive.

From the above lemma, we deduce:

Lemma 2.15. *Let $p : M \to N$ be a surjection in $\widehat{\mathcal{N}(G)}_{\mathcal{L}}$. Then*

1. *p is essential if and only if the induced map $t^*_{M/\mathcal{L}} \to t^*_{N/\mathcal{L}}$ is an isomorphism.*

2. *If p is a small extension, then p is not essential if and only if p has a section.*

Proof.   1. Assume that p is essential. Since $t^*_{M/\mathcal{L}} \to t^*_{N/\mathcal{L}}$ is surjective, so must $M \to t^*_{N/\mathcal{L}}$ be. Using the semisimplicity of $\mathrm{Rep}(G)$ (since G is reductive), this map has a section in $\widehat{\mathrm{Rep}}(G)$. Let its image be V, and let $q : L_{\mathcal{L}}(V) \to M$ be the map determined by the inclusion $V \hookrightarrow M$. Now, pq is surjective, since it induces an isomorphism on cotangent spaces. Therefore q is surjective, so $V \to t^*_{M/\mathcal{L}}$ is surjective, as required. The converse is immediate.

2. If p is not essential, construct the pro-Lie algebra $L_{\mathcal{L}}(V)$ as in the previous part, and let $N' \hookrightarrow M$ be its image. Then, by comparing tangent spaces, we see that $N' \to N$ is a surjection. Since p is not essential, $t^*_{M/\mathcal{L}} \to t^*_{N/\mathcal{L}}$ is not injective, so $\ker p \to t^*_{M/\mathcal{L}}$ is non-zero, hence an embedding, since $\ker p$ is absolutely irreducible. The image of N' in $t^*_{M/\mathcal{L}}$ has zero intersection with the image of $\ker p$. Therefore $\ker p$ and N' have zero intersection so $N' \times \ker p$ is a sub-pro-Lie algebra of M with the same cotangent space, so $M = N' \times \ker p$, and $N' \cong N$, which gives the section of p.

$\square$

Definition 2.16. A pro-couple (L, ξ) is a hull for F if the induced map $h_L \to F$ is étale.

Lemma 2.17. *Suppose F is a functor such that*

$$F(V\epsilon \oplus W\epsilon) \xrightarrow{\sim} F(V\epsilon) \times F(W\epsilon)$$

for $V, W \in \mathrm{Rep}(G)$. Then $F(V\epsilon)$ has a canonical vector space structure, and the tangent space functor

$$
\begin{aligned}
t_F : \mathrm{Rep}(G) &\to \mathrm{Vect} \\
V &\mapsto F(V\epsilon)
\end{aligned}
$$

is additive.

Definition 2.18. Given $F : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Set}$, let $N' \to N$ and $N'' \to N$ be morphisms in $\mathcal{N}(G)_{\mathcal{L}}$, and consider the map:

(†) $F(N' \times_N N'') \to F(N') \times_{F(N)} F(N'')$.

We make the following definitions for properties of F:

(H1) (†) is a surjection whenever $N'' \to N$ is a small extension.

(H2) (†) is a bijection whenever $N = 0$ and $N'' = V\epsilon$, for an irreducible G-representation V.

(H4) (†) is a bijection whenever $N' = N''$ and $N' \to N$ is a small extension.

Remark 2.19. These conditions are so named for historical reasons, following [Sch68]. The missing condition (H3) concerned finite-dimensionality of tangent spaces, which is irrelevant to our (weaker) notion of pro-representability.

Lemma 2.20. *Let* $F : \mathcal{N}(G) \to \mathrm{Set}$ *satisfy (H1) and (H2), and* $M \to N$ *be a semi-small extension in* $\widehat{\mathcal{N}}(G)$. *Given a surjection* $M \twoheadrightarrow M_\alpha$, *with* $M_\alpha \in \mathcal{N}(G)$, *let* $N_\alpha = M_\alpha *_M N$. *If* $\xi \in \hat{F}(N)$ *has the property that for all such surjections, the image* $\xi_\alpha \in F(N_\alpha)$ *of* ξ *lifts to* $F(M_\alpha)$, *then* ξ *lifts to* $\hat{F}(M)$.

Proof. Since $M \twoheadrightarrow N$ is semi-small, $M_\alpha \twoheadrightarrow N_\alpha$ is also. Let $I := \ker(M \to N)$ and $I_\alpha := \ker(M_\alpha \to N_\alpha)$. Observe that we have canonical isomorphisms $M_\alpha \times I_\alpha \cong M_\alpha \times_{N_\alpha} M_\alpha$, and let T_α be the fibre of $F(M_\alpha \to F(N_\alpha))$ over ξ_α. Since F satisfies (H2), we have a map

$$F(M_\alpha) \times t_F(I_\alpha) \to F(M_\alpha) \times_{F(N_\alpha)} F(M_\alpha),$$

so $t_F(I_\alpha)$ acts on T_α. From (H1) it follows that this action is transitive (and if F also satisfied (H4) then T_α would be a principal homogeneous space under this action). Let $K_\alpha \subset t_F(I_\alpha)$ be the stabiliser of T_α.

We wish to construct a compatible system $\eta_\alpha \in T_\alpha$. Let $\eta'_\alpha \in T_\alpha$ be any element, and assume that we have constructed a compatible system $\eta_\beta \in T_\beta$, for all strict epimorphisms $M_\alpha \twoheadrightarrow M_\beta$. Then $\eta_\beta = v_\beta(\eta'_\alpha)$, for a unique $v_\beta \in t_F(I_\beta)/K_\beta$. Observe that

$$t_F(I_\alpha)/K_\alpha \to \varprojlim_\beta t_F(I_\beta)/K_\beta$$

is surjective, and that the v_β form an element of the right-hand side. Lift to $v \in t_F(I_\alpha)/K_\alpha$, and let $\eta_\alpha := v(\eta'_\alpha)$. The construction proceeds inductively (since every poset can be enriched to form a totally ordered set, and we have satisfied the hypotheses for transfinite induction). $\square$

Proposition 2.21. *Let* (L, ξ), (L', ξ') *be hulls of* F. *Then there exists an isomorphism* $u : L \to L'$ *such that* $F(u)(\xi) = \xi'$.

Proof. We wish to lift ξ and ξ' to $u \in h_{L'}(L)$ and $u' \in h_L(L')$. We may apply Lemma 2.20 to the the functor $h_{L'}$ and the successive semi-small extensions $L_{n+1} \to L_n$, where $L_n = L/(\Gamma_n L)$. By smoothness of $h_{L'} \to F$, we obtain successive lifts of $\xi_n \in F(L_n)$ to $u_n \in h_{L'}(L_n)$. Let $u = \varprojlim u_n$, and construct u' similarly. We therefore obtain $u : (L', \xi) \to (L, \xi)$ and $u' : (L, \xi) \to (L', \xi)$, inducing identity on cotangent spaces. Therefore uu' induces the identity on $t^*_{L/\mathcal{L}}$, so is an automorphism, by Lemma 2.13. $\square$

Proposition 2.22. 1. *Let* $M \to N$ *be a morphism in* $\widehat{\mathcal{N}(G)}_\mathcal{L}$. *Then* $h_N \to h_M$ *is smooth if and only if* $N \cong L_M(V)$, *for some* $V \in \mathrm{Rep}(G)$.

2. *If $F \to E$ and $E \to H$ are smooth morphisms of functors, then the composition $F \to H$ is smooth.*

3. *If $u : F \to E$ and $v : E \to H$ are morphisms of functors such that u is surjective and vu is smooth, then v is smooth.*

Proof. If $N = M * L(V)$, then $h_N(\mathfrak{g}) = h_M(\mathfrak{g}) \times \text{Hom}(V, \mathfrak{g})$, which is smooth over h_M, since $\text{Rep}(G)$ is a semisimple category. Conversely, assume that $h_N \to h_M$ is smooth, and let $V = t^* N/M$. Let $N' = L_M(V)$, and observe that, by choosing a lift of V to N, we obtain a morphism $f : N' \to N$, inducing an isomorphism on relative cotangent spaces.

