K-theory and real Deligne cohomology

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K-theory

A a ring, $\coprod_n \mathrm{GL}_n(A)$ a groupoid.

- Monoidal structure $g \oplus h = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$.
- Algebraic K-theory $K^{\mathrm{alg}}(A)_{\mathbb{R}}$ parametrises additive maps from $\coprod_n B\mathrm{GL}_n(A)$ to real chain complexes.
- If $A = C(X)$, topological K-theory $K^{\mathrm{top}}(C(X))$ replaces discrete group $\mathrm{GL}_n(A)$ with topological group $\mathrm{GL}_n(X)$.

Abelian quotients

- $\mathrm{GL}_n(A)^{\mathrm{ab}} = \mathbb{G}_m(A) = A^\times$.
- For discrete topology δ, $B(\mathbb{C}^\times)^\delta$ is huge.
- Usual topology gives
  $B\mathbb{C}^\times \simeq BS^1 = K(\mathbb{Z}, 2)$.
- Idea: encode more subtle topological features of the Lie group.

$K_1(\mathbb{C})$

- $\mathrm{H}^1(B(\mathbb{C}^\times)^\delta, \mathbb{R}) = \mathrm{Hom}((\mathbb{C}^\times)^\delta, \mathbb{R})$ (group homomorphisms), so $K_1^{\mathrm{alg}}(\mathbb{C})_{\mathbb{R}} = \mathbb{C}^\times \otimes_{\mathbb{Z}} \mathbb{R}$.
- $\mathrm{H}^1(B\mathbb{C}^\times, \mathbb{R}) = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{R}^\delta) = 0$, so $K_1^{\mathrm{top}}(\mathbb{C}) = 0$.
- Lie group perspective: $\mathrm{Hom}(\mathbb{C}^\times, \mathbb{R}) = \mathrm{Hom}(\mathbb{C}^\times/S^1, \mathbb{R}) \cong \mathbb{R}$, so seek theory $K^?$ with

$$K_1^?(\mathbb{C}) = \mathbb{C}^\times/S^1 \cong \mathbb{R}.$$

Pro-Banach K-theory

- A Fréchet ($\approx$ pro-Banach) algebra.
- Motivating example: holomorphic functions on complex manifold.
- $K_{\mathrm{Ban}}(A)$ universal for additive $\mathcal{C}^\infty$ maps

$$\coprod_n B\mathrm{GL}_n(A) \to V$$

to complexes V of pro-Banach spaces.

More precisely:

- A map $K_{\mathrm{Ban}}(A) \to V$ is a system

$$K^{\mathrm{alg}}(\mathcal{C}^\infty(Z, A)) \to \mathcal{C}^\infty(Z, V)$$

functorial in suitable Fréchet manifolds Z.

- Homotopy end

$$\int_Z^h \underline{\mathrm{Hom}}_{\mathbb{R}}(K^{\mathrm{alg}}(\mathcal{C}^\infty(Z, A))_{\mathbb{R}}, \mathcal{C}^\infty(Z, V)).$$

A cleaner approach

- $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ a monoid under composition.
- Frölicher: the category of Fréchet manifolds embeds in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$-acts, by

$$Z \rightsquigarrow \mathcal{C}^\infty(\mathbb{R}, Z).$$

- Thus $K_{\mathrm{Ban}}(A)$ determined by action of $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ on $K^{\mathsf{alg}}(\mathcal{C}^\infty(\mathbb{R}, A))$.

Real Deligne cohomology

- holomorphic de Rham complex

$$\Omega_X^\bullet = (\mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \dots)$$

- Hodge filtration $F^p\Omega_X^\bullet = \Omega_X^{\geqslant p}$.
- Real Deligne complex $\mathbb{R}_\mathcal{D}(p)$ is homotopy fibre product

$$(2\pi i)^p\mathbb{R} \times^h_{\Omega_X^\bullet} F^p\Omega_X^\bullet.$$

- Equivalently $\mathbb{R}_\mathcal{D}(p)$ is

$$(2\pi i)^p\mathbb{R} \to \mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{p-1}.$$

Beilinson's regulator

- $r\colon\ K_i^{\mathrm{alg}}(X)_{\mathbb{R}} \to \prod_p \mathrm{H}_{\mathcal{D}}^{2p-i}(X, \mathbb{R}(p))$
- Conjecture (Beilinson):
  For X smooth proper over $\mathbb{C}$, with integral model $X_{\mathbb{Z}}/\mathbb{Z}$,

$$K_i^{\mathrm{alg}}(X_{\mathbb{Z}})_{\mathbb{R}} \to \prod_p \mathrm{H}_{\mathcal{D}}^{2p-i}(X, \mathbb{R}(p))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$$

  is an isomorphism for $i \geqslant 2$.
  [$i = 0, 1$ subtle, deep.]

$$\mathrm{H}^0_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(p)) = \begin{cases} (2\pi i)^p\mathbb{R} & p \leqslant 0 \\ 0 & p < 0. \end{cases}$$

$$\mathrm{H}^1_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(p)) = \begin{cases} \mathbb{C}/(2\pi i)^p\mathbb{R} & p > 0 \\ 0 & p \leqslant 0. \end{cases}$$

$$r\colon\ K_{2p-1}(\mathbb{C}) \to \mathbb{C}/(2\pi i)^p\mathbb{R}$$
$$r\colon\ K_0(\mathbb{C}) \to \mathbb{R}.$$

r is Borel's regulator (up to constant).

