Shifted Poisson structures

J.P.Pridham

Derived Geometry

- Setting for this talk: differential geometry ($\mathcal{C}^\infty$ functions).
- $\exists$ version for analytic geometry (over $\mathbb{C}, \mathbb{R}, \mathbb{Q}_p, \mathbb{Q}((t)), \ldots$),
- and for algebraic geometry (char. 0).

We enhance manifolds in two directions:

- Derived enhancements (e.g. derived critical loci).
- Stacky enhancements (e.g. non-singular Lie algebroids and Lie groupoids, NQ-manifolds).

Derived enhancements

A *derived manifold* $X = (X^0, \mathscr{O}_{X,\bullet})$ is given by

- a manifold X^0 (then let $\mathscr{O}_{X,0} := \mathscr{O}_{X^0}$),
- a chain complex $\mathscr{O}_{X,0} \xleftarrow{\delta} \mathscr{O}_{X,1} \xleftarrow{\delta} \ldots$ of sheaves on X^0 (i.e. $\delta \circ \delta = 0$)
- a graded-commutative ($ba = (-1)^{\deg a \deg b} ab$) multiplication $\mathscr{O}_{X,i} \otimes \mathscr{O}_{X,j} \to \mathscr{O}_{X,i+j}$, with δ a derivation;
- need $\mathscr{O}_{X,\#} \cong \mathscr{O}_{X,0} \otimes_{\mathbb{R}} \mathrm{Symm}(V)$ locally on X^0, for finite-dimensional graded v.s. V.
- Set $\mathcal{C}^\infty(X, \mathbb{R}) := \Gamma(X^0, \mathscr{O}_X)$.

$f : X \to Y$ an equivalence if quasi-isomorphism, i.e. $\mathrm{H}_*\mathcal{C}^\infty(Y, \mathbb{R}) \cong \mathrm{H}_*\mathcal{C}^\infty(X, \mathbb{R})$.

Example: derived vanishing locus

- Y a manifold, V a vector bundle, $s\colon Y \to V$ a smooth section.
- Functions $\mathcal{C}^\infty(X)$ for $X := \mathbf{R}s^{-1}\{0\}$ given by

  $\mathcal{C}^\infty(Y,\mathbb{R}) \xleftarrow{s} \mathcal{C}^\infty(Y,V^*) \xleftarrow{\lrcorner s} \mathcal{C}^\infty(Y,\Lambda^2 V^*) \ldots.$
- $\mathrm{H}_0\mathcal{C}^\infty(X,\mathbb{R}) = \mathcal{C}^\infty(s^{-1}\{0\},\mathbb{R})$, but X has more structure.
- Sub-example $\mathrm{DCrit}(f) = \mathbf{R}(df)^{-1}\{0\}$ for $f\colon Y \to \mathbb{R}$ smooth.
- If Y has local co-ords y_i, then $X = \mathrm{DCrit}(f)$ has local co-ords $y_i \in \mathscr{O}_{X,0}$, $\eta_i \in \mathscr{O}_{X,1}$ with

$$\delta a = \sum\nolimits_i \frac{\partial f}{\partial y_i}\frac{\partial a}{\partial \eta_i}.$$

(Higher) Lie algebroids

An *NQ manifold* $X = (X_0, \mathscr{O}_X^\bullet)$ is given by

- a manifold X_0 (set $\mathscr{O}_X^0 := \mathscr{O}_{X_0}$),
- a cochain complex $\mathscr{O}_X^0 \xrightarrow{Q} \mathscr{O}_X^1 \xrightarrow{Q} \ldots$ of sheaves on X_0,
- graded-commutative multiplication $\mathscr{O}_X^i \otimes \mathscr{O}_X^j \to \mathscr{O}_X^{i+j}$, with Q a derivation;
- $\mathscr{O}_X^\# \cong \mathscr{O}_X^0 \otimes_{\mathbb{R}} \mathrm{Symm}(V)$ locally on X_0, for finite-dimensional graded v.s. V.
- Set $\mathcal{C}^\infty(X) := \Gamma(X_0, \mathscr{O}_X)$.

In contrast with derived manifolds, cohomology isomorphisms are *not* equivalences for these.

