TANNAKA DUALITY FOR ENHANCED TRIANGULATED CATEGORIES

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Abstract. We develop Tannaka duality theory for dg categories. To any dg functor from a dg category $\mathcal{A}$ to finite-dimensional complexes, we associate a dg coalgebra C via a Hochschild homology construction. When the dg functor is faithful, this gives a quasi-equivalence between the derived dg categories of $\mathcal{A}$-modules and of C-comodules. When $\mathcal{A}$ is Morita fibrant (i.e. an idempotent-complete pre-triangulated category), it is thus quasi-equivalent to the derived dg category of compact C-comodules. We give several applications for pro-algebraic homotopy types associated to various cohomology theories, and for motivic Galois groups.

Introduction

Tannaka duality in Joyal and Street's formulation ([JS, §7, Theorem 3]) characterises abelian k-linear categories $\mathcal{A}$ with exact faithful k-linear functors ω to finite-dimensional k-vector spaces as categories of finite-dimensional comodules of coalgebras C. When $\mathcal{A}$ is a rigid tensor category and ω monoidal, C becomes a Hopf algebra (so $\operatorname{Spec} C$ is a group scheme), giving the duality theorem of [DMOS, Ch. II].

The purpose of this paper is to extend these duality theorems to dg categories. Various derived versions of Tannaka duality have already been established, notably [Toë1, Wal, FI, Lur, Iwa]. However, those works usually require the presence of t-structures, and all follow [DMOS, Ch. II] in restricting attention to monoidal derived categories, then take higher stacks as the derived generalisation of group schemes.

Our viewpoint does not require the dg categories to have monoidal structures, and takes dg coalgebras as the dual objects. Arbitrary dg coalgebras are poorly behaved (for instance, quasi-isomorphism does not imply Morita equivalence), but they perfectly capture the behaviour of arbitrary dg categories without t-structures. Even in the presence of monoidal structures, we consider more general dg categories than heretofore, and our dg coalgebras then become dg bialgebras, in which case our results give dg enhancements and strengthenings of Ayoub's weak Tannaka duality from [Ayo3]. A similar strengthening has appeared in [Iwa], but without the full description needed for applications to motives (see Remarks 2.11 and 2.17).

The first crucial observation we make is that in the Joyal–Street setting, the dual coalgebra C to $\omega\colon \mathcal{A} \to \mathrm{FDVect}$ is given by the Hochschild homology group

$$\omega^\vee \otimes_{\mathcal{A}} \omega = \mathrm{HH}_0(\mathcal{A}, \omega^\vee \otimes_k \omega),$$

where $\omega^\vee\colon \mathcal{A}^{\mathrm{opp}} \to \mathrm{FDVect}$ sends X to the dual $\omega(X)^\vee$. The natural generalisation of the dual coalgebra to dg categories is then clear: given a k-linear dg category $\mathcal{A}$ and a k-linear dg functor ω to finite-dimensional complexes, we put a dg coalgebra structure

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C on the Hochschild homology complex

$$\omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega \simeq \mathrm{CC}_\bullet(\mathcal{A}, \omega^\vee \otimes_k \omega).$$

In order to understand the correct generalisation of the dg fibre functor ω, we look to Morita (or Morita–Takeuchi) theory. In the underived setting, if ω is representable by an object $G \in \mathcal{A}$, the condition that ω be exact and faithful amounts to requiring that G be a projective generator for $\mathcal{A}$. This means that in the dg setting, $\underline{\mathrm{Hom}}(G, -)$ should be a dg fibre functor if and only if G is a derived generator. In other words, $\underline{\mathrm{Hom}}(G, -)$ must reflect acyclicity of complexes, so we consider dg functors ω from $\mathcal{A}$ to finite-dimensional complexes which are faithful in the sense that $\omega(X)$ is acyclic only if X is acyclic, for X in the derived category $\mathcal{D}(\mathcal{A})$.

Corollary 2.12 gives a derived analogue of [JS, §7, Theorem 3]. When ω is faithful, this gives a quasi-equivalence between the dg enhancements $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ and $\mathcal{D}_{\mathrm{dg}}(C)$ of the derived categories (of the first kind) of $\mathcal{A}$ and C. This comparison holds for all dg categories; in particular, replacing $\mathcal{A}$ with any subcategory of compact generators of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ will yield a dg coalgebra C with the same property. Our derived analogue of an abelian category is a Morita fibrant dg category: when $\mathcal{A}$ is such a dg category, we have a quasi-equivalence between $\mathcal{A}$ and the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(C)$ on compact objects.

Crucially, Theorem 2.9 gives a further generalisation to non-faithful dg functors ω, showing that the dg derived category $\mathcal{D}_{\mathrm{dg}}(C)$ of C-comodules is quasi-equivalent to a derived quotient $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega)$ of the dg derived category $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ generated by $\mathcal{A}$. This has many useful applications to scenarios where $\mathcal{A}$ arises as a quotient of a much simpler dg category $\mathcal{B}$, allowing us to compute C directly from $\mathcal{B}$ and ω.

Section 1 contains the key constructions used throughout the paper. After recalling the Hochschild homology complex $\mathrm{CC}_\bullet(\mathcal{A}, F)$ of a dg category $\mathcal{A}$ with coefficients in a $\mathcal{A}$-bimodule F, we study the dg coalgebra $C_\omega(\mathcal{A}) := \mathrm{CC}_\bullet(\mathcal{A}, \omega^\vee \otimes_k \omega)$.

We then introduce the notion of universal coalgebras of $\mathcal{A}$, which are certain resolutions D of $\mathcal{A}(-,-)$ as a $\otimes_{\mathcal{A}}$-coalgebra in $\mathcal{A}$-bimodules. The canonical choice is the Hochschild complex $\mathrm{CC}_\bullet(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes_k h_{\mathcal{A}})$ of the Yoneda embedding. For any universal coalgebra D, a dg fibre functor ω gives a dg coalgebra $C := \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$, and a tilting module $P := D \otimes_{\mathcal{A}} \omega$. When $(\mathcal{A}, \boxtimes)$ is a tensor dg category, we consider universal bialgebras, which are universal coalgebras equipped with compatible multiplication with respect to $\boxtimes$, the Hochschild complex again being one such. In this case, a tensor dg functor ω makes C into a dg bialgebra.

The main results of the paper are in Section 2. For C and P a dg coalgebra and tilting module as above, there is a left Quillen dg functor $- \otimes_{\mathcal{A}} P$ from the category of dg $\mathcal{A}$-modules to the category of dg C-comodules (Lemma 2.5). The functors $\mathcal{D}(C) \to \mathcal{D}(\mathcal{A}) \to \mathcal{D}(C)$ then form a retraction (Proposition 2.7). Theorem 2.9 establishes quasi-equivalences

$$\mathcal{D}_{\mathrm{dg}}(C) \to (\ker \omega)^\perp \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega)$$

of dg enhancements of derived categories, which simplify to the equivalences of Corollary 2.12 when ω is faithful. Remark 2.11 relates this to Ayoub's weak Tannaka duality, with various consequences for describing motivic Galois groups given in §2.5. Proposition 2.21 ensures that the equivalences preserve tensor structures when present, and Example 2.22 applies this to motivic Galois groups.

The main drawback of the Hochschild construction for the dg coalgebra is that it always creates terms in negative cochain degrees. This means that quasi-isomorphisms of such dg coalgebras might not be derived Morita equivalences, and that we cannot rule out negative homotopy groups for dg categories of cohomological origin.

In Section 3, we give an alternative presentation of the Hochschild construction which associates non-negative dg coalgebras to hearts of t-structures (Corollary 3.19, Propositions 3.7, 3.9). In this setting, the correspondence between dg categories and dg coalgebras can be understood as a form of Koszul duality (Proposition 3.16). Via duality of the commutative and Lie operads, dg tensor categories then correspond to dg Hopf algebras (Corollary 3.29, Proposition 3.27). Example 3.20 then explains how these results combine with Ayoub's calculations to show that existence of a motivic t-structure would characterise Voevodsky's motives over a number field as the derived category of Nori's abelian category of mixed motives, implying the $K(\pi, 1)$ conjecture. §3.5 explains how our constructions generalise Moriya's Tannakian dg categories.

Section 4 is mostly concerned with applications to the real relative Malcev homotopy types of a manifold X. Lemma 4.4 equates the dg category of derived connections on X with the pre-triangulated category generated by the de Rham dg category of semisimple local systems. Corollary 4.9 then equates this with the dg category of representations of the schematic homotopy type $G(X, x)^{\mathrm{alg}}$. §4.3 looks at the universal bialgebra, which avoids choices of basepoint and can be thought of as the sheaf of functions on the space of algebraic paths. In §4.4, we establish analogues for $\mathbb{Q}_\ell$ relative Malcev homotopy types of a scheme, and §4.5 discusses motivic generalisations.

In the appendix, we give technical details for constructing monoidal dg functors giving rise to the motivic Galois groups of Example 2.22, and show that, in the case of Betti cohomology, this construction gives a dg functor non-canonically quasi-isomorphic to the usual cohomology dg functor (Corollary A.19).

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Notational conventions. Fix a commutative ring k. When the base is not specified, $\otimes$ will mean $\otimes_k$. When k is a field, we write Vect_k for the category of all vector spaces over k, and FDVect_k for the full subcategory of finite-dimensional vector spaces.

We will always use the symbol $\cong$ to denote isomorphism, while $\simeq$ will be equivalence, quasi-isomorphism or quasi-equivalence.

Contents

1. Hochschild homology of a DG category

Definition 1.1. A k-linear dg category $\mathcal{C}$ is a category enriched in cochain complexes of k-modules, so has objects $\mathrm{Ob}\mathcal{C}$, cochain complexes $\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y)$ of morphisms, associative multiplication

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(y,z)\otimes_k \underline{\mathrm{Hom}}_{\mathcal{C}}(x,y)\to \underline{\mathrm{Hom}}_{\mathcal{C}}(x,z)$$

and identities $\mathrm{id}_x\in \underline{\mathrm{Hom}}_{\mathcal{C}}(x,x)^0$.

Given a dg category $\mathcal{C}$, we will write $\mathrm{Z}^0\mathcal{C}$ and $\mathrm{H}^0\mathcal{C}$ for the categories with the same objects as $\mathcal{C}$ and with morphisms

$$\mathrm{Hom}_{\mathrm{Z}^0\mathcal{C}}(x,y):=\mathrm{Z}^0\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y),$$
$$\mathrm{Hom}_{\mathrm{H}^0\mathcal{C}}(x,y):=\mathrm{H}^0\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y).$$

When we refer to limits or colimits in a dg category $\mathcal{C}$, we will mean limits or colimits in the underlying category $\mathrm{Z}^0\mathcal{C}$.

Definition 1.2. Given a dg category $\mathcal{C}$ and objects x,y, write $\mathcal{C}(x,y):=\underline{\mathrm{Hom}}_{\mathcal{C}}(y,x)$.

Definition 1.3. A dg functor $F\colon \mathcal{A}\to\mathcal{B}$ is said to be a quasi-equivalence if $\mathrm{H}^0F\colon \mathrm{H}^0\mathcal{A}\to \mathrm{H}^0\mathcal{B}$ is an equivalence of categories, with $\mathcal{A}(X,Y)\to\mathcal{B}(FX,FY)$ a quasi-isomorphism for all objects $X,Y\in\mathcal{A}$.

Definition 1.4. We follow [Kel2, 2.2] in writing $\mathcal{C}_{\mathrm{dg}}(k)$ for the dg category of chain complexes over k, where $\underline{\mathrm{Hom}}(U,V)^i$ consists of graded k-linear morphisms $U\to V[i]$, and the differential is given by $df=d\circ f\mp f\circ d$.

