ON ℓ-ADIC PRO-ALGEBRAIC AND RELATIVE PRO-ℓ FUNDAMENTAL GROUPS

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Abstract. We recall ℓ-adic relative Malcev completions and relative pro-ℓ completions of pro-finite groups and homotopy types. These arise when studying unipotent completions of fibres or of normal subgroups. Several new properties are then established, relating to ℓ-adic analytic moduli and comparisons between relative Malcev and relative pro-ℓ completions. We then summarise known properties of Galois actions on the pro-$\mathbb{Q}_\ell$-algebraic geometric fundamental group and its big Malcev completions. For smooth varieties in finite characteristics ($\neq \ell$), these groups are determined as a Galois representations by cohomology of semisimple local systems. Olsson's non-abelian tale-crystalline comparison theorem gives slightly weaker results for varieties over ℓ-adic fields, since the non-abelian Hodge filtration cannot be recovered from cohomology.

Introduction

Given a topological group Γ, an affine group scheme R over $\mathbb{Q}_\ell$, and a continuous Zariski-dense representation $\rho : \Gamma \to R(\mathbb{Q}_\ell)$, the relative Malcev completion $\Gamma^{R,\mathrm{Mal}}$ is the universal pro-unipotent extension $\Gamma^{R,\mathrm{Mal}} \to R$ equipped with a continuous homomorphism $\Gamma \to \Gamma^{R,\mathrm{Mal}}(\mathbb{Q}_\ell)$ extending ρ. Relative Malcev completion was introduced by Hain in [Hai3] for discrete groups, and was extended to pro-finite groups in [Pri3], although the similar notion of weighted completion had already appeared in [HM2]. In Section 1, we recall its main properties and establish several new results.

Relative Malcev completion simultaneously generalises both (unipotent) Malcev completion (take $R = 1$) and the deformation theory of $\mathbb{Q}_\ell$-representations (by restricting the unipotent extensions — see §1.1.3). As an example of its power, consider a semi-direct product $\Gamma = \Delta \ltimes \Lambda$. Unless the action of Δ on Λ is nilpotent, the Malcev completion of Γ can destroy Λ. However, Example 1.18 shows that for suitable R we have

$$\Gamma^{R,\mathrm{Mal}} = \Delta^{R,\mathrm{Mal}} \ltimes \Lambda^{1,\mathrm{Mal}},$$

where $\Lambda^{1,\mathrm{Mal}}$ is the (unipotent) Malcev completion of Λ.

Although relative Malcev completion is right-exact, it is not left-exact. However, there is a theory of relative Malcev homotopy types and higher homotopy groups, as developed in [Pri5] and summarised in §1.2. There is a long exact sequence of homotopy (Theorem 1.17), allowing us to describe the Malcev completion of the kernel of a surjection $\Gamma \twoheadrightarrow \Delta$ in terms of the relative Malcev homotopy types of Γ and Δ. §1.2.3 then establishes criteria for these higher homotopy groups to vanish.

In Section 2, we introduce a new notion, that of relative Malcev completion $\Gamma^{\rho,\mathrm{Mal}}_{\mathbb{Z}_\ell}$ over $\mathbb{Z}_\ell$. This is a canonical $\mathbb{Z}_\ell$-form of the $\mathbb{Q}_\ell$-scheme $\Gamma^{\rho,\mathrm{Mal}}$, and is a strictly finer invariant from which we can recover analytic moduli spaces of Γ-representations over $\mathbb{Q}_\ell$ (Proposition 2.16), rather than just local deformations.

A similar notion to relative Malcev completion is the relative pro-ℓ completion $\Gamma^{(\ell),\rho}$ of [HM3]. For a surjective homomorphism $\bar{\rho} : \Gamma \to \bar{R}$ of pro-finite groups, $\Gamma^{(\ell),\bar{\rho}}$ is the universal pro-ℓ extension of $\bar{R}$ equipped with a continuous homomorphism $\Gamma \to \Gamma^{(\ell),\rho}$

The author is supported by the Engineering and Physical Sciences Research Council [grant number EP/F043570/1].

extending $\bar{\rho}$. It turns out (Proposition 2.9) that $\Gamma^{(\ell,\rho)}$ is in fact the relative $\mathbb{F}_\ell$-Malcev completion of Γ over $\bar{R}$, where the latter is regarded as a pro-finite group scheme over $\mathbb{F}_\ell$.

However, specialisation of $\Gamma^{\rho,\mathrm{Mal}}_{\mathbb{Z}_\ell}$ to $\mathbb{F}_\ell$ does not recover $\Gamma^{(\ell),\rho}$ in general. Instead, we need to look at the specialisation of the relative Malcev homotopy type over $\mathbb{Z}_\ell$, with a universal coefficient theorem giving the required data (Proposition 2.20). In this sense, the homotopy type over $\mathbb{Z}_\ell$ acts as a bridge between relative Malcev and relative pro-ℓ completions.

In the final section, we summarise the main implications of [Pri3], [Ols] and [Pri5] for relative $\mathbb{Q}_\ell$-Malcev completions $\pi_1^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}}$ of geometric fundamental groups. For smooth quasi-projective varieties in finite characteristic, these can be recovered as Galois representations from cohomology of semisimple local systems (Propositions 3.3 and 3.4). Over ℓ-adic local fields, there are similar results (Theorem 3.8) using Olsson's non-abelian étale-crystalline comparison theorem, but for a full description it is necessary to understand the non-abelian Hodge filtration as well. Over global fields K, we just a have a weight filtration on $\pi_1^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}}$, which splits $\mathrm{Gal}(\bar{K}_\mathfrak{p}/K_\mathfrak{p})$-equivariantly for each prime $\mathfrak{p}$ of good reduction (§3.4).

Notation. For any affine scheme Z, we write $O(Z) := \Gamma(Z, \mathscr{O}_Z)$.

1. Relative Malcev completion

Fix a topological group Γ, an affine group scheme R over $\mathbb{Q}_\ell$, and a continuous Zariski-dense representation $\rho : \Gamma \to R(\mathbb{Q}_\ell)$, where $R(\mathbb{Q}_\ell)$ is given the ℓ-adic topology. Explicitly, [DMOS] Ch. II shows that R can be expressed as a filtered inverse limit $R = \varprojlim R_\alpha$ of linear algebraic groups. Each $R_\alpha(\mathbb{Q}_\ell)$ has a canonical ℓ-adic topology (induced by any embedding $R_\alpha \hookrightarrow \mathrm{GL}_n$), and we define $R(\mathbb{Q}_\ell)$ to be the topological space $\varprojlim R_\alpha(\mathbb{Q}_\ell)$.

The following definition appears in [Hai3] for Γ discrete, and in [Pri3] for Γ pro-finite:

Definition 1.1. Define the Malcev completion $\Gamma^{\rho,\mathrm{Mal}}$ (or $\Gamma^{R,\mathrm{Mal}}$) of Γ relative to ρ to be the universal diagram

$$\Gamma \xrightarrow{\breve{\rho}} \Gamma^{\rho,\mathrm{Mal}}(\mathbb{Q}_\ell) \xrightarrow{p} R(\mathbb{Q}_\ell),$$

with $p : \Gamma^{\rho,\mathrm{Mal}} \to R$ a pro-unipotent extension, and the composition $p \circ \breve{\rho}$ equal to ρ.

If R^{red} is the maximal pro-reductive quotient of R, then $R \to R^{\mathrm{red}}$ is a pro-unipotent extension, so there is a morphism $\Gamma^{R^{\mathrm{red}},\mathrm{Mal}} \to R$. This must itself be a pro-unipotent extension, so we see that $\Gamma^{R^{\mathrm{red}},\mathrm{Mal}} = \Gamma^{R,\mathrm{Mal}}$. For this reason, from now on we will (unless otherwise stated) assume that R is pro-reductive.

Example 1.2. If $R = 1$, then $\Gamma^{1,\mathrm{Mal}}$ is just the (pro-unipotent) Malcev completion of Γ.

Example 1.3. Take Γ^{red} to be universal among Zariski-dense morphisms $\Gamma \to R(\mathbb{Q}_\ell)$ to pro-reductive affine group schemes, and set $R = \Gamma^{\mathrm{red}}$. Then $\Gamma^{R,\mathrm{Mal}} = \Gamma^{\mathrm{alg}}$, the pro-algebraic (or Hochschild–Mostow) completion of Γ. The morphism $\Gamma \to \Gamma^{\mathrm{alg}}(\mathbb{Q}_\ell)$ is universal among continuous morphisms from Γ to affine group schemes over $\mathbb{Q}_\ell$.

In fact, we can describe $O(\Gamma^{\mathrm{red}})$ explicitly: if T is the set of isomorphism classes of irreducible representations of Γ over $\bar{\mathbb{Q}}_\ell$, then $O(\Gamma^{\mathrm{red}}) \otimes \bar{\mathbb{Q}}_\ell \cong \bigoplus_{V \in T} \mathrm{End}(V)$ as a vector space. For example, $O(\hat{\mathbb{Z}}^{\mathrm{red}}) = \bar{\mathbb{Q}}_\ell[\bar{\mathbb{Z}}_\ell{}^*]^{\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$.

Note that since $\mathbb{Q}_\ell$ is of characteristic 0, there is a Levi decomposition

$$\Gamma^{R,\mathrm{Mal}} \cong R \ltimes \mathrm{R_u}(\Gamma^{R,\mathrm{Mal}}),$$

unique up to conjugation by $\mathrm{R_u}(\Gamma^{R,\mathrm{Mal}})$, where $\mathrm{R_u}(\Gamma^{R,\mathrm{Mal}}) := \ker(\Gamma^{R,\mathrm{Mal}} \to R)$ is the pro-unipotent radical.

Lemma 1.4. *The affine group scheme* $\mathrm{R_u}(\Gamma^{R,\mathrm{Mal}})$ *is determined its tangent space* $\mathfrak{r}_u(\Gamma^{R,\mathrm{Mal}})$ *at* 1*, regarded as a pro-(finite dimensional nilpotent) Lie algebra.*

Proof. This is true for all pro-unipotent group schemes U. Given an inverse system $V = \{V_\alpha\}$ of vector spaces, write $V\hat{\otimes}A := \varprojlim_\alpha(V_\alpha \otimes A)$. We may thus regard the Lie algebra $\mathfrak{u}$ of U as an affine scheme, with A-valued points given by $\mathfrak{u}(A) := \mathfrak{u}\hat{\otimes}A$, for any $\mathbb{Q}_\ell$-algebra A. The Lie algebra structure of $\mathfrak{u}(A)$ over A then allows us to define a group $\exp(\mathfrak{u}(A))$ with the same elements as $\mathfrak{u}(A)$, but with multiplication given by the Baker–Campbell–Hausdorff formula.

Exponentiation then gives a canonical isomorphism $U \cong \exp(\mathfrak{u})$ of group schemes; for unipotent U this is standard, and the pro-unipotent case follows by taking inverse limits. □

1.1. **Properties of relative Malcev completion.** We now summarise various properties of relative Malcev completion.

1.1.1. *Representations.*

Lemma 1.5. *The functor* $\mathrm{FDRep}(\Gamma^{R,\mathrm{Mal}}) \to \mathrm{FDRep}(\Gamma)$ *from finite-dimensional* $\Gamma^{R,\mathrm{Mal}}$*-representations to continuous finite-dimensional* Γ*-representations is full and faithful. A* Γ*-representation* V *lies in the essential image of this functor if and only if its semisimplification* V^{ss} *is an* R*-representation — in other words, if the morphism* $\Gamma \to \mathrm{GL}(V^{ss})$ *factors through* $\rho : \Gamma \to R$*.*

Proof. Take an algebraic morphism $\Gamma^{R,\mathrm{Mal}} \to \mathrm{GL}(V)$, and define a decreasing filtration S^pV on V by $S^pV = (\mathrm{R_u}\Gamma^{R,\mathrm{Mal}} - 1)^pV$. Since $\mathrm{R_u}\Gamma^{R,\mathrm{Mal}}$ is pro-unipotent, either $S^pV = 0$ or $\dim S^{p+1}V < \dim S^pV$. Thus the filtration is Hausdorff, and $\mathrm{gr}_S V$ is the semisimplification of V. Since $\mathrm{gr}_S V$ is an R-representation, this establishes essential surjectivity.

Full faithfulness just follows because the map $\Gamma \to \Gamma^{R,\mathrm{Mal}}(\mathbb{Q}_\ell)$ is Zariski-dense — if it were not, then the Zariski closure of its image would have the same universal property, giving a contradiction. □

In particular, $\mathrm{FDRep}(\Gamma^{\mathrm{red}})$ consists of continuous semisimple finite-dimensional Γ-representations, while $\mathrm{FDRep}(\Gamma^{\mathrm{alg}})$ consists of all continuous finite-dimensional Γ-representations.

An arbitrary $\Gamma^{R,\mathrm{Mal}}$-representation V is by definition an $O(\Gamma^{R,\mathrm{Mal}})$-comodule. This is the same as saying that V is a sum of finite-dimensional $\Gamma^{R,\mathrm{Mal}}$-representations (as is true for all affine group schemes — see [DMOS] Ch. II).

1.1.2. *Cohomology.* The following is [Pri3] Lemma 2.3:

Lemma 1.6. *For any finite-dimensional* R*-representation* V*, the canonical maps*

$$\mathrm{H}^i(\Gamma^{R,\mathrm{Mal}}, V) \to \mathrm{H}^i(\Gamma, V),$$

are bijective for $i = 0, 1$ *and injective for* $i = 2$*.*

Note that the long exact sequence of cohomology then implies that the same is true for all finite-dimensional $\Gamma^{R,\mathrm{Mal}}$-representations V.

