Smooth functions on algebraic K-theory

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Beilinson's regulator

- $r\colon\ K_i^{\mathrm{alg}}(X)\otimes_{\mathbb{Z}}\mathbb{R}\to\prod_p \mathrm{H}_{\mathcal{D}}^{2p-i}(X,\mathbb{R}(p))$
- Conjecture (Beilinson):
  For X smooth proper over $\mathbb{C}$, with integral model $X_{\mathbb{Z}}/\mathbb{Z}$,

$$K_i^{\mathrm{alg}}(X_{\mathbb{Z}})\otimes_{\mathbb{Z}}\mathbb{R}\to\bigoplus_p \mathrm{H}_{\mathcal{D}}^{2p-i}(X,\mathbb{R}(p))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$$

  is an isomorphism for $i\geqslant 2$.
  [$i=0,1$ subtle (height pairing); $\mathbb{Z}$ coeffs related to L-functions]

Algebraic K-theory

A a ring, $\coprod_n \mathrm{GL}_n(A)$ a groupoid.

- Monoidal structure $g \oplus h = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$.
- $\mathbb{R} \oplus K^{\mathrm{alg}}_{>0}(A)_{\mathbb{R}}$ represents additive maps $(f(g \oplus h) = f(g) + f(h))$ from $\coprod_n B\mathrm{GL}_n(A)$ to $\mathbb{R}$-linear spaces.
- Non-connective delooping $\mathbb{K}^{\mathrm{alg}}(A)_{\mathbb{R}}$ satisfies Zariski descent and

$$\mathbb{K}^{\mathrm{alg}}(\mathbb{P}^1_A)_{\mathbb{R}} \simeq \mathbb{K}^{\mathrm{alg}}(A)_{\mathbb{R}} \oplus \mathbb{K}^{\mathrm{alg}}(A)_{\mathbb{R}}.$$

- $\cong$ Bloch's higher Chow groups.

Real Deligne cohomology

- holomorphic de Rham complex

$$\Omega_X^\bullet = (\mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \dots)$$

- Hodge filtration $F^p\Omega_X^\bullet = \Omega_X^{\geqslant p}$.
- Real Deligne complex $\mathbb{R}_{\mathcal{D}}(p)$ is homotopy fibre product

$$(2\pi i)^p\mathbb{R} \times^h_{\Omega_X^\bullet} F^p\Omega_X^\bullet.$$

- Equivalently $\mathbb{R}_{\mathcal{D}}(p)$ is

$$(2\pi i)^p\mathbb{R} \to \mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{p-1}.$$

$$\mathrm{H}^0_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(p)) = \begin{cases} (2\pi i)^p\mathbb{R} & p \leqslant 0 \\ 0 & p < 0. \end{cases}$$

$$\mathrm{H}^1_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(p)) = \begin{cases} \mathbb{C}/(2\pi i)^p\mathbb{R} & p > 0 \\ 0 & p \leqslant 0. \end{cases}$$

$$r\colon\ K_{2p-1}(\mathbb{C}) \to \mathbb{C}/(2\pi i)^p\mathbb{R}$$
$$r\colon\ K_0(\mathbb{C}) \to \mathbb{R}.$$

r is Borel's regulator (up to constant).

Topological K-theory

- $C(X) = C(X, \mathbb{R})$ continuous fns on X.
- Topological K-theory $K^{\text{top}}(C(X))$ looks at topological group $\mathrm{GL}_n(X)$ instead of underlying abstract group.
- Only depends on $\coprod_n |B\mathrm{GL}_n(X)|$ (hence on X) up to homotopy.
- Bott element $\beta\colon\ K^{\text{top}}(C(X)) \to K^{\text{top}}(C(X))_{[2]}$ required to be invertible.