We write $\mathrm{per}_{\mathrm{dg}}(k)$ for the full dg subcategory of finite rank cochain complexes of projective k-modules. Beware that this category is not closed under quasi-isomorphisms, so does not include all perfect complexes in the usual sense.

The following is adapted from [Mit, §12] and [Kel1, 1.3]:

Definition 1.5. Take a small k-linear dg category $\mathcal{A}$ and an $\mathcal{A}$-bimodule

$$F\colon \mathcal{A} \times \mathcal{A}^{\mathrm{opp}} \to \mathcal{C}_{\mathrm{dg}}(k),$$

(i.e. a k-bilinear dg functor). Define the homological Hochschild complex

$$\underline{\mathrm{CC}}_\bullet(\mathcal{A}, F)$$

(a simplicial diagram of cochain complexes) by

$$\underline{\mathrm{CC}}_n(\mathcal{A}, F) := \bigoplus_{X_0,\ldots,X_n \in \mathrm{Ob}\mathcal{A}} \mathcal{A}(X_0, X_1) \otimes_k \mathcal{A}(X_1, X_2) \otimes_k \ldots \otimes_k \mathcal{A}(X_{n-1}, X_n) \otimes_k F(X_n, X_0),$$

with face maps

$$\partial_i(a_1 \otimes \ldots a_n \otimes f) = \begin{cases} a_2 \otimes \ldots a_n \otimes (f \circ a_1) & i = 0 \\ a_1 \otimes \ldots a_{i-1} \otimes (a_i \circ a_{i+1}) \otimes a_{i+2} \otimes \ldots \otimes a_n \otimes f & 0 < i < n \\ a_1 \otimes \ldots a_{n-1} \otimes (a_n \circ f) & i = n \end{cases}$$

and degeneracies

$$\sigma_i(a_1 \otimes \ldots a_n \otimes f) = (a_1 \otimes \ldots \otimes a_i \otimes \mathrm{id} \otimes a_{i+1} \otimes \ldots \otimes a_n \otimes f).$$

Definition 1.6. Define the total Hochschild complex

$$\mathrm{CC}(\mathcal{A}, F)$$

by first regarding $\underline{\mathrm{CC}}_\bullet(\mathcal{A}, F)$ as a chain cochain complex with chain differential $\sum_i (-1)^i \partial_i$, then taking the total complex

$$(\mathrm{Tot}\, \underline{\mathrm{CC}}_\bullet(\mathcal{A}, F)^\bullet)^n = \bigoplus_i \underline{\mathrm{CC}}_i(\mathcal{A}, F)^{n+i},$$

with differential given by the cochain differential $\pm$ the chain differential.

There is also a quasi-isomorphic normalised version

$$N\mathrm{CC}(\mathcal{A}, F),$$

given by replacing $\underline{\mathrm{CC}}_i$ with $\underline{\mathrm{CC}}_i / \sum_j \sigma_j \underline{\mathrm{CC}}_{i-1}$.

Remark 1.7. Note that $\mathrm{H}^i\mathrm{CC}(\mathcal{A}, F)^\bullet = \mathrm{HH}_{-i}(\mathcal{A}, F)$, which is a Hochschild *homology* group. We have, however, chosen cohomological gradings because our motivating examples will all have $\mathrm{H}^{<0} = 0$.

1.1. **The Tannakian envelope.** Fix a small k-linear dg category $\mathcal{A}$ and a k-linear dg functor $\omega\colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$.

Remark 1.8. If k is a field and we instead have a dg functor $\omega\colon \mathcal{A} \to h\mathrm{FDCh}_k$ to the dg category of cohomologically finite-dimensional complexes (i.e. perfect complexes in the usual sense), we can reduce to the setting above. We could first take a cofibrant replacement $\tilde{\mathcal{A}} \to \mathcal{A}$ of $\mathcal{A}$ in Tabuada's model structure ([Tab2], as adapted in [Kel2, Theorem 4.1]) on dg categories. Because k is a field, the inclusion $\mathrm{per}_{\mathrm{dg}}(k) \to h\mathrm{FDCh}_k$ is a quasi-equivalence, so the composite dg functor $\omega\colon \tilde{\mathcal{A}} \to h\mathrm{FDCh}_k$ is homotopy equivalent to a dg functor $\omega'\colon \tilde{\mathcal{A}} \to \mathrm{per}_{\mathrm{dg}}(k)$.

Definition 1.9. Define the Tannakian dual $C_\omega(\mathcal{A})$ by

$$C_\omega(\mathcal{A}) := \mathrm{CC}(\mathcal{A}, \omega \otimes \omega^\vee),$$

where the $\mathcal{A}$-bimodule

$$\omega \otimes \omega^\vee\colon \mathcal{A} \times \mathcal{A}^{\mathrm{opp}} \to \mathrm{per}_{\mathrm{dg}}(k)$$

is given by

$$\omega \otimes \omega^\vee(x, y) = (\omega x) \otimes_k (\omega y)^\vee.$$

Similarly, write $NC_\omega(\mathcal{A}) := N\mathrm{CC}(\mathcal{A}, \omega \otimes \omega^\vee)$.

Proposition 1.10. *The cochain complexes $C_\omega(\mathcal{A}), NC_\omega(\mathcal{A})$ have the natural structure of coassociative counital dg coalgebras over k.*

Proof. We may rewrite

$$\underline{\mathrm{CC}}_n(\mathcal{A}, \omega \otimes \omega^\vee) = \bigoplus_{X_0,\ldots,X_n \in \mathrm{Ob}\mathcal{A}} \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_1, X_2) \otimes \ldots \otimes \mathcal{A}(X_{n-1}, X_n) \otimes \omega X_n \otimes \omega(X_0)^\vee,$$

as

$$\bigoplus_{X_0,\ldots,X_n \in \mathrm{Ob}\mathcal{A}} \omega(X_0)^\vee \otimes \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_1, X_2) \otimes \ldots \otimes \mathcal{A}(X_{n-1}, X_n) \otimes \omega X_n.$$

There is a comultiplication Δ on the bicomplex

$$\underline{\mathrm{CC}}_\bullet(\mathcal{A}, \omega \otimes \omega^\vee),$$

with

$$\Delta \colon \underline{\mathrm{CC}}_{m+n}(\mathcal{A}, \omega \otimes \omega^\vee) \to \underline{\mathrm{CC}}_m(\mathcal{A}, \omega \otimes \omega^\vee) \otimes_k \underline{\mathrm{CC}}_n(\mathcal{A}, \omega \otimes \omega^\vee)$$

being the map

$$\begin{aligned}
&(\omega X_0)^\vee \otimes \mathcal{C}(X_0, X_1) \otimes \ldots \otimes \mathcal{C}(X_{m+n-1}, X_{m+n}) \otimes (\omega X_{m+n}) \to \\
&[(\omega X_0)^\vee \otimes \mathcal{C}(X_0, X_1) \otimes \ldots \otimes \mathcal{C}(X_{m-1}, X_m) \otimes (\omega X_m)] \\
&\otimes [(\omega X_m)^\vee \otimes \mathcal{C}(X_m, X_{m+1}) \otimes \ldots \otimes \mathcal{C}(X_{m+n-1}, X_{m+n}) \otimes (\omega X_{m+n})]
\end{aligned}$$

given by tensoring with

$$\mathrm{id}_X \in (\omega X_m) \otimes (\omega X_m)^\vee.$$

Now,

$$\begin{aligned}
&(\partial_i \otimes \mathrm{id}) \circ \Delta_{m+1,n}(x \otimes c_1 \otimes \ldots \otimes c_{m+n+1} \otimes y) \\
&= \begin{cases} \Delta_{m,n}\partial_i(x \otimes c_1 \otimes \ldots \otimes c_{m+n+1} \otimes y) & i \le m, \\ x \otimes c_1 \otimes \ldots \otimes c_m \otimes (\omega c_{m+1}) \otimes c_{m+2} \otimes \ldots \otimes c_{m+n+1} \otimes y & i = m+1; \end{cases}
\end{aligned}$$

$$\begin{aligned}
&(\mathrm{id} \otimes \partial_i) \circ \Delta_{m,n+1}(x \otimes c_1 \otimes \ldots \otimes c_{m+n+1} \otimes y) \\
&= \begin{cases} \Delta_{m,n}\partial_{i+m}(x \otimes c_1 \otimes \ldots \otimes c_{m+n} \otimes y) & i > 0, \\ x \otimes c_1 \otimes \ldots \otimes c_m \otimes (\omega c_{m+1}) \otimes c_{m+2} \otimes \ldots \otimes c_{m+n+1} \otimes y & i = 0. \end{cases}
\end{aligned}$$

Thus the differential $d = \sum (-1)^i \partial_i$ has the property that

$$[(d \otimes \mathrm{id} + (-1)^m \mathrm{id} \otimes d) \circ \Delta]_{m,n} = \Delta_{m,n} \circ d.$$

In other words, Δ is a chain map with respect to d, so passes to a comultiplication on $C_\omega(\mathcal{A}) = \mathrm{Tot}\, \underline{\mathrm{CC}}_\bullet(\mathcal{A}, \omega \otimes \omega^\vee)$. The properties of the ∂_i above also ensure that the comultiplication descends to $NC_\omega(\mathcal{A})$. $\square$

It is immediately clear that the constructions are functorial in the following sense:

Lemma 1.11. *For any k-linear dg functor $F\colon \mathcal{B} \to \mathcal{A}$, there is an induced morphism $C_{\omega\circ F}(\mathcal{B}) \to C_\omega(\mathcal{A})$ of dg coalgebras, which also induces a morphism on the normalisations.*

In §2.4, we will combine this lemma with Theorem 2.9 to show that $C_\omega(\mathcal{A})$ is essentially invariant under quasi-equivalent choices of $\mathcal{A}$ and quasi-isomorphic choices of ω, so that the choice in Remark 1.8 does not affect the output.

1.2. **The universal coalgebra and tilting modules.**

1.2.1. *Background terminology.* Following the conventions of [Kel2, 3.1], we will write $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ for the dg category of k-linear dg functors $\mathcal{A}^{\mathrm{opp}} \to \mathcal{C}_{\mathrm{dg}}(k)$ to chain complexes over k. Observe that when $\mathcal{A}$ has a single object $*$ with $\mathcal{A}(*,*) = A$, $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ is equivalent to the category of A-modules in complexes. We write $\mathcal{C}(\mathcal{A})$ for the (non-dg) category $\mathrm{Z}^0\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ of dg $\mathcal{A}$-modules.

An object P of $\mathcal{C}(\mathcal{A})$ is cofibrant (for the projective model structure) if every surjective quasi-isomorphism $L \to P$ has a section. The full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ on cofibrant objects is denoted $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$. This is the idempotent-complete pre-triangulated category (in the sense of [BK4, Definition 3.1]) generated by $\mathcal{A}$ and closed under filtered colimits. We write $\mathcal{D}(\mathcal{A})$ for the derived category $\mathrm{H}^0\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ of dg $\mathcal{A}$-modules — this is equivalent to the localisation of $\mathcal{C}(\mathcal{A})$ at quasi-isomorphisms. Thus $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ is a dg enhancement of the triangulated category $\mathcal{D}(A)$.

Definition 1.12. Define $\mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \subset \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ to be the full subcategory on compact objects, i.e those X for which

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(X, -)$$

preserves filtered colimits. Explicitly, $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ consists of objects arising as direct summands of finite complexes of objects of the form $h_X[n]$, for $X \in \mathcal{A}$, where h is the Yoneda embedding.

When $\mathcal{A}$ has a single object $*$ with $\mathcal{A}(*,*) = A$, then $h_*[n]$ corresponds to the A-module $A[n]$. Since projective modules are direct summands of free modules, Definitions 1.12 and 1.4 are thus consistent.