We may regard $O(R)$ as an R-representation via left multiplication. Applying the Hochschild-Serre spectral sequence (as in [Pri3] Lemma 2.6) to the morphism $\Gamma^{R,\mathrm{Mal}} \to R$ then gives canonical isomorphisms

$$\mathrm{H}^i(\mathrm{R_u}\Gamma^{R,\mathrm{Mal}}, \mathbb{Q}_\ell) \cong \mathrm{H}^i(\Gamma^{R,\mathrm{Mal}}, O(R)).$$

As observed in [Pri3] Lemma 2.7, there are canonical isomorphisms

$$\mathrm{H}^*(\mathrm{R_u}\Gamma^{R,\mathrm{Mal}}, \mathbb{Q}_l) \cong \mathrm{H}^*(\mathfrak{r}_u\Gamma^{R,\mathrm{Mal}}, \mathbb{Q}_l).$$

These results combine to show that there is a presentation of $\mathfrak{r}_u\Gamma^{R,\mathrm{Mal}}$ with generators dual to $\mathrm{H}^1(\Gamma^{R,\mathrm{Mal}}, O(R))$ and relations dual to $\mathrm{H}^2(\Gamma^{R,\mathrm{Mal}}, O(R))$. If $\mathrm{H}^*(\Gamma, -)$ commutes with filtered direct limits, we then have a presentation with generators $\mathrm{H}^1(\Gamma, \rho^{-1}O(R))^\vee$ and relations $\mathrm{H}^2(\Gamma, \rho^{-1}O(R))^\vee$.

1.1.3. *Deformations.* We now show how relative Malcev completions naturally encode all the information about deformations of a representation.

Take a representation $\rho : \Gamma \to \mathrm{GL}(V)$. We consider the formal scheme F_ρ, defined for any Artinian local $\mathbb{Q}_\ell$-algebra A with residue field $\mathbb{Q}_\ell$ by

$$F_\rho(A) = \mathrm{Hom}(\Gamma, \mathrm{GL}(V \otimes A)) \times_{\mathrm{Hom}(\Gamma,\mathrm{GL}(V))} \{\rho\}.$$

Now, $\mathrm{GL}(V \otimes A) = \mathrm{GL}(V) \ltimes (1 + \mathrm{End}(V) \otimes \mathfrak{m}(A))$, where $\mathfrak{m}(A)$ is the maximal ideal of A. If R is the Zariski closure of the image of ρ (which need not be reductive), then

$$\begin{aligned} F_\rho(A) &= \mathrm{Hom}(\Gamma, R \ltimes (1 + \mathrm{End}(V) \otimes \mathfrak{m}(A))) \times_{\mathrm{Hom}(\Gamma,R)} \rho \\ &= \mathrm{Hom}(\Gamma^{R,\mathrm{Mal}}, R \ltimes (1 + \mathrm{End}(V) \otimes \mathfrak{m}(A))) \times_{\mathrm{Hom}(\Gamma,R)} \rho, \end{aligned}$$

since $R \ltimes (1 + \mathrm{End}(V) \otimes \mathfrak{m}(A))$ is a unipotent extension of R.

Applying $\log : 1 + \mathrm{End}(V) \otimes \mathfrak{m}(A) \to \mathrm{End}(V) \otimes \mathfrak{m}(A)$, we see that F_ρ is a formal subscheme contained in the germ at 0 of $O(\Gamma^{R,\mathrm{Mal}}) \otimes \mathrm{End}(V)$, defined by the conditions

$$\exp(f) \cdot \rho \in F_\rho(A) \iff f(a \cdot b) = f(a) \star (\mathrm{ad}_{\rho(a)}(f(b)))$$

for $a, b \in \Gamma^{R,\mathrm{Mal}}$, where $\star$ is the Baker–Campbell–Hausdorff product $a \star b = \log(\exp(a) \cdot \exp(b))$.

Remark 1.7. Note that the same formulae hold if we replace R with any larger quotient of Γ^{alg}. In particular, this means that we can recover F_ρ directly from Γ^{alg}.

There is also a natural conjugation action of the group $1+\mathrm{End}(V)\otimes\mathfrak{m}(A)$ on $O(\Gamma^{R,\mathrm{Mal}})\otimes \mathrm{End}(V) \otimes \mathfrak{m}(A)$, so we can even recover the formal stack

$$A \mapsto [F_\rho(A)/(1 + \mathrm{End}(V) \otimes \mathfrak{m}(A))]$$

of representations modulo infinitesimal inner automorphisms.

Remark 1.8. If we wished to consider representations to an arbitrary linear algebraic group G, then the formulae above adapt, replacing $\mathrm{End}(V)$ with the Lie algebra $\mathfrak{g}$, and $1 + \mathrm{End}(V) \otimes \mathfrak{m}(A)$ with $\ker(G(A) \to G(\mathbb{Q}_\ell)) = \exp(\mathfrak{g} \otimes \mathfrak{m}(A))$.

1.2. **Higher homotopy groups.** Relative Malcev completion was developed in [Pri5] for any pointed pro-finite homotopy type (X, x). Examples of pro-finite homotopy types are the classifying space $B\Gamma$ of a pro-finite group Γ, or Artin and Mazur's pointed étale homotopy type $(Y_{\acute{e}t}, \bar{y})$ of a connected Noetherian scheme Y (as in [AM] or [Fri]). In particular, note that

(1) $\pi_1(B\Gamma) = \Gamma$;
(2) $\pi_n(B\Gamma) = 0$ for all $n > 1$;
(3) $\pi_1(Y_{\acute{e}t}, \bar{y}) = \pi_1^{\acute{e}t}(Y, \bar{y})$;
(4) $\mathrm{H}^*(Y_{\acute{e}t}, F) = \mathrm{H}^*_{\acute{e}t}(Y, F)$ for all finite $\pi_1^{\acute{e}t}(Y, \bar{y})$-representations F in abelian groups.

The first stage in the construction of relative Malcev completion is to form (as in [Pri5] §1) a simplicial pro-finite group $\widehat{G(X, x)}$, based on Kan's loop group construction ([Kan]). It has the following properties:

(1) $\pi_n(\widehat{G(X, x)}) = \pi_{n+1}(X, x)$ for all $n \geq 0$;
(2) $\mathrm{H}^*(\widehat{G(X, x)}, F) = \mathrm{H}^*(X, F)$ for all finite $\pi_1(X, x)$-representations in abelian groups;
(3) for all n, the pro-finite group $\widehat{G(X, x)}$ is freely generated.

In particular, this means that for any pro-finite group Γ, $\widehat{G(B\Gamma)}$ is a free simplicial resolution of Γ in pro-finite groups.

Definition 1.9. Take a pointed pro-finite homotopy type (X, x), a pro-reductive affine group scheme R over $\mathbb{Q}_\ell$, and a Zariski-dense map $\rho : \pi_1(X, x) \to R(\mathbb{Q}_\ell)$. Define the relative Malcev homotopy type $(X, x)^{R,\mathrm{Mal}}$ (or $(X, x)^{\rho,\mathrm{Mal}}$) to be the simplicial affine group scheme over $\mathbb{Q}_\ell$ given by $\widehat{G(X, x)}^{R,\mathrm{Mal}}$ (as in [Pri5] Definition 3.20 and Lemma 1.17).

Define relative Malcev homotopy groups $\varpi_n(X, x)^{R,\mathrm{Mal}}$ by

$$\varpi_n(X, x)^{R,\mathrm{Mal}} := \pi_{n-1}\widehat{G(X, x)}^{R,\mathrm{Mal}}.$$

Relative Malcev homotopy types have the following properties:

(1) $\varpi_1(X, x)^{R,\mathrm{Mal}} = \pi_1(X, x)^{R,\mathrm{Mal}}$;
(2) For $n > 1$, $\varpi_n(X, x)^{R,\mathrm{Mal}}$ is a commutative pro-unipotent group scheme;
(3) For any finite-dimensional $\pi_1(X, x)^{R,\mathrm{Mal}}$-representation V, the map

$$\mathrm{H}^*(X^{R,\mathrm{Mal}}, V) \to \mathrm{H}^*(X, V)$$

(coming from the morphism $\widehat{G(X, x)} \to \widehat{G(X, x)}^{R,\mathrm{Mal}}$) is an isomorphism;
(4) There is a conjugation action of $\varpi_1(X, x)^{R,\mathrm{Mal}}$ on $\varpi_n(X, x)^{R,\mathrm{Mal}}$;
(5) For $m, n > 1$, there is a graded Lie bracket

$$[-, -] : \varpi_m(X, x)^{R,\mathrm{Mal}} \times \varpi_n(X, x)^{R,\mathrm{Mal}} \to \varpi_{m+n-1}(X, x)^{R,\mathrm{Mal}}$$

— the Whitehead bracket.

Remark 1.10. In [Pri2], a category $s\mathcal{E}(R)$ was introduced to model relative Malcev homotopy types over R. Its objects are simplicial diagrams $G_\bullet$ of pro-unipotent extensions $G_n \to R$. A morphism $f : G \to H$ in $s\mathcal{E}(R)$ is said to be a weak equivalence if it induces isomorphisms $\pi_n G \to \pi_n H$ on homotopy groups for all n.

Letting $\mathrm{Ho}_*(s\mathcal{E}(R))$ be the category obtained by formally inverting all weak equivalences in $s\mathcal{E}(R)$, we get a homotopy category of pointed relative Malcev homotopy types, as studied in [Pri4] Theorem 3.28. For unpointed relative Malcev homotopy types, we define $\mathrm{Ho}(s\mathcal{E}(R))$ to have the same objects as $s\mathcal{E}(R)$, but with

$$\mathrm{Hom}_{\mathrm{Ho}(s\mathcal{E}(R))}(G, H) := \mathrm{Hom}_{\mathrm{Ho}_*(s\mathcal{E}(R))}(G, H)/(\mathrm{R_u}H_0),$$

where $\mathrm{R_u}H_0$ acts by conjugation. As we will see in Theorem 1.16, the functor $(X, x) \mapsto \widehat{G(X, x)}^{R,\mathrm{Mal}}$ descends to a functor from unpointed pro-finite homotopy types to $\mathrm{Ho}(s\mathcal{E}(R))$.

A related result is [Pri2] Corollary 3.57, which shows that $\mathrm{Ho}(s\mathcal{E}(R))$ forms a full subcategory of Toën's unpointed schematic homotopy types ([Toë]) over BR. The same argument shows that $\mathrm{Ho}_*(s\mathcal{E}(R))$ forms a full subcategory within pointed schematic homotopy types over BR.

Definition 1.11. Given a graded vector space V, let $\mathrm{Lie}(V)$ be the free graded Lie algebra generated by V, so for a of degree i and b of degree j,

$$[a, b] = (-1)^{ij+1}[b, a].$$

Let $\mathrm{Lie}_r(V) \subset \mathrm{Lie}(V)$ consist of elements of bracket length r in V, so $\mathrm{Lie}(V) = \bigoplus_{r>0} \mathrm{Lie}_r(V)$.

For the ease of stating the next two results, we introduce the notation that Π_1 is the Lie algebra of $\mathrm{R_u}\varpi_1(X, x)^{R,\mathrm{Mal}}$, while $\Pi_n := \varpi_n(X, x)^{R,\mathrm{Mal}}$ for all $n > 1$. These are pro-finite-dimensional vector spaces — note that taking continuous duals $(\varprojlim_\alpha V_\alpha)^\vee = \varinjlim_\alpha V_\alpha^\vee$ gives a contravariant equivalence from pro-finite-dimensional vector spaces to arbitrary vector spaces.

Proposition 1.12. *There is a convergent Adams spectral sequence (in pro-finite-dimensional vector spaces)*

$$E^1_{pq} = (\mathrm{Lie}_{-p}(\tilde{\mathrm{H}}^{*+1}(X, \rho^{-1}O(R))^\vee))_{p+q} \implies \Pi_{p+q},$$

where $\tilde{\mathrm{H}}$ *denotes reduced cohomology.*

Moreover, the differential

$$\begin{aligned} d^1_{-1,q} \colon \tilde{\mathrm{H}}^q(X, \rho^{-1}O(R))^\vee &\to (\bigwedge^2 \tilde{\mathrm{H}}^{*+1}(X, \rho^{-1}O(R))^\vee)_{q-2} \\ &= ((\mathrm{Symm}^2 \tilde{\mathrm{H}}^*(X, \rho^{-1}O(R)))^q)^\vee \end{aligned}$$

is dual to the cup product on $\tilde{\mathrm{H}}^*(X, \rho^{-1}O(R))$.

Proof. This is [Pri2] Proposition 1.12 — the spectral sequence is induced by studying the lower central series filtration on the pro-unipotent radical $\mathrm{R_u}\widehat{G(X,x)}^{R,\mathrm{Mal}}$.

Beware that $\rho^{-1}O(R)$ is here regarded as an ind-object of finite-dimensional local systems, with cohomology calculated accordingly (this is only an issue if $\mathrm{H}^*(X, -)$ does not preserve filtered colimits). □

Theorem 1.13. *There is a canonical convergent reverse Adams spectral sequence*

$$E_1^{pq} = (\mathrm{Symm}^p(\Pi_{*-1}))^{p+q} \implies \mathrm{H}^{p+q}(X, \rho^{-1}O(R)),$$

where Symm *is the symmetric functor on graded vector spaces.*

Proof. This is [Pri2] Theorem 1.53. □

Finally, these combine to give a Hurewicz theorem:

Corollary 1.14. *Let V be the (ind-)local system on X corresponding to the $\pi_1(X, x)$-representation $O(\varpi_1(X, x)^{R,\mathrm{Mal}})$. Then for $n \geq 1$, the following conditions are equivalent*

(1) $\varpi_i(X)^{R,\mathrm{Mal}} = 0$ *for all* $2 \leq i < n$;
(2) $\mathrm{H}^i(X, V) = 0$ *for all* $2 \leq i < n$,

and if either of these conditions holds, then $\varpi_n(X)^{R,\mathrm{Mal}} \cong \mathrm{H}^n(X, V)^\vee$. In particular, this always holds for $n = 2$.