$K_1(\mathbb{C})$

- $\mathrm{H}^1(B(\mathbb{C}^\times)^\delta, \mathbb{R}) = \mathrm{Hom}((\mathbb{C}^\times)^\delta, \mathbb{R})$ (group homomorphisms), so $K_1^{\mathrm{alg}}(\mathbb{C})_\mathbb{R} = \mathbb{C}^\times \otimes_\mathbb{Z} \mathbb{R}$.
- $\mathrm{H}^1(B\mathbb{C}^\times, \mathbb{R}) = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{R}^\delta) = 0$ (as $\mathbb{C}^\times$ connected), so $K_1^{\mathrm{top}}(\mathbb{C}) = 0$.
- Lie group perspective: $\mathrm{Hom}(\mathbb{C}^\times, \mathbb{R}) = \mathrm{Hom}(\mathbb{C}^\times/S^1, \mathbb{R}) \cong \mathbb{R}$, so seek theory $K^?$ with

$$K_1^?(\mathbb{C}) = \mathbb{C}^\times/S^1 \cong \mathbb{R}.$$

K-theory presheaf

- $\mathsf{Aff}_{\mathbb{C}}$ affine $\mathbb{C}$-schemes.
- For any $\mathbb{C}$-scheme X, presheaf $\underline{K}(X)$ on $\mathsf{Aff}_{\mathbb{C}}$:

$$\underline{K}(X)(U) := \mathbb{K}^{\mathrm{alg}}(X \times U).$$

- Similarly, for any dg category $\mathcal{A}$ over $\mathbb{C}$,

$$\underline{K}(\mathcal{A})(U) := \mathbb{K}^{\mathrm{alg}}(\mathcal{A} \otimes_{\mathbb{C}} O(U)),$$

with $\underline{K}(X) = \underline{K}(\mathrm{Perf}_X)$.

- Any functor F on $\mathsf{Aff}_{\mathbb{C}}$ has enriched left Kan extension $\mathrm{Lan}\, F$ defined on l.f.p. spectral presheaves on $\mathsf{Aff}_{\mathbb{C}}$.
- Tautology: $\mathbb{K}^{\mathrm{alg}}(\mathcal{A}) \simeq \mathrm{Lan}\, \underline{K}(\mathcal{A})(\mathbb{C})$.
- Blanc's semi-topological K-theory

$$K^{st}(\mathcal{A}) := \mathrm{Lan}\, \mathrm{Sing}(\underline{K}(\mathcal{A})),$$

  $\mathrm{Sing}(U)$ the singular simplices on $U(\mathbb{C})$.
- Blanc's $K^{top}(\mathcal{A}) := K^{st}(\mathcal{A})[\beta^{-1}]$, with $K^{top}(\mathrm{Perf}_X) \simeq K^{top}(X(\mathbb{C}))$ for $\mathbb{C}$-schemes X.

Blanc's analogue of $\mathrm{H}^*_{\mathcal{D}}$

- Chern character to periodic cyclic homology $\mathrm{ch}\colon K^{top}(\mathcal{A}) \to \mathrm{HP}^{\mathbb{C}}(\mathcal{A})$.
- By Feigin–Tsygan, cyclic homology $\mathrm{HC}^{\mathbb{C}}(\mathrm{Perf}_X) \simeq \bigoplus_p \mathbf{R}\Gamma(X, \Omega^\bullet_X/F^p)^{[2p-2]}$ for X smooth.
- Canonical map $\mathrm{HP} \to \mathrm{HC}^{[2]}$.
- So consider $K^{top}(\mathcal{A})_{\mathbb{R}} \times^h_{\mathrm{HC}^{\mathbb{C}}(\mathcal{A})^{[2]}} 0$.
- Gives $\bigoplus_p \mathbf{R}\Gamma_{\mathcal{D}}(p)^{[2p]}$ when $\mathcal{A} = \mathrm{Perf}_X$.

Our philosophy

- $\mathbb{K}^{\mathrm{alg}}$ from discrete sets $U(\mathbb{C})^{\delta}$.
- K^{st} from homotopy types $\mathrm{Sing}(U)$.
- Seek something in between.
- $U(\mathbb{C})^{\delta}$ recovers discontinuous functions.
- $\mathrm{Sing}(U)$ recovers Betti cohomology.
- Want smooth functions, for Lie group perspective on $\mathrm{GL}(U)$.
- Take compactly supported distributions, and write $\mathcal{C}^{\infty}(F, \mathbb{R})' := \mathrm{Lan}\, \mathcal{C}^{\infty}(F, \mathbb{R})'$.