As explained in [Kel2, 4.5], $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ is the idempotent-complete pre-triangulated envelope or hull of $\mathcal{A}$, in the sense of [BK4, §3]. Note that in [Kel1, §2], pre-triangulated categories are called exact DG categories.

By [Tab1, Theorem 5.1], there is a Morita model structure on k-linear dg categories. Weak equivalences are dg functors $\mathcal{A} \to \mathcal{B}$ which are derived Morita equivalences in the sense that

$$\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{B})$$

is a quasi-equivalence. The dg functor $\mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ is fibrant replacement in this model structure.

Note that a dg category $\mathcal{A}$ is an idempotent-complete pre-triangulated category if and only if the natural embedding $\mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ is a quasi-equivalence. This is equivalent to saying that $\mathcal{A}$ is Morita fibrant (i.e. fibrant in the Morita model structure), or triangulated in the terminology of [TV, Definition 2.4].

1.2.2. *Universal coalgebras.*

Definition 1.13. Recall (e.g. from [Toë3, Remark 8.5]) that there is a monoidal structure $\otimes_{\mathcal{A}}$ on the dg category $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$, given by

$$(F \otimes_{\mathcal{A}} G)(X, Y) = F(X, -) \otimes_{\mathcal{A}} G(-, Y),$$

for $X \in \mathcal{A}, Y \in \mathcal{A}^{\mathrm{opp}}$. The unit of the monoidal structure is the dg functor $\mathrm{id}_{\mathcal{A}}$, given by

$$\mathrm{id}_{\mathcal{A}}(X, Y) = \mathcal{A}(X, Y).$$

Take a k-linear dg category $\mathcal{A}$, and $D \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$ a coassociative $\otimes_{\mathcal{A}}$-coalgebra, with the co-unit $D \to \mathrm{id}_{\mathcal{A}}$ a quasi-isomorphism. We regard this as being a universal coalgebra associated to $\mathcal{A}$.

Example 1.14. If the k-complexes $\mathcal{A}(X, Y)$ are all cofibrant (automatic when k is a field), a canonical choice for D is the Hochschild complex

$$\mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$$

of the Yoneda embedding $h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}} \colon \mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$. Explicitly, $\mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$ is the total complex of the chain complex

$$\bigoplus_{X_0 \in \mathcal{A}} X_0^{\mathrm{opp}} \otimes X_0 \leftarrow \bigoplus_{X_0, X_1 \in \mathcal{A}} X_0^{\mathrm{opp}} \otimes \mathcal{A}(X_0, X_1) \otimes X_1 \leftarrow \ldots,$$

where we write X and X^{opp} for the images of X under the Yoneda embeddings $\mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, $\mathcal{A}^{\mathrm{opp}} \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}})$.

The $\otimes_{\mathcal{A}}$-coalgebra structure is given by the formulae of Proposition 1.10, noting that

$$Y \otimes_{\mathcal{A}} X^{\mathrm{opp}} = \mathcal{A}(Y, X),$$

so $\mathrm{id}_X \in X \otimes_{\mathcal{A}} X^{\mathrm{opp}}$.

The normalised version of the Hochschild complex $N\mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$ provides another choice for D, which is more canonical in some respects.

If we write $L = \bigoplus_{X \in \mathcal{A}} X^{\mathrm{opp}} \otimes_k X$, then L is a $\otimes_{\mathcal{A}}$-coalgebra in $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$. The counit is just the composition $\bigoplus_X \mathcal{A}(-, X) \otimes_k \mathcal{A}(X, -) \to \mathcal{A}(-, -)$, and comultiplication comes from $\mathrm{id}_X \in X \otimes_{\mathcal{A}} X^{\mathrm{opp}}$. Then D is the total complex of the simplicial diagram given by $\underbrace{L \otimes_{\mathcal{A}} L \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} L}_{n+1}$ in level n, so D is just the Čech nerve of the $\otimes_{\mathcal{A}}$-comonoid L.

Definition 1.15. Say that a coassociative $\otimes_{\mathcal{A}}$-coalgebra $D \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$ is ind-compact if it can be expressed as a filtered colimit $D \cong \varinjlim_i D_i$ in the underlying category $\mathrm{Z}^0\mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$, with each D_i a coassociative $\otimes_{\mathcal{A}}$-coalgebra which is compact as an object of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$.

Note that if $\mathcal{A}$ itself is a field, then the fundamental theorem of coalgebras ([JS, Proposition 7.1]) says that all $\otimes_{\mathcal{A}}$-coalgebras are filtered colimits (i.e. nested unions) of finite-dimensional coalgebras, so are ind-compact.

Example 1.16. If k is a field, then the $\otimes_{\mathcal{A}}$-coalgebra $\mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$ is ind-compact. We construct the exhaustive system of compact subcoalgebras as follows. The indexing set will consist of triples (S, n, V) with S a finite subset of $\mathrm{Ob}\mathcal{A}$, $n \in \mathbb{N}_0$ and $V(X, Y) \subset \mathcal{A}(X, Y)$ a collection of finite-dimensional cochain complexes for $X, Y \in S$.

For $X', Y' \in S$, we now let $V^{(i)}(X', Y') \subset \mathcal{A}(X', Y')$ be the cochain complex generated by strings of length at most 2^i in elements of V. We now define $D_{(S,n,V)} \subset D$ to be the total complex of

$$\begin{aligned}\bigoplus_{X_0 \in S} X_0^{\mathrm{opp}} \otimes X_0 \leftarrow & \bigoplus_{X_0, X_1 \in S} X_0^{\mathrm{opp}} \otimes V^{(n-1)}(X_0, X_1) \otimes X_1 \\ \leftarrow & \bigoplus_{X_0, X_1, X_2 \in S} X_0^{\mathrm{opp}} \otimes V^{(n-2)}(X_0, X_1) \otimes V^{(n-2)}(X_1, X_2) \otimes X_2 \\ \leftarrow \ldots \leftarrow & \bigoplus_{X_0, \ldots X_n \in S} X_0^{\mathrm{opp}} \otimes V^{(0)}(X_0, X_1) \otimes \ldots \otimes V^{(0)}(X_{n-1}, X_n) \otimes X_n.\end{aligned}$$

This is indeed a complex because multiplication in $\mathcal{A}$ gives boundary maps $V^{(n-i)}(X, Y) \otimes V^{(n-i)}(Y, Z) \to V^{(n-i-1)}(X, Z)$, and it is a subcoalgebra because $V^{(n-i-j)} \subset V^{(n-i)} \cap V^{(n-j)}$. The indexing set becomes a poset by saying $(S, m, U) \subset (T, n, V)$ whenever $S \subset T$, $m \leq n$ and $U \subset V$. Thus we have a filtered colimit

$$D = \varinjlim_{(S,n,V)} D_{(S,n,V)},$$

of the required form.

1.2.3. *Tilting modules.* Given $\omega\colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$, define the tilting module P by $P := D \otimes_{\mathcal{A}} \omega \in \mathcal{C}(\mathcal{A}^{\mathrm{opp}})$; this is cofibrant and has a natural quasi-isomorphism $P \to \omega$. Also set $Q \in \mathcal{C}(\mathcal{A})$ by $Q := \omega^\vee \otimes_{\mathcal{A}} D$ and set $C := \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$. Note that the natural transformation $\mathrm{id}_{\mathcal{A}} \to \omega \otimes_k \omega^\vee$ makes C into a dg coalgebra over k:

$$\begin{aligned} C = \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \to{} & \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \\ &= \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \\ &\to \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \otimes_k \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega \\ &= C \otimes_k C. \end{aligned}$$

Likewise, P becomes a right C-comodule and Q a left C-comodule.

Also note that because D is a cofibrant replacement for $\mathrm{id}_{\mathcal{A}}$, we have

$$C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \mathrm{id}_{\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{A}} \omega \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega.$$

For a chosen exhaustive system $D = \varinjlim_i D_i$ of an ind-compact coalgebra, we also write $P_i := D_i \otimes_{\mathcal{A}} \omega$, $C_i := \omega^\vee \otimes_{\mathcal{A}} D_i \otimes_{\mathcal{A}} \omega$ and $Q_i := \omega^\vee \otimes_{\mathcal{A}} D_i$. Each C_i is a dg coalgebra, with P_i (resp. Q_i) a right (resp. left) C_i-comodule.

Example 1.17. When $D = \mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$, observe that

$$\begin{aligned} C &= \mathrm{CC}(\mathcal{A}, \omega \otimes \omega), \\ P &= \mathrm{CC}(\mathcal{A}, h_{\mathcal{A}} \otimes \omega), \\ Q &= \mathrm{CC}(\mathcal{A}, \omega^\vee \otimes h_{\mathcal{A}^{\mathrm{opp}}}), \end{aligned}$$

so C is just the dg coalgebra $C_\omega(\mathcal{A})$ of Definition 1.9.

1.2.4. *Preduals.*

Definition 1.18. Given $M \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, define the predual $M' \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}})$ to be the dg functor $M'\colon \mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(k)$ given by $M'(Y) = \underline{\mathrm{Hom}}_{\mathcal{C}_{\mathrm{dg}}(\mathcal{A})}(M, Y)$. Note that this construction is only quasi-isomorphism invariant for $M \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$.

Observe that for $X \in \mathcal{A}$ and the Yoneda embeddings h, we have $h'_X = h_{X^{\mathrm{opp}}}$, giving isomorphisms $N \otimes_{\mathcal{A}} h_X \cong N(X) \cong \underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{opp}}}(h'_X, N)$ for all $N \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}})$. Passing to finite complexes and arbitrary colimits in $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, this gives us a natural transformation

$$N \otimes_{\mathcal{A}} M \to \underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{opp}}}(M', N)$$

for all $M \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$; this is necessarily an isomorphism when $M \in \mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ because both sides preserve finite complexes and direct summands.

1.3. **Monoidal categories.** In order to recover the setting of [DMOS, Ch. II], we now introduce monoidal structures. For the purposes of this subsection $(\mathcal{A}, \boxtimes, \mathbb{1})$ is a strictly monoidal dg category, so we have k-linear dg functors $\mathbb{1} \colon k \to \mathcal{A}$ and $\boxtimes \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, such that if we also write $\mathbb{1}$ for the image of the unique object in k,

$$(X \boxtimes Y) \boxtimes Z = X \boxtimes (Y \boxtimes Z), \quad \mathbb{1} \boxtimes X = X, \quad X \boxtimes \mathbb{1} = \mathbb{1}.$$

Definition 1.19. Say that a dg functor $\omega \colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$ is lax monoidal if it is equipped with natural transformations

$$\mu_{XY} \colon \omega(X) \otimes \omega(Y) \to \omega(X \boxtimes Y), \quad \epsilon \colon k \to \omega(\mathbb{1})$$

satisfying associativity and unitality conditions.

It is said to be strict (resp. strong, resp. quasi-strong) if μ and η are equalities (resp. isomorphisms, resp. quasi-isomorphisms).

Remark 1.20. The hypothesis that $\mathcal{A}$ and ω be strictly monoidal is of course very strong. All the results of this section will be straightforwardly functorial with respect to isomorphisms, though not always with respect to quasi-isomorphisms, so we could replace $\mathcal{A}$ with any equivalent dg category (quasi-equivalent does not suffice). Thus the results will also apply to strongly monoidal dg categories and functors, where the equalities above are replaced by isomorphisms in such a way that $\mathrm{Z}^0\mathcal{A}$ becomes a strongly monoidal category and $\omega \colon \mathrm{Z}^0\mathcal{A} \to \mathrm{Z}^0\mathrm{per}_{\mathrm{dg}}(k)$ a strongly monoidal functor. This condition is satisfied by Example 2.22, our main motivating example.