Borel's proof

- $\mathrm{Hom}(K_i(\mathbb{Z}), \mathbb{R}) \cong P\mathrm{H}^i(B\mathrm{GL}(\mathbb{Z}), \mathbb{R})$ (primitives $f(M \oplus N) = f(M) + f(N)$).
- $P\mathrm{H}^*(B\mathrm{GL}(\mathbb{Z}), \mathbb{R}) \approx P\mathrm{H}^*(B\mathrm{SL}(\mathbb{Z}), \mathbb{R})$.
- Borel's theorem: $\mathrm{H}^*(B\mathrm{SL}(\mathbb{Z}), \mathbb{R}) \approx \mathrm{H}^*_{\mathrm{cts}}(B\mathrm{SL}(\mathbb{R}), \mathbb{R})$ (continuous group coho).
- Also rings of integers in number fields.

Van Est isomorphism

- Symmetric spaces $M_n = \mathrm{SO}_n \backslash \mathrm{SL}_n(\mathbb{R})$ contractible.
- $\mathrm{H}^*_{\mathrm{cts}}(B\mathrm{SL}(\mathbb{R}), \mathbb{R}) \simeq A^*(M, \mathbb{R})^{\mathrm{GL}(\mathbb{R})}$ (invariant differential forms on M).
- Lie algebra coho $\mathrm{H}^i(\mathfrak{sl}(\mathbb{R}), \mathfrak{so}; \mathbb{R})$.
- $\rightsquigarrow \mathrm{H}^*_{\mathcal{D}}(\mathrm{Spec}\,\mathbb{C}, \mathbb{R}(p))^{\mathrm{Gal}(\mathbb{C}/R)}$.

Generalisation

- IDEA:
  Copy all the stages except Borel's theorem.

$\underline{\mathrm{Hom}}(K_{\mathrm{Ban}}(A), V) \cong P\mathbf{R}\Gamma_{\mathrm{diff}}(B\mathrm{GL}(A), V)$, for V Banach, $\mathbf{R}\Gamma_{\mathrm{diff}}$ differentiable cohomology.
(cf. $\mathrm{Hom}(K_i(A), \mathbb{R}) \cong P\mathrm{H}^i(B\mathrm{GL}(A), \mathbb{R})$).

Back to complex manifolds

- Let $A = \mathscr{O}_X(U)$, U a polydisc.
- Then $U_n(\mathbb{C}) \to \mathrm{GL}_n(A)$ a $\mathcal{C}^\infty$ deformation retract (iterated integrals).
- Thus $M_n = U_n(\mathbb{C})\backslash\mathrm{GL}_n(A)$ contractible;

$$\mathbf{R}\Gamma_{\mathrm{diff}}(B\mathrm{GL}(A), V) \cong A^\bullet(M, V)^{\mathrm{GL}(A)}.$$

- This is real continuous Lie algebra cohomology

$$E^\bullet_{\mathsf{cts},\mathbb{R}}(\mathfrak{gl}(A), \mathfrak{u}; V).$$

- $E^{\bullet}_{\text{cts},\mathbb{R}}(\mathfrak{gl}(A), \mathfrak{u}; -)$ represented by continuous Lie algebra homology

$$\hat{E}^{\mathbb{R}}_{\bullet}(\mathfrak{gl}(A), \mathfrak{u})$$

- $K_{\mathrm{Ban}}(A)$ is primitives.
- By Loday–Quillen, cyclic homology $\hat{\mathrm{C}}_{\bullet}(A)$ gives primitive part of $\hat{E}^{\mathbb{R}}_{\bullet}(\mathfrak{gl}(A))$.

Deligne cohomology appears

- For $A = \mathscr{O}_X(U)$, Fréchet version of HKR gives
$$\hat{\mathrm{C}}_\bullet(A) \simeq \bigoplus_{p>0} (\Omega_A^{2p-\bullet}/F^{p+1}).$$
- Contribution of $\mathfrak{u}$ to $P\hat{E}_\bullet^{\mathbb{R}}(\mathfrak{gl}(A), \mathfrak{u})$ gives copies of $(2\pi i)^p\mathbb{R}$.
- Thus $K_{\mathrm{Ban}}(A) \simeq \bigoplus_{p\geqslant 0} \mathbb{R}_{X,\mathcal{D}}(p)^{2p-\bullet}$ as pro-Banach complexes.

- Sheafify (with connectivity apologies!):

$$K_{\mathrm{Ban}}(X) := \mathbf{R}\Gamma(X, K_{\mathrm{Ban}}(\mathscr{O}_X))$$

(truncation gives absolute Hodge).

- Thus for X compact,

$$K_{\mathrm{Ban}}(X) \simeq \bigoplus_{p \geqslant 0} \mathbf{R}\Gamma_{\mathcal{D}}(X, \mathbb{R}(p))^{2p-\bullet}$$

(Hodge theory says topology on RHS doesn't matter).