Corrigendum to “Unifying derived deformation theories”

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The definition of geometric weak equivalences in Definitions 2.7 and 4.11 is incorrect. In fact, taking p to be the identity morphism on B, it follows that the definition given is equivalent to being an isomorphism. There is consequently an error in the final line of the proof of Lemma 2.10, so the “if” implication in the lemma is false as stated; this affects the proof of Theorem 2.10. The relevant definitions should be modified as follows.

Definition (2.6)**.** Given a quasi-smooth map $E \xrightarrow{p} B$ in $sc\mathrm{Sp}$, and a morphism $X \to B$ in $sc\mathrm{Sp}$, define $[X,E]_B$ to be the coequaliser

$$\mathrm{Hom}_{sc\mathrm{Sp}\downarrow B}(X, E^{\Delta^1} \times_{B^{\Delta^1}} B) \rightrightarrows \mathrm{Hom}_{sc\mathrm{Sp}\downarrow B}(X,E) \longrightarrow [X,E]_B\,,$$

where $sc\mathrm{Sp}\downarrow B$ is the slice category of objects over B.

Note that Definition 2.19 (which allows more general E) is consistent with this definition of $[X,E]_B$, since $E^{\Delta^1} \times_{B^{\Delta^1}} B$ is a path object for E over B. Also note that for $f : X \to B$, we have $[X,E]_B \cong [X, f^*E]_X$, where $f^*E = E \times_B X$.

Definition (4.10)**.** Given a quasi-smooth map $E \xrightarrow{p} B$ in $sDG\mathrm{Sp}$, and $X \in sDG\mathrm{Sp}$, define $[X,E]_B$ to be the coequaliser

$$\mathrm{Hom}_{sDG\mathrm{Sp}\downarrow B}(X, E^{\Delta^1} \times_{B^{\Delta^1}} B) \rightrightarrows \mathrm{Hom}_{sDG\mathrm{Sp}\downarrow B}(X,E) \longrightarrow [X,E]_B\,.$$

We now correct Definitions 2.7 and 4.11:

Definition. A map $f : X \to Y$ in either of the categories $sc\mathrm{Sp}$ or $sDG\mathrm{Sp}$ is said to be a geometric weak equivalence if for all quasi-smooth maps $p : E \to Y$, the map

$$f^* : [Y,E]_Y \to [X,E]_Y$$

is an isomorphism.

Taking these revised definitions, the proof of Lemma 2.10 is now correct as stated. However, it is no longer immediate that the class of geometric weak equivalences has the two out of three property, so the following lemmas should be inserted before Theorem 2.14:

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Lemma. *Given a pushout $U \to V$ of a morphism in J, and a quasi-smooth map $f : X \to U$, there exists a quasi-smooth map $g : Y \to V$ and an isomorphism*

$$\phi : X \cong Y \times_V U$$

over U. The pair (Y, ϕ) is unique up to unique isomorphism over U.

Proof. Write $V = \operatorname{Spf} A$, $U = \operatorname{Spf} B$ and $X \times_U \operatorname{Spf} k = \operatorname{Spf} R$; set $M = \ker(A \to B)$. Since f is quasi-smooth, it is a Reedy fibration. Adapting [22] Proposition 10.6 to pro-Artinian rings, we see that the obstruction to lifting f to a Reedy fibration $g : Y \to V$ lies in

$$\mathrm{H}^2(X/U \hat{\otimes} \operatorname{Tot}^{\Pi} N(M \hat{\otimes} R)),$$

with notation as in Lemma 2.40. If the obstruction is zero, then the same proposition gives that the isomorphism class of liftings is isomorphic to

$$\mathrm{H}^1(X/U \hat{\otimes} \operatorname{Tot}^{\Pi} N(M \hat{\otimes} R))$$

Since $U \to V$ is a pushout of a morphism in J, it follows that $\mathrm{H}_*(\operatorname{Tot}^{\Pi} NM) = 0$ (as this is true for morphisms in J). By the Künneth formula,

$$\mathrm{H}_* \operatorname{Tot}^{\Pi} N(M \hat{\otimes} R)) \cong \mathrm{H}_*(\operatorname{Tot}^{\Pi} NM) \hat{\otimes} H_*(\operatorname{Tot}^{\Pi} NR),$$

so $\mathrm{H}^n(X/U \hat{\otimes} \operatorname{Tot}^{\Pi} N(M \hat{\otimes} R)) = 0$ for all n.

Thus f has a unique lift to a Reedy fibration $g : T \to V$. By looking at cohomology, we see that any quasi-smooth deformation of a smooth map in $c\mathrm{Sp}$ is necessarily smooth. Thus the partial matching maps $Y_n \to M_{\Lambda^n_k} Y \times_{M_{\Lambda^n_k} V} V_n$ are smooth in $c\mathrm{Sp}$, so g is quasi-smooth in $sc\mathrm{Sp}$. □

Lemma. *Take a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $sc\mathrm{Sp}$. If any two of f, g, gf are geometric weak equivalences, then so is the third.*

Proof. If f and one of g, gf are geometric weak equivalences, then this is immediate. This leaves the case where gf and g are assumed geometric weak equivalences and we wish to show that f is also.

Take a quasi-smooth map $p : E \to Y$; we need to show that $[Y, E]_Y \to [X, E]_Y$ is an isomorphism. If g is quasi-smooth, then gp is also quasi-smooth, and by hypothesis we have isomorphisms $[X, E]_Z \cong [Z, E]_Z \cong [Y, E]_Z$. Now, g satisfies the conditions of Lemma 2.10, hence so does $E \times_Z Y \to E$. From the proof of Lemma 2.10, it thus follows that $[Y, E \times_Z Y]_Y \cong [Y, E]_Y$ and similarly for X, so the isomorphism above becomes $[X, E]_Y \cong [Y, E]_Y$, as required.

If g is now any geometric weak equivalence, note that we may use the small object argument to factorise g as $Y \xrightarrow{g'} Z' \xrightarrow{g''} Z$, where g' is a relative J-cell and g'' is in J-inj (i.e. quasi-smooth). Applying the result of the first paragraph to g, g', g'', we see that g'' must be a geometric weak equivalence. Applying the second paragraph to the diagram $X \xrightarrow{g'f} Z' \xrightarrow{g''} Z$, we deduce that $g'f$ must also be a geometric weak equivalence. Replacing g by g', we have therefore reduced to the case where g is a relative J-cell. We may apply the lemma above inductively to obtain a quasi-smooth map $\tilde{p} : \tilde{E} \to Z$ with $g^* \tilde{E} \cong E$. By hypothesis, we have isomorphisms $[X, \tilde{E}]_Z \cong [Z, \tilde{E}]_Z \cong [Y, \tilde{E}]_Z$, or equivalently $[X, E]_Y \cong [Y, E]_Y$, as required. □