Let $N'_n = N'/(\Gamma_{n+1} N')$, and observe that $N'_n \to N'_{n-1}$ is a semi-small extension, as is $N'_1 \to t^*_{N'/M}$. We have a canonical map $N \to t^*_{N'/M}$ arising from the isomorphism $t^*_{N'/M} \cong t^*_{N/M}$. Since h_N satisfies (H1) and (H2), we may now apply Lemma 2.20 to construct a map $g : N \to N'$ lifting this. Therefore the compositions fg and gf induce the identity on cotangent spaces, so are isomorphisms, and $N' \cong N$, as required.

The remaining statements follow by formal arguments. $\square$

Remark 2.23. For the proposition above, it is essential that G be reductive, since we need the exactness of Hom on $\text{Rep}(G)$ to ensure that $L(V)$ is smooth.

Theorem 2.24. *1. F has a hull if and only if F has properties (H1) and (H2).*

2. F is moreover pro-representable if and only if F has the additional property (H4).

Proof. This is essentially [Sch68], Theorem 2.11, *mutatis mutandis*. Since t_F defines a left-exact functor on $\text{Rep}(G)$, by (H2), let it be pro-represented by $W \in \widehat{\text{Rep}}(G)$, and let $\mathfrak{h} = L_{\mathcal{L}}(W)$. The hull $\mathfrak{g}$ will be a quotient of $\mathfrak{h}$, which will be constructed as an inverse limit of semi-small extensions of W. Let $\mathfrak{g}_2 = W$, and $\xi_2 \in \hat{F}(W)$ the canonical element corresponding to the pro-representation of t_F. Assume that we have constructed $(\mathfrak{g}_q, \xi_q)$. We wish to find a semi-small extension $\mathfrak{g}_{q+1} \to \mathfrak{g}_q$, maximal among those quotients of $\mathfrak{h}$ which admit a lift of ξ_q.

Given quotients $M, N \in \mathcal{N}(G)$ of $\mathfrak{h}$, we write

$$M \wedge N := M *_{\mathfrak{h}} N, \quad \text{and} \quad M \vee N := M \times_{M \wedge N} N.$$

Note that $M \wedge N$ is then maximal among those Lie algebras which are dominated by both M and N, while $M \vee N$ is minimal among those quotients of $\mathfrak{h}$ which dominate both M and N. Next, observe that a set $\{M_\alpha \in \mathcal{N}(G)\}_{\alpha \in I}$ of quotients of $\mathfrak{h}$ corresponds to a quotient $\mathfrak{h} \twoheadrightarrow \varprojlim_\alpha M_\alpha$ of $\mathfrak{h}$ if and only if the following two conditions hold:

(Q1) If $M_\alpha \twoheadrightarrow N$ is a surjection, for any $\alpha \in I$ and any $N \in \mathcal{N}(G)$, then $N \cong M_\beta$, for some $\beta \in I$.

(Q2) Given any $\alpha, \beta \in I$, $M_\alpha \vee M_\beta = M_\gamma$, for some $\gamma \in I$.

We will now form such a set of quotients by considering those $\mathfrak{h} \twoheadrightarrow M$ satisfying:

1. $M \twoheadrightarrow M *_{\mathfrak{h}} \mathfrak{g}_q$ is a semi-small extension.

2. The image of ξ_q in $F(M *_{\mathfrak{h}} \mathfrak{g}_q)$ lifts to $F(M)$.

It is immediate that this set satisfies (Q1). To see that it satisfies (Q2), take quotients M, N satisfying these conditions. It is clear that $M \vee N \twoheadrightarrow (M \vee N) *_{\mathfrak{h}} \mathfrak{g}_q$ is a semi-small extension, since $M \vee N$ is a sub-Lie algebra of $M \times N$. To see that ξ_q lifts to $F(M \vee N)$, let $x \in F(M), y \in F(N)$ be lifts of ξ_q. Now, as in the proof of Lemma 2.20, the fibre of $F(N)$ over ξ_q surjects onto the fibre of $F(M \wedge N)$ over ξ_q. Therefore, we may assume that x and y have the same image in $F(M \wedge N)$. Now (H1) provides the required lift:

$$F(M \vee N) \twoheadrightarrow F(M) \times_{F(M \wedge N)} F(N).$$

Let $\mathfrak{g}_{q+1} \in \widehat{\mathcal{N}}(G)$ be the quotient defined by this collection. By Lemma 2.20, it follows that ξ_q lifts to $\hat{F}(\mathfrak{g}_{q+1})$. Let $\mathfrak{g} := \varprojlim \mathfrak{g}_q$, with $\xi := \varprojlim \xi_q$.

It remains to show that this is indeed a hull for F. By construction, $h_{\mathfrak{g}} \to F$ is unramified, so we must show it is smooth. Let $p : (N', \eta') \to (N, \eta)$ be a morphism of couples in $\mathcal{N}(G)$, with p a small extension, $N = N'/I$, and assume we are given $u : (\mathfrak{g}, \xi) \to (N, \eta)$. We must lift u to a morphism $(\mathfrak{g}, \xi) \to (N'\eta')$. It will suffice to find a morphism $u' : \mathfrak{g} \to A'$ such that $pu' = u$, since we may then use transitivity of the action of $t_F(I)$ on $F(p)^{-1}(\eta)$.

For some q, u factors as $(\mathfrak{g}, \xi) \to (\mathfrak{g}_q, \xi_q) \to (N, \eta)$. By smoothness of $\mathfrak{h}$, we may choose a morphism w making the following diagram commute. We wish to construct the morphism v:

If the small extension pr_1 has a section, then v obviously exists. Otherwise, by Lemma 2.15, pr_1 is essential, so w is a surjection. (H1) then provides a lift of ξ_q to $F(\mathfrak{g}_q \times_N N')$, so by the construction of $\mathfrak{g}_{q+1}$, w factors through $\mathfrak{g}_{q+1}$, and so v must exist. This completes the proof that $h_{\mathfrak{g}} \to F$ is a hull.

If F also satisfies (H4), then we may use induction on the length of N to show that $h_{\mathfrak{g}}(N) \cong F(N)$, using the observation in the proof of Lemma 2.20 that all non-empty fibres over small extensions $I \to N' \to N$ are principal homogeneous $t_F(I)$-spaces.

Necessity of the conditions follows by a formal argument. $\qquad\square$

Definition 2.25. $F : \mathcal{N}(G)_{\mathcal{L}} \to$ Set is homogeneous if

$$\eta : F(N' \times_N N'') \to F(N') \times_{F(N)} F(N'')$$

is an isomorphism for every $N' \twoheadrightarrow N$.

Note that a homogeneous functor satisfies conditions (H1), (H2) and (H4).

Definition 2.26. $F : \mathcal{N}(G)_{\mathcal{L}} \to$ Set is a deformation functor if:

1. η is surjective whenever $N' \twoheadrightarrow N$.

2. η is an isomorphism whenever $N = 0$.

Note that a deformation functor satisfies conditions (H1) and (H2).

The following results are adapted from[Man99]:

Definition 2.27. Given $F : \mathcal{N}_{\mathcal{L},k} \to \mathrm{Set}$, an obstruction theory (O, o_e) for F consists of an additive functor $O : \mathrm{Rep}(G) \to \mathrm{Vect}$, the obstruction space, together with obstruction maps $o_e : F(N) \to O(I)$ for each small extension

$$e : 0 \to I \to L \to N \to 0,$$

such that:

1. If $\xi \in F(N)$ can be lifted to $F(L)$ then $o_e(\xi) = 0$.

2. For every morphism $\alpha : e \to e'$ of small extensions, we have $o_{e'}(\alpha(\xi)) = O(\alpha)(v_e(\xi))$, for all $\xi \in F(N)$.