Example: quotient Lie algebroid

- Y a manifold, G a Lie group acting on Y, with Lie algebra $\mathfrak{g}$.
- Functions $\mathscr{O}_X$ for $X := [Y/\mathfrak{g}]$ given by

$$\mathscr{O}_Y \xrightarrow{Q} \mathscr{O}_Y \otimes \mathfrak{g}^* \xrightarrow{Q} \mathscr{O}_Y \otimes \Lambda^2 \mathfrak{g}^* \xrightarrow{Q} \dots$$

  on $X^0 := Y$, with Chevalley–Eilenberg differential Q given by co-action.
- These give nice resolution of Lie groupoid (differentiable stack) $[Y/G]$ as

$$[Y/\mathfrak{g}] \Leftarrow [Y \times G/\mathfrak{g}^{\oplus 2}] \Lleftarrow [Y \times G^2/\mathfrak{g}^{\oplus 3}] \dots$$

Combining derived and stacky structures

- Things of the form $X = (X^0_0, \mathscr{O}^{\bullet}_{X,\bullet})$ (double complex).
- Chains encode derived structure, cochains encode stacky structure.
- Examples of form $[Y/\mathfrak{g}]$ for $\mathfrak{g}$-equivariant derived manifold Y.
- Derived Hamiltonian reduction (Calaque, Safronov) is $[\mathbf{R}\mu^{-1}(0)/G]$, for $\mu\colon\ Y \to \mathfrak{g}^*$ Hamiltonian, so infinitesimally given by $[\mathbf{R}\mu^{-1}(0)/\mathfrak{g}]$.
- **Do not** try to combine structures in a single $\mathbb{Z}$-grading — too much information lost.

Example

Functions on the derived Hamiltonian reduction $[\mathbf{R}\mu^{-1}(0)/\mathfrak{g}]$ look like

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathscr{O}_Y \otimes \Lambda^2\mathfrak{g} & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \Lambda^2\mathfrak{g} \otimes \mathfrak{g}^* & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \Lambda^2\mathfrak{g} \otimes \Lambda^2\mathfrak{g}^* & \xrightarrow[Q]{} & \dots \\
{\scriptstyle \lrcorner\mu}\downarrow & & {\scriptstyle \lrcorner\mu}\downarrow & & {\scriptstyle \lrcorner\mu}\downarrow & & \\
\mathscr{O}_Y \otimes \mathfrak{g} & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \mathfrak{g} \otimes \mathfrak{g}^* & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \mathfrak{g} \otimes \Lambda^2\mathfrak{g}^* & \xrightarrow[Q]{} & \dots \\
{\scriptstyle \mu}\downarrow & & {\scriptstyle \mu}\downarrow & & {\scriptstyle \mu}\downarrow & & \\
\mathscr{O}_Y & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \mathfrak{g}^* & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \Lambda^2\mathfrak{g}^* & \xrightarrow[Q]{} & \dots
\end{array}
$$

n-shifted Poisson structures I

- On a derived manifold X, an n-shifted Poisson structure consists of smooth p-derivations $\{\pi_p\}_{p\geqslant 2}$ with

$$\pi_p\colon \mathscr{O}_{X,k_1}\times\mathscr{O}_{X,k_2}\times\ldots\times\mathscr{O}_{X,k_p}\to\mathscr{O}_{X,\sum k_i+pn+p-n-2}$$

  such that $(\mathscr{O}_{X[-n]},\delta,\pi)$ becomes an L_∞-algebra.
- When $\pi_p=0\,\forall p>2$, just get an n-shifted Lie bracket π_2 w.r.t. which δ a derivation.
- Quasi-isos can introduce higher π_p terms.
- Equivalences of Poisson structures come from suitable L_∞-isomorphisms.

(-1)-shifted structure on DCrit

- For $f\colon Y \to \mathbb{R}$, consider $X := \mathrm{DCrit}(f)$.
- Functions $\mathscr{O}_X$ given by

$$\mathscr{O}_Y \xleftarrow{\lrcorner df} \mathscr{T}_Y \xleftarrow{\lrcorner df} \Lambda^2 \mathscr{T}_Y \xleftarrow{\lrcorner df} \ldots$$

  on $X^0 := Y$, for tangent sheaf $\mathscr{T}_Y$.
- Canonical Poisson structure has
  $\pi_2(a, v) = v(a)$ for $a \in \mathscr{O}_Y$, $v \in \mathscr{T}_Y$,
  $\pi_p = 0$ for $p > 2$.
- In co-ordinates, $\pi_2(b, c) = \sum_i (\frac{\partial b}{\partial y_i}\frac{\partial c}{\partial \eta_i} + \frac{\partial b}{\partial \eta_i}\frac{\partial c}{\partial y_i})$.

n-shifted Poisson structures II [Pri17]

- On an NQ manifold X, an n-shifted Poisson structure consists of smooth p-derivations $\{\pi_p\}_{p \geqslant 2}$ with

$$\pi_p \colon \mathscr{O}_X^{k_1} \times \mathscr{O}_X^{k_2} \times \ldots \times \mathscr{O}_X^{k_p} \to \mathscr{O}_X^{\sum k_i - pn - p + n + 2}$$

  such that $(\mathscr{O}_X^{[n]}, Q, \pi)$ becomes an L_∞-algebra.