Theorem (P)

For $\mathbb{C}$-dg categories $\mathcal{A}$ of geometric origin,

$$\mathcal{C}^\infty(\underline{K}(\mathcal{A}), \mathbb{R})' \simeq K^{st}(\mathcal{A})_{\mathbb{R}} \times^h_{\mathrm{HC}^{\mathbb{C}}(\mathcal{A})^{[2]}} 0.$$

So for X smooth, $\mathcal{C}^\infty(\underline{K}(X), \mathbb{R})'$ is

$$\mathrm{cone}(K^{st}(X)_{\mathbb{R}} \to \bigoplus_p \mathbf{R}\Gamma(X, \Omega^\bullet/F^p)^{[2p]})^{[-1]}.$$

[This is $\bigoplus_p \mathbf{R}\Gamma_{\mathcal{D}}(X, \mathbb{R}(p))^{[2p]}$ "mod" β, so $\mathbb{R}.U(\mathbb{C}) \hookrightarrow \mathcal{C}^\infty(U, \mathbb{R})'$ yields regulator.]

Summary of proof (details follow)

- Simpson–de Rham stack $\underline{K}(\mathcal{A})_{\mathrm{dR}}$.
- Riemann–Hilbert

$$\mathcal{C}^\infty(\underline{K}(\mathcal{A})_{\mathrm{dR}}, \mathbb{R})' \simeq K^{st}(\mathcal{A}) \wedge H\mathbb{R}.$$

- Goodwillie implies equivalence of cones:

$$\underline{K}(\mathcal{A})_{\mathbb{Q}} \times^h_{\underline{K}(\mathcal{A})_{\mathbb{Q},\mathrm{dR}}} 0 \simeq (\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A}) \times^h_{\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A})_{\mathrm{dR}}} 0)^{[1]}.$$

- Calculation gives

$$\mathcal{C}^\infty(\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A}), \mathbb{R})' \simeq \mathrm{HC}^{\mathbb{C}}(\mathcal{A}),$$
$$\mathcal{C}^\infty(\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A})_{\mathrm{dR}}, \mathbb{R})' \simeq 0.$$

Simpson's de Rham stack

- Given a presheaf F on Aff_R with $\mathbb{Q} \subset R$, define $F_{\mathrm{dR}}(U) := F(U^{\mathrm{red}})$.
- Natural transformation $F \to F_{\mathrm{dR}}$.
- So named because for Y smooth, $\mathbf{R}\Gamma(Y_{\mathrm{dR}}, \mathscr{O}) \simeq \mathbf{R}\Gamma(Y, \Omega^{\bullet}_{Y/R})$.
- Quasi-coherent sheaves on Y_{dR} correspond to sheaves with flat connection on Y.
- Gives Hartshorne's algebraic de Rham cohomology for singular schemes.

Riemann–Hilbert

- For $Y/\mathbb{C}$ regarded as presheaf on $\mathrm{Aff}_{\mathbb{C}}$, Hartshorne gives

$$\mathcal{C}^\infty(Y_{\mathrm{dR}}, \mathbb{R}) \simeq \mathbf{R}\Gamma(Y(\mathbb{C}), \mathbb{R}), \text{ so}$$
$$\mathcal{C}^\infty(Y_{\mathrm{dR}}, \mathbb{R})' \simeq \mathrm{Sing}(Y) \wedge H\mathbb{R}$$

  (singular $\mathbb{R}$-chains).
- Thus for any presheaf F,

$$\mathcal{C}^\infty(F_{\mathrm{dR}}, \mathbb{R})' \simeq \mathrm{Lan}\,\mathrm{Sing}(F) \wedge H\mathbb{R}.$$

- Hence $\mathcal{C}^\infty(\underline{K}(\mathcal{A})_{\mathrm{dR}}, \mathbb{R})' \simeq K^{st}(\mathcal{A})_{\mathbb{R}}$.

Goodwillie's comparison

- If $B_{\geqslant 0}$ a dg $\mathbb{Q}$-algebra and I a nilpotent dg ideal, with $C = B/I$, then

$$K^{\mathrm{alg}}(B)_{\mathbb{Q}} \times^h_{K^{\mathrm{alg}}(C)_{\mathbb{Q}}} 0 \simeq (\mathrm{HC}^{\mathbb{Q}}(B) \times^h_{\mathrm{HC}^{\mathbb{Q}}(C)} 0)^{[1]}.$$

- Extends to connective qu-coh dgas $\mathscr{E}$ on semi-separated schemes by descent.
- May therefore take $\mathcal{A} = \mathrm{Perf}(\mathscr{E})$, or its semi-orthogonal summands (Orlov's geometric noncommutative schemes).
- Then extends to $\mathbb{K}$ by Bass delooping.