An obstruction theory (O, o_e) is called complete if $\xi \in F(N)$ can be lifted to $F(L)$ whenever $o_e(\xi) = 0$.

Proposition 2.28. *(Standard Smoothness Criterion) Given $\phi : F \to E$, with $(O, o_e) \xrightarrow{\phi'} (P, p_e)$ a compatible morphism of obstruction theories, if (O, o_e) is complete, $O \xrightarrow{\phi'} P$ injective, and $t_F \to t_E$ surjective, then ϕ is smooth.*

Proof. [Man99], Proposition 2.17. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

For functors $F : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Set}$ and $E : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Grp}$, we say that E acts on F if we have a functorial group action $E(N) \times F(N) \xrightarrow{*} F(N)$, for each N in $\mathcal{N}(G)$. The quotient functor F/E is defined by $(F/E)(N) = F(N)/E(N)$.

Proposition 2.29. *If $F : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Set}$, a deformation functor, and $E : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Grp}$ a smooth deformation functor, with E acting on F, then $D := F/E$ is a deformation functor, and if $\nu : t_E \to t_F$ denotes $h \mapsto h * 0$, then $t_D = \mathrm{coker}\,\nu$, and the universal obstruction theories of D and F are isomorphic.*

Proof. [Man99], Lemma 2.20. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Proposition 2.30. *For $F : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Set}$ homogeneous, and $E : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Grp}$ a deformation functor, given $a, b \in F(L)$, define $\mathrm{Iso}(a, b) : \mathcal{N}_{L,k} \to \mathrm{Set}$ by*

$$\mathrm{Iso}(a, b)(L \xrightarrow{f} N) = \{g \in E(N) | g * f(a) = f(b)\}.$$

Then $\mathrm{Iso}(a, b)$ is a deformation functor, with tangent space $\ker \nu$ and, if E is moreover smooth, complete obstruction space $\mathrm{coker}\,\nu = t_D$.

Proof. [Man99], Proposition 2.21. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Proposition 2.31. *If E, E' are smooth deformation functors, acting on F, F' respectively, with F, F' homogeneous, $\ker \nu \to \ker \nu'$ surjective, and $\mathrm{coker}\,\nu \to \mathrm{coker}\,\nu'$ injective, then $F/E \to F'/E'$ is injective.*

Proof. [Man99], Corollary 2.22. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

This final result does not have an analogue in [Man99], but proves extremely useful:

Corollary 2.32. *If $F : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Set}$ and $E : \mathcal{N}(G)_{\mathcal{L}} \to \mathrm{Grp}$ are deformation functors, with E acting on F, let $D := F/E$, then:*

1. *If E is smooth, then η_D is surjective for every $M \twoheadrightarrow N$ (i.e. D is a deformation functor).*

2. *If F is homogeneous and $\ker \nu = 0$, then η_D is injective for every $M \twoheadrightarrow N$.*

Thus, in particular, F/E will be homogeneous if F is homogeneous, E is a smooth deformation functor and $\ker \nu = 0$.

To summarise the results concerning the pro-representability of the quotient $D = F/E$, we have:

1. If F is a deformation functor and E a smooth deformation functor, then D has a hull.

2. If F is homogeneous and E a smooth deformation functor, with $\ker \nu = 0$, then D is pro-representable.

3 Twisted differential graded algebras

Throughout this section, we will adopt the conventions of [DMOS82] concerning tensor categories. In particular, the associativity isomorphisms will be denoted

$$\phi_{UVW} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W,$$

and the commutativity isomorphisms

$$\psi_{UV} : U \otimes V \to V \otimes U.$$

Definition 3.1. The category DGVect of graded real vector spaces $\bigoplus_{i \geq 0} V^i$ is a tensor category, with the obvious tensor product

$$(U \otimes V)^n = \bigoplus_{i+j=n} U^i \otimes V^j,$$

with differential $d|_{U^i \otimes V^j} = d_{U^i} \otimes \mathrm{id} + (-1)^j \mathrm{id} \otimes d_{V^j}$. The associativity map is the obvious one, while the commutativity map is

$$\begin{aligned} \psi_{UV} : U \otimes V &\to V \otimes U, \\ u \otimes v &\mapsto (-1)^{ij} v \otimes u, \end{aligned}$$

for $u \in U^i, v \in V^j$.

Definition 3.2. A (real) DGA over a tensor category $\mathcal{C}$ is defined to be an additive functor $A : \mathcal{C} \to \mathrm{DGVect}$, equipped with a multiplication

$$\mu_{UV} : A(U) \otimes A(V) \to A(U \otimes V),$$

functorial in U and V, such that

1. Associativity. The following diagram commutes:

$$AU \otimes (AV \otimes AW) \xrightarrow{\text{id} \otimes \mu} AU \otimes A(V \otimes W) \xrightarrow{\mu} A(U \otimes (V \otimes W))$$

$$\downarrow \phi \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow A\phi$$

$$(AU \otimes AV) \otimes AW \xrightarrow{\mu \otimes \text{id}} A(U \otimes V) \otimes AW \xrightarrow{\mu} A((U \otimes V) \otimes W).$$

2. Commutativity. The following diagram commutes:

$$AU \otimes AV \xrightarrow{\mu} A(U \otimes V)$$

$$\downarrow \psi \qquad\qquad\qquad \downarrow A\psi$$

$$AV \otimes AU \xrightarrow{\mu} A(V \otimes U).$$

Remark 3.3. Note that, if we take $\mathcal{C}$ to be the category of finite-dimensional complex vector spaces, then giving a DGA A over $\mathcal{C}$ is equivalent to giving the differential graded algebra $A(\mathbb{R})$, which motivates the terminology.

Definition 3.4. We say that a DGA A over $\mathcal{C}$ is flat if for every exact sequence $0 \to U \to V \to W \to 0$ in $\mathcal{C}$, the sequence $0 \to AU \to AV \to AW \to 0$ is exact.

Lemma 3.5. *Given a finite-dimensional Lie algebra L with a G-action, the graded vector space $A(L)$ has the natural structure of a differential graded Lie algebra.*

Proof. It suffices to define the Lie bracket. Let it be the composition

$$A(L) \otimes A(L) \xrightarrow{\mu} A(L \otimes L) \xrightarrow{A([,])} A(L).$$

the associativity, commutativity and compatibility axioms are easily verified. $\square$

Definition 3.6. Given a flat DGA A over $\text{Rep}(G)$, the Maurer-Cartan functor $\text{MC}_A : \mathcal{N}(G) \to \text{Set}$ is defined by

$$\text{MC}_A(N) = \{x \in A(N)^1 | dx + \frac{1}{2}[x, x] = 0\}.$$

Observe that for $\omega \in A(N)^1$,

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \Rightarrow (d + \text{ad}_\omega) \circ (d + \text{ad}_\omega) = 0,$$

so $(A(N), [,], d + \text{ad}_\omega)$ is a DGLA.

Definition 3.7. Define the gauge functor $\text{G}_A : \mathcal{N}(G) \to \text{Grp}$ by

$$\text{G}_A(N) = \exp(A(N)^0),$$

noting that nilpotence of N implies nilpotence of $A(N)^0$.