Proof. If $\varpi_1(X^{R,\mathrm{Mal}}) = 1$, then $V = \mathbb{Q}_\ell$ and these results follow by studying the Adams and reverse Adams spectral sequences. For the general result, we replace $X^{R,\mathrm{Mal}}$ with its universal cover $\widetilde{X^{R,\mathrm{Mal}}}$. Explicitly, $\widetilde{X^{R,\mathrm{Mal}}}$ is the homotopy fibre of $X^{R,\mathrm{Mal}}$ over $\varpi_1(X, x)^{R,\mathrm{Mal}}$, given by taking any free resolution of the kernel $\ker(\widehat{G(X,x)}^{R,\mathrm{Mal}} \to \varpi_1(X, x)^{R,\mathrm{Mal}})$.

Now, $\varpi_1(\widetilde{X^{R,\mathrm{Mal}}}) = 1$, and the Hochschild-Serre spectral sequence gives $\mathrm{H}^*(\widetilde{X^{R,\mathrm{Mal}}}, \mathbb{Q}_\ell) = \mathrm{H}^*(X, V)$, so the general results follow from the simply connected case. □

Beware that $\varpi_n(B\Gamma)^{R,\mathrm{Mal}}$ can be non-zero for $n > 1$ — determining when this happens is the purpose of §1.2.3.

1.2.1. *Equivariant cochains.* Given a pro-finite homotopy type (X, x) with $\pi_1(X, x) = \Gamma$, and a continuous Γ-representation Λ in finite rank $\mathbb{Z}_\ell$-modules, [Pri5] Definition 1.21 constructs a cosimplicial ℓ-adic sheaf $\mathscr{C}^\bullet(\Lambda)$. This is an acyclic resolution of Λ, so gives a cosimplicial $\mathbb{Z}_\ell$-module $\mathrm{C}^\bullet(X, \Lambda) := \Gamma(X, \mathscr{C}^\bullet(\Lambda))$ with the property that $\mathrm{H}^*\mathrm{C}^\bullet(X, \Lambda) = \mathrm{H}^*(X, \Lambda)$.

In particular, if $X = Y_{\text{ét}}$, the étale homotopy type of a Noetherian scheme Y, then $\mathrm{C}^\bullet(Y_{\text{ét}}, \Lambda)$ is a model for the ℓ-adic étale Godement resolution of Y with coefficients in Λ. If $X = B\Gamma$, then $\mathrm{C}^\bullet(B\Gamma, \Lambda)$ is just the continuous group cohomology complex

$$\mathrm{C}^n(B\Gamma, \Lambda) = \mathrm{Hom}_{\mathrm{Top}}(\Gamma^n, \Lambda),$$

with its usual operations.

Definition 1.15. Take a pro-finite homotopy type X with $\pi_1(X,x) = \Gamma$, an affine group scheme R over $\mathbb{Q}_\ell$, and a representation $\rho : \Gamma \to R(\mathbb{Q}_\ell)$. Given a finite-dimensional R-representation V, choose a Γ-equivariant $\mathbb{Z}_\ell$-lattice $\Lambda \subset V$, and define the cosimplicial vector space $\mathrm{C}^\bullet(X, \rho^{-1}V)$ by

$$\mathrm{C}^\bullet(X, \rho^{-1}V) = \mathrm{C}^\bullet(X, \rho^{-1}\Lambda) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

If $U = \bigcup_\alpha U_\alpha$ is a nested union of finite-dimensional R-representations, define

$$\mathrm{C}^\bullet(X, \rho^{-1}U) := \bigcup_\alpha \mathrm{C}^\bullet(X, \rho^{-1}U_\alpha).$$

In particular this applies when $U = O(R)$, in which case Proposition 2.2 will provide us with a canonical choice $O(R_{\mathbb{Z}_\ell})$ of lattice.

Theorem 1.16. *The relative Malcev homotopy type $(X,x)^{R,\mathrm{Mal}}$ is determined up to pointed homotopy (i.e. up to unique isomorphism in the category $\mathrm{Ho}_*(s\mathcal{E}(R))$ of Remark 1.10) by the quasi-isomorphism class of the augmented R-equivariant cosimplicial algebra*

$$\mathrm{C}^\bullet(X, \rho^{-1}O(R)) \xrightarrow{x^*} O(R).$$

Up to unpointed homotopy (i.e. up to unique isomorphism in the category $\mathrm{Ho}(s\mathcal{E}(R))$ of Remark 1.10), $(X,x)^{R,\mathrm{Mal}}$ is determined by the quasi-isomorphism class of $\mathrm{C}^\bullet(X, \rho^{-1}O(R))$.

In particular, the relative Malcev homotopy groups $\varpi_n(X,x)^{R,\mathrm{Mal}}$ are functorially determined by the augmented cosimplicial algebra, while the unaugmented algebra determines the $\varpi_n(X,x)^{R,\mathrm{Mal}}$ up to conjugation by $\mathrm{R_u}\varpi_1(X,x)^{R,\mathrm{Mal}}$.

Proof. This is [Pri5] Theorem 3.30, and makes use of a bar construction from cosimplicial algebras to simplicial Lie algebras.

We now sketch a demonstration of how to recover $\varpi_1(X,x)^{R,\mathrm{Mal}}$ in the unpointed case. First note that for a unipotent R-equivariant group scheme U,

$$\mathrm{Hom}(\varpi_1(X,x)^{R,\mathrm{Mal}}, R \ltimes U)_R/U$$

is the set of (isomorphism classes of) $(R \ltimes U)(\mathbb{Q}_\ell)$-torsors T on X for which $T \times_{(R\ltimes U)(\mathbb{Q}_\ell)} R(\mathbb{Q}_\ell)$ is the $R(\mathbb{Q}_\ell)$-torsor T_0 associated to ρ. Here, $\mathrm{Hom}(G_1, G_2)_R$ denotes morphisms over R.

Now, the construction of $\mathrm{C}^\bullet(X, G)$ extends to non-abelian pro-finite groups G, with group cohomology $\mathrm{H}^1(X, G)$ given by $\mathrm{H}^1(X,G) = \mathrm{Z}^1(X,G)/\mathrm{C}^0(X,G)$, where

$$\mathrm{Z}^1(X,G) = \{\omega \in \mathrm{C}^1(X,G) \,:\, \partial^1\omega = (\partial^2\omega)(\partial^0\omega) \in \mathrm{C}^2(X,G)\}$$

(the cocycle condition), and $\mathrm{C}^0(X,G)$ acts by setting $g(\omega) := (\partial^1 g)\omega(\partial^0 g)^{-1}$.

These formulae can be extended from pro-finite groups to ℓ-adic Lie groups, and the set of torsors we want is then

$$\{\omega \in \mathrm{Z}^1(X, R \ltimes U) \,:\, \omega \mapsto T_0 \in \mathrm{Z}^1(X,R)\}/\mathrm{C}^0(X,U).$$

If $\mathfrak{u}$ is the Lie algebra of U, regarded as an R-representation, then it turns out that this is just

$$\{\omega \in \exp(\mathrm{C}^1(X, \rho^{-1}\mathfrak{u})) \,:\, \partial^1\omega = (\partial^2\omega) \star (\partial^0\omega)\}/\exp(\mathrm{C}^0(X, \rho^{-1}\mathfrak{u})),$$

where $\star$ is the Baker–Campbell–Hausdorff product

Since $\mathrm{C}^\bullet(X, \rho^{-1}\mathfrak{u}) = \mathrm{C}^\bullet(X, \rho^{-1}O(R)) \otimes^R \mathfrak{u}$, we have recovered $\mathrm{Hom}(\varpi_1(X,x)^{R,\mathrm{Mal}}, R \ltimes U)_R/U$ from $\mathrm{C}^\bullet(X, \rho^{-1}O(R))$ functorially in U, which amounts to determining $\varpi_1(X,x)^{R,\mathrm{Mal}}$ up to conjugation by $\mathrm{R_u}\varpi_1(X,x)^{R,\mathrm{Mal}}$. □

1.2.2. *The long exact sequence of homotopy.*

Theorem 1.17. *Take a morphism $f:(X,x)\to(Y,y)$ of pro-finite homotopy types which is surjective on fundamental groups. Assume that the homotopy fibre F of f over $\{y\}$ has finite-dimensional cohomology groups $\mathrm{H}^i(F,\mathbb{Q}_\ell)$, and let R be the reductive quotient of the Zariski closure of the homomorphism $\pi_1(Y,y)\to\prod_i \mathrm{GL}(H^i(F,\mathbb{Q}_\ell))$. Then the (unipotent) Malcev homotopy type $(F,x)^{1,\mathrm{Mal}}$ is the homotopy fibre of*

$$(X,x)^{R,\mathrm{Mal}}\to(Y,y)^{R,\mathrm{Mal}}.$$

In particular, there is a long exact sequence

$$\ldots\to\varpi_n(F,x)^{1,\mathrm{Mal}}\to\varpi_n(X,x)^{R,\mathrm{Mal}}\to\varpi_n(Y,y)^{R,\mathrm{Mal}}\to\varpi_{n-1}(X,x)^{1,\mathrm{Mal}}\to$$
$$\ldots\to\varpi_1(F,x)^{1,\mathrm{Mal}}\to\varpi_1(X,x)^{R,\mathrm{Mal}}\to\varpi_1(Y,y)^{R,\mathrm{Mal}}\to 1.$$

Proof. This is a special case of [Pri5] Theorem 3.32. □

If f is a fibration, then the homotopy fibre is just the fibre. One case when this happens is for nerves $B\Delta\to B\Gamma$ of morphisms of pro-finite groups, in which case the (homotopy) fibre is just $B\ker(\Gamma\to\Delta)$. Other cases are when f is the étale homotopy type of a geometric fibration of schemes (in the sense of [Fri] Definition 11.4) — these include smooth projective morphisms, and smooth quasi-projective morphisms where the divisor is transverse to f.

Example 1.18. If $\Gamma=\Delta\ltimes\Lambda$ is a semi-direct product of pro-finite groups with $\mathrm{H}^*(\Lambda,\mathbb{Q}_\ell)$ finite-dimensional, then we may apply the theorem with $X=B\Gamma$, $Y=B\Delta$ and $F=B\Lambda$. Since $\Gamma\to\Delta$ has a section, the connecting homomorphism $\varpi_2(\Delta)^{R,\mathrm{Mal}}\to\Lambda^{1,\mathrm{Mal}}$ is necessarily 0, so we get

$$\Gamma^{R,\mathrm{Mal}}\cong\Delta^{R,\mathrm{Mal}}\ltimes\Lambda^{1,\mathrm{Mal}}.$$

In fact, this even remains true if we take R to be the reductive quotient of the Zariski closure of the homomorphism $\Delta\to\mathrm{GL}(H^1(\Lambda,\mathbb{Q}_\ell))$ ([Pri1] Lemma 4.6).

Example 1.19. For a case where higher homotopy can affect fundamental groups, consider the symplectic group $\Delta:=\mathrm{Sp}_g(\mathbb{Z}_\ell)$ for $g\ge 2$; this has $\Delta^{\mathrm{alg}}=\mathrm{Sp}_g$, which is reductive. Letting $R=\mathrm{Sp}_g$, we get $\mathrm{H}^2(\Delta,O(R))\cong\mathbb{Q}_\ell$ (as effectively calculated in [Hai2], [Hai1] and [HM3]). Thus Corollary 1.14 implies that $\varpi_2(B\Delta)^{R,\mathrm{Mal}}=\mathbb{G}_a$.

For any surjective map $\Gamma\twoheadrightarrow\Delta$ whose kernel Λ has $\mathrm{H}^1(\Lambda,\mathbb{Q}_\ell)$ finite-dimensional, this gives us an exact sequence

$$\mathbb{G}_a\to\Lambda^{1,\mathrm{Mal}}\to\Gamma^{R,\mathrm{Mal}}\to\mathrm{Sp}_g\to 1,$$

confirming the observation in [HM3] Proposition 6.2 that $\ker(\Lambda^{1,\mathrm{Mal}}\to\Gamma^{R,\mathrm{Mal}})$ is at most 1-dimensional, and proving that it is indeed central. Examples of this form arise from taking Γ to be a group (such as a Galois group or the mapping class group) acting on cohomology of a genus g curve.

Example 1.20. If we set $Y=B\pi_1(X,x)$, then F will be the universal covering space of X, a simply connected space with $\pi_n(F,x)=\pi_n(X,x)$ for $n\ge 2$ (if X is an étale homotopy type, these are Artin–Mazur étale homotopy groups). When each $\mathrm{H}^i(F,\mathbb{Q}_\ell)$ is finite-dimensional, Theorem 1.17 gives a long exact sequence

$$\ldots\to\pi_n(X,x)\otimes_{\hat{\mathbb{Z}}}\mathbb{Q}_\ell\to\varpi_n(X,x)^{R,\mathrm{Mal}}\to\varpi_n(B\pi_1(X,x))^{R,\mathrm{Mal}}\to\pi_{n-1}(F,x)\otimes_{\hat{\mathbb{Z}}}\mathbb{Q}_\ell\to$$
$$\ldots\to\pi_2(X,x)\otimes_{\hat{\mathbb{Z}}}\mathbb{Q}_\ell\to\varpi_2(X,x)^{R,\mathrm{Mal}}\to\varpi_2(B\pi_1(X,x))^{R,\mathrm{Mal}}\to 0;$$

see [Pri5] Theorem 3.40 for a refinement.