- [CPT+17] approach different, but almost certainly equivalent.

2-shifted Poisson structures on $[Y/\mathfrak{g}]$

- Functions $\mathscr{O}_X$ given by

$$\mathscr{O}_Y \xrightarrow{Q} \mathscr{O}_Y \otimes \mathfrak{g}^* \xrightarrow{Q} \mathscr{O}_Y \otimes \Lambda^2 \mathfrak{g}^* \xrightarrow{Q} \dots$$

- Look for 2-shifted Poisson structures.
- Multiderivations determined on generators, so only non-zero term is $\pi_2 \colon \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathscr{O}_Y$.
- Jacobi identities reduce to

$$\{\pi_2 \in (S^2 \mathfrak{g} \otimes \mathscr{O}_Y)^{\mathfrak{g}} \ : \ [\pi_2, \mathscr{O}_Y] = 0 \subset \mathfrak{g} \otimes \mathscr{O}_Y\}$$

- When $Y = *$, this is just the set of Casimirs

$$\pi_2 \in (S^2 \mathfrak{g})^{\mathfrak{g}}.$$

- No equivalences to worry about.

2-shifted Poisson structures on BG

- Structures pull back along tangent quasi-isos.
- For BG, need to find compatible system on

$$[*/\mathfrak{g}] \Leftarrow [G/\mathfrak{g}^{\oplus 2}] \Lleftarrow [G^2/\mathfrak{g}^{\oplus 3}] \ldots$$

  (simplicial diagram of Lie algebroids).
- Just need 2-Poisson structure on $[*/\mathfrak{g}]$ whose pullbacks to $[G/\mathfrak{g}^{\oplus 2}]$ agree, as no equivalences.
- Set of 2-shifted Poisson structures is then

$$(S^2\mathfrak{g})^G \subset (S^2\mathfrak{g})^{\mathfrak{g}}.$$

1-shifted Poisson structures on $[Y/\mathfrak{g}]$

- Multiderivations determined on generators, so only possible non-zero terms are

$$\pi_2\colon\ \mathfrak{g}^* \times \mathfrak{g}^* \to \mathscr{O}_Y \otimes \mathfrak{g}^*, \quad \pi_2\colon\ \mathfrak{g}^* \times \mathscr{O}_Y \to \mathscr{O}_Y$$
$$\pi_3\colon\ \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^* \to \mathscr{O}_Y.$$

- Safronov [Saf17]: this is just quasi-Lie bialgebroid, with 2-differential $\pi_2 \in (\Lambda^2\mathfrak{g} \otimes \mathscr{O}_Y) \oplus (\mathfrak{g} \otimes \mathscr{T}_Y)$ and curvature $\pi_3 \in \Lambda^3\mathfrak{g} \otimes \mathscr{O}_Y$.
- Isomorphisms given by twists $\lambda \in \Lambda^2\mathfrak{g} \otimes \mathscr{O}_Y$.
- Roytenberg [Roy02]: quasi-Lie bialgebroid $\mathcal{L}$ gives Courant algebroid $\mathcal{L} \oplus \mathcal{L}^*$.

1-shifted Poisson structures on $[Y/G]$

- Reduces to finding compatible system on simplicial diagram

$$[Y/\mathfrak{g}] \Leftarrow [Y \times G/\mathfrak{g}^{\oplus 2}] \Lleftarrow [Y \times G^2/\mathfrak{g}^{\oplus 3}] \dots$$

  of Lie algebroids.
- Need Poisson structure on $[Y/\mathfrak{g}]$ whose pullbacks to $[Y \times G/\mathfrak{g}^{\oplus 2}]$ are isomorphic, with isomorphism satisfying cocycle condition on $[Y \times G^2/\mathfrak{g}^{\oplus 3}]$.
- [Saf17]: for source-connected Lie groupoid, 1-shifted Poisson structures are precisely quasi-Poisson structures.
- also see [IPLGX12], [BCLX18].

n-shifted Poisson structures III [Pri17]

- Derived and stacky structures $\mathscr{O}^{\bullet}_{X,\bullet}$.
- An n-shifted Poisson structure consists of smooth p-derivations

$$\{\pi_p \in (\hat{\mathrm{Tot}}\,(\mathscr{T}_X^{\otimes p}))_{pn+p-n-2}\}_{p\geqslant 2},$$

  where $(\hat{\mathrm{Tot}}\,V)_m = \bigoplus_{k<0} V^k_{m+k} \oplus \prod_{k\geqslant 0} V^k_{m+k}$, making

$$(\hat{\mathrm{Tot}}\,\mathscr{O}_{X[-n]}, Q \pm \delta, \pi)$$

  an L_∞-algebra.
- Be careful with double complexes!