1.3.1. *The Tannakian envelope for strongly monoidal functors.*

Definition 1.21. Given dg functors $F \colon \mathcal{B} \to \mathrm{per}_{\mathrm{dg}}(k)$, $G \colon \mathcal{C} \to \mathrm{per}_{\mathrm{dg}}(k)$, define the external tensor product

$$F \odot G \colon \mathcal{B} \otimes \mathcal{C} \to \mathrm{per}_{\mathrm{dg}}(k)$$

by $(F \odot G)(X \otimes Y) := F(X) \otimes_k G(Y)$.

Lemma 1.22. *For dg categories $\mathcal{B}$,$\mathcal{C}$ and k-linear dg functors $F \colon \mathcal{B} \to \mathrm{per}_{\mathrm{dg}}(k)$, $G \colon \mathcal{C} \to \mathrm{per}_{\mathrm{dg}}(k)$, the dg coalgebras of Proposition 1.10 have canonical quasi-isomorphisms*

$$C_F(\mathcal{B}) \otimes_k C_G(\mathcal{C}) \to C_{F \odot G}(\mathcal{B} \otimes \mathcal{C}),$$
$$NC_F(\mathcal{B}) \otimes_k NC_G(\mathcal{C}) \to NC_{F \odot G}(\mathcal{B} \otimes \mathcal{C}).$$

These maps are symmetric on interchanging $(\mathcal{B}, F)$ and $(\mathcal{C}, G)$, and the construction is associative in the sense that it induces a unique map

$$C_F(\mathcal{B}) \otimes_k C_G(\mathcal{C}) \otimes_k C_H(\mathcal{D}) \to C_{F \odot G \odot H}(\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}),$$

and similarly for NC.

Proof. First observe that we have canonical isomorphisms

$$\underline{\mathrm{CC}}_m(\mathcal{B}\otimes\mathcal{C},(F\odot G)^\vee\otimes(F\odot G))\cong\underline{\mathrm{CC}}_m(\mathcal{B},F)\otimes_k\underline{\mathrm{CC}}_m(\mathcal{C},G)$$

for all m. These isomorphisms are clearly compatible with the comultiplication maps $\Delta\colon\ \underline{\mathrm{CC}}_{m+n}\to\underline{\mathrm{CC}}_m\otimes\underline{\mathrm{CC}}_n$ and with the simplicial operations.

Now the Eilenberg–Zilber shuffle product of [Qui, I.4.2–3] gives a symmetric associative quasi-isomorphism from $C_F(\mathcal{B})\otimes_k C_G(\mathcal{C})$ to the total complex of the simplicial cochain complex $m\mapsto\underline{\mathrm{CC}}_m(\mathcal{B},F)\otimes_k\underline{\mathrm{CC}}_m(\mathcal{C},G)$, which is compatible with the comultiplications. Combined with the isomorphisms above, this gives

$$C_F(\mathcal{B})\otimes_k C_G(\mathcal{C})\to C_{F\odot G}(\mathcal{B}\otimes\mathcal{C}),$$

and similarly on normalisations. □

Proposition 1.23. *If $\omega\colon\ \mathcal{A}\to\mathrm{per}_{\mathrm{dg}}(k)$ is strongly monoidal, the monoidal structures endow the dg coalgebras $C_\omega(\mathcal{A}),NC_\omega(\mathcal{A})$ with the natural structure of unital dg bialgebras. These are graded-commutative whenever $\boxtimes$ and ω are symmetric.*

Proof. Define a dg functor $\boxtimes_*\omega$ on $\mathcal{A}\otimes\mathcal{A}$ by $(\boxtimes_*\omega)(X\otimes Y):=\omega(X\boxtimes Y)$. We may apply Lemma 1.11 to the dg functor $\boxtimes$ to obtain a morphism

$$C_{\boxtimes_*\omega}(\mathcal{A}\otimes\mathcal{A})\to C_\omega(\mathcal{A}).$$

Strong monoidality of ω gives $\boxtimes_*\omega\cong\omega\odot\omega$ and hence $C_{\boxtimes_*\omega}(\mathcal{A}\otimes\mathcal{A})\cong C_{\omega\odot\omega}(\mathcal{A}\otimes\mathcal{A})$. Lemma 1.22 then provides a dg coalgebra quasi-isomorphism

$$C_\omega(\mathcal{A})\otimes_k C_\omega(\mathcal{A})\to C_{\omega\odot\omega}(\mathcal{A}\otimes\mathcal{A}),$$

which completes the construction of the associative multiplication. This product is moreover commutative whenever $\boxtimes$ and ω are symmetric, and induces a product on $NC_\omega(\mathcal{A})$ similarly.

Applying Lemma 1.11 to the unit $\mathbb{1}$ similarly induces morphisms

$$k=C_{\mathrm{id}}(k)\cong C_{\omega\circ\mathbb{1}}(k)\to C_\omega(\mathcal{A}),$$

and unitality of ω and $\mathbb{1}$ ensures that this is a unit for the multiplication above. □

Remark 1.24. In the scenario considered in [DMOS, Ch. II], the tensor category was rigid in the sense that it admitted strong duals, or equivalently internal Homs. Then the Tannaka dual bialgebra $\mathrm{HH}_0(\mathcal{A},\omega^\vee\otimes_k\omega)$ became a Hopf algebra.

If our dg category $\mathcal{A}$ has strong duals, then we may define an involution ρ on $C_\omega(\mathcal{A})$ by combining the isomorphism $C_\omega(\mathcal{A})^{\mathrm{opp}}\cong C_{\omega^\vee}(\mathcal{A}^{\mathrm{opp}})$ with the map $C_{\omega^\vee}(\mathcal{A}^{\mathrm{opp}})\to C_\omega(\mathcal{A})$ induced by applying Lemma 1.11 to the duality functor.

The condition that ρ be an antipode on a bialgebra C is that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle(\rho\otimes\mathrm{id})} \\ k & \xrightarrow{1} & C \end{array}\qquad\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle(\mathrm{id}\otimes\rho)} \\ k & \xrightarrow{1} & C \end{array}$$

commute.

On the bialgebra $\pi_0\underline{\mathrm{CC}}_\bullet(\mathcal{A},\omega^\vee\otimes\omega)$, it turns out that ρ defines an antipode, making the bialgebra into a Hopf algebra and recovering the construction of [DMOS, II.2].

However, the dg bialgebras $C_\omega(\mathcal{A}), NC_\omega(\mathcal{A})$ are far from being dg Hopf algebras. This is easily seen by looking at
$$\underline{\mathrm{CC}}_0(\mathcal{A}, \omega^\vee \otimes \omega)^\vee = \prod_{X \in \mathcal{A}} \mathrm{End}(\omega X).$$
The antipodal condition above reduces to saying that for all $f \in \underline{\mathrm{CC}}_0(\mathcal{A}, \omega^\vee \otimes \omega)^\vee$, we require that $\omega(\epsilon_X) \circ f_{X^* \boxtimes X} = f_1 \circ \omega(\epsilon_X)$, for $\epsilon_X \colon X^* \boxtimes X \to \mathbb{1}$ the duality transformation. There are few dg categories $\mathcal{A}$ for which this holds, so ρ seldom makes $\underline{\mathrm{CC}}_0(\mathcal{A}, \omega^\vee \otimes \omega)$ into a Hopf algebra. However, the condition above automatically holds for all Hochschild 0-cocycles, which is why $\pi_0 \underline{\mathrm{CC}}_\bullet(\mathcal{A}, \omega^\vee \otimes \omega)$ is a Hopf algebra.

In §3.4 we will see a context where a variant of the Hochschild complex does have a suitable antipode, and hence the structure of a Hopf algebra.

1.3.2. *The Tannakian envelope for lax monoidal functors.* If the dg functor ω is only quasi-strong, it is too much to expect that $C_\omega(\mathcal{A})$ will be a bialgebra in general. A dg bialgebra is a monoid in dg coalgebras, and it is not usually possible to strictify algebraic and coalgebraic structures simultaneously, so $C_\omega(\mathcal{A})$ should be a form of strong homotopy monoid in dg coalgebras.

We will now construct the structures enriching $C_\omega(\mathcal{A})$ for any lax monoidal dg functor ω. When ω is quasi-strong, Corollary 2.16 will ensure that this indeed gives a form of strong homotopy monoid.

Definition 1.25. Define I to be the category on two objects $0, 1$ with a unique non-identity morphism $0 \to 1$. Define K to be the category whose objects are $I, \{0\}, \{1\}$ and whose non-identity morphisms are the inclusions $\{0\}, \{1\} \to I$. Thus the objects of K are categories in their own right.

As in [Pri4, Definition 1.1], we will write Δ_{**} for the subcategory of the ordinal number category Δ consisting of morphisms which fix the initial and final vertices. This has a monoidal structure given by setting $\mathbf{m} \otimes \mathbf{n} = \mathbf{m} + \mathbf{n}$. As in [Pri4, Definition 1.2], Δ_{**} is opposite to the augmented ordinal number category, so an up-to-homotopy monoid structure on $C \in \mathcal{C}$ in the sense of [Lei] is a colax monoidal functor $M \colon \Delta_{**} \to \mathcal{C}$ for which the maps $M(i + j) \to M(i) \otimes M(j)$ are weak equivalences and $M(\mathbf{1}) = C$.

We now adapt [Pri4, Definition 3.10]:

Definition 1.26. Define $\mathbb{K} \colon \Delta_{**} \to \mathrm{Cat}$ to be the category-valued lax monoidal functor given on objects by $\mathbf{0} \mapsto *$, $\mathbf{n} \mapsto K^{n-1}$ and on morphisms by
$$\begin{aligned}
\mathbb{K}(\partial^i)(k_1, \ldots, k_n) &= (k_1, \ldots, k_{i-1}, \{1\}, k_i, \ldots, k_n);\\
\sigma^i(k_1, \ldots, k_n) &= (k_1, \ldots, k_{i-1}, m(k_i, k_{i+1}), k_{i+2}, \ldots, k_n);\\
\sigma^0(k_1, \ldots, k_n) &= (k_2, \ldots, k_n);\\
\sigma^n(k_1, \ldots, k_n) &= (k_1, \ldots, k_{n-1}),
\end{aligned}$$
where $m \colon K^2 \to K$ is the symmetric functor determined by $m(\{0\}, k) = \{0\}$, $m(\{1\}, k) = k$, $m(I, I) = I$. The monoidal structure on $\mathbb{K}$ is given by the maps
$$K^{m-1} \times K^{n-1} \cong K^{m-1} \times \{0\} \times K^{n-1} \subset K^{m+n-1}.$$

Proposition 1.27. *If $\omega \colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$ is lax monoidal with $\omega(\mathbb{1}) = k$, the monoidal structures give rise to a colax monoidal functor $\mathbb{C}_\omega(\mathcal{A})$ from the opposite $\Gamma(\mathbb{K}^{\mathrm{opp}})^{\mathrm{opp}}$ of the Grothendieck construction $\Gamma(\mathbb{K}^{\mathrm{opp}})$ of $\mathbb{K}^{\mathrm{opp}}$ to the category of dg coalgebras over k, with $\mathbb{C}_\omega(\mathcal{A})(\mathbf{1}) = C_\omega(\mathcal{A})$. There is a similar construction for $NC_\omega(\mathcal{A})$.*

Proof. We prove this for $C_\omega(\mathcal{A})$, the proof for $NC_\omega(\mathcal{A})$ being entirely similar. On the category $\mathcal{A}^{\otimes 2}$, ω induces dg fibre functors $\omega \odot \omega$ and $\boxtimes_*\omega$. The transformation between them gives a dg fibre functor $\underline{\omega}$ on $\mathcal{A}^{\otimes 2} \times I$ with $(X, Y, 0) \mapsto \omega(X) \otimes \omega(Y)$ and $(X, Y, 1) \mapsto \omega(X \boxtimes Y)$.