1.2.3. *Relative goodness.* We now establish criteria for the higher relative Malcev homotopy groups of $B\Gamma$ to vanish.

Definition 1.21. Say that Γ is B_n relative to a continuous Zariski-dense map $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ if $\varpi_i(B\Gamma)^{R,\mathrm{Mal}} = 0$ for all $1 < i \le n$. We say that Γ is good relative to ρ if it is B_n for all n.

By [Pri5] Examples 3.38, the following are good relative to all representations: free pro-finite groups, finitely generated nilpotent pro-finite groups, and étale fundamental groups of smooth projective curves over algebraically closed fields.

Proposition 1.22. *For $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ as above, the following are equivalent:*

(1) Γ *is B_n relative to ρ.*
(2) $\mathrm{H}^i(\Gamma, O(\Gamma^{R,\mathrm{Mal}})) = 0$ *(for $\rho^{-1}O(\Gamma^{R,\mathrm{Mal}}$ interpreted as an ind-finite-dimensional representation) for all $0 < i \le n$.*
(3) *For all finite-dimensional $\Gamma^{R,\mathrm{Mal}}$-representations, $\mathrm{H}^i(\Gamma^{R,\mathrm{Mal}}, V) \to \mathrm{H}^i(\Gamma, V)$ is an isomorphism for all $i \le n$, and injective for $i = n+1$.*
(4) *For all finite-dimensional $\Gamma^{R,\mathrm{Mal}}$-representations, $\mathrm{H}^i(\Gamma^{R,\mathrm{Mal}}, V) \to \mathrm{H}^i(\Gamma, V)$ is surjective for all $i \le n$.*
(5) *For all finite-dimensional $\Gamma^{R,\mathrm{Mal}}$-representations V, all $1 < i \le n$, and any element $x \in \mathrm{H}^i(\Gamma, V)$, there exists an embedding $V \hookrightarrow W_x$ of finite-dimensional $\Gamma^{R,\mathrm{Mal}}$-representations, with x lying in the kernel of $\mathrm{H}^i(\Gamma, V) \to \mathrm{H}^i(\Gamma, W_x)$.*

Proof. This is based on [KPT] Lemma 4.15.

($1 \iff 2$) This follows immediately from the Hurewicz theorem (Corollary 1.14).

($2 \implies 3$) As a $\Gamma^{R,\mathrm{Mal}}$-representation, $V \otimes O(\Gamma^{R,\mathrm{Mal}})$ is injective, so there is a cosimplicial injective resolution $V \otimes O(W\Gamma^{R,\mathrm{Mal}})$ (as in [Pri2] Example 1.45) given by $O(W\Gamma^{R,\mathrm{Mal}}) = O(\Gamma^{R,\mathrm{Mal}})^{\otimes n+1}$ in level n. This gives us spectral sequences

$$E_1^{ij} = \mathrm{H}^i(\Gamma, V \otimes O(W\Gamma^{R,\mathrm{Mal}})^j) \implies \mathrm{H}^{i+j}(\Gamma, V).$$

By hypothesis (2), $E^{ij} = 0$ for $0 < i \le n$. Since $\mathrm{H}^0(\Gamma, -) = \mathrm{H}^0(\Gamma^{R,\mathrm{Mal}}, -)$ and $V \otimes O(W\Gamma^{R,\mathrm{Mal}})$ is an injective resolution, the complex $E_1^{0\bullet}$ computes $\mathrm{H}^*(\Gamma^{R,\mathrm{Mal}}, V)$. Thus $E_2^{0j} = \mathrm{H}^j(\Gamma^{R,\mathrm{Mal}}, V)$, and $E_2^{ij} = 0$ for $0 < i \le n$, implying (3).

($3 \implies 4$) This is immediate.

($4 \implies 2$) Since $O(\Gamma^{R,\mathrm{Mal}})$ is injective as a $\Gamma^{R,\mathrm{Mal}}$-representation, (4) implies that $\mathrm{H}^i(\Gamma, O(\Gamma^{R,\mathrm{Mal}})) = 0$ for $0 < i \le n$.

($4 \implies 5$) By (4), $x \in \mathrm{H}^i(\Gamma, V)$ lifts to $\tilde{x} \in \mathrm{H}^i(\Gamma^{R,\mathrm{Mal}}, V)$. Write $V \otimes O(\Gamma^{R,\mathrm{Mal}}) = \varinjlim_\alpha W_\alpha$ as a union of finite-dimensional subrepresentations. Thus the image of $\tilde{x}$ in $\varinjlim_\alpha \mathrm{H}^i(\Gamma^{R,\mathrm{Mal}}, W_\alpha)$ is 0, so for some W_α, the image of x in $\mathrm{H}^i(\Gamma, W_\alpha)$ is 0.

($5 \implies 4$) We prove this by induction on i, the case $i = 0$ being trivial. Choosing some $x \in \mathrm{H}^i(\Gamma, V)$, it follows from the long exact sequence of cohomology that x lies in the image of the connecting homomorphism $\mathrm{H}^{i-1}(\Gamma, W_x/V) \to \mathrm{H}^i(\Gamma, V)$; let y lie in the pre-image of x. By induction, there exists $\tilde{y} \in \mathrm{H}^{i-1}(\Gamma^{R,\mathrm{Mal}}, W_x/V)$ lying over y. Thus the image of $\tilde{y}$ in $\mathrm{H}^i(\Gamma^{R,\mathrm{Mal}}, V)$ lies over x, giving the required surjectivity.

□

Thus super rigid groups Γ cannot be good for any representation. This is because we necessarily have $\Gamma^{R,\mathrm{Mal}} = R$ for any R as above, so $\mathrm{H}^*(\Gamma^{R,\mathrm{Mal}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$, whereas $\mathrm{H}^*(\Gamma, \mathbb{Q}_\ell)$ has non-trivial higher cohomology. Examples of super rigid groups are $\mathrm{Sp}_g(\mathbb{Z}_\ell)$ for $g \ge 2$, and $\mathrm{SL}_n(\mathbb{Z}_\ell)$ for $n \ge 3$. For these examples, the respective pro-algebraic completions are Sp_g and SL_n (since every $\mathrm{Sp}_g(\mathbb{Z}_\ell)$-representation is an algebraic Sp_g-representation, and likewise for SL_n).

1.3. **Weighted completion.** A closely related notion to relative Malcev completion is that of the weighted completion of $\rho : \Gamma \to R(\mathbb{Q}_\ell)$, developed in [HM2]. This assumes the extra data of a morphism $\mathbb{G}_m \to R$, and agrees with the relative completion if $\mathrm{R_u}(\Gamma^{R,\mathrm{Mal}})^{\mathrm{ab}}$ is of strictly negative weights for the $\mathbb{G}_m$-action. If not, the weighted completion is the largest quotient G of $\Gamma^{R,\mathrm{Mal}}$ on which $\mathrm{R_u}(G)$ is of strictly negative weights.

As shown in [HM1] §7, representations of the weighted completion correspond to Γ-representations equipped with a well-behaved weight filtration.

Example 1.23. Let Γ be the Galois group of the maximal algebraic extension of $\mathbb{Q}$ unramified outside ℓ. Then for the cyclotomic character $\xi : \Gamma \to \mathbb{G}_m(\mathbb{Q}_\ell)$, Hain and Matsumoto proved ([HM2] Theorem 7.3) that the pro-unipotent radical $\mathrm{R_u}(G)$ of weighted completion G of Γ is freely generated by Soulé elements $s_1, s_3, s_5, \ldots$.

In Section 3, we will be establishing weight filtrations on relative completions $\varpi_1(X_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}}$ of geometric fundamental groups, with the pro-unipotent radical being of strictly negative weights. These will therefore correspond to weighted completions whenever the $\mathbb{G}_m$-action is an inner action.

In general, relative completion and weighted completion tend to be applied to different types of groups. As we saw in Theorem 1.17, relative completions of geometric fundamental groups arise when studying fibrations. Galois actions respect the weight filtrations of §3, and often the action on the graded group $\mathrm{gr}_W(\varpi_1(X_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}})$ is pro-reductive and algebraic, so we can set S to be the Zariski closure of $\mathrm{Gal} \to \mathrm{Aut}(\mathrm{gr}_W(\varpi_1(X_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}}))$. Assuming that the canonical weight map $\mathbb{G}_m \to \mathrm{Aut}(\mathrm{gr}_W(\varpi_1(X_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}}))$ factors through S, it then follows that the weighted completion of Gal over S acts on $\varpi_1(X_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}}$.

Remark 1.24. An alternative way to look at weighted completion is to use affine monoid schemes rather than group schemes. Tannakian theory shows that for any exact tensor category $\mathcal{C}$ (not necessarily containing duals) fibred over $\mathbb{Q}_\ell$ vector spaces, there is an affine monoid scheme M such that $\mathrm{FDRep}(M)$is equivalent to $\mathcal{C}$. For instance, the subcategory of $\mathrm{FDRep}(\mathbb{G}_m)$ generated by $\{\mathbb{Q}(n)\}_{n\le 0}$ is just the multiplicative monoid $\mathbb{A}^1$.

Since we just want to work with the category generated by $\{\mathbb{Q}(n)\}_{n<0}$, we can go further, and require that our monoid M contains an element 0, with the property that $0 \cdot g = g \cdot 0 = 0$ for all $g \in M$. Then we define M-representations to be multiplicative morphisms $M \to \mathrm{End}(V)$ preserving 0 and 1, so $\mathbb{A}^1$-representations are strictly negatively weighted vector spaces. For affine group schemes G, the corresponding monoid is then just $G \sqcup \{0\}$.

In the scenario of Theorem 1.17, we would then replace R with the Zariski closure R' of $\mathbb{A}^1 \cup R$ in $\mathrm{End}(\mathrm{H}^{>0}(F, \mathbb{Q}_\ell))$ (for $\mathbb{A}^1$ acting according to the weights on cohomology). Weighted completion of $\pi_1(Y, y)$ can then be interpreted as a kind of (monoidal) relative completion over R'.

An even more efficient choice would be to set R' as the Zariski closure of the monoid $\{0\} \cup \pi_1(Y, y)$, so $\mathrm{FDRep}(R')$ would be the exact tensor subcategory of $\mathrm{FDRep}(\pi_1(Y, y))$ generated by $\mathrm{H}^{>0}(F, \mathbb{Q}_\ell)$, meaning that we only consider local systems of geometric origin (and not their duals).

2. Relative Malcev completion over $\mathbb{Z}_\ell$ and $\mathbb{F}_\ell$

In this section, we introduce canonical $\mathbb{Z}_\ell$-forms for relative Malcev completion, and show how this recovers finer invariants of the fundamental group.

2.1. **Forms defined over $\mathbb{Z}_\ell$.** Now assume that our topological group Γ is compact (e.g. pro-finite).

Definition 2.1. Given a set S, a scheme X over a ring A, and a map $f : S \to X(A)$, say that f is Z-dense if there is no closed subscheme $Y \subsetneq X$ with $f(S) \subset Y(A)$. If A is a field, note that this is equivalent to saying that X is reduced and f is Zariski-dense.

Proposition 2.2. *Given a continuous Zariski-dense group homomorphism*

$$\phi : \Gamma \to G(\mathbb{Q}_\ell)$$

to an affine group scheme G over $\mathbb{Q}_\ell$, there is a model $G_{\mathbb{Z}_\ell}$ for G over $\mathbb{Z}_\ell$, unique subject to the conditions

1. *$\phi : \Gamma \to G_{\mathbb{Z}_\ell}(\mathbb{Q}_\ell)$ factors through $G(\mathbb{Z}_\ell)$;*
2. *If we set $\bar{G} := G_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$, then the morphism $\bar{\phi} : \Gamma \to \bar{G}(\mathbb{F}_\ell)$ (given by reduction* mod ℓ*) is Z-dense.*

Proof. Define a valuation on $O(G)$ by setting $\|f\| = \max_{\gamma\in\Gamma} |f(\phi\gamma)|$, noting that this is well-defined because Γ is compact. The first condition above says that for all $f \in O(G_{\mathbb{Z}_\ell})$, $\|f\| \leq 1$.

The second condition says that the morphism $\psi : O(G_{\mathbb{Z}_\ell}) \to \mathrm{Hom}_{\mathrm{Top}}(\Gamma, \mathbb{F}_\ell)$ to the set of continuous maps $\Gamma \to \mathbb{F}_\ell$ is injective. Considering $\ker \psi$, this is equivalent to saying that

$$\{f \in O(G_{\mathbb{Z}_\ell}) \,:\, \|f\| < 1\} \subset \ell O(G_{\mathbb{Z}_\ell}).$$

The conditions thus force us to set

$$O(G_{\mathbb{Z}_\ell}) = \{f \in O(G) \,:\, \|f\| \leq 1\},$$

and it is straightforward to check that this is indeed a Hopf algebra over $\mathbb{Z}_\ell$. □

Beware that unlike most models over $\mathbb{Z}_\ell$, the affine group scheme $G_{\mathbb{Z}_\ell}$ is seldom of finite type, even when G is so (as we will see in Example 2.5).

Note that we may apply this construction to the representation $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ considered earlier, and even to the universal representation $\breve{\rho}; \Gamma \to \Gamma^{R,\mathrm{Mal}}(\mathbb{Q}_\ell)$. Since the topology on $G(\mathbb{Q}_\ell)$ is totally disconnected, the image of ϕ is pro-finite, so these maps all factor through the pro-finite completion $\hat{\Gamma}$ of the topological group Γ.