We may now define the DGLA $(A(N))_d$ as in [Man99]:

$$(A(N))_d^i = \begin{cases} (A(N))^1 \oplus \mathbb{R}d & i = 1 \\ (A(N))^i & i \neq 1, \end{cases}$$

with

$$d_d(d) = 0, \quad [d, d] = 0, \quad [d, a]_d = da, \quad \forall a \in (A(N)).$$

Lemma 3.8. $\exp(A(N)^0)$ *commutes with* $[,]$ *when acting on* $(A(N))_d$ *via the adjoint action.*

Corollary 3.9. *Since* $\exp(A(N)^0)$ *preserves* $(A(N)^1) + d \subset (A(N))_d$ *under the adjoint action, and*

$$x \in \mathrm{MC}_A(N) \Leftrightarrow [x + d, x + d] = 0,$$

the adjoint action of $\exp(A(N)^0)$ *on* $A(N)^1 + d$ *induces an action of* $\mathrm{G}_A(N)$ *on* $\mathrm{MC}_A(N)$, *which we will call the gauge action.*

Definition 3.10. $\mathrm{Def}_A = \mathrm{MC}_A/\mathrm{G}_A$, the quotient being given by the gauge action $\alpha(x) = \mathrm{ad}_\alpha(x + d) - d$. Observe that G_A and MC_A are homogeneous. Define the deformation groupoid $\mathfrak{Def}_A$ to have objects MC_A, and morphisms given by G_A.

Now, $t_{\mathrm{G}_A}(V) = A^0(V)$, and $t_{\mathrm{MC}_A}(V) = \mathrm{Z}^1(A(V))$, with action

$$\begin{aligned} t_{\mathrm{G}_A} \times t_{\mathrm{MC}_A} &\to t_{\mathrm{MC}_A}; \\ (b, x) &\mapsto x + db, \text{ so} \end{aligned}$$

$$t_{\mathrm{Def}_A}(V) = \mathrm{H}^1(A(V)).$$

Lemma 3.11. $\mathrm{H}^2(A)$ *is a complete obstruction space for* MC_A.

Proof. Given a small extension

$$e : 0 \to I \to N \to M \to 0,$$

and $x \in \mathrm{MC}_A(M)$, lift x to $\tilde{x} \in A^1(N)$, and let

$$h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in A^2(N).$$

In fact, $h \in A^2(I)$, as $dx + \frac{1}{2}[x, x] = 0$.

Now,

$$dh = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h - \frac{1}{2}[\tilde{x}, \tilde{x}], \tilde{x}] = [h, \tilde{x}] = 0,$$

since $[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$ and $[I, N] = 0$. Let

$$o_e(x) = [h] \in \mathrm{H}^2(A(I)).$$

This is well-defined: if $y = \tilde{x} + z$, for $z \in A^1(K)$, then

$$dy + \frac{1}{2}[y, y] = d\tilde{x} + dz + \frac{1}{2}[\tilde{x}, \tilde{x}] + \frac{1}{2}[z, z] + [\tilde{x}, z] = h + dz,$$

as $[I, N] = 0$.

This construction is clearly functorial, so it follows that $(\mathrm{H}^2(A), o_e)$ is a complete obstruction theory for MC_A. $\square$

Now Proposition 2.29 implies the following:

Theorem 3.12. Def_A *is a deformation functor,* $t_{\mathrm{Def}_A} \cong \mathrm{H}^1(A)$*, and* $\mathrm{H}^2(A)$ *is a complete obstruction theory for* Def_A*.*

The other propositions of Section 2 can be used to prove:

Theorem 3.13. *If* $\phi : A \to B$ *is a morphism of DGAs over* $\mathrm{Rep}(G)$*, and*

$$\mathrm{H}^i(\phi) : \mathrm{H}^i(A) \to \mathrm{H}^i(B)$$

are the induced maps on cohomology, then:

1. *If* $\mathrm{H}^1(\phi)$ *is bijective, and* $\mathrm{H}^2(\phi)$ *injective, then* $\mathrm{Def}_A \to \mathrm{Def}_B$ *is étale.*

2. *If also* $\mathrm{H}^0(\phi)$ *is surjective, then* $\mathrm{Def}_A \to \mathrm{Def}_B$ *is an isomorphism.*

3. *Provided condition 1 holds,* $\mathfrak{Def}_A \to \mathfrak{Def}_B$ *is an equivalence of functors of groupoids if and only if* $\mathrm{H}^0(\phi)$ *is an isomorphism.*

Proof. [Man99], Theorem 3.1, mutatis mutandis. $\square$

Theorem 3.14. *If* $\mathrm{H}^0(A) = 0$*, then* Def_A *is homogeneous.*

Proof. Proposition 2.32. $\square$

Thus, in particular, a quasi-isomorphism of DGAs gives an isomorphism of deformation functors and of deformation groupoids.

Remark 3.15. The category of DGAs over $\mathrm{Rep}(G)$ is, in fact, equivalent to the category of G-equivariant differential graded algebras. Given a DGA A over $\mathrm{Rep}(G)$, we consider the structure sheaf $O(G)$ of G, regarded as a G-representation via the left action. Then $O(G) \in \mathrm{ind}(\mathrm{Rep}(G))$, and we therefore set $B = A(O(G))$, which has a DGA structure arising from the algebra structure on $O(G)$, and a G-action given by the right action on $O(G)$. Conversely, given a G-equivariant DGA B, we define $A(V) := B \otimes^G V$, the subspace of G-invariants of $B \otimes V$.

By Lemma 1.3, the vector space $O(G) \otimes^G V$ is isomorphic to V, with the G-action on $O(G)$ coming from the right action of G. This implies that the functors above define an equivalence. This equivalence will mean that the twisted DGA considered in Section 6 is a model for the schematic homotopy type considered in [KPT08].

4 Relative Malcev completions

Definition 4.1. Given a group Γ with a representation $\rho_0 : \Gamma \to G$ to a reductive real pro-algebraic group, define the functor

$$\mathfrak{R}_{\rho_0} : \mathcal{N}(G) \to \mathrm{Grpd}$$

of deformations of ρ_0 so that the objects of $\mathfrak{R}_{\rho_0}(\mathfrak{u})$ are representations

$$\rho : \Gamma \to \exp(\mathfrak{u}) \rtimes G$$

lifting ρ_0, and isomorphisms are given by the conjugation action of the unipotent group $\exp(\mathfrak{u})$ on $\exp(\mathfrak{u}) \rtimes G$. Explicitly, $u \in \exp(\mathfrak{u})$ maps ρ to $u\rho u^{-1}$.

Lemma 4.2. *1. The functor R_{ρ_0} of isomorphism classes of $\mathfrak{R}_{\rho_0}$ is a deformation functor, with tangent space $V \mapsto \mathrm{H}^1(\Gamma, \rho_0^\sharp V)$ and obstruction space $V \mapsto \mathrm{H}^2(\Gamma, \rho_0^\sharp V)$.*

2. Given $\omega, \omega' \in \mathfrak{R}_{\rho_0}(\mathfrak{g})$, the functor on $\mathcal{N}(G)_{\mathfrak{g}}$ given by

$$\mathfrak{u} \mapsto \mathrm{Iso}_{\mathfrak{R}_{\rho_0}(\mathfrak{u})}(\omega, \omega')$$

is homogeneous, with tangent space

$$V \mapsto \mathrm{H}^0(\Gamma, \rho_0^\sharp V)$$

and obstruction space

$$V \mapsto \mathrm{H}^1(\Gamma, \rho_0^\sharp V).$$

Proposition 4.3. *Let Γ^{alg} be the pro-algebraic completion of Γ, and $\rho_0 : \Gamma \to \Gamma^{\mathrm{red}}$ its reductive quotient. Then the Lie algebra $\mathcal{L}(\mathrm{R}_{\mathrm{u}}(\Gamma^{\mathrm{alg}}))$ of the pro-unipotent radical of Γ^{alg}, equipped with its Γ^{red}-action as in Theorem 1.7, is a hull for the functor R_{ρ_0}.*

Proof. By definition,

$$\begin{aligned}
R_{\rho_0}(U) &= \mathrm{Hom}(\Gamma, U \rtimes \Gamma^{\mathrm{red}})_{\rho_0}/U \\
&= \mathrm{Hom}(\Gamma^{\mathrm{alg}}, U \rtimes \Gamma^{\mathrm{red}})_{\rho_0}/U.
\end{aligned}$$

If we now fix a Levi decomposition $\Gamma^{\mathrm{alg}} \cong \mathrm{R}_{\mathrm{u}}(\Gamma^{\mathrm{alg}}) \rtimes \Gamma^{\mathrm{red}}$, we may rewrite this as

$$\mathrm{Hom}(\mathrm{R}_{\mathrm{u}}(\Gamma^{\mathrm{alg}}) \rtimes \Gamma^{\mathrm{red}}, U \rtimes \Gamma^{\mathrm{red}})_{\rho_0}/U.$$

There is a natural map

$$f : \mathrm{Hom}_{\hat{\mathcal{N}}(\Gamma^{\mathrm{red}})}(\mathrm{R}_{\mathrm{u}}(\Gamma^{\mathrm{alg}}), U) \to \mathrm{Hom}(\mathrm{R}_{\mathrm{u}}(\Gamma^{\mathrm{alg}}) \rtimes \Gamma^{\mathrm{red}}, U \rtimes \Gamma^{\mathrm{red}})_{\rho_0}/U,$$

and we need to show that this map is surjective, and an isomorphism on tangent spaces.