On derived Hamiltonian reduction $[\mathbf{R}\mu^{-1}(0)/\mathfrak{g}]$, Poisson structure on $\mathscr{O}_Y$ combines with pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$ to give canonical 0-shifted Poisson structure:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathscr{O}_Y \otimes \Lambda^2\mathfrak{g} & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \Lambda^2\mathfrak{g} \otimes \mathfrak{g}^* & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \Lambda^2\mathfrak{g} \otimes \Lambda^2\mathfrak{g}^* & \xrightarrow[Q]{} & \ldots \\
\downarrow{\scriptstyle \lrcorner\mu} & & \downarrow{\scriptstyle \lrcorner\mu} & & \downarrow{\scriptstyle \lrcorner\mu} & & \\
\mathscr{O}_Y \otimes \mathfrak{g} & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \mathfrak{g} \otimes \mathfrak{g}^* & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \mathfrak{g} \otimes \Lambda^2\mathfrak{g}^* & \xrightarrow[Q]{} & \ldots \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \\
\mathscr{O}_Y & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \mathfrak{g}^* & \xrightarrow[Q]{} & \mathscr{O}_Y \otimes \Lambda^2\mathfrak{g}^* & \xrightarrow[Q]{} & \ldots,
\end{array}
$$

Hamiltonian ensures $Q \pm \delta$ a Lie derivation here.

de Rham complexes

- Take derived manifold $X = (X^0, \mathscr{O}_{X,\bullet})$
- 1-forms $\Omega^1_{X,\bullet}$ (a chain complex).
- Exterior powers give p-forms $\Omega^p_{X,\bullet}$.
- de Rham differential $d\colon \Omega^p_{X,\bullet} \to \Omega^{p+1}_{X,\bullet}$.
- Take product total complex for de Rham complex

$$\mathrm{DR}(X)^i := \prod_p (\Omega^p_X)_{p-i},$$

differential $d \pm \delta$ (Koszul signs).

- Hodge filtration $F^p\mathrm{DR}(X) = \prod \Omega_X^{\geqslant p}$.
- Closed form $\omega \in F^p\mathrm{DR}(X)^i$ consists of $(\omega_p, \omega_{p+1}, \ldots)$,

$$\omega_n \in (\Omega_X^n)_{n-i},$$
$$d\omega_n = \delta\omega_{n+1}.$$

- Similar formulae for NQ manifold $X = (X_0, \mathscr{O}_X^\bullet)$, replacing δ with Q and changing signs.
- For derived NQ manifold $X = (X_0, \mathscr{O}_{X,\bullet}^\bullet)$, note Ω_X^p is a double complex, so have to take

$$\mathrm{DR}(X)^i := \prod_{p,j} (\Omega_X^p)_{p+j-i}^j.$$

n-shifted pre-symplectic structures

- $\omega \in \mathrm{Z}^{n+2}F^2\mathrm{DR}(X)$ [KV08, PTVV13].
- Explicitly, $\omega = \sum_{p \geqslant 2} \omega_p$, with

$$\delta\omega_2 = 0, \quad d\omega_p = \delta\omega_{p+1}.$$

- For NQ manifolds, replace δ with Q.
- Equivalences given by chain homotopies; equivalence classes $\mathrm{H}^{n+2}F^2$.
- Symplectic if non-degenerate:

$$\omega_2^\sharp \colon \mathrm{H}_*\mathscr{T}_X \xrightarrow{\simeq} \mathrm{H}_{*-n}\Omega^1_X.$$

Examples

- Symplectic structure on smooth manifold is 0-shifted (no higher terms).
- Derived critical locus is (-1)-shifted symplectic.
- Lie groupoid $B\mathrm{GL}_n$ is 2-shifted symplectic.
- Classifying stack $\mathrm{map}(X, B\mathrm{GL}_n)$ of vector bundles on X is $(2-d)$-shifted symplectic for $d = \dim X$ whenever $\Omega_X^d \cong \mathscr{O}_X$ [PTVV13].