Lemma 2.3. *Given $\phi : \Gamma \to G(\mathbb{Q}_\ell)$ as above and an affine group scheme H over $\mathbb{Z}_\ell$, morphisms $\phi : G_{\mathbb{Z}_\ell} \to H$ correspond to morphisms $\psi : G \to H \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ for which $\psi\phi(\Gamma) \subset H(\mathbb{Z}_\ell)$.*

Proof. A Hopf algebra map $\psi^\sharp : O(H) \to O(G_{\mathbb{Z}_\ell})$ is determined by the corresponding map $\psi^\sharp : O(H) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to O(G)$. Now, $\psi^\sharp$ preserves the $\mathbb{Z}_\ell$-models if and only if $\|\psi^\sharp(f)\| \leq 1$ for all $f \in: O(H)$. This is equivalent to saying that $f(\psi\phi\gamma) \in \mathbb{Z}_\ell$ for all $\gamma \in \Gamma$, or equivalently that $\psi\phi(\Gamma) \subset H(\mathbb{Z}_\ell)$. □

In particular, if we take $H = \mathrm{GL}_n$, this describes $G_{\mathbb{Z}_\ell}$-representations in finite free $\mathbb{Z}_\ell$-modules. It also implies that $\Gamma^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell} \to R_{\mathbb{Z}_\ell}$ is the universal pro-unipotent extension under Γ, i.e. the relative Malcev $\mathbb{Z}_\ell$-completion.

Proposition 2.4. *Assume we have a $\mathbb{Z}_\ell$-model H for an affine group scheme G over $\mathbb{Q}_\ell$, and a* surjective *continuous group homomorphism $\phi : \Gamma \to H(\mathbb{Z}_\ell)$ for which the induced map $\phi : \Gamma \to H(\mathbb{Q}_\ell) = G(\mathbb{Q}_\ell)$ is Zariski-dense.*

Then $G_{\mathbb{Z}_\ell}$ is the affine scheme over $\mathbb{Z}_\ell$ given on $\mathbb{Z}_\ell$-algebras A by

$$G_{\mathbb{Z}_\ell}(A) = H(W(A)) \times_{H(A)^{\mathbb{N}_0}} H(A),$$

where $W = W_{\ell^\infty}$ is the Witt vector functor, $w : W(A) \to A^{\mathbb{N}_0}$ is the ghost component morphism, and $H(A) \to H(A)^{\mathbb{N}_0}$ is the diagonal map.[1]

[1]Correction (5/6/19): as in [MRT], this should instead say $G_{\mathbb{Z}_\ell}(A) = H(W(A))^F$, the fixed points of Frobenius.

Explicitly, $O(G_{\mathbb{Z}_\ell})$ is the smallest $\mathbb{Z}_\ell$-subalgebra of $O(G)$ containing $O(H)$ and closed under the operations

$$f \mapsto w^{-1}(f, f, \ldots)_n$$

for all $n \geq 0$, where $w^{-1} : O(G)^{\mathbb{N}_0} \to W(O(G))$ is inverse to w.[2]

Proof. First observe that the functor $G_{\mathbb{Z}_\ell}$ above preserves arbitrary limits. Write $H = \varprojlim H_\alpha$ as a filtered limit of finitely generated affine group schemes, and set $G_{\alpha,n,\mathbb{Z}_\ell}(A) := H_\alpha(W_n(A)) \times_{H_\alpha(A)^{[0,n]}} H_\alpha(A)$. Thus the functor $G_{\alpha,n,\mathbb{Z}_\ell}$ commutes with filtered colimits and arbitrary limits, so is represented by a finitely generated affine group scheme. Since $G_{\mathbb{Z}_\ell} = \varprojlim_{\alpha,n} G_{\alpha,n,\mathbb{Z}_\ell}$, it is also an affine group scheme over $\mathbb{Z}_\ell$.

We need to show that $G_{\mathbb{Z}_\ell}$ satisfies the conditions of Proposition 2.2. The first observation to make is that for $\mathbb{Q}_\ell$-algebras A, the map $w : W(A) \to A^{\mathbb{N}_0}$ is an isomorphism, so $G_{\mathbb{Z}_\ell}(A) = H(A) = G(A)$. Thus $G_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = G$.

We now have to check that $\phi(\Gamma) \subset G_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell)$ in $G(\mathbb{Q}_\ell)$. We know that $w : W(\mathbb{Z}_\ell) \to \mathbb{Z}_\ell^{\mathbb{N}_0}$ is injective (as ℓ is not a zero divisor in $\mathbb{Z}_\ell$), so $W(\mathbb{Z}_\ell) \times_{\mathbb{Z}_\ell^{\mathbb{N}_0}} \mathbb{Z}_\ell \subset \mathbb{Z}_\ell$. However, the ghost component integrality lemma (e.g. [Haz] Lemma 17.6.1) shows that $(b, b, b, \ldots)$ lies in the image of $w : W(\mathbb{Z}_\ell) \to \mathbb{Z}_\ell^{\mathbb{N}_0}$. This means that $W(\mathbb{Z}_\ell) \times_{\mathbb{Z}_\ell^{\mathbb{N}_0}} \mathbb{Z}_\ell = \mathbb{Z}_\ell$, and hence $G_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell) = H(\mathbb{Z}_\ell)$.

The last check is that $\phi(\Gamma) \to \bar{G}(\mathbb{F}_\ell)$ is Z-dense. For this, we first make the definition $\bar{G}_{\alpha,n} := G_{\alpha,n,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$ (for $G_{\alpha,n,\mathbb{Z}_\ell}$ as above) and note that for $n \geq 1$ and A an $\mathbb{F}_\ell$-algebra,[3]

$$W_n(A) \times_{(A)^{[0,n]}} A = W_n(\{a \in A \,:\, a^\ell = a\}).$$

If $\operatorname{Spec} A$ is connected, this is just $W_n(\mathbb{F}_\ell) = \mathbb{Z}/\ell^n$, so $\bar{G}_{\alpha,n}$ is the finite group scheme $H_\alpha(\mathbb{Z}/\ell^n)$. This means that $\bar{G}$ is pro-finite, so $\phi(\Gamma) \to \bar{G}(\mathbb{F}_\ell)$ is Z-dense if and only if it is surjective. But $\bar{G}(\mathbb{F}_\ell) = H(W(\mathbb{F}_\ell)) = H(\mathbb{Z}_\ell)$, and we have surjectivity by hypothesis.

Finally, for the description of $O(G_{\mathbb{Z}_\ell})$, let $D_n(f) := w^{-1}(f, f, \ldots)_n$, and let B be the smallest $\mathbb{Z}_\ell$-subalgebra of $O(G)$ containing $O(H)$ and closed under the operations D_n. Then B is a Hopf algebra over $\mathbb{Z}_\ell$, and for any ring homomorphism $f : O(H) \to \mathbb{Z}_\ell$, there is a unique compatible homomorphism $\tilde{f} : B \to \mathbb{Z}_\ell$, determined by the conditions that $\tilde{f}(D_n a) = D_n f(a)$. Thus $H' := \operatorname{Spec} B$ satisfies the same conditions as H. Now, there is a canonical element

$$\omega \in H'(W(B)) \times_{H'(B)^{\mathbb{N}_0}} H'(B) = G_{\mathbb{Z}_\ell}(B),$$

given by $a \mapsto (\underline{D}a, a)$ for $a \in B$. This amounts to giving a section of $G_{\mathbb{Z}_\ell} \to H'$, so we must have $H' = G_{\mathbb{Z}_\ell}$. ∎

Example 2.5. For $\Gamma = \mathbb{Z}_\ell$ and $G = \mathbb{G}_a$, with $\phi : \mathbb{Z}_\ell \to \mathbb{G}_a(\mathbb{Q}_\ell)$ the standard inclusion $\mathbb{Z}_\ell \hookrightarrow \mathbb{Q}_\ell$, the $\mathbb{Z}_\ell$ form is given by

$$G_{\mathbb{Z}_\ell} = \mathbb{W} \times_{\mathbb{G}_a^{\mathbb{N}_0}} \mathbb{G}_a,$$

where $\mathbb{W}$ is the Witt vector group scheme $\mathbb{W}(A) = W(A)$.[4]

2.2. **Relative Malcev completion over $\mathbb{F}_\ell$.** In fact, relative Malcev completion can be defined over any field, and we now replace $\mathbb{Q}_\ell$ with $\mathbb{F}_\ell$. Assume that we have an affine group scheme $\bar{R}$ over $\mathbb{F}_\ell$ and a continuous Z-dense representation $\bar{\rho} : \Gamma \to \bar{R}(\mathbb{F}_\ell)$, where $\bar{R}(\mathbb{F}_\ell)$ is given the pro-discrete topology. Explicitly, [DMOS] Ch. II shows that $\bar{R}$ can be expressed as a filtered inverse limit $\bar{R} = \varprojlim \bar{R}_\alpha$ of linear algebraic groups. Each $\bar{R}_\alpha(\mathbb{F}_\ell)$ is given the discrete topology, and we define $\bar{R}(\mathbb{F}_\ell)$ to be the topological space $\varprojlim R_\alpha(\mathbb{F}_\ell)$. In particular, this implies that $\bar{R}(\mathbb{F}_\ell)$ is a pro-finite topological group.

[2]The corrected statement is to take closure under the operation $f \mapsto \frac{f^p - f}{p}$.
[3]This expression is incorrect. The corrected statement is that $W(A)^F = W(\{a \in A \,:\, a^\ell = a\})$.
[4]The corrected statement is that $G_{\mathbb{Z}_\ell} = \mathbb{W}^F$.

Definition 2.6. Define the Malcev completion $(\Gamma)^{\bar{\rho},\mathrm{Mal}}$ (or $\Gamma^{\bar{R},\mathrm{Mal}}$) of Γ relative to $\bar{\rho}$ to be the universal diagram

$$\Gamma \to \Gamma^{\bar{\rho},\mathrm{Mal}}(\mathbb{F}_\ell) \xrightarrow{p} \bar{R}(\mathbb{F}_\ell),$$

with $p : \Gamma^{\bar{\rho},\mathrm{Mal}} \to \bar{R}$ a pro-unipotent extension of affine group schemes over $\mathbb{F}_\ell$, and the composition equal to ρ.

There are various ways to prove that this universal object exists. Since the analogue of Lemma 1.5 must also hold over $\mathbb{F}_\ell$, we can characterise the category of $\Gamma^{\bar{\rho},\mathrm{Mal}}$-representations in terms of Γ and $\bar{\rho}$. The Tannakian formalism of [DMOS] Ch. II then uniquely determines $\Gamma^{\bar{\rho},\mathrm{Mal}}$.

Definition 2.7. Given a pro-finite group $\Gamma = \varprojlim \Gamma_\alpha$, define the associated affine group scheme $\Gamma_{\mathbb{F}_\ell}$ over $\mathbb{F}_\ell$ by $\Gamma_{\mathbb{F}_\ell} := \varprojlim(\Gamma_\alpha \times \operatorname{Spec} \mathbb{F}_\ell)$. Explicitly, $O(\Gamma_{\mathbb{F}_\ell})$ consists of continuous functions from Γ to $\mathbb{F}_\ell$, and $\Gamma_{\mathbb{F}_\ell}(U) = \Gamma$ for any connected affine scheme U.

Definition 2.8. Say that an affine group scheme G over $\mathbb{F}_\ell$ is pro-finite if the map $G(\mathbb{F}_\ell)_{\mathbb{F}_\ell} \to G$ is an isomorphism.

This is equivalent to saying that G is a filtered inverse limit of group schemes of the form $F \times \operatorname{Spec} \mathbb{F}_\ell$, for F finite.

From now on, assume that Γ is compact.

Proposition 2.9. *The group schemes $\bar{R}$ and $\Gamma^{\bar{\rho},\mathrm{Mal}}$ are pro-finite. Moreover, $\Gamma^{\bar{\rho},\mathrm{Mal}}(\mathbb{F}_\ell) \to \bar{R}(\mathbb{F}_\ell)$ is the relative pro-ℓ completion $\Gamma^{(\ell),\rho}$ of Γ over $\bar{R}(\mathbb{F}_\ell)$, in the sense of* [HM3].

Proof. The image of $\bar{\rho} : \Gamma \to \bar{R}(\mathbb{F}_\ell)$ is compact and totally disconnected, hence pro-finite. Thus $\bar{\rho}(\Gamma)_{\mathbb{F}_\ell}$ is a closed subscheme of $\bar{R}$ containing the image of Γ, so must equal $\bar{R}$, since $\bar{\rho}$ is Z-dense. The same holds for $\Gamma^{\bar{\rho},\mathrm{Mal}}$ (where the corresponding map is Z-dense by universality).

Now, observe that a pro-finite group scheme over $\mathbb{F}_\ell$ is pro-unipotent if and only if it is a pro-ℓ group. Thus

$$\Gamma \to \Gamma^{\bar{\rho},\mathrm{Mal}}(\mathbb{F}_\ell) \xrightarrow{p} \bar{R}(\mathbb{F}_\ell)$$

is the universal such diagram with p a pro-ℓ extension; in other words, this says that $\Gamma^{\bar{\rho},\mathrm{Mal}}(\mathbb{F}_\ell) = \Gamma^{(\ell),\rho}$. □

Proposition 2.10. *Given an affine group scheme $G_{\mathbb{Z}_\ell}$ over $\mathbb{Z}_\ell$, arising from a continuous Zariski-dense group homomorphism $\phi : \Gamma \to G(\mathbb{Q}_\ell)$ as in Proposition 2.2, set $\bar{G} := G_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$. Then $\phi : \Gamma \to G_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell)$ is surjective, and $\bar{G} = \phi(\Gamma)_{\mathbb{F}_\ell}$.*

Proof. By Proposition 2.2, $\bar{\phi} : \Gamma \to \bar{G}(\mathbb{F}_\ell)$ is Z-dense, so Proposition 2.9 implies that $\bar{G} = G_{\mathbb{Z}_\ell}(\mathbb{F}_\ell)_{\mathbb{F}_\ell}$. we therefore need to show that the maps

$$\phi(\Gamma) \to G_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell) \to G_{\mathbb{Z}_\ell}(\mathbb{F}_\ell)$$

are isomorphisms. Since the first map is injective and the composition surjective, it suffices to show that $G_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell) \to G_{\mathbb{Z}_\ell}(\mathbb{F}_\ell)$ is injective.