For surjectivity, take

$$\rho : \mathrm{R}_{\mathrm{u}}(\Gamma^{\mathrm{alg}}) \rtimes \Gamma^{\mathrm{red}} \to U \rtimes \Gamma^{\mathrm{red}}$$

lifting ρ_0. Since Γ^{red} is reductive, $\rho(\Gamma^{\mathrm{red}}) \leq U \rtimes \Gamma^{\mathrm{red}}$ must be reductive. But the composition

$$\rho(\Gamma^{\mathrm{red}}) \hookrightarrow U \rtimes \Gamma^{\mathrm{red}} \twoheadrightarrow \Gamma^{\mathrm{red}}$$

is a surjection, and Γ^{red} is also the reductive quotient of $U \rtimes \Gamma^{\mathrm{red}}$, so $\rho(\Gamma^{\mathrm{red}})$ is a maximal reductive subgroup. By the Levi decomposition theorem, maximal reductive subgroups are conjugate under the action of U, so there exists $u \in U$ such that $\mathrm{ad}_u\rho(\Gamma^{\mathrm{red}}) = \Gamma^{\mathrm{red}}$. Now we may replace ρ by $\mathrm{ad}_u\rho$, since they define the same element of $R_{\rho_0}(U)$. As $\mathrm{ad}_u\rho$ preserves Γ^{red}, its restriction $\mathrm{R}_{\mathrm{u}}(\Gamma^{\mathrm{alg}}) \to U$ is Γ^{red}-equivariant, so $\mathrm{ad}_u\rho$ lies in the image of f.

To see that f induces an isomorphism on tangent spaces, we need to show that it is injective whenever U is abelian. This is immediate, since the conjugation action of U on U is then trivial. $\qquad\square$

Remark 4.4. In the terminology of [**?**], $\Gamma^{\mathrm{alg}} \to \Gamma^{\mathrm{red}}$ is the relative Malcev completion of the representation $\Gamma \to \Gamma^{\mathrm{red}}$, so we can regard this section as studying Malcev completions of arbitrary Zariski-dense reductive representations.

Definition 4.5. Given a homomorphism $\theta : G \to H$ of algebraic groups, with H reductive, define $\theta_\sharp : \widehat{\mathcal{N}}(G) \to \widehat{\mathcal{N}}(H)$ to be left adjoint to the restriction map $\theta^\sharp : \widehat{\mathcal{N}}(H) \to \widehat{\mathcal{N}}(G)$, so that

$$\mathrm{Hom}_{\widehat{\mathcal{N}}(G)}(\theta_\sharp L, N) \cong \mathrm{Hom}_{\widehat{\mathcal{N}}(H)}(L, \theta^\sharp N).$$

This left adjoint must exist, since the functor on the right satisfies Schlessinger's conditions.

Lemma 4.6. *If $\Gamma = \Delta \rtimes \Lambda$, such that the adjoint action of Λ on the pro-unipotent completion $\Delta \otimes \mathbb{R}$ is reductive, then*

$$\bar{\rho}_\sharp \mathrm{R_u}(\Gamma^{\mathrm{alg}}) \cong (\Delta \otimes \mathbb{R}) \times \mathrm{R_u}(\Lambda^{\mathrm{alg}}) \in \exp(\widehat{\mathcal{N}}(\Lambda^{\mathrm{red}})),$$

where we write ρ for the composition $\Gamma \to \Lambda^{\mathrm{red}}$, and $\bar{\rho}$ for the quotient representation $\Gamma^{\mathrm{red}} \to \Lambda^{\mathrm{red}}$.

Proof. We use the fact that $\bar{\rho}_\sharp \mathrm{R_u}(\Gamma^{\mathrm{alg}})$ pro-represents the functor $U \mapsto \mathrm{Hom}(\Gamma, U \rtimes \Lambda^{\mathrm{red}})_\rho$, for $U \in \exp(\mathcal{N}(\Lambda^{\mathrm{red}}))$.

A homomorphism $\Gamma \to U \rtimes \Lambda^{\mathrm{red}}$ lifting ρ gives rise to a map $\Delta \otimes \mathbb{R} \to U$, since U is unipotent. It is then clear that

$$\bar{\rho}_\sharp \mathrm{R_u}(\Gamma^{\mathrm{alg}}) \cong (\Delta \otimes \mathbb{R}) \rtimes \mathrm{R_u}(\Lambda^{\mathrm{alg}}).$$

Finally, observe that $\mathrm{R_u}(\Lambda^{\mathrm{alg}})$ acts trivially on $\Delta \otimes \mathbb{R}$, since the action is reductive. $\quad\square$

5 Principal homogeneous spaces

Fix a connected differentiable manifold X. Let $\varpi_1^{\mathrm{red}}(X, x)$ be the reductive quotient of the pro-algebraic real completion of $\pi_1(X, x)$, so that $\mathrm{Rep}(\varpi_1^{\mathrm{red}}(X, x))$ can be regarded as the category of real semisimple $\pi_1(X, x)$-representations. Given such a representation V, let $\mathbb{V}$ denote the corresponding semisimple local system.

Definition 5.1. We may then define a DGA over $\mathrm{Rep}(\varpi_1^{\mathrm{red}}(X, x))$ by

$$A(V) := \Gamma(X, \mathbb{V} \otimes \mathscr{A}^\bullet),$$

where $\mathscr{A}^\bullet$ is the sheaf of real C^∞ forms on X. The multiplication is given by

$$
\begin{aligned}
A(V) \otimes A(W) \quad &\cong \quad \Gamma(X \times X, p_1^*(\mathbb{V} \otimes \mathscr{A}^\bullet) \otimes p_2^*(\mathbb{W} \otimes \mathscr{A}^\bullet)) \\
&\xrightarrow{\Delta^\sharp} \quad \Gamma(X, (\mathbb{V} \otimes \mathscr{A}^\bullet) \otimes (\mathbb{W} \otimes \mathscr{A}^\bullet)) \\
&\to \quad \Gamma(X, (\mathbb{V} \otimes \mathbb{W}) \otimes \mathscr{A}^\bullet),
\end{aligned}
$$

the first map being the Künneth isomorphism, where $\Delta : X \to X \times X$ is the diagonal map, and $p_1, p_2 : X \times X \to X$ the projection maps. The final isomorphism is the composition of the multiplication on $\mathscr{A}^\bullet$ with the relevant associativity and commutativity isomorphisms.

The aim of this section is to prove that the groupoids $\mathfrak{Def}_A(\mathfrak{g})$ are functorially equivalent to the groupoid of $\exp(g)$-torsors, where g is the sheaf of Lie algebras associated to $\mathfrak{g}$.