- Because $O(U) \to O(U)^{\mathrm{red}}$ nilpotent, cones agree:

$$\underline{K}(\mathcal{A})_{\mathbb{Q}} \times^h_{\underline{K}(\mathcal{A})_{\mathbb{Q},\mathrm{dR}}} 0 \simeq (\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A}) \times^h_{\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A})_{\mathrm{dR}}} 0)^{[1]}.$$

- Since $\mathcal{C}^\infty(F_{\mathrm{dR}}, \mathbb{R})$ depends only on $F_{\mathbb{R}} = F \wedge H\mathbb{R}$, results so far replace $\mathcal{C}^\infty(\underline{K}(\mathcal{A}), \mathbb{R})'$ with htpy exact sequence

$$K^{st}(\mathcal{A})_{\mathbb{R}} \to \mathcal{C}^\infty(\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A}))'^{[2]} \to \mathcal{C}^\infty(\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A})_{\mathrm{dR}})'^{[2]}$$

- Only remains to calculate $\mathcal{C}^\infty(\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A}), \mathbb{R})$, $\mathcal{C}^\infty(\underline{\mathrm{HC}}^{\mathbb{Q}}(\mathcal{A})_{\mathrm{dR}}, \mathbb{R})$.

The calculation

- Given f.d. $\mathbb{C}$-vector space V, write $\underline{V} := V \otimes_{\mathbb{C}} \mathbb{G}_a$, the presheaf $U \mapsto V \otimes O(U)$ on $\mathrm{Aff}_{\mathbb{C}}$.
- $\underline{\mathrm{HC}}^{\mathbb{Q}}(A)$ formed from cyclic quotients of tensors $\underline{A} \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} \underline{A}$. (Tensor product of affine spaces!)
- Can rewrite these tensors as $H\underline{A} \wedge \ldots \wedge H\underline{A} \wedge H\mathbb{Q}$, for Eilenberg–MacLane spectrum H.

Lemma

For f.d. complexes $V^{(1)}, \ldots, V^{(r)}$ over $\mathbb{C}$,

$$\mathcal{C}^\infty(H\underline{V}^{(1)} \wedge \ldots \wedge H\underline{V}^{(r)}, \mathbb{R})' \simeq V^{(1)} \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} V^{(r)}.$$

Idea of proof.

For connected simplicial V, $\mathrm{Symm}_{\mathbb{R}}(V^\vee) \xrightarrow{\sim} \mathcal{C}^\infty(V, \mathbb{R})$, via algebraic/$\mathcal{C}^\infty$ Hodge–de Rham spectral sequences

$$\mathrm{Symm}_{\mathbb{R}}(V^\vee) \otimes \Lambda^\bullet V^\vee \implies \mathbb{R}$$

$$\mathcal{C}^\infty(V, \mathbb{R}) \otimes \Lambda^\bullet V^\vee \implies \mathbb{R}. \qquad \square$$

- Thus $\mathcal{C}^\infty(\underline{\mathsf{HC}}^{\mathbb{Q}}(\mathcal{A}), \mathbb{R})' \simeq \mathsf{HC}^{\mathbb{R}}(\mathcal{A})$, which is $\mathsf{HC}^{\mathbb{C}}(\mathcal{A})$ as $\mathbb{R} \to \mathbb{C}$ étale.
- $\mathcal{C}^\infty((\underline{HV}^{(1)} \times \ldots \times \underline{HV}^{(r)})_{\mathsf{dR}}, \mathbb{R}) \simeq \mathbb{R}$ (de Rham cohomology), so we have $\mathcal{C}^\infty((\underline{HV}^{(1)} \wedge \ldots \wedge \underline{HV}^{(r)})_{\mathsf{dR}}, \mathbb{R}) \simeq 0$.
- Thus $\mathcal{C}^\infty(\underline{\mathsf{HC}}^{\mathbb{Q}}(\mathcal{A})_{\mathsf{dR}})' \simeq 0$.
- Hence the desired equivalence

$$\mathcal{C}^\infty(\underline{K}(\mathcal{A}), \mathbb{R})' \simeq K^{st}(\mathcal{A})_{\mathbb{R}} \times^h_{\mathsf{HC}^{\mathbb{C}}(\mathcal{A})^{[2]}} 0.$$