If $\epsilon^2 = 0$, there is a ring isomorphism

$$\begin{aligned} (\mathbb{Z}/\ell^{n+1}) \times_{\mathbb{F}_\ell} \mathbb{F}_\ell[\epsilon] \quad &\cong \quad (\mathbb{Z}/\ell^{n+1}) \times_{(\mathbb{Z}/\ell^n)} (\mathbb{Z}/\ell^{n+1}) \\ a + b\epsilon \quad &\mapsto \quad (a, a + \ell^n b). \end{aligned}$$

Thus

$$G_{\mathbb{Z}_\ell}(\mathbb{Z}/\ell^{n+1}) \times_{G_{\mathbb{Z}_\ell}(\mathbb{Z}/\ell^n)} G_{\mathbb{Z}_\ell}(\mathbb{Z}/\ell^{n+1}) \cong G_{\mathbb{Z}_\ell}(\mathbb{Z}/\ell^{n+1}) \times_{\bar{G}(\mathbb{F}_\ell)} \bar{G}(\mathbb{F}_\ell[\epsilon]).$$

Now, since $\bar{G}$ is pro-finite, $\bar{G}(\mathbb{F}_\ell[\epsilon]) \cong \bar{G}(\mathbb{F}_\ell)$, so we have shown that $G_{\mathbb{Z}_\ell}(\mathbb{Z}/\ell^{n+1}) \to G_{\mathbb{Z}_\ell}(\mathbb{Z}/\ell^n)$ is injective. Since $G_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell) = \varprojlim_n G_{\mathbb{Z}_\ell}(\mathbb{Z}/\ell^n)$, this completes the proof. □

Proposition 2.11. *If* Γ *is a freely generated pro-finite group, then the natural morphism*

$$\Gamma^{(\ell),\rho} \to \Gamma^{\rho,\mathrm{Mal}}(\mathbb{Q}_\ell)$$

is injective.

Proof. Let K be the kernel of $\Gamma^{(\ell),\rho} \to \rho(\Gamma)$, and form the lower central series L^nK by setting $L^1K = K$, $L^{n+1}K = [K, L^nK_n]$. Then $\Gamma^{(\ell),\rho} = \varprojlim_n \Gamma/L^nK$, and this maps to $(\Gamma^{\rho,\mathrm{Mal}}/L^n\mathrm{R_u}(\Gamma^{\rho,\mathrm{Mal}}))(\mathbb{Q}_\ell)$. It therefore suffices to show that the associated graded map

$$\prod_{n\geq 1} \mathrm{gr}_L^n K \to \prod_{n\geq 1} \mathrm{gr}_L^n \mathrm{R_u}(\Gamma^{\rho,\mathrm{Mal}})$$

is injective.

Now, $\mathrm{gr}_L\mathrm{R_u}(\Gamma^{\rho,\mathrm{Mal}}) \cong \mathrm{gr}_L\mathfrak{r}_u(\Gamma^{\rho,\mathrm{Mal}})$, which is the free pro-(nilpotent finite-dimensional) Lie algebra generated by $\mathrm{H}^1(\Gamma, \rho^{-1}O(R))^\vee$. Meanwhile, $\mathrm{gr}_L K$ is a Lie ring (topologically) generated by $K/[K,K]$. Thus it suffices to show that the map $K/[K,K] \to \mathrm{H}^1(\Gamma^{\rho,\mathrm{Mal}}, \rho^{-1}O(R))^\vee$ is injective.

It follows from the Hochschild–Serre spectral sequence for $\Gamma \to \rho(\Gamma)$ that

$$K/[K,K] = \mathrm{H}_1(\Gamma^{(\ell),\rho}, \mathbb{Z}_\ell[\rho(\Gamma)]),$$

which is just $\mathrm{H}_1(\Gamma, \mathbb{Z}_\ell[\rho(\Gamma)])$.

Meanwhile, if we write $O(R) = \varinjlim V_\alpha$ for V_α finite-dimensional, then

$$\mathrm{H}^1(\Gamma^{\rho,\mathrm{Mal}}, \rho^{-1}O(R)) = \varinjlim_\alpha \mathrm{H}^1(\Gamma, V_\alpha),$$

so

$$\mathrm{H}^1(\Gamma^{\rho,\mathrm{Mal}}, \rho^{-1}O(R))^\vee = \varprojlim_\alpha \mathrm{H}_1(\Gamma, V_\alpha^\vee) = \mathrm{H}_1(\Gamma, O(R)^\vee),$$

where we regard $O(R)^\vee$ as a pro-finite-dimensional Γ-representation.

Since $\mathbb{Z}_\ell[\rho(\Gamma)]$ embeds into $O(R)^\vee$, we now apply the long exact sequence of homology (with coefficients in pro-abelian groups). As Γ is free, H_2 is identically 0, so $\mathrm{H}_1(\Gamma^{(\ell),\rho}, \mathbb{Z}_\ell[\rho(\Gamma)]) \hookrightarrow \mathrm{H}_1(\Gamma, O(R)^\vee)$, as required. □

Corollary 2.12. *If* Γ *is a freely generated pro-finite group, then*

$$\Gamma^{\rho,\mathrm{Mal}}_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = \Gamma^{\bar{\rho},\mathrm{Mal}}.$$

Proof. By Proposition 2.10, $\Gamma^{\rho,\mathrm{Mal}}_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = \rho(\Gamma)_{\mathbb{F}_\ell}$. By Proposition 2.11, this is $\Gamma^{(\ell),\rho}_{\mathbb{F}_\ell}$, which is $\Gamma^{\bar{\rho},\mathrm{Mal}}$ by Proposition 2.9. □

2.3. **The ℓ-adic analytic moduli space of representations.** The space $\mathrm{Hom}(\Gamma, \mathrm{GL}(V))$ has the structure of an ℓ-adic analytic space. As a set, it is just $\mathrm{Hom}(\Gamma^{\mathrm{alg}}, \mathrm{GL}(V))$, and §1.1.3 shows how infinitesimal neighbourhoods in this space can be recovered from Γ^{alg}. The purpose of this section is to show how the full analytic structure can be recovered from the $\mathbb{Z}_\ell$-form of Γ^{alg}.

Definition 2.13. Given a Zariski-dense morphism $\phi : \Gamma \to G(\mathbb{Q}_\ell)$, define $\widehat{O(G)}$ to be the completion of $O(G)$ with respect to the valuation $\|.\|$ from the proof of Proposition 2.2. Explicitly,

$$\widehat{O(G)} = \varprojlim_n O(G)/\ell^n O(G_{\mathbb{Z}_\ell}).$$

Lemma 2.14. *The canonical morphism* $\widehat{O(G)} \to \mathrm{Hom}_{\mathrm{Top}}(\phi(\Gamma), \mathbb{Q}_\ell)$ *is an isomorphism.*

Proof. We can rewrite this ring homomorphism as
$$(\varprojlim_n O(G_{\mathbb{Z}_\ell})/\ell^n O(G_{\mathbb{Z}_\ell})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to (\varprojlim_n \mathrm{Hom}_{\mathrm{Top}}(\phi(\Gamma), \mathbb{Z}/\ell^n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$
since Γ is compact. It therefore suffices to show that the maps
$$\ell^n O(G_{\mathbb{Z}_\ell})/\ell^{n+1} O(G_{\mathbb{Z}_\ell}) \to \mathrm{Hom}_{\mathrm{Top}}(\phi(\Gamma), \ell^n\mathbb{Z}/\ell^{n+1}\mathbb{Z})$$
are isomorphisms. But Proposition 2.10 gives $O(G_{\mathbb{Z}_\ell}) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell = \mathrm{Hom}_{\mathrm{Top}}(\phi(\Gamma), \mathbb{F}_\ell)$, yielding the required isomorphisms. □

Definition 2.15. Define $\mathcal{R}(G_{\mathbb{Z}_\ell}, \mathrm{GL}(V))$ to be the subset of $\widehat{O(G)} \otimes \mathrm{End}(V)$ consisting of f such that
$$\begin{aligned} \mu(f) &= m(f \otimes f) \in O(\widehat{G \times G}) \otimes \mathrm{End}(V) \\ \varepsilon(f) &= 1 \in \mathrm{End}(V), \end{aligned}$$
where $\mu : O(G) \to O(G) \otimes O(G)$ is the comultiplication, $\varepsilon : O(G) \to \mathbb{Q}_\ell$ the co-unit, and $m : \mathrm{End}(V) \otimes \mathrm{End}(V) \to \mathrm{End}(V)$ multiplication.

Proposition 2.16. $\mathcal{R}(G_{\mathbb{Z}_\ell}, \mathrm{GL}(V))$ *has the natural structure of an ℓ-adic analytic space, isomorphic to* $\mathrm{Hom}(\phi(\Gamma), \mathrm{GL}(V))$.

Proof. This follows immediately from Lemma 2.14. □

Remark 2.17. Note that $\mathrm{Hom}(G, \mathrm{GL}(V))$ is just $\mathcal{R}(G_{\mathbb{Z}_\ell}, \mathrm{GL}(V)) \cap (O(G) \otimes \mathrm{End}(V))$. If $G = \Gamma^{\mathrm{alg}}$ (or any affine group scheme, such as $\hat{\Gamma}$, for which the map $\hat{\Gamma} \to G(\mathbb{Q}_\ell)$ is injective), then this shows that the analytic spaces $\mathrm{Hom}(\Gamma, \mathrm{GL}(V))$ can be recovered directly from $G_{\mathbb{Z}_\ell}$.

If $G = \Gamma^{R,\mathrm{Mal}}$, then for any $\psi \in \mathrm{Hom}(\Gamma^{R,\mathrm{Mal}}, \mathrm{GL}(V))$, the results from §1.1.3 show that the space $\mathcal{R}(G_{\mathbb{Z}_\ell}, \mathrm{GL}(V))$ contains an open neighbourhood of ψ in $\mathrm{Hom}(\Gamma, \mathrm{GL}(V))$.

2.4. **Homotopy types over $\mathbb{F}_\ell$.** For any field k, [KPT] develops a theory of schematic homotopy types over k, using simplicial affine group schemes over k. In many respects, these behave like schematic homotopy types over fields of characteristic 0, except that we no longer have Levi decompositions or the correspondence between unipotent group schemes and nilpotent Lie algebras. This means that although there is not an explicit analogue of Theorem 1.16, equivariant cochains still determine the homotopy type ([KPT] Proposition 3.26).

Definition 2.18. Take a pro-finite homotopy type (X, x), an affine group scheme $\bar{R}$ over $\mathbb{F}_\ell$, and a continuous Z-dense representation $\bar{\rho} : \pi_1(X, x) \to \bar{R}(\mathbb{F}_\ell)$. Define the relative Malcev homotopy type $(X, x)^{\bar{R},\mathrm{Mal}}$ of (X, x) over $\bar{R}$ to be the simplicial affine group scheme
$$\widehat{G(X,x)}^{\bar{R},\mathrm{Mal}} = \widehat{G(X,x)}^{(\ell),\bar{\rho}}$$
(the identification following from Proposition 2.9).

Definition 2.19. Define relative Malcev homotopy groups by
$$\varpi_n(X, x)^{\bar{R},\mathrm{Mal}} := \pi_{n-1}\widehat{G(X,x)}^{\bar{R},\mathrm{Mal}}.$$

Observe that, since relative Malcev completion is right exact,
$$\varpi_1(X, x)^{\bar{R},\mathrm{Mal}} = \pi_1(X, x)^{\bar{R},\mathrm{Mal}}.$$
Theorem 1.17 is also true for relative Malcev homotopy types over $\mathbb{F}_\ell$, since the proof only involves the Hochschild-Serre spectral sequence. Thus the long exact sequence of homotopy allows us to interpret the failure of relative completion to be left exact (as observed in [HM3]) in terms of the non-vanishing of $\varpi_2(B\Gamma)^{\bar{R},\mathrm{Mal}}$.

Now take a Zariski-dense continuous homomorphism $\rho : \pi_1(X,x) \to R(\mathbb{Q}_\ell)$, form $R_{\mathbb{Z}_\ell}$ as in Proposition 2.2, and set $\bar{R} := R_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$. We cannot recover $\pi_1(X,x)^{(\ell),\bar{\rho}}$ from the relative Malcev completion over $\mathbb{Z}_\ell$, since the latter annihilates elements of $\ker \rho$ which are not infinitely ℓ-divisible. However, the following proposition implies that we can recover $\pi_1(X,x)^{(\ell),\bar{\rho}}$ from the $\mathbb{Z}_\ell$ form $\widehat{G(X,x)}^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell}$ of the homotopy type, given by applying Proposition 2.2 levelwise. This can be interpreted as saying that information about non-divisible elements of $\ker \rho$ is encoded by higher homotopy over $\mathbb{Z}_\ell$.

Proposition 2.20. *For $\rho : \pi_1(X,x) \to R(\mathbb{Q}_\ell)$ as above,*

$$\widehat{G(X,x)}^{\bar{R},\mathrm{Mal}} = \widehat{G(X,x)}^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell.$$

This gives an exact sequence

$$\begin{aligned} 0 \to O(\varpi_1(X,x)^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell}) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \to & O(\pi_1(X,x)^{\bar{R},\mathrm{Mal}}) \\ & \to \mathrm{H}^1(O(\widehat{G(X,x)}^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell})) \xrightarrow{\ell} \mathrm{H}^1(O(\widehat{G(X,x)}^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell})). \end{aligned}$$

Proof. Since $\widehat{G_n(X,x)}$ is freely generated as a pro-finite group, it satisfies the hypotheses of Corollary 2.12, giving $\widehat{G_n(X,x)}^{\bar{R},\mathrm{Mal}} = \widehat{G_n(X,x)}^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$ for all n.