Definition 5.2. Given a locally constant sheaf $\mathbb{G}$ of groups on X, define $\mathfrak{B}(\mathbb{G})$, the category of $\mathbb{G}$-torsors (or principal homogeneous $\mathbb{G}$-spaces) to consist of sheaves of sets $\mathbb{B}$ on X, together with a multiplication $\mathbb{G} \times \mathbb{B} \to \mathbb{B}$ such that $g \cdot (h \cdot b) = (gh) \cdot b$, with stalks $\mathbb{B}_x$ isomorphic (as $\mathbb{G}_x$-spaces) to $\mathbb{G}_x$.

Lemma 5.3. *There is a canonical morphism $\mathbb{B} : \mathfrak{Def}_A(\mathfrak{g}) \to \mathfrak{B}(\exp(g))$, functorial in $\mathfrak{g} \in \mathcal{N}(\varpi_1^{\mathrm{red}}(X, x))$.*

Proof. Given $\omega \in \mathrm{MC}_A$, let
$$\mathbb{B}_\omega := D^{-1}(\omega),$$
where
$$D : \exp(g \otimes \mathscr{A}^0) \ \to \ g \otimes \mathscr{A}^1$$
$$\alpha \ \mapsto \ d\alpha \cdot \alpha^{-1}.$$

Then $\mathbb{B}_\omega$ is a principal $\exp(\mathfrak{g})$-sheaf on X. $\qquad\square$

Lemma 5.4.   *1. The functor $\mathfrak{g} \mapsto B(\exp(g))$, the set of isomorphism classes of $\mathfrak{B}(\exp(g))$, is a deformation functor with tangent space*
$$V \mapsto \mathrm{H}^1(X, \mathbb{V}),$$
and obstruction space
$$V \mapsto \mathrm{H}^2(X, \mathbb{V}).$$

2. Given $\omega, \omega' \in \mathfrak{B}(\exp(g))$, the functor on $\mathcal{N}(\varpi_1^{\mathrm{red}}(X, x))_\mathfrak{g}$ given by
$$\mathfrak{h} \mapsto \mathrm{Iso}_{\mathfrak{B}(\exp(h))}(\omega, \omega')$$
is homogeneous, with tangent space
$$V \mapsto \mathrm{H}^0(X, \mathbb{V})$$
and obstruction space
$$V \mapsto \mathrm{H}^1(X, \mathbb{V}).$$

Proof. Take an cover $\{U_i\}$ of X by open discs. Then a $\mathbb{V}$-torsor $\mathbb{B}$ is determined by fixing isomorphisms $\mathbb{V}|_{U_i} \cong \mathbb{B}|_{U_i}$ and specifying transition maps in $\mathbb{V}_{U_i \cap U_j}$ satisfying the cocycle condition. The result follows by considering isomorphism classes of these data. $\qquad\square$

Theorem 5.5. *The functor $\mathbb{B}$ is an equivalence of groupoids.*

Proof. We begin by proving essential surjectivity. The morphism $\mathrm{Def}_A(\mathfrak{g}) \to B(\exp(g))$ induces an isomorphism on tangent and obstruction spaces $\mathrm{H}^i(X, \mathbb{V})$, so is étale (by Proposition 2.28). Now, $\mathrm{Iso}(\omega, \omega') \to \mathrm{Iso}(\mathbb{B}_\omega, \mathbb{B}_{\omega'})$ on $\mathcal{N}(G)_\mathfrak{g}$ is similarly étale, so must be an isomorphism, both functors being pro-representable. $\qquad\square$

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We will look at an algebraic interpretation of the groupoids we have been considering.

Lemma 5.6. *If X is a connected differentiable manifold and $\Gamma = \pi_1(X, x)$ is its fundamental group, then there is a canonical equivalence of groupoids*

$$\mathbb{B} : \mathfrak{R}_{\rho_0}(\mathfrak{g}) \to \mathfrak{B}(\exp(\mathfrak{g})),$$

for $\rho_0 : \pi_1(X, x) \to \varpi_1^{\mathrm{red}}(X, x)$, and $\mathfrak{g} \in \mathcal{N}(\varpi_1^{\mathrm{red}}(X, x))$.

Proof. Let $\tilde{X} \xrightarrow{\pi} X$ be the universal covering space of X, on which Γ acts. Then, associated to any representation $\rho : \Gamma \to H$, we have the H-torsor

$$\mathbb{B}_{\rho, H} := (\pi_* H)^{\Gamma, \rho}.$$

Associated to any $\rho : \Gamma \to \exp(\mathfrak{g}) \rtimes G$ lifting ρ_0, we have a representation $\rho : \Gamma \to H$, where $H = \exp(\mathfrak{g}) \rtimes \Gamma$. This gives rise to the H-torsor $\mathbb{B}_{\rho, H}$. Let $\mathbb{B}_\rho := \mathbb{B}_{\rho, H}/\Gamma$ be the quotient sheaf under the Γ-action (using $\Gamma \leq H$). It follows that this is an $\exp(\mathfrak{g})$-torsor.

Finally to see that $\mathbb{B}$ defines an equivalence, observe that the maps on tangent and obstruction spaces are

$$\mathrm{H}^i(\Gamma, V) \to \mathrm{H}^i(X, \mathbb{V}),$$

which are isomorphisms for $i = 0, 1$, and injective for $i = 2$. The equivalence then follows from Proposition 2.28. $\square$

6 Hodge theory

Let X be a compact connected Kähler manifold, and let $\varpi_1^{\mathrm{red}}(X, x)$ be the reductive pro-algebraic completion of $\pi_1(X, x)$. Recall that the DGA A is defined over $\mathrm{Rep}(\varpi_1^{\mathrm{red}}(X, x))$ by

$$A(V) := \Gamma(X, \mathbb{V} \otimes \mathscr{A}^\bullet).$$

Since $\mathrm{Rep}(\varpi_1^{\mathrm{red}}(X, x))$ is a semisimple category, note that all DGAs over $\mathrm{Rep}(\varpi_1^{\mathrm{red}}(X, x))$ are flat.

Theorem 6.1. *The DGA A is formal, i.e. weakly equivalent to its cohomology DGA.*

Proof. As in [Sim92] §1, we have operators $D'.D''$ on $A \otimes \mathbb{C}$, defined in terms of the harmonic metric K. Define $d^{\mathrm{c}} := iD' - iD''$; this is a real operator, and satisfies $dd^{\mathrm{c}} + d^{\mathrm{c}}d = 0$. We then have the following morphisms of DGAs:

$$\mathrm{H}_{d^{\mathrm{c}}}(A)(V) \leftarrow \mathrm{Z}_{d^{\mathrm{c}}}(A)(V) \to (A(V)),$$

where $\mathrm{Z}_{d^{\mathrm{c}}}(A)(V)^n = \ker(d^{\mathrm{c}} : A(V)^n \to A(V)^{n+1})$, with differential d, and $\mathrm{H}_{d^{\mathrm{c}}}(A)(V)$ also has differential d. Since $d^{\mathrm{c}}(a \cup b) = (d^{\mathrm{c}}a) \cup b + (-1)^{\deg a}a \cup (db)$, these are indeed both DGAs. It follows from [Sim92] Lemmas 2.1 and 2.2, using the dd^{c} lemma instead of the $\partial\bar{\partial}$ lemma, that these morphisms are quasi-isomorphisms, and that $d = 0$ on $\mathrm{H}_{d^{\mathrm{c}}}(A)(V)$. $\square$

Remark 6.2. It follows from Remark 3.15 that this is equivalent to [KPT08] Corollary 2.1.3, which states that the complex schematic homotopy type is formal.