Iterating this construction gives us dg fibre functors $\underline{\omega}\colon \mathcal{A}^{\otimes n} \times I^{n-1} \to \mathrm{per}_{\mathrm{dg}}(k)$ for all n. At a vertex of I^{n-1}, the corresponding fibre functor is given by writing ω for each 0 co-ordinate and $\boxtimes_*$ for each 1 co-ordinate, then appending a final ω and introducing $\odot$s as separators. For $I^{n-1} \in K^{n-1}$, we define

$$\mathbb{C}_\omega(\mathcal{A})(n, I^{n-1}) := C_{\underline{\omega}}(\mathcal{A} \times I^{n-1}).$$

Now, any object $k \in K^{n-1}$ is a subcategory of I^{n-1}, and we may define $\mathbb{C}_\omega(\mathcal{A})(n, k) := C_{\underline{\omega}|_k}(\mathcal{A} \times k)$. Lemma 1.11 then gives morphisms $\mathbb{C}_\omega(\mathcal{A})(n, k) \to \mathbb{C}_\omega(\mathcal{A})(n, l)$ for each morphism $k \to l$ in K^{n-1}. This defines $\mathbb{C}_\omega(\mathcal{A})$ on the subcategories $\mathbb{K}(n) \subset \Gamma(\mathbb{K}^{\mathrm{opp}})^{\mathrm{opp}}$, and it remains to define the images of the cosimplicial morphisms ∂^i, σ^i and the monoidal structure.

The fibred dg functor $(\mathcal{A} \otimes \mathcal{A}, \omega \boxtimes \omega) \xrightarrow{\boxtimes} (\mathcal{A}, \omega)$ induces fibred dg functors $(\mathcal{A}^{\otimes n+1} \times \partial^i(k), \underline{\omega}|_{\partial^i(k)}) \to (\mathcal{A}^{\otimes n} \times k, \underline{\omega}|_k)$ for all i, n and $k \in K^{n-1}$. By Lemma 1.11, this gives a morphism

$$\partial_i\colon\ \mathbb{C}_\omega(\mathcal{A})(n + 1, \partial^i(k)) \to \mathbb{C}_\omega(\mathcal{A})(n, k)$$

of dg coalgebras, which we define to be the image of $\partial_i\colon\ (\mathbf{n+1}, \partial^i(k)) \to (\mathbf{n}, k)$.

Substituting the unit $\mathbb{1} \in \mathcal{A}$ in either factor induces fibred dg functors $(\mathcal{A}, \omega) \to (\mathcal{A} \otimes \mathcal{A}, \omega \boxtimes \omega)$ and $(\mathcal{A}, \omega) \to (\mathcal{A} \otimes \mathcal{A}, \omega \odot \omega)$. Similar arguments show that these induce morphisms

$$\sigma_i\colon\ \mathbb{C}_\omega(\mathcal{A})(n - 1, \sigma^i(k)) \to \mathbb{C}_\omega(\mathcal{A})(n, k).$$

To define the colax monoidal structure, we need morphisms

$$\mathbb{C}_\omega(\mathcal{A})(m, k) \otimes \mathbb{C}_\omega(\mathcal{A})(n, l) \to \mathbb{C}_\omega(\mathcal{A})(m + n, k \times \{0\} \times l),$$

but these are just given by Lemma 1.22. □

Remark 1.28. We can give $\Gamma(\mathbb{K}^{\mathrm{opp}})$ the structure of a relative category by setting a morphism to be a weak equivalence when its image in Δ_{**} is the identity. Since K has a final object, its nerve is contractible, so the projection map $\Gamma(\mathbb{K}^{\mathrm{opp}}) \to \Delta_{**}$ is a weak equivalence of relative categories in the sense of [BK1].

If ω is quasi-strong, then Corollary 2.16 will imply that $\mathbb{C}_\omega$ sends all weak equivalences to derived Morita equivalences. If we let $\mathcal{C}$ be the relative category of dg k-coalgebras with weak equivalences given by derived Morita equivalences, then Proposition 1.27 gives a colax monoidal functor $(\int \mathbb{K}^{\mathrm{opp}})^{\mathrm{opp}} \to \mathcal{C}$ of relative categories. By contrast, an up-to-homotopy monoid is just a colax monoidal functor $\Delta_{**}^{\mathrm{opp}} \to \mathcal{C}$ of relative categories. Since $\int \mathbb{K}^{\mathrm{opp}}$ is weakly equivalent to Δ_{**}, Proposition 1.27 can thus be regarded as giving $C_\omega(\mathcal{A})$ the structure of a monoid up to coherent homotopy whenever ω is quasi-strong.

If the monoidal structure on the pair $(\mathcal{A}, \omega)$ is moreover symmetric, then structures above can be adapted by replacing Δ with the category of finite sets, thus incorporating symmetries by not restricting to non-decreasing maps, and giving rise to a homotopy coherent symmetric monoid.

1.3.3. *The universal bialgebra.* The monoidal structure $\boxtimes$ on $\mathcal{A}$ induces a monoidal structure on $\mathcal{A}^{\mathrm{opp}}$, which we also denote by $\boxtimes$. There is also a monoidal structure $\boxtimes^2$ on $\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A}$, given by $(X \otimes Y) \boxtimes^2 (X' \otimes Y') := (X \boxtimes X') \otimes (Y \boxtimes Y')$.

Definition 1.29. Define the dg functor

$$\boxtimes\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \otimes \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$$

as follows. The dg functor $\boxtimes\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ induces a dg functor $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}\otimes\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$, which we compose with the dg functor $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \otimes \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \xrightarrow{\odot} \mathcal{D}_{\mathrm{dg}}(\mathcal{A}\otimes\mathcal{A})$ given by $(M\odot N)(X\otimes Y) := M(X)\otimes_k N(Y)$ for $X, Y \in \mathcal{A}$. Define $\boxtimes\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}})\otimes\mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}})$ and $\boxtimes^2\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A}) \otimes \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A})$ similarly.

In simpler terms, $\boxtimes$ is just given by extending the dg functor on $\mathcal{A}$ to finite complexes, filtered colimits and direct summands.

Definition 1.30. Define $\boxtimes_*\colon \mathcal{C}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A}\otimes\mathcal{A})$ by setting

$$(\boxtimes_* M)(X\otimes Y) = M(X\boxtimes Y)$$

for $X, Y \in \mathcal{A}$.

Define $\boxtimes_*\colon \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}) \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A}^{\mathrm{opp}})$ and $\boxtimes^2_*\colon \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A}) \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A}\otimes\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A})$ similarly.

Remark 1.31. For $S, T \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ and $M \in \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, note that we have a natural isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}_{\mathrm{dg}}(\mathcal{A})}(S\boxtimes T, M) \cong \underline{\mathrm{Hom}}_{\mathcal{C}_{\mathrm{dg}}(\mathcal{A}\otimes\mathcal{A})}(S\otimes_k T, \boxtimes_* M).$$

This isomorphism is tautological when $S = h_X, T = h_Y$ for $X, Y \in \mathcal{A}$, noting that

$$h_{X\otimes Y}(U\otimes V) = \mathcal{A}(X,U)\otimes_k \mathcal{A}(Y,V) = h_X \otimes_k h_Y.$$

The general case follows by passing to complexes and direct summands.

The same observation holds for any monoidal dg category, and hence to $(\mathcal{A}^{\mathrm{opp}}, \boxtimes)$ and $(\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A}, \boxtimes^2)$.

Lemma 1.32. *The unit* $\mathrm{id}_{\mathcal{A}} \in \mathcal{C}(\mathcal{A}\otimes\mathcal{A}^{\mathrm{opp}})$ *is equipped with a canonical associative multiplication*

$$\mathrm{id}_{\mathcal{A}} \otimes_k \mathrm{id}_{\mathcal{A}} \to \boxtimes^2_* \mathrm{id}_{\mathcal{A}},$$

which is commutative whenever $\boxtimes$ *is symmetric. The unit for this multiplication is*

$$\mathrm{id}_{\mathbb{1}} \in \mathcal{A}(\mathbb{1},\mathbb{1}) = \mathrm{id}_{\mathcal{A}}(\mathbb{1},\mathbb{1}) = \mathbb{1}^2_* k,$$

for $\mathbb{1}^2_*\colon \mathcal{C}_{\mathrm{dg}}(k) \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A})$.

Proof. Evaluated at $X\otimes Y\otimes X'\otimes Y' \in \mathcal{A}\otimes\mathcal{A}^{\mathrm{opp}}\otimes\mathcal{A}\otimes\mathcal{A}^{\mathrm{opp}}$, this is just the map

$$\mathcal{A}(X,Y)\otimes_k \mathcal{A}(X',Y') \to \mathcal{A}(X\boxtimes X', Y\boxtimes Y')$$

induced by the bilinearity of $\boxtimes$. □

Definition 1.33. We say that a universal coalgebra D (in the sense of §1.2.2) is a universal bialgebra with respect to $\boxtimes$ if is equipped with an associative multiplication $D\otimes_k D \to \boxtimes^2_* D$ and unit $k \to D(\mathbb{1},\mathbb{1})$. These are required to be compatible with the coalgebra structure, in the sense that the comultiplication and co-unit

$$D \to D\otimes_{\mathcal{A}} D \quad D \to \mathrm{id}_{\mathcal{A}}$$

must be morphisms of associative unital $\boxtimes^2$-algebras.

When $\boxtimes$ is symmetric, we say that D is a universal commutative bialgebra if the multiplication $D\otimes_k D \to \boxtimes^2_* D$ is commutative.

Remark 1.34. Since universal coalgebras are required to be objects of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$, we may apply Remark 1.31 to rephrase the algebra structure on D to be an associative multiplication $D \boxtimes^2 D \to D$ and a unit $\mathbb{1} \otimes \mathbb{1} \to D$.

Example 1.35. Under the conditions of Example 1.16 (e.g. when k is a field), the Hochschild complexes

$$\mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}}) \quad N\mathrm{CC}(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$$

associated to the Yoneda embedding $h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}} \colon \mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$ are universal bialgebras, commutative whenever $\boxtimes$ is symmetric.

The coalgebra structure is given in Example 1.16, and the multiplication and unit are given by the formulae of Proposition 1.23.

Lemma 1.36. *Given a universal bialgebra D and a strong monoidal dg functor ω, the dg coalgebra $C := \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$ becomes a unital associative dg bialgebra, which is commutative whenever D is commutative and ω symmetric.*

Proof. Since $\omega, \omega^\vee$ are strong monoidal dg functors, we have an isomorphism

$$\omega^\vee \otimes_{\mathcal{A}} (D \boxtimes^2 D) \otimes_{\mathcal{A}} \omega \cong (\omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega) \otimes_k (\omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega),$$

so the multiplication $D \boxtimes^2 D \to D$ gives $C \otimes_k C \to C$. Likewise,

$$\omega^\vee \otimes_{\mathcal{A}} (\mathbb{1} \otimes \mathbb{1}) \otimes_{\mathcal{A}} \omega = \omega^\vee(\mathbb{1}) \otimes_k \omega(\mathbb{1}) \cong k,$$

so the unit gives $k \to C$. Compatibility of the algebra and coalgebra structures follows from the corresponding results for D. $\square$

Remark 1.37. When $\mathcal{A}$ is a neutral Tannakian category, taking duals gives an equivalence $\mathcal{A}^{\mathrm{opp}} \simeq \mathcal{A}$. Then $\mathrm{id}_{\mathcal{A}} \in \mathcal{C}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$ corresponds to the ring of functions on Deligne's fundamental groupoid $G(\mathcal{A}) \in \mathcal{C}(\mathcal{A} \otimes \mathcal{A})^{\mathrm{opp}}$ from [Del1, 6.13]. Since $\mathrm{id}_{\mathcal{A}} = \mathrm{H}^0(D)$, we thus think of D as being the ring of functions on the path space of $\mathcal{A}$.