This gives an exact sequence

$$0 \to O(\widehat{G(X,x)}^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell}) \xrightarrow{\ell} O(\widehat{G(X,x)}^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell}) \to O(\widehat{G(X,x)}^{\bar{R},\mathrm{Mal}}) \to 0.$$

Applying the long exact sequence of cohomology gives the required result, since $\mathrm{H}^0 O(\widehat{G(X,x)}^{\bar{R},\mathrm{Mal}}) = O(\pi_1(X,x)^{\bar{R},\mathrm{Mal}})$. □

Note that Proposition 1.13 relates $\mathrm{H}^*(O(\widehat{G(X,x)}^{R,\mathrm{Mal}}_{\mathbb{Z}_\ell})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ to the homotopy groups $\varpi_*(X,x)^{R,\mathrm{Mal}}$.

3. Geometric fundamental groups

In this section, we will describe geometric fundamental groups as Galois representations. All relative Malcev completions will be over $\mathbb{Q}_\ell$.

X_0 will be a connected variety over a field k, with $\bar{k}$ an algebraic closure of k, and we write $X = X_0 \otimes_k \bar{k}$. Assume that we have a point $x \in X_0(k)$, with associated geometric point $\bar{x} \in X(\bar{k})$.

3.1. **Weight filtrations.** If X_0 is smooth and quasi-projective, with smooth compactification $j : X_0 \to \bar{X}_0$, then [Pri5] Definition 4.37 gives an associated Leray filtration W (there denoted by J) on $\varpi_1^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}}$, and indeed on $\varpi_n^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}}$ for all n. Explicitly (as in [Pri5] Corollary 6.15), we have a sequence

$$\ldots \le W_{-r}\varpi_*^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}} \le \ldots \le W_0\varpi_*^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}} = \varpi_*^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}}$$

of closed subgroup schemes, with

$$[W_{-r}\varpi_m^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}}, W_{-s}\varpi_n^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}}] \le W_{-r-s}\varpi_{m+n-1}^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}}.$$

If R is a quotient of $\varpi_1^{\text{ét}}(X,\bar{x})$, then the filtration W has the additional property that

$$W_{-1}\varpi_n^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}} = \ker(\varpi_n^{\text{ét}}(X,\bar{x})^{R,\mathrm{Mal}} \to \varpi_n^{\text{ét}}(\bar{X},\bar{x})^{R,\mathrm{Mal}}).$$

The construction of W is based on the idea that the R-equivariant cosimplicial algebra $\mathrm{C}^\bullet(X,\rho^{-1}O(R))$ of §1.2.1 is quasi-isomorphic to the diagonal of the bicosimplicial algebra

$$\mathrm{C}^\bullet(\bar{X}, j_*\mathscr{C}^\bullet(\rho^{-1}O(R))),$$

and that good truncations of $j_*\mathscr{C}^\bullet(\rho^{-1}O(R))$ give an increasing Leray filtration

$$W_0 = \mathrm{C}^\bullet(\bar{X}, j_*\rho^{-1}O(R)) \subset \ldots \subset W_\infty \simeq \mathrm{C}^\bullet(X, \rho^{-1}O(R))$$

by R-equivariant cosimplicial complexes, with $W_i \cdot W_j \subset W_{i+j}$. This filtration is essentially the same as the weight filtration of [Del1] Proposition 3.1.8.

In [Pri5] Theorem 4.22, the bar construction is used to transfer this filtration to a filtration $W_0 \geq W_{-1} \geq \ldots$ by (simplicial) subgroup schemes on the homotopy type $(X_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}}$ and homotopy groups $\varpi_n^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}$, satisfying the conditions above. The rough idea is to adapt Theorem 1.16 to give a functor on negatively filtered Lie algebras, replacing the cosimplicial Lie algebra $\mathrm{C}^\bullet(X, \rho^{-1}\mathfrak{u})$ with

$$W_0\mathrm{C}^\bullet(X, j_*\mathscr{C}^\bullet(\rho^{-1}\mathfrak{u})) := \sum_{i\geq 0} W_i\mathrm{C}^\bullet(X, j_*\mathscr{C}^\bullet(\rho^{-1}O(R))) \otimes^R W_{-i}\mathfrak{u}.$$

Studying the spectral sequence of Proposition 1.12 shows that when R is a quotient of $\varpi_1^{\text{ét}}(X, \bar{x})$, the Leray filtration on $\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}$ is given by

$$\begin{aligned} W_{-1}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} &= \ker(\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} \to \varpi_1^{\text{ét}}(\bar{X}, \bar{x})^{R,\mathrm{Mal}}) \\ W_{-n}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} &= [W_{-1}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}, W_{1-n}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}] \end{aligned}$$

for $n \geq 2$.

In fact, décalage gives another filtration, $\mathrm{Dec}\, W$, on $\mathrm{C}^\bullet(\bar{X}, j_*\mathscr{C}^\bullet(\rho^{-1}O(R)))$, and this is the true weight filtration (cf. [Mor] or [Del2]), in the sense that $\mathrm{H}^a(\bar{X}, \mathbf{R}^b j_*\rho^{-1}O(R))$ has weight $a + 2b$ with respect to $\mathrm{Dec}\, W$, but only weight b with respect to W.

Via the bar construction, this also induces a filtration $(\mathrm{Dec}\, W)_0 \geq (\mathrm{Dec}\, W)_{-1} \geq \ldots$ on the homotopy type and homotopy groups. Beware, however, that décalage does not commute with the bar construction. Studying the spectral sequence of Proposition 1.12 then gives that when R is a quotient of $\varpi_1^{\text{ét}}(X, \bar{x})$,

$$\begin{aligned} (\mathrm{Dec}\, W)_{-1}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} &= \mathrm{R_u}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} \\ (\mathrm{Dec}\, W)_{-2}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} &= \ker(\mathrm{R_u}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} \to (\mathrm{R_u}\varpi_1^{\text{ét}}(\bar{X}, \bar{x})^{R,\mathrm{Mal}})^{\mathrm{ab}}), \end{aligned}$$

with the lower terms determined inductively by the condition that for $n \geq 3$, the subgroup $(\mathrm{Dec}\, W)_{-n}\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}$ is the smallest closed normal subgroup containing $[(\mathrm{Dec}\, W)_{1-n}, (\mathrm{Dec}\, W)_{-1}]$ and $[(\mathrm{Dec}\, W)_{2-n}, (\mathrm{Dec}\, W)_{-2}]$. This filtration is analogous to the weight filtration of [Pri4] Theorem 5.14, which however is only defined for smooth proper complex varieties.

The following sections give circumstances in which the filtration $\mathrm{Dec}\, W$ splits canonically.

3.2. **Finite characteristic, $\ell \neq p$.** In this section, we assume that k is a finite field of characteristic $p \neq \ell$. Fix a Galois-equivariant Zariski-dense representation $\rho : \pi_1^{\text{ét}}(X, \bar{x}) \to R(\mathbb{Q}_\ell)$, where R is a pro-reductive affine group scheme equipped with an algebraic $\mathrm{Gal}(\bar{k}/k)$-action. In other words, $\mathrm{Gal}(\bar{k}/k)^{\mathrm{alg}} \ltimes R$ is a quotient of $\varpi_1^{\text{ét}}(X_0, \bar{x})$.

Example 3.1. To see how such groups R arise naturally, assume that $f_0 : Y_0 \to X_0$ is a smooth proper morphism with connected fibres. Let R be the Zariski closure of the map

$$\pi_1^{\text{ét}}(X, \bar{x}) \to \prod_n \mathrm{Aut}((\mathbf{R}^n f_*^{\text{ét}}\mathbb{Q}_\ell)_{\bar{x}});$$

then R is a pro-reductive affine group scheme satisfying the hypotheses.

Example 3.2. The universal case is given by letting G be the image of the homomorphism $\pi_1^{\text{ét}}(X, \bar{x})^{\mathrm{alg}} \to \pi_1^{\text{ét}}(X_0, \bar{x})^{\mathrm{alg}}$, then setting $R := G^{\mathrm{red}}$, the pro-reductive quotient. In that case, [Pri3] Lemma 1.3 implies that $G = \pi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}$, and that $\pi_1^{\text{ét}}(X_0, \bar{x})^{\mathrm{alg}} = G \rtimes \mathrm{Gal}(\bar{k}/k)^{\mathrm{alg}}$.

The following is [Pri3] Theorem 2.10; see [Pri5] Theorem 6.10 for a generalisation to higher homotopy groups.

Proposition 3.3. *If X is smooth and proper over $\bar{k}$, then there is a unique Galois-equivariant isomorphism*

$$\varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}} \cong R \ltimes \exp(\text{Fr}(\text{H}^1(X,\rho^{-1}O(R))^\vee)/\sim),$$

where Fr *is the free pro-(finite-dimensional nilpotent) Lie algebra functor, and $\sim$ is generated by*

$$\text{H}^2(X,\rho^{-1}O(R))^\vee \xrightarrow{\cup^\vee} \bigwedge^2 \text{H}^1(X,\rho^{-1}O(R))^\vee,$$

the map dual to the cup product.

Proof (sketch). The Galois action on R gives the sheaf $\rho^{-1}O(R)$ the natural structure of a sheaf on X_0. Lafforgue's Theorem ([Laf] Theorem VII.6 and Corollary VII.8), combined with the description of Example 1.3, shows that $\rho^{-1}O(R)$ is pure of weight 0.

By [Del3] Corollaries 3.3.4 – 3.3.6, $\text{H}^n(X,\rho^{-1}O(R))$ is thus pure of weight n, so the spectral sequence of Proposition 1.12 thus degenerates at E_2. This gives a description of all homotopy groups in terms of $\text{H}^*(X,\rho^{-1}O(R))$.

Explicitly, write $\mathfrak{u}$ for the Lie algebra of $\text{R}_{\text{u}}\varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}}$, and note that $\mathfrak{u}^{\text{ab}} \cong \text{H}^1(X,\rho^{-1}O(R))^\vee$, which is pure of weight 1. Now, $\text{H}^2(\mathfrak{u},\mathbb{Q}_\ell) \cong \text{H}^2(X,\rho^{-1}O(R))$, which is pure of weight 2, so the only possible relation defining $\mathfrak{u}$ is

$$\cup^\vee : \text{H}^2(X,\rho^{-1}O(R))^\vee \to \bigwedge^2 \text{H}^1(X,\rho^{-1}O(R))^\vee \subset \text{Fr}(\text{H}^1(X,\rho^{-1}O(R))^\vee).$$

□

This can be used to construct examples of groups which cannot be fundamental groups of any smooth proper variety in finite characteristic (e.g. [Pri3] Example 2.30).

The following specialises [Pri5] Theorem 6.15 and Corollary 6.16 to the case of fundamental groups.

Proposition 3.4. *Assume that $X = \bar{X} - D$ for $\bar{X}$ smooth and proper over $\bar{k}$, with $D \subset \bar{X}$ a divisor locally of normal crossings. If ρ has tame monodromy around the components of D, then there is a Galois-equivariant isomorphism*

$$\varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}} \cong R \ltimes \exp(\text{Fr}(\text{H}^1(\bar{X},j_*\rho^{-1}O(R))^\vee \oplus \text{H}^0(\bar{X},\mathbf{R}^1 j_*\rho^{-1}O(R))^\vee)/\sim),$$

where $\sim$ is generated by

$$\begin{aligned}
\text{H}^2(\bar{X},j_*\rho^{-1}O(R))^\vee &\xrightarrow{(d_2^\vee,\cup^\vee)} \text{H}^0(\bar{X},\mathbf{R}^1 j_*\rho^{-1}O(R))^\vee \oplus \bigwedge^2 \text{H}^1(X,j_*\rho^{-1}O(R))^\vee,\\
\text{H}^1(\bar{X},\mathbf{R}^1 j_*\rho^{-1}O(R))^\vee &\xrightarrow{(\cup^\vee)} \text{H}^0(\bar{X},\mathbf{R}^1 j_*\rho^{-1}O(R))^\vee \otimes \text{H}^1(X,j_*\rho^{-1}O(R))^\vee,\\
\text{H}^0(\bar{X},\mathbf{R}^2 j_*\rho^{-1}O(R))^\vee &\xrightarrow{(\cup^\vee)} \bigwedge^2 \text{H}^0(\bar{X},\mathbf{R}^1 j_*\rho^{-1}O(R))^\vee.
\end{aligned}$$

Here $\cup^\vee$ is the map dual to the cup product, and $d_2^\vee$ is dual to the differential d_2 on the E_2 sheet of the Leray spectral sequence.

Proof (sketch). Again, the Frobenius action on $\rho^{-1}O(R)$ is pure of weight 0, so [Del3] Corollaries 3.3.4 – 3.3.6 imply that $\text{H}^a(\bar{X},\mathbf{R}^b j_*\rho^{-1}O(R))$ is pure of weight $a+2b$. This means that the Leray spectral sequence

$$E_2^{ab} = \text{H}^a(\bar{X},\mathbf{R}^b j_*\rho^{-1}O(R)) \implies \text{H}^{a+b}(X,\rho^{-1}O(R))$$

degenerates at E_3.