Corollary 6.3. *The Lie algebra* $\mathcal{L}(R_u(\varpi_1(X,x)))$ *associated to* $R_u(\varpi_1(X,x))$ *is quadratically presented (i.e. defined by equations of bracket length 2) as an element of* $\widehat{\mathcal{N}}(\varpi_1^{\mathrm{red}}(X,x))$, *and has a weight decomposition (as a pro-vector space), unique up to inner automorphism.*

Proof. The functor $\mathrm{MC}_{\mathrm{H}(A)}$ is homogeneous, hence pro-representable, and $\mathrm{MC}_{\mathrm{H}(A)} \to \mathrm{Def}_{\mathrm{H}(A)}$ is étale, so $\mathrm{MC}_{\mathrm{H}(A)}$ is pro-represented by a hull for $\mathrm{Def}_{\mathrm{H}(A)}$. By Theorem 6.1, $\mathrm{Def}_{\mathrm{H}(A)}$ is isomorphic to Def_A, which by Theorem 5.5 and Lemma 5.6 is isomorphic to R_{ρ_0}. By Lemma 4.3, this has hull $R_u(\varpi_1(X,x))$. Therefore $R_u(\varpi_1(X,x))$ pro-represents $\mathrm{MC}_{\mathrm{H}(A)}$, by the uniqueness of hulls.

Now,
$$\mathrm{MC}_{\mathrm{H}(A)}(\mathfrak{g}) = \{\omega \in \mathrm{H}^1(X, \mathfrak{g}) \,|\, [\omega, \omega] = 0 \in \mathrm{H}^2(X, \mathfrak{g})\}$$

As in Remark 3.15, we may replace A by the $\varpi_1^{\mathrm{red}}(X,x)$-equivariant DGA $B := A(O(\varpi_1^{\mathrm{red}}(X,x)))$. Letting $\mathbb{O}$ denote the ind-local system on X associated to the representation $O(\varpi_1^{\mathrm{red}}(X,x))$, $H(A)$ then corresponds to the $\varpi_1^{\mathrm{red}}(X,x)$-equivariant DGA

$$H(B)^n := \mathrm{H}^n(X, \mathbb{O}).$$

The ind-local system $\mathbb{O}$ was defined using the left $\varpi_1^{\mathrm{red}}(X,x)$-action on $O(\varpi_1^{\mathrm{red}}(X,x))$, and the $\varpi_1^{\mathrm{red}}(X,x)$-action on B is then defined using the right action.

Therefore $H(A)^n(V) = \mathrm{H}^n(X, \mathbb{O}) \otimes^{\varpi_1^{\mathrm{red}}(X,x)} V$, so

$$\mathrm{MC}_{\mathrm{H}(A)}(\mathfrak{g}) = \{\omega \in \mathrm{H}^1(X, \mathbb{O}) \otimes^{\varpi_1^{\mathrm{red}}(X,x)} \mathfrak{g} \,|\, [\omega, \omega] = 0 \in \mathrm{H}^2(X, \mathbb{O}) \otimes^{\varpi_1^{\mathrm{red}}(X,x)} \mathfrak{g}\}.$$

Let
$$H_i := \mathrm{H}^i(X, \mathbb{O})^\vee \in \widehat{\mathrm{Rep}}(\varpi_1^{\mathrm{red}}(X,x)).$$

There are canonical isomorphisms

$$\mathrm{Hom}_{\widehat{\mathcal{N}}(\varpi_1^{\mathrm{red}}(X,x))}(L(H_1), \mathfrak{g}) \cong \mathrm{Hom}_{\widehat{\mathrm{Rep}}(\varpi_1^{\mathrm{red}}(X,x))}(H_1, \mathfrak{g}) \cong \mathrm{H}^1(X, \mathfrak{g}),$$

where L denotes the free pro-nilpotent Lie algebra functor.

Now, the cup product

$$\mathrm{H}^1(X, \mathbb{O}) \otimes \mathrm{H}^1(X, \mathbb{O}) \xrightarrow{\cup} \mathrm{H}^1(X, \mathbb{O})$$

gives a coproduct

$$\Delta : H_2 \to H_1 \hat{\otimes} H_1 \to \overset{2}{\hat{\bigwedge}} H_1 \subset L(H_1).$$

Finally, observe that $\mathrm{MC}_{\mathrm{H}(A)}(\mathfrak{g})$ is isomorphic to the set

$$\{\omega \in \mathrm{Hom}_{\widehat{\mathcal{N}}(\varpi_1^{\mathrm{red}}(X,x))}(L(H_1), \mathfrak{g}) \,|\, \omega \circ \Delta(H_2) = 0\},$$

so

$$\mathcal{L}(R_u(\varpi_1(X,x))) \cong L(H_1)/\Delta(H_2)$$

is a quadratic presentation.

If we set H_1 to have weight -1, and H_2 to have wight -2, then $\mathcal{L}(R_u(\varpi_1(X,x)))$ has a canonical weight decomposition arising from those on H_1 and H_2, since Δ preserves

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the weights. Note that a weight decomposition on a pro-finite-dimensional vector space is an infinite product, rather than an infinite direct sum. This decomposition is only unique up to inner automorphism, since the hull morphism is; this is equivalent to saying that we have not made a canonical choice of Levi decomposition. □

Corollary 6.4. *Let G be an arbitrary reductive real algebraic group, acting on a real unipotent algebraic group U defined by homogeneous equations, i.e. $\mathfrak{u} \cong \mathrm{gr}\mathfrak{u}$ as Lie algebras with G-actions. If*

$$\rho_2 : \pi_1(X, x) \to (U/[U, [U, U]]) \rtimes G$$

is a Zariski-dense representation, then

$$\rho_1 : \pi_1(X, x) \to (U/[U, U]) \rtimes G$$

lifts to a representation

$$\rho : \pi_1(X, x) \to U \rtimes G.$$

Proof. Observe that these representations correspond to $\pi_1(X, x)$-equivariant homomorphisms $R_u(\varpi_1(X, x)) \to U$. Let $\mathfrak{g}$ be the Lie algebra associated to $R_u(\varpi_1(X, x))$. We must show that the surjective map

$$\rho_1 : \mathfrak{g} \to \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$$

lifts to $\mathfrak{u}$. Let $\mathfrak{g} = L(V)/\langle W \rangle$, for $W \subset \wedge^2 V$. Since $V \cong \mathrm{gr}_1\mathfrak{g} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, the morphism ρ_1 gives us a map $\theta : L(V) \to \mathrm{gr}\mathfrak{u}$, so it will suffice to show that θ annihilates W. But $\wedge^2\mathfrak{g} \to \mathrm{gr}_2\mathfrak{u}$ must send $u \wedge v$ to $[\rho_1(u), \rho_1(v)] = [\rho_2(u), \rho_2(v)]$, which annihilates W, as required. □

Corollary 6.5. *For semisimple $\pi_1(X, x)$-representations $V_1, \ldots V_n$, with $n \geq 3$, the Massey products*

$$\mathrm{H}^1(\pi_1(X, x), V_1) \otimes \mathrm{H}^1(\pi_1(X, x), V_2) \otimes \ldots \otimes \mathrm{H}^1(V_n) \dashrightarrow \mathrm{H}^2(\pi_1(X, x), V_1 \otimes V_2 \otimes \ldots \otimes V_n)$$

are all zero.

Proof. This follows from the observation that these maps all arise as quotients of higher obstruction maps for quotients of $L(V_1 \oplus V_2 \oplus \ldots \oplus V_n)$. Alternatively, it can be deduced directly from Theorem 6.1, which implies that all the higher Massey products are zero on the cohomology of X with semisimple coefficients. □

Remarks 6.6. Note that Corollary 6.3 implies the results on the fundamental group of [DGMS75], of [GM88] and of [**?**]. The pro-unipotent completion $\varpi_1(X, x) \otimes \mathbb{R}$ studied in [DGMS75] is just the maximal quotient of $R_u(\varpi_1(X, x))$ on which $\pi_1(X, x)$ acts trivially.