Symplectic versus Poisson

- Classical case: 2-form ω is symplectic iff inverse π is Poisson.
- Standard proof uses Darboux theorem (cotangent bundle) — only partially generalises to shifted setting.
- Instead, we look to generalise

$$\pi^\flat \circ \omega^\sharp \circ \pi^\flat = \pi^\flat \colon \Omega^1 \to \mathscr{T}.$$

Details of the comparison

- Poisson structure π gives contraction $\mu(-, \pi)$ from de Rham to Poisson cohomology (cf. [KSM90] classically).
- π also gives element

$$\sigma(\pi) := \sum_{p \geqslant 2} (p-1)\pi_p$$

  in Poisson cohomology.
- Corresponding symplectic form ω is solution of

$$\mu(\omega, \pi) \simeq \sigma(\pi).$$

- For honest isomorphism (not equivalence), [KV08] solve this as Legendre transformation. Otherwise [Pri17].

Lagrangians

- Take (X, ω) n-shifted symplectic.
- Lagrangian structure on $f\colon L \to X$ is homotopy $\lambda\colon f^*\omega \simeq 0$, i.e.

$$\lambda \in F^2\mathrm{DR}(L)^{n+1} \ : \ (d \pm \delta \pm Q)\lambda = f^*\omega,$$

such that $(\omega_2, \lambda_2)^\sharp$ gives l.e.s.

$$\ldots \mathrm{H}_*\mathscr{T}_L \to \mathrm{H}_{*-n}f^*\Omega^1_X \to \mathrm{H}_{*-n}\Omega^1_L \to \mathrm{H}_{*-1}\mathscr{T}_L \ldots$$

- Lagrangian corresponds to non-degenerate co-isotropic [MS18]. This means L has $(n-1)$-Poisson structure on which X acts.

Lagrangian "intersections"

- If (L_i, λ_i) Lagrangian over (X, ω), then derived fibre product
$$(L_1 \times^h_X L_2, \lambda_1 - \lambda_2)$$
is $(n-1)$-shifted symplectic.
- Intersection in 0-shifted:
$$\begin{array}{ccc} \mathrm{DCrit}(f) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ Y & \xrightarrow{(\mathrm{id},df)} & T^*Y. \end{array}$$
- Intersection in 1-shifted:
$$\begin{array}{ccc} [\mathbf{R}\mu^{-1}\{0\}/G] & \longrightarrow & [\{0\}/G] \\ \downarrow & & \downarrow \\ [Y/G] & \xrightarrow{\mu} & [\mathfrak{g}^*/G]. \end{array}$$

References I

F. Bonechi, N. Ciccoli, C. Laurent-Gengoux, and P. Xu, *Shifted Poisson structures on differentiable stacks*, ArXiv e-prints (2018).

D. Calaque, T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted Poisson structures and deformation quantization*, J. Topol. **10** (2017), no. 2, 483–584, arXiv:1506.03699v4 [math.AG].

David Iglesias-Ponte, Camille Laurent-Gengoux, and Ping Xu, *Universal lifting theorem and quasi-Poisson groupoids*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 3, 681–731. MR 2911881

Yvette Kosmann-Schwarzbach and Franco Magri, *Poisson–Nijenhuis structures*, Ann. Inst. H. Poincaré Phys. Théor. **53** (1990), no. 1, 35–81. MR 1077465 (92b:17026)

H. M. Khudaverdian and Th. Th. Voronov, *Higher Poisson brackets and differential forms*, Geometric methods in physics, AIP Conf. Proc., vol. 1079, Amer. Inst. Phys., Melville, NY, 2008, arXiv:0808.3406v2 [math-ph], pp. 203–215. MR 2757715

Valerio Melani and Pavel Safronov, *Derived coisotropic structures II: stacks and quantization*, Selecta Math. (N.S.) **24** (2018), no. 4, 3119–3173, arXiv:1704.03201 [math.AG]. MR 3848017

References II

- J. P. Pridham, *Shifted Poisson and symplectic structures on derived N-stacks*, J. Topol. **10** (2017), no. 1, 178–210, arXiv:1504.01940v5 [math.AG].
- T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328, arXiv: 1111.3209v4 [math.AG]. MR 3090262
- Dmitry Roytenberg, *Quasi-Lie bialgebroids and twisted Poisson manifolds*, Lett. Math. Phys. **61** (2002), no. 2, 123–137. MR 1936572
- Pavel Safronov, *Poisson reductions as a coisotropic intersection*, arXiv:1509.08081v1 [math.AG], 2015.
- P. Safronov, *Poisson-Lie structures as shifted Poisson structures*, arXiv: 1706.02623v2 [math.AG], 2017.