1.3.4. *Tilting modules.*

Lemma 1.38. *Given a universal bialgebra D and a strong monoidal dg functor ω, the tilting module $P := D \otimes_{\mathcal{A}} \omega$ becomes a monoid in $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}})$ with respect to $\boxtimes$, which is commutative whenever D is commutative and ω symmetric.*

Moreover, the co-action $P \to P \otimes_k C$ of §1.2.3 is an algebra morphism in the sense that the diagram

$$\begin{tikzcd} P \boxtimes P \arrow[r] \arrow[d] & (P \otimes_k C) \boxtimes (P \otimes_k C) \arrow[r, equal] & (P \boxtimes P) \otimes_k (C \otimes_k C) \arrow[d] \\ P \arrow[rr] & & (P \otimes_k C) \end{tikzcd}$$

commutes, where the horizontal maps are co-action and the vertical maps are multiplication.

Proof. Since ω is a strong monoidal dg functor, we have an isomorphism

$$(D \boxtimes^2 D) \otimes_{\mathcal{A}} \omega \cong (D \otimes_{\mathcal{A}} \omega) \boxtimes (D \otimes_{\mathcal{A}} \omega),$$

which gives the required multiplication, with existence of the unit coming from the isomorphism $(\mathbb{1} \otimes \mathbb{1}) \otimes_{\mathcal{A}} \omega \cong \mathbb{1} \otimes_k \omega(\mathbb{1}) \cong \mathbb{1}$.

The final statement follows from compatibility of the algebra and coalgebra structures for D. $\square$

2. Comodules

From now on, k will be a field. Throughout this section, we will fix a small k-linear dg category $\mathcal{A}$, a k-linear dg functor $\omega\colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$, and a universal coalgebra $D \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$ in the sense of §1.2.2. We write $C := \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$ and $P := D \otimes_{\mathcal{A}} \omega$ for the associated dg coalgebra and tilting module.

2.1. **The Quillen adjunction.**

2.1.1. *Model structure on dg comodules.*

Definition 2.1. Let $\mathcal{C}_{\mathrm{dg}}(C)$ be the dg category of right C-comodules in cochain complexes over k. Write $\mathcal{C}(C)$ for the underlying category $\mathrm{Z}^0\mathcal{C}_{\mathrm{dg}}(C)$ of right C-comodules in cochain complexes, and $\mathcal{D}(\mathcal{C})$ for the homotopy category given by formally inverting quasi-isomorphisms.

Proposition 2.2. *There is a closed model structure on $\mathcal{C}(C)$ in which weak equivalences are quasi-isomorphisms and cofibrations are injections. Fibrations are surjections with kernel K such that*

(1) *the graded module $K^{\#}$ underlying K is injective as a comodule over the graded coalgebra $C^{\#}$ underlying C, and*
(2) *for all acyclic N, $\underline{\mathrm{Hom}}_C(N, K)$ is acyclic.*

Proof. This is described in [Pos, Remark 8.2], as the model structure "of the first kind". For ease of reference, we summarise the arguments here.

As in [Pos, Theorem 8.1], the lifting properties follow from the statement that $\underline{\mathrm{Hom}}_C(E, I) \simeq 0$ whenever I is fibrant and either E or I is acyclic. For E, this is tautologous. For I, note that the identity morphism in $\underline{\mathrm{Hom}}_C(I, I)$ is then a coboundary, so we have a contracting homotopy h with $[d, h] = \mathrm{id}$, implying that $\underline{\mathrm{Hom}}_C(E, I) \simeq 0$ for all E.

To establish factorisation, we first observe that we can embed any comodule M into a quasi-isomorphic $C^{\sharp}$-injective comodule using a bar resolution

$$\bigoplus_{n \geq 0} M \otimes C^{\otimes n+1}[-n].$$

Fibrant replacement then follows from a triangulated argument, [Pos, Lemma 1.3]. The key step is given in [Pos, Lemma 5.5], where Brown representability gives a right adjoint to the functor from the coderived category to the derived category. □

Remark 2.3. We might sometimes wish to consider multiple dg fibre functors. Given a set $\{\omega_x\}_{x \in X}$ of dg fibre functors, we can consider the coalgebroid C on objects X given by $C(x, y) := \omega_x^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega_y$, with comultiplication $C(x, y) \to C(x, z) \otimes_k C(z, y)$ and counit $C(x, x) \to k$ defined by the usual formulae.

There is also a category $\mathcal{C}(C)$ of right C-comodules in cochain complexes, with such a comodule M consisting of cochain complexes $M(x)$ for each $x \in X$, together with a distributive action $M(y) \to M(x) \otimes C(x, y)$. The proof of Proposition 2.2 then adapts to give a closed model structure on the category $\mathcal{C}(C)$, noting that bar resolutions

$$y \mapsto \bigoplus_{x \in X^{n+1}} M(x_0) \otimes C(x_0, x_1) \otimes \ldots \otimes C(x_{n-1}, x_n) \otimes C(x_n, y)[-n]$$

still exist in this setting.

Definition 2.4. Given a left C-comodule M and a right C-comodule N, set the cotensor product $N \otimes^C M$ to be kernel of the map

$$(\mu_N \otimes_k \mathrm{id}_M - \mathrm{id}_N \otimes_k \mu_M)\colon\ N \otimes_k M \to N \otimes_k C \otimes_k M,$$

where μ denotes the C-coaction. Note that this is denoted by $N \square_C M$ in [Pos, 2.1].

2.1.2. *The Quillen adjunction.*

Lemma 2.5. *The adjunction*

$$\begin{tikzcd} \mathcal{C}(\mathcal{A}) \arrow[r, shift left, "-\otimes_{\mathcal{A}} P"] \arrow[r, phantom, "\perp"] & \mathcal{C}(C) \arrow[l, shift left, "\underline{\mathrm{Hom}}_C(P,-)"] \end{tikzcd}$$

is a Quillen adjunction.

Proof. It suffices to show that $- \otimes_{\mathcal{A}} P$ sends (trivial) generating cofibrations to (trivial) cofibrations. Generating cofibrations are of the form $X \otimes_k U \to X \otimes_k V$ for $X \in \mathcal{A}$ and $U \hookrightarrow V$ finite-dimensional cochain complexes. Now, $(X \otimes_k U) \otimes_{\mathcal{A}} P = U \otimes_k P(X)$, and $\otimes_k P(X)$ preserves both injections and quasi-isomorphisms. $\square$

Definition 2.6. Denote the co-unit of the Quillen adjunction by

$$\varepsilon_N\colon\ \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P \to N.$$

2.1.3. *The retraction.* From now on, we assume that our chosen $\otimes_{\mathcal{A}}$-coalgebra D is ind-compact.

Proposition 2.7. *The counit*

$$\varepsilon_N\colon\ \mathbf{R}\underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P \to N$$

of the derived adjunction $(- \otimes_{\mathcal{A}} P) \dashv \mathbf{R}\underline{\mathrm{Hom}}_C(P, -)$ is an isomorphism in the derived category $\mathcal{D}(C)$ for all N.

Proof. For any C-comodule N, we have the following isomorphisms

$$\begin{aligned} \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P &= \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} (\varinjlim_i P_i) \\ &\cong \varinjlim_i (\underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P_i) \\ &\cong \varinjlim_i \underline{\mathrm{Hom}}_{\mathcal{A}}(P'_i, \underline{\mathrm{Hom}}_C(P, N)) \\ &\cong \varinjlim_i \underline{\mathrm{Hom}}_C(P'_i \otimes_{\mathcal{A}} P, N), \end{aligned}$$

where P'_i is the predual of P_i, as in Definition 1.18.

The co-unit ε_N induces C-comodule morphisms

$$\underline{\mathrm{Hom}}_C(P'_i \otimes_{\mathcal{A}} P, N) = \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P_i \to N$$

for all i, and hence C_i-comodule morphisms

$$\underline{\mathrm{Hom}}_C(P'_i \otimes_{\mathcal{A}} P, N) \to N \otimes^C C_i.$$

Now, since C_i is finite-dimensional, we have $N \otimes^C C_i \cong \underline{\mathrm{Hom}}_C(C_i^\vee, N)$, so ε induces C-bicomodule morphisms

$$\alpha_i\colon\ C_i^\vee \to P'_i \otimes_{\mathcal{A}} P,$$

compatible with the transition maps $C_i \to C_j$, $P_i \to P_j$.

As P_i' is cofibrant, the quasi-isomorphism $P \to \omega$ induces a quasi-isomorphism $\beta_i\colon P_i' \otimes_{\mathcal{A}} P \to P_i' \otimes_{\mathcal{A}} \omega = C_i^{\vee}$ of cochain complexes. Now, α_i is equivalent to the coaction map $P_i \to P_i \otimes_k C_i \to P \otimes_k C_i$, so $\beta_i \circ \alpha_i$ is equivalent to the coaction map $P_i \to \omega \otimes_k C_i$. This is equivalent to the isomorphism $P_i' \otimes_{\mathcal{A}} \omega = C_i^{\vee}$, so $\beta_i \circ \alpha_i$ is the identity.

Therefore the α_i are all quasi-isomorphisms, so for N fibrant, the map

$$\underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P_i \to N \otimes^C C_i$$

is a quasi-isomorphism. Since filtered colimits commute with finite limits, this gives a quasi-isomorphism

$$\begin{aligned}\underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P &= \varinjlim_i \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P_i \\ &\to \varinjlim_i N \otimes^C C_i \\ &\cong N \otimes^C (\varinjlim_i C_i) \\ &= N \otimes^C C = N.\end{aligned}$$

□

2.2. **Tannakian comparison.** We now show how for our chosen ind-compact universal coalgebra $D \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$ and dg functor $\omega\colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$, the tilting module $P = D \otimes_{\mathcal{A}} \omega$ can give rise to a comparison between the derived category of $\mathcal{A}$-modules and the derived category of comodules of $C = \omega^{\vee} \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$. This is analogous to derived Morita theory (comparing two derived categories of modules) or Morita–Takeuchi theory (comparing two derived categories of comodules).

Definition 2.8. Write $\ker \omega$ for the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ consisting of objects X with $\omega(X) := \omega \otimes_{\mathcal{A}} X$ quasi-isomorphic to 0.

Recall from [Dri, §12.6] that the right orthogonal complement $(\ker \omega)^{\perp} \subset \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ is the full dg subcategory consisting of those X for which $\underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{opp}}}(M, X) \simeq 0$ for all $M \in \ker \omega$.

Theorem 2.9. *For the constructions of $C \simeq \omega^{\vee} \otimes^{\mathbf{L}}_{\mathcal{A}} \omega$ and the tilting module P above, the derived adjunction $(- \otimes_{\mathcal{A}} P) \dashv \mathbf{R}\underline{\mathrm{Hom}}_C(P, -)$ gives rise to a quasi-equivalence between the dg categories $(\ker \omega)^{\perp}$ and $\mathcal{D}_{\mathrm{dg}}(C)$. Moreover, the map $(\ker \omega)^{\perp} \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega)$ to the dg quotient is a quasi-equivalence.*

Proof. Functorial cofibrant and fibrant replacement give us composite dg functors $U\colon \mathcal{D}_{\mathrm{dg}}(C) \xrightarrow{\underline{\mathrm{Hom}}_C(P,-)} \mathcal{C}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ and $F\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \xrightarrow{\otimes_{\mathcal{A}} P} \mathcal{C}_{\mathrm{dg}}(C) \to \mathcal{D}_{\mathrm{dg}}(C)$, and these will yield the quasi-equivalence.