The problem considered in [GM88] (and generalised in [Sim92]) is to fix a reductive representation $\rho_0 : \pi_1(X, x) \to G(\mathbb{R})$, and consider lifts $\rho : \pi_1(X, x) \to G(A)$, for Artinian rings A. The hull of this functor is the functor

$$A \mapsto \mathrm{Hom}_{\pi_1(X, x)}(R_u(\varpi_1(X, x)), \exp(\mathfrak{g} \otimes \mathfrak{m}_A)),$$

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where $\mathfrak{g}$ is the Lie algebra of G, regarded as the adjoint representation. It follows that this hull then has generators $\mathrm{Hom}_{\pi_1(X,x)}(\mathfrak{g}, H_1)$, and relations

$$\mathrm{Hom}_{\pi_1(X,x)}(\mathfrak{g}, H_2) \to S^2 \mathrm{Hom}_{\pi_1(X,x)}(\mathfrak{g}, H_1)$$

given by composing the coproduct and the Lie bracket.

The statement of Corollary 6.3 is equivalent to saying that the relative Malcev completion of any Zariski-dense representation $\rho : \pi_1(X,x) \to G(\mathbb{R})$, for G reductive, is quadratically presented. In [**?**] Theorem 13.14, this is proved only for those ρ which are polarised variations of Hodge structure, and no consequences are given.

Proposition 6.7. *If $\pi_1(X,x) = \Delta \rtimes \Lambda$, with Λ acting reductively on the pro-unipotent completion $\Delta \otimes \mathbb{R}$, then $\Delta \otimes \mathbb{R}$ is quadratically presented.*

Proof. By Lemma 4.6, we know that

$$\rho_\sharp R_u(\varpi_1(X,x)) \cong (\Delta \otimes \mathbb{R}) \times R_u(\Lambda^{\mathrm{alg}}) \in \exp(\widehat{\mathcal{N}}(\Lambda^{\mathrm{red}})),$$

for $\rho : \varpi_1^{\mathrm{red}}(X,x) \to \Lambda^{\mathrm{red}}$. From Theorem 6.3, we know that $R_u(\varpi_1(X,x))$ is quadratically presented, hence so is $\rho_\sharp R_u(\varpi_1(X,x))$.

Now, to give a quadratic presentation for a Lie algebra $\mathfrak{g}$ is equivalent to giving a homomorphism $\theta : \mathrm{gr}(\mathfrak{g}) \to \mathfrak{g}$ from its associated graded Lie algebra such that $\mathrm{gr}(\theta) : \mathrm{gr}(\mathfrak{g}) \to \mathrm{gr}(\mathfrak{g})$ is the identity, provided that $\mathrm{gr}(\mathfrak{g})$ is quadratic as a graded Lie algebra. Now, if $\mathfrak{g} \oplus \mathfrak{h}$ is quadratically presented, then so is $\mathrm{gr}(\mathfrak{g} \oplus \mathfrak{h}) = \mathrm{gr}(\mathfrak{g}) \oplus \mathrm{gr}(\mathfrak{h})$, hence so is $\mathrm{gr}(\mathfrak{g})$. Taking the composition

$$\mathrm{gr}(\mathfrak{g}) \to \mathrm{gr}(\mathfrak{g}) \oplus \mathrm{gr}(\mathfrak{h}) \xrightarrow{\theta} \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}$$

then gives a quadratic presentation for $\mathfrak{g}$.

Combining these results, we see that $\Delta \otimes \mathbb{R}$ must be quadratically presented. $\square$

Remark 6.8. Since [Sim92] Lemma 4.5 states that properly rigid reductive representations underlie variations of Hodge structure, Proposition 6.7 can be deduced directly from [**?**] whenever the composition $\pi_1(X,x) \to \Lambda \to \mathrm{Aut}(\Delta \otimes \mathbb{R})$ is properly rigid and reductive.

Example 6.9. Let $\mathfrak{h} = \mathbb{R}^2 \oplus \mathbb{R}$, with Lie bracket $[\mathfrak{h}, \mathbb{R}] = 0$ and $[u, v] = u \wedge v \in \mathbb{R}$ for $u, v \in \mathbb{R}^2$, so $\exp(\mathfrak{h})$ is isomorphic to the real three-dimensional Heisenberg group. The Campbell-Baker-Hausdorff formula enables us to regard $\exp(\mathfrak{h})$ as the group with underlying set $\mathfrak{h}$ and product $a \cdot b = a + b + \frac{1}{2}[a,b]$, since all higher brackets vanish. It then follows that the lattice

$$H := \exp(\mathbb{Z}^2 \oplus \frac{1}{2}\mathbb{Z}),$$

is closed under this multiplication, so forms a discrete group, with $H \otimes \mathbb{R} = \exp(\mathfrak{h})$. Now, $\mathrm{SL}_2(\mathbb{Z})$ acts on H by the formula:

$$A(v, w) := (Av, (\det A)w) = (Av, w),$$

for $v \in \mathbb{Z}^2$ and $w \in \frac{1}{2}\mathbb{Z}$.

Let $\Lambda := \mathbb{Z}^2 = \mathbb{Z}a \oplus \mathbb{Z}b$ act on H via the homomorphism $\vartheta : \Lambda \to \mathrm{SL}_2(\mathbb{Z})$ given by

$$\vartheta(a) = \vartheta(b) = M := \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

Then the Zariski closure of this representation is isomorphic to $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^*$, being the torus in $\mathrm{SL}_2(\mathbb{R})$ containing M, so the action of Λ on $\mathfrak{h}$ is reductive. Since $\mathfrak{h}$ is not quadratically presented, the group $\Gamma := H \rtimes \mathbb{Z}^2$ cannot be the fundamental group of any compact Kähler manifold.

Note that [DGMS75] cannot be used to exclude this group: since the commutator $[a,(v,0)] = (Mv - v, 0)$ and $M - I$ is non-singular, $[\Gamma, \Delta]$ is of finite index in Δ, so $\Gamma \otimes \mathbb{R} = \Lambda \otimes \mathbb{R}$, which is quadratically presented.

Furthermore, this result cannot be obtained by substituting Remark 6.8 and [**?**] Theorem 13.14 for Corollary 6.3, since ϑ is not rigid.

Alternatively, we could use Corollary 6.4 to prove that Γ is not a Kähler group. Let $G = \mathbb{G}_m(\mathbb{R})$, $\mathfrak{u} = L(\mathbb{R}^2)$ and $U = \exp(\mathfrak{u})$. Observe that $\mathfrak{h} \cong \mathfrak{u}/[\mathfrak{u},[\mathfrak{u},\mathfrak{u}]]$, and let

$$\rho_2 : H \rtimes \Lambda \to \exp(\mathfrak{h}) \rtimes \mathbb{G}_m(\mathbb{R}),$$

be given by combining the standard embedding with ϑ.

Since all triple commutators vanish in H, this does not lift to a representation

$$\rho : H \rtimes \Lambda \to U \rtimes \mathbb{G}_m(\mathbb{R}).$$

References

[DGMS75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[GM88] William M. Goldman and John J. Millson. The deformation theory of representations of fundamental groups of compact Kähler manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (67):43–96, 1988.

[Gro95] Alexander Grothendieck. Technique de descente et théorèmes d'existence en géométrie algébrique. II. Le théorème d'existence en théorie formelle des modules. In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 195, 369–390. Soc. Math. France, Paris, 1995.

[HM69] G. Hochschild and G. D. Mostow. Pro-affine algebraic groups. *Amer. J. Math.*, 91:1127–1140, 1969.

[KPT08] L. Katzarkov, T. Pantev, and B. Toën. Schematic homotopy types and non-abelian Hodge theory. *Compos. Math.*, 144(3):582–632, 2008. arXiv math.AG/0107129 v5.

[Man99] Marco Manetti. Deformation theory via differential graded Lie algebras. In *Algebraic Geometry Seminars, 1998–1999 (Italian) (Pisa)*, pages 21–48. Scuola Norm. Sup., Pisa, 1999. arXiv math.AG/0507284.

[Pri04] J. P. Pridham. The structure of the pro-l-unipotent fundamental group of a smooth variety. arXiv math.AG/0401378 v2, 2004.

[Sch68] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

[Sim92] Carlos T. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, (75):5–95, 1992.