Substituting the terms $\text{H}^a(\bar{X},\mathbf{R}^b j_*\rho^{-1}O(R))^\vee$ into the Adams spectral sequence of Proposition 1.12, the relations above turn out to be the only maps compatible with both the Frobenius weights and the Leray filtration W on $\varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}}$. □

Note that the filtration W (resp. $\mathrm{Dec}\, W$) on $\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}$ is then determined by the conditions that $W_{-1}R = (\mathrm{Dec}\, W)_{-1}R = 1$, and that $\mathrm{H}^a(\bar{X}, \mathbf{R}^b j_* \rho^{-1} O(R))^\vee$ is contained in W_{-b} (resp. $(\mathrm{Dec}\, W)_{-a-2b}$), but not in W_{-b-1} (resp. $(\mathrm{Dec}\, W)_{-a-2b-1}$). Thus $\mathrm{Dec}\, W$ is precisely the filtration by weights of Frobenius on the Lie algebra of $\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}$.

Proposition 3.4 can be used to construct examples of groups which cannot be fundamental groups of any smooth quasi-projective variety in finite characteristic (e.g. [Pri3] Example 2.31).

3.3. **Mixed characteristic, $\ell = p$.** In this section, we will assume that X_0 is a connected variety of good reduction over a local field K, with residue field k.

Explicitly, let V be a complete discrete valuation ring, with residue field k (finite, of characteristic p), and fraction field K (of characteristic 0). Let $\bar{k}, \bar{K}$ be the algebraic closures of k, K respectively, and $\bar{V}$ the algebraic closure of V in $\bar{K}$. Write K_0 for the fraction field of $W(k)$.

Assume that we have a scheme $X_V = \bar{X}_V - D_V$ over V, with $\bar{X}_V$ smooth and proper, D_V a normal crossings divisor, and $X_0 = X_V \otimes_V K$. Also fix a basepoint $x_V \in X_V(V)$, giving $x \in X_V(K) = X_0(K)$ and $\bar{x} \in X_0(\bar{K})$. Write $X := X_0 \otimes_K \bar{K}$.

3.3.1. *Crystalline étale sheaves.* We now introduce crystalline étale sheaves, as in [Fal] V(f), [Ols] §6.13, or [AI].

Definition 3.5. Say that a smooth $\mathbb{Q}_p$-sheaf $\mathbb{V}$ on X_K is *crystalline* if it is associated to a filtered convergent F-isocrystal on $(\bar{X}_V, D_V)$.

This means that there exists a filtered convergent F-isocrystal E, and a collection of isomorphisms
$$\iota_U : \mathbb{V} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(\hat{U}) \to E(B_{\mathrm{cris}}(\hat{U}))$$
for $U \to X_V$ étale, compatible with the filtrations and semi-linear Frobenius automorphisms, and with morphisms over X, so that ι becomes an isomorphism of étale presheaves. Here, $B_{\mathrm{cris}}(\hat{U})$ is formed by applying Fontaine's construction to the p-adic completion $\hat{U}$ of U.

By [Pri5] Proposition 7.8, the category of crystalline $\mathbb{Q}_p$-sheaves is closed under extensions and subquotients, and the isocrystal associated to a crystalline $\mathbb{Q}_p$-sheaf is essentially unique. More precisely, association gives a fully faithful functor D_{cris}^X from crystalline $\mathbb{Q}_p$-sheaves to filtered convergent F-isocrystals.

3.3.2. *Structure of fundamental groups.* Now fix a Galois-equivariant Zariski-dense representation $\rho : \pi_1^{\text{ét}}(X, \bar{x}) \to R(\mathbb{Q}_\ell)$, where R is a pro-reductive affine group scheme equipped with an algebraic $\mathrm{Gal}(\bar{k}/k)$-action.

Assume that $D_{\mathrm{cris}}^X \rho^{-1} O(R)$ is an ind-object in the category of ι-pure overconvergent F-isocrystals. This is equivalent to saying that for every R-representation V, the corresponding sheaf $\mathbb{V}$ on $X_{\bar{K}}$ can be embedded in the pullback of a crystalline étale sheaf $\mathbb{U}$ on X_K, associated to an ι-pure overconvergent F-isocrystal on $(\bar{X}_k, D_k)/K$. Also note that this implies that $O(R)$ is a crystalline Galois representation for which the Frobenius action on $D_{\mathrm{cris}} O(R)$ is ι-pure.

Example 3.6. To see how these hypotheses arise naturally, assume that $f_0 : Y_0 \to X_0$ is a smooth proper morphism with connected components, for Y of good reduction. Let R be the Zariski closure of the map
$$\pi_1^{\text{ét}}(X, \bar{x}) \to \prod_n \mathrm{Aut}((\mathbf{R}^n f_*^{\text{ét}} \mathbb{Q}_p)_{\bar{x}}),$$
so R is a pro-reductive affine group scheme. By [Fal], $\mathbf{R}^n f_*^{\text{ét}} \mathbb{Q}_p$ is associated to $\mathbf{R}^n f_{\bar{k},*}^{\mathrm{cris}} \mathscr{O}_{Y_{\bar{k}},\mathrm{cris}}$, which by [Ked] Theorem 6.6.2 is ι-pure. Thus the semisimplifications of

the R-representations $(\mathbf{R}^n f_*^{\text{ét}}\mathbb{Q}_p)_{\bar{x}}$ are direct sums of ι-pure representations. Since these generate the Tannakian category of R-representations, the hypotheses are satisfied.

We may write $F := Y \times_{f,X,\bar{x}} \operatorname{Spec}\bar{K}$, and Theorem 1.17 then shows that the homotopy fibre of

$$Y_{\text{ét}}^{R,\text{Mal}} \to X_{\text{ét}}^{R,\text{Mal}}$$

over $\bar{x}$ is the unipotent Malcev homotopy type $F_{\text{ét}}^{1,\text{Mal}}$.

Definition 3.7. From now on, write $B := B_{\text{cris}}(V)$, with $B^\sigma \subset B$ the invariants under Frobenius.

Theorem 3.8. *For R as above, there is a Galois-equivariant isomorphism*

$$\begin{aligned} &\varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}} \otimes_{\mathbb{Q}_p} B^\sigma \\ \cong\ & (R \ltimes \exp(\mathrm{Fr}(\mathrm{H}^1(\bar{X}, j_*\rho^{-1}O(R))^\vee \oplus \mathrm{H}^0(\bar{X}, \mathbf{R}^1 j_*\rho^{-1}O(R))^\vee)/\sim)) \otimes_{\mathbb{Q}_p} B^\sigma \quad , \end{aligned}$$

of affine group schemes over B^σ, where the relations $\sim$ are defined as in Proposition 3.4.

Proof. This is [Pri5] Theorem 7.35 (or alternatively [Ols] Theorem 7.22 when X is projective), which also has corresponding results for higher homotopy groups and indeed the whole homotopy type. [Ols] 6.8 introduces a ring $\tilde{B} \supset B_{\text{cris}}(V)$ equipped with a Hodge filtration and Galois action. The proof then proceeds by using the results of [Ols], which give a weak equivalence

$$X_{\bar{K},\text{ét}}^{R,\text{Mal}} \otimes_{\mathbb{Q}_p} \tilde{B} \sim X_{\bar{k},\text{cris}}^{D_{\text{cris}}R,\text{Mal}} \otimes_{K_0} \tilde{B},$$

preserving Hodge filtrations and Galois actions. Here, $X_{\bar{k},\text{cris}}^{D_{\text{cris}}R,\text{Mal}}$ is a relative Malcev crystalline homotopy type over K_0; representations of its fundamental group are isocrystals, and its cohomology is crystalline cohomology.

This implies that $O(\varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}})$ is crystalline as a Galois representation, and that

$$D_{\text{cris}}O(\varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}}) = O(\varpi_1^{\text{cris}}(X_{\bar{k}},\bar{x})^{D_{\text{cris}}R,\text{Mal}}).$$

Now, if we write $\mathscr{E}(R) := D_{\text{cris}}^X\rho^{-1}O(R)$, then replacing [Del3] with [Ked], Proposition 3.4 adapts to show that

$$\begin{aligned} &\varpi_1^{\text{cris}}(X_{\bar{k}},\bar{x})^{D_{\text{cris}}R,\text{Mal}} \\ \cong\ & D_{\text{cris}}R \ltimes \exp(\mathrm{Fr}(\mathrm{H}^1_{\text{cris}}(\bar{X}, j_*\mathscr{E}(R))^\vee \oplus \mathrm{H}^0(\bar{X}, \mathbf{R}^1 j_*\mathscr{E}(R) \quad)^\vee)/\sim), \end{aligned}$$

for $\sim$ defined as in Proposition 3.4. This isomorphism is Frobenius-equivariant, but need not respect the Hodge filtration.

The final step is to tensor this isomorphism with B_{cris} and to take Frobenius-invariants, using the comparison above to replace crystalline fundamental groups and cohomology with étale fundamental groups and cohomology. ☐

In fact, [Pri5] Theorem 7.35 also shows that the isomorphism of Theorem 3.8 also holds without having to tensor with B^σ, but at the expense of Galois-equivariance.

Remark 3.9. Although Theorem 3.8 is weaker than Proposition 3.4, it is more satisfactory in one important respect. Proposition 3.4 effectively shows that relative Malcev fundamental groups over $\mathbb{Q}_\ell$ carry no more information than cohomology, whereas to recover relative Malcev fundamental groups over $\mathbb{Q}_p$, we still need to identify $\varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}} \subset \varpi_1^{\text{ét}}(X,\bar{x})^{R,\text{Mal}} \otimes_{\mathbb{Q}_p} B^\sigma$. This must be done by describing the Hodge filtration on $\varpi_1^{\text{cris}}(X_{\bar{k}},\bar{x})^{D_{\text{cris}}R,\text{Mal}}$, which is not determined by cohomology (since it is not Frobenius-equivariant). Thus the Hodge filtration is the only really new structure on the relative Malcev fundamental group.

Remark 3.10. There is a similar Archimedean phenomenon established in [Pri4] §2. If X is a smooth proper variety over $\mathbb{C}$, with R a real affine group scheme and $\rho : \pi_1(X(\mathbb{C}), x) \to R(\mathbb{R})$ Zariski-dense, then we can study the relative Malcev completion $\varpi_1(X, x)^{R,\mathrm{Mal}}$ of the topological fundamental group $\pi_1(X(\mathbb{C}), x)$. If all R-representations underlie variations of Hodge structure, then [Pri4] Theorems 5.14 and 4.20 show that the Hopf algebra $O(\varpi_1(X, x)^{R,\mathrm{Mal}})$ is a sum of real mixed Hodge structures.

If we define $B(\mathbb{R}) := \mathbb{C}[t]$ to be of weight 0, with Hodge filtration given by $\mathrm{Fil}^n B(\mathbb{R}) = (t - i)^n B(\mathbb{R})$, and with σ denoting complex conjugation, then by [Pri4] Theorem 4.21, there is an equivariant isomorphism

$$\varpi_1(X(\mathbb{C}), x)^{R,\mathrm{Mal}} \otimes_{\mathbb{R}} B(\mathbb{R})^\sigma \cong (R \ltimes \exp(\mathrm{Fr}(\mathrm{H}^1(X(\mathbb{C}), \rho^{-1}O(R))^\vee)/\sim)) \otimes_{\mathbb{R}} B(\mathbb{R})^\sigma$$

preserving Hodge and weight filtrations, for $\sim$ as in Theorem 3.3.

If X is the complex form of a real variety X_0, then (by [Pri4] Remark 2.15) this isomorphism is moreover $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant, where the non-trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $B(\mathbb{R})$ as the $\mathbb{C}$-algebra homomorphism determined by $t \mapsto -t$.

3.4. **Global fields.** We now summarise how the previous sections provide information over global fields. Given a smooth quasi-projective variety $X_0 = \bar{X}_0 - D_0$ over a number field K, §3.1 gives a filtration $\mathrm{Dec}\, W$ on $\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}}$. Assume that $\mathrm{Gal}(\bar{K}/K)$ acts algebraically on R, and that the Zariski-dense representation $\rho : \pi_1^{\text{ét}}(X, \bar{x}) \to R(\mathbb{Q}_\ell)$ is Galois-equivariant.

Theorem 3.11. *For each prime $\mathfrak{p} \nmid \ell$ of $\bar{K}$ at which $(\bar{X}, D)$ has potentially good reduction and tame monodromy round the divisor, there is a weight decomposition*

$$\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} = \prod_{n \le 0} {}_{\mathfrak{p}}\mathcal{W}_n \varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}},$$

splitting the true weight filtration $\mathrm{Dec}\, W$*. These decompositions are conjugate under the action of* $\mathrm{Gal}(\bar{K}/K)$*, in the sense that*

$$g({}_{\mathfrak{p}}\mathcal{W}_*) = {}_{g\mathfrak{p}}\mathcal{W}_*.$$

If $\mathfrak{p} \mid \ell$ is a prime at which $(\bar{X}, D)$ has potentially good reduction, and $\rho^{-1}O(R)$ is a potentially crystalline $\mathbb{Q}_\ell$-sheaf associated to a sum of ι-pure overconvergent F-isocrystals, then there is a weight decomposition

$$\varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}^\sigma = \prod_{n \le 0} {}_{\mathfrak{p}}\mathcal{W}_n \varpi_1^{\text{ét}}(X, \bar{x})^{R,\mathrm{Mal}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}^\sigma,$$

of affine schemes over B_{cris}^σ, splitting the true weight filtration $\mathrm{Dec}\, W$*. These decompositions are conjugate under the action of* $\mathrm{Gal}(\bar{K}/K)$*.*

Proof. This just combines Proposition 3.4 (using smooth specialisation to compare special and generic fibres) and Theorem 3.8, assigning R the weight 0, $\mathrm{H}^1(\bar{X}, j_*\rho^{-1}O(R))^\vee$ the weight -1 and $\mathrm{H}^0(\bar{X}, \mathbf{R}^1 j_*\rho^{-1}O(R))^\vee$ the weight -2. □

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