First observe that for $K \in \ker \omega$, we have quasi-isomorphisms

$$K \otimes_{\mathcal{A}} P \simeq \omega(K) \simeq 0$$

of cochain complexes, since P is a resolution of ω and K is cofibrant.

For any $N \in \mathcal{D}_{\mathrm{dg}}(C)$, Proposition 2.7 gives that the counit

$$\varepsilon_N\colon \underline{\mathrm{Hom}}_C(P, N) \otimes_{\mathcal{A}} P \to N$$

of the adjunction $(- \otimes_{\mathcal{A}} P) \dashv \underline{\mathrm{Hom}}_C(P, -)$ is a quasi-isomorphism. Thus for any $K \in \ker \omega$, we have

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{A}}(K, \mathbf{R}\underline{\mathrm{Hom}}_C(P, N)) \simeq \mathbf{R}\underline{\mathrm{Hom}}_C(K \otimes_{\mathcal{A}} P, N) \simeq \mathbf{R}\underline{\mathrm{Hom}}_C(0, N) \simeq 0,$$

so UN (the cofibrant replacement of $\mathbf{R}\underline{\mathrm{Hom}}_C(P,N)$) lies in $(\ker\omega)^\perp$.

Thus F provides a retraction of $(\ker\omega)^\perp$ onto $\mathcal{D}_{\mathrm{dg}}(C)$, and in particular $U\colon\ \mathcal{D}_{\mathrm{dg}}(C)\to(\ker\omega)^\perp$ is a full and faithful dg functor.

For any $M\in\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$, we now consider the unit

$$\eta_M\colon\ M\to\mathbf{R}\underline{\mathrm{Hom}}_C(P,M\otimes_{\mathcal{A}}P)=\underline{\mathrm{Hom}}_C(P,FM)$$

of the adjunction. On applying $\otimes_{\mathcal{A}}P$, this becomes a quasi-isomorphism, with quasi-inverse $\varepsilon_{M\otimes_{\mathcal{A}}P}$, so $\omega\otimes^{\mathbf{L}}_{\mathcal{A}}(\eta_M)$ is a quasi-isomorphism. Since M is cofibrant, the map η_M lifts to a map

$$\tilde{\eta}_M\colon\ M\to UFM$$

of cofibrant objects, with

$$\mathrm{cone}(\tilde{\eta}_M)\in\ker\omega,\qquad\text{i.e. }F\mathrm{cone}(\tilde{\eta}_M)\simeq 0.$$

The dg subcategory $\ker\omega$ is thus right admissible in the sense of [Dri, §12.6], because we have the morphism

$$M\xrightarrow{\tilde{\eta}_M}UFM$$

for all $M\in\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$, with $UFM\in(\ker\omega)^\perp$ and $\mathrm{cocone}(\tilde{\eta}_M)\in\ker\omega$.

In particular, this implies that if $M\in(\ker\omega)^\perp$, the map $\tilde{\eta}_M\colon\ M\to UFM$ is a quasi-isomorphism, so $U\colon\ \mathcal{D}_{\mathrm{dg}}(C)\to(\ker\omega)^\perp$ is essentially surjective and hence a quasi-equivalence.

As observed in [Dri, §12.6], the results of [BK3, §1] and [Ver, §I.2.6] show that right admissibility is equivalent to saying that $(\ker\omega)^\perp\to\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker\omega)$ is an equivalence. □

Remark 2.10. Note that Theorem 2.9 implies that for any choices D,D' of ind-compact $\otimes_{\mathcal{A}}$-coalgebra resolution of $\mathrm{id}_{\mathcal{A}}$, the associated coalgebras C,C' are derived Morita equivalent. Given a quasi-isomorphism $D\to D'$, we then have a derived Morita equivalence $C\to C'$, which is *a fortiori* a quasi-isomorphism.

It might therefore seem curious that $D\to\mathrm{id}_{\mathcal{A}}$ is only required to be a quasi-isomorphism. However, any quasi-isomorphism to the trivial coalgebra $\mathrm{id}_{\mathcal{A}}$ is automatically a Morita equivalence. The reason for this is that fibrant replacement in the category of D-comodules is given by the coaction $M\to M\otimes D$, so the forgetful dg functor from D-comodules to $\mathrm{id}_{\mathcal{A}}$-comodules is a quasi-equivalence.

Remark 2.11. In [Ayo3], Ayoub establishes a weak Tannaka duality result for any monoidal functor $f\colon\ \mathcal{M}\to\mathcal{E}$ of monoidal categories equipped with a (non-monoidal) right adjoint g. He sets $H:=fg(\mathbb{1})$, shows (Theorem 1.21) that H has the natural structure of a biunital bialgebra, and then (Propositions 1.28 and 1.55) proves that f factors through the category of H-comodules, and that H is universal with this property.

We may compare this with our setting by taking $\mathcal{M}=\mathcal{D}(\mathcal{A})$ and $\mathcal{E}=\mathcal{D}(k)$, the derived categories of $\mathcal{A}$ and k. In this case, Ayoub's formula for the coalgebra underlying H is defined provided $\mathcal{A}$ and f are k-linear, without requiring that $\mathcal{D}(\mathcal{A})$ be monoidal.

We can take f to be $\otimes^{\mathbf{L}}_{\mathcal{A}}\omega$, which has right adjoint $\mathbf{R}\underline{\mathrm{Hom}}_k(\omega,-)$ (with the same reasoning as Lemma 2.5). Thus

$$H\simeq\mathbf{R}\underline{\mathrm{Hom}}_k(\omega,k)\otimes^{\mathbf{L}}_{\mathcal{A}}\omega=\omega^\vee\otimes^{\mathbf{L}}_{\mathcal{A}}\omega,$$

which is the image $[C]\in\mathcal{D}(k)$ of our dg coalgebra $C\in\mathcal{C}_{\mathrm{dg}}(k)$.

One reason our duality results in Theorem 2.9 give a comparison rather than just universality is that we use the dg category of C-comodules in $\mathcal{C}_{\mathrm{dg}}(k)$. Instead, [Ayo3, Proposition 1.55] just looks at C-comodules in the derived category $\mathcal{D}(k)$ — in other words, (weak) homotopy comodules without higher coherence data. Likewise, his bialgebra H is only defined as a (weak) homotopy bialgebra.

To recover Ayoub's weak universality from Theorem 2.9 in this setting, first observe that there is a forgetful functor from $\mathcal{D}(C)$ to the category $\mathrm{CoMod}([C])$ of $[C]$-comodules in $\mathcal{D}(k)$. The equivalence $\mathcal{D}(\mathcal{A})/(\ker\omega) \simeq \mathcal{D}(C)$ then ensures that $\omega\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(k)$ factors through $\mathrm{CoMod}([C])$. Likewise, if $\omega\colon \mathrm{per}(\mathcal{A}) \to \mathcal{D}(k)$ factored through $\mathrm{CoMod}(B)$ for some other coalgebra $B \in \mathcal{D}(k)$, Theorem 2.9 would give an exact functor $\mathcal{D}(C) \to \mathrm{CoMod}(B)$ fibred over $\mathcal{D}(k)$. The image of C would be a B-comodule structure on $[C] \in \mathcal{D}(k)$, compatible with the coalgebra structure of $[C]$ via the image of the comultiplication $C \otimes C \to C$, giving a morphism $[C] \to B$ of coalgebras in $\mathcal{D}(k)$.

In [Iwa, Theorem 4.14], Iwanari effectively gives a refinement of Ayoub's Tannaka duality. Starting from a monoidal ∞-functor of stable ∞-categories (a generalisation of dg categories), he constructs a derived affine group scheme G (thus incorporating higher coherence data), and shows that its ∞-category $\mathrm{Rep}(G)$ of representations has a universal property, but without the characterisation $\mathrm{Rep}(G) \simeq (\ker\omega)^{\perp}$ of Theorem 2.9 above — this characterisation will be essential in our comparisons of categories of motives in Remark 2.17, and in Examples 2.22 and 3.20.

2.3. **Tannakian equivalence.** When ω is faithful, we now have statements about the idempotent-complete pre-triangulated envelope $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ of $\mathcal{A}$, and its closure $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) = \mathrm{ind}(\mathrm{per}_{\mathrm{dg}}(\mathcal{A}))$ under filtered colimits:

Corollary 2.12. *Assume that $\omega\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{C}_{\mathrm{dg}}(k)$ is faithful in the sense that $\ker\omega$ is the category of acyclic $\mathcal{A}$-modules, and take the tilting module P and dg coalgebra $C \simeq \omega^{\vee} \otimes^{\mathbf{L}}_{\mathcal{A}} \omega$ as above. Then the derived adjunction $(- \otimes_{\mathcal{A}} P) \dashv \mathbf{R}\underline{\mathrm{Hom}}_C(P, -)$ of Theorem 2.9 gives rise to a quasi-equivalence between the dg categories $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ and $\mathcal{D}_{\mathrm{dg}}(C)$.*

Moreover, the idempotent-complete pre-triangulated category $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ generated by $\mathcal{A}$ is quasi-equivalent to the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(C)_{\mathrm{cpt}} \subset \mathcal{D}_{\mathrm{dg}}(C)$ on objects which are compact in $\mathcal{D}(C)$. If $\mathcal{A}$ is Morita fibrant, this gives a quasi-equivalence $\mathcal{A} \simeq \mathcal{D}_{\mathrm{dg}}(C)_{\mathrm{cpt}}$.

Proof. Since $\ker\omega$ consists only of acyclic modules, $(\ker\omega)^{\perp} = \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$, and we apply Theorem 2.9. For the second part, note that the quasi-equivalence $- \otimes_{\mathcal{A}} P\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(C)$ preserves filtered colimits, so $- \otimes_{\mathcal{A}} P\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(C)$ preserves and reflects compact objects. Since $\mathrm{H}^0\mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ is the full subcategory on compact objects, the same must be true of its image in $\mathcal{D}(C)$. Finally, if $\mathcal{A}$ is Morita fibrant, then $\mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ is a quasi-equivalence. □

Example 2.13. Let $A = k[\epsilon]$ with $\epsilon^2 = 0$ (the dual numbers), and let ω be the A-module $k = k[\epsilon]/\epsilon$. This is faithful because for any cofibrant complex M of A-modules, we have $\omega(M) = M/\epsilon M$, and a short exact sequence

$$0 \to \omega(M) \xrightarrow{\epsilon} M \to \omega(M) \to 0.$$

A model for the cofibrant dg $\otimes_A$-coalgebra D in A-bimodules is given by

$$D^{-n} = A \otimes (k\xi_n) \otimes A$$

for $n \geq 0$, with comultiplication given by

$$\Delta(a \otimes \xi_n \otimes b) = \sum_{i+j=n} a \otimes \xi_i \otimes 1 \otimes \xi_j \otimes b \in A \otimes (k\xi_i) \otimes A \otimes (k\xi_j) \otimes A,$$

and counit $a \otimes \xi_0 \otimes b \mapsto ab \in A$. The differential is determined by

$$d(1 \otimes \xi_1 \otimes 1) = (\epsilon \otimes \xi_0 \otimes 1) - (1 \otimes \xi_0 \otimes \epsilon).$$

We therefore get $C := k \otimes_A D \otimes_A k = k\langle\xi\rangle$, the free dg coalgebra on generator $\xi = \xi_1$ in degree -1, with $d\xi = 0$. The tilting module P is given by $A\langle\xi\rangle$, with left multiplication by A, right comultiplication by C, and $d\xi = \epsilon$.

Thus the dg category of cofibrant A-modules is equivalent to the dg category of fibrant $k\langle\xi\rangle$-comodules. Contrast this with [Kel1, Example 2.5], which uses the tilting module P to give a derived Morita equivalence between the category of all finitely generated A-modules and the dg category of perfect $k\langle\xi\rangle^\vee$-modules.

Remark 2.14. If we have a finite set $\{\omega_x \colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)\}_{x \in X}$ of dg fibre functors, we can form a dg coalgebroid on objects X by $C(x,y) = \omega_x \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega_y$, and then Theorem 2.9 adapts to give an equivalence between dg C-comodules and $(\bigcap_{x\in X} \ker \omega_x)^\perp$, using Remark 2.3. When the ω_x are jointly faithful, Corollary 2.12 will thus adapt to give a quasi-equivalence between $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ and $\mathcal{D}_{\mathrm{dg}}(C)$.

Beware that if we had infinitely many dg fibre functors, the proof of Theorem 2.9 would no longer adapt, because the expression $N \otimes^C C_i$ in the proof of Proposition 2.7 would then be an infinite limit.

This also raises the question of a generalisation to faithful dg fibre functors $\omega \colon \mathcal{A} \to \mathcal{C}$ to more general categories. The obvious level of generality would replace $\mathrm{per}_{\mathrm{dg}}(k)$ with some rigid tensor category $\mathcal{C}$ over k. In order to proceed further, we would need an extension of Theorem 2.9 to deal with $\mathcal{C}$-coalgebras. In particular, generalisations would be required of the relevant model structures on comodules in [Pos, 8.2].

2.4. **Homotopy invariance.**

Corollary 2.15. *Given a k-linear dg functor $\omega \colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$ and a k-linear quasi-equivalence $F \colon \mathcal{B} \to \mathcal{A}$, the morphism*

$$C_{(\omega\circ F)}(\mathcal{B}) \to C_\omega(\mathcal{A})$$

of dg coalgebras induced by F is a derived Morita equivalence, so a fortiori *a quasi-isomorphism.*

Proof. By Theorem 2.9, the dg functor $\mathcal{D}_{\mathrm{dg}}(C_{(\omega\circ F)}(\mathcal{B})) \to \mathcal{D}_{\mathrm{dg}}(C_\omega(\mathcal{A}))$ is quasi-equivalent to $\mathcal{D}_{\mathrm{dg}}(\mathcal{B})/(\ker(\omega \circ F)) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega)$, which is a quasi-equivalence because F is so. Therefore we have a derived Morita equivalence of dg coalgebras. □

Corollary 2.16. *Given a natural quasi-isomorphism $\eta \colon \omega \to \omega'$ of k-linear dg functors $\omega, \omega' \colon \mathcal{A} \to \mathrm{per}_{\mathrm{dg}}(k)$, there is a span of morphisms between $C_\omega(\mathcal{A})$ and $C_{\omega'}(\mathcal{A})$ which are derived Morita equivalences.*

Proof. If we let I be the category with objects $0, 1$ and a unique non-identity morphism $\partial \colon 0 \to 1$, then η defines a k-linear dg functor

$$\underline{\eta} \colon \mathcal{A} \times I \to \mathrm{per}_{\mathrm{dg}}(k)$$

determined by $\underline{\eta}|_{\mathcal{A}\times 0} = \omega$, $\underline{\eta}|_{\mathcal{A}\times 1} = \omega'$ and $\underline{\eta}(\partial \colon X \times 0 \to X \times 1) = \eta_X \colon \omega(X) \to \omega'(X)$.

Lemma 1.11 combined with the functors $0, 1 \to I$ then gives us morphisms

$$C_\omega(\mathcal{A}) \to C_{\underline{\eta}}(\mathcal{A} \times I) \leftarrow C_{\omega'}(\mathcal{A})$$

of dg coalgebras; we need to show these are derived Morita equivalences.

By Theorem 2.9, this is equivalent to showing that the functors

$$\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega) \xrightarrow{0^*} \mathcal{D}_{\mathrm{dg}}(\mathcal{A} \times I)/(\ker \underline{\eta}) \xleftarrow{1^*} \mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega')$$

are quasi-equivalences.

For all $X \in \mathcal{A}$, the cone c_X of $h_{(X,0)} \to h_{(X,1)}$ lies in $\ker \underline{\eta}$ because η is a quasi-isomorphism. For $M \in (c_X)^\perp$, this implies that the maps $M(X, 1) \to M(X, 0)$ are all quasi-isomorphisms. For $M, N \in \mathcal{D}_{\mathrm{dg}}(\mathcal{A} \times I)$, the complex $\underline{\mathrm{Hom}}_{\mathcal{A}\times I}(M, N)$ is quasi-isomorphic to the cocone of

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M(0), N(0)) \times \underline{\mathrm{Hom}}_{\mathcal{A}}(M(1), N(1)) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(M(1), N(0)),$$

so the dg functors

$$0^*, 1^* \colon \mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A} \times I)/(\{c_X \;:\; X \in \mathcal{A}\})$$

are quasi-equivalences, as is their retraction 0_* given by $M \mapsto M(0)$.

We now just observe (by checking on representables $h_{(X,i)}$) that there is a natural quasi-isomorphism from $\underline{\eta}$ to the dg functor $M \mapsto \mathrm{cone}(\omega M(1) \to \omega M(0) \oplus \omega' M(1))$. Since $\omega \to \omega'$ is a quasi-isomorphism, this is quasi-isomorphic to the dg functor $\omega \circ 0_*$, so $\ker \underline{\eta} = \ker(\omega \circ 0_*)$. $\square$

In particular, this implies that the choice in Remark 1.8 does not affect the output, and that the constructions of §1.3.2 associate strong homotopy monoids to quasi-strong monoidal functors.

2.5. **Example: motives.** Our main motivating example comes from the derived category of motives.

As explained in [Ayo2, §3], there is a projective model structure on the category $\mathcal{M}$ of symmetric T-spectra in presheaves of k-linear complexes on the category Sm/S of smooth S-schemes. By [Ayo1, Definitions 4.3.6 et 4.5.18], this has a left Bousfield localisation $\mathcal{M}_{\mathbb{A}^1}$, the projective $(\mathbb{A}^1, \text{ét})$-local model structure, whose homotopy category is Voevodsky's triangulated category of motives over S whenever S is normal. These model categories are defined in terms of cochain complexes, so have the natural structure of dg model categories.

Write $\mathcal{M}_{\mathrm{dg}}, \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}$ for the full dg subcategories on fibrant cofibrant objects — this ensures that $\mathrm{Ho}(\mathcal{M}) \simeq \mathrm{H}^0\mathcal{M}_{\mathrm{dg}}$ and similarly for $\mathcal{M}_{\mathbb{A}^1}$. Take $\mathcal{M}_{\mathrm{dg},c}, \mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}$ to be the full subcategories of $\mathcal{M}_{\mathrm{dg}}, \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}$ on homotopically compact objects. For similar constructions along these lines, see [BV].

By [CD1, Proposition 2.1.6], any choice of k-linear stable cohomology theory over S gives a commutative ring object $\mathscr{E}$ in $\mathcal{M}$, with the property that

$$E := \underline{\mathrm{Hom}}_{\mathcal{M}}(-, \mathscr{E}) \colon \mathcal{M}_{\mathrm{dg}}^{\mathrm{opp}} \to \mathcal{D}_{\mathrm{dg}}(k)$$

represents the cohomology theory. In particular, the functor is stable and $\mathbb{A}^1$-invariant, so by [CD1, Theorem 1], E gives a dg functor from $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}^{\mathrm{opp}}$ to cohomologically finite complexes.

Remark 1.8 allows us to replace this with a dg functor

$$\tilde{E} \colon \tilde{\mathcal{M}}_{\mathrm{dg},\mathbb{A}^1,c}^{\mathrm{opp}} \to \mathrm{per}_{\mathrm{dg}}(k)$$

for some cofibrant replacement $\tilde{\mathcal{M}}_{\mathrm{dg},\mathbb{A}^1,c}$ of $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}$, and we can then form the dg coalgebra $C := C_{\tilde{E}^\vee}(\tilde{\mathcal{M}}_{\mathrm{dg},\mathbb{A}^1,c})$. By Corollary 2.16, this construction is essentially independent of the choice $\tilde{E}$ of replacement.

Theorem 2.9 then gives a quasi-equivalence

$$\mathcal{D}_{\mathrm{dg}}(C) \simeq \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}/(\ker E)$$

between the dg category of C-comodules and the dg enhancement of the triangulated category of motives modulo homologically acyclic motives. If E' is the composition of E with the derived localisation dg functor $\tilde{\mathcal{M}}_{\mathrm{dg},c} \to \tilde{\mathcal{M}}_{\mathrm{dg},\mathbb{A}^1,c}$, and $C' := C_{\tilde{E'}^\vee}(\tilde{\mathcal{M}}_{\mathrm{dg},c})$ this also gives

$$\mathcal{D}_{\mathrm{dg}}(C) \simeq \mathcal{M}_{\mathrm{dg}}/(\ker E') \simeq \mathcal{D}_{\mathrm{dg}}(C').$$

One consequence of the existence of a motivic t-structure over S would be that $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}$ lies in the right orthogonal complement $(\ker E)^\perp$, in which case $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}$ would be quasi-equivalent to a full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(C)$.

These constructions can all be varied by replacing $\mathcal{M}$ with the category $\mathcal{M}^{\mathrm{eff}}$ of presheaves of k-linear complexes on Sm/S with its projective model structure. This has a left Bousfield localisation $\mathcal{M}_{\mathbb{A}^1}$, by [Ayo1, Definition 4.4.33], and the homotopy category of $\mathcal{M}^{\mathrm{eff}}_{\mathbb{A}^1}$ is Voevodsky's triangulated category of effective motives when S is normal, as in [Ayo2, Appendix B]. There are natural dg enhancements, and we denote their restrictions to fibrant cofibrant objects by $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}, \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}$.

Composing E with the suspension functor $\Sigma^\infty \colon \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1} \to \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}$ defines a dg functor $E\colon (\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1})^{\mathrm{opp}} \to \mathcal{D}_{\mathrm{dg}}(k)$, which is simply given by $\underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{eff}}}(-, \mathscr{E}(0))$. We now get a dg coalgebra $C^{\mathrm{eff}} := C_{\tilde{E}^\vee}(\tilde{\mathcal{M}}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1,c})$ with

$$\mathcal{D}_{\mathrm{dg}}(C^{\mathrm{eff}}) \simeq \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}/(\ker E) \simeq \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}/(\ker E').$$

A set of compact generators of $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}$ is given by the set of presheaves $k(X)$ for smooth S-varieties X. If $\mathcal{A}$ is the full subcategory on these generators and E' has finite-dimensional values on $\mathcal{A}$, then we get a Morita equivalence between C and $C_{E'}(\mathcal{A})$. Note that the latter is just given by the total complex of

$$n \mapsto \bigoplus_{X_0,\ldots X_n} E'(X_0) \otimes k(X_0(X_1) \times X_1(X_2) \times \ldots \times X_{n-1}(X_n)) \otimes E'(X_n)^\vee,$$

where we write $X(Y) = \mathrm{Hom}_S(Y, X)$, with the dg coalgebra structure of Proposition 1.10.

Example 2.17 (Ayoub and Nori's motivic Galois groups). We now compare this with Ayoub's construction of a motivic Galois group from [Ayo3, §2]. For a number field F, he applies his Tannaka duality construction to the Betti realisation functor

$$\mathrm{Ho}(E)^\vee \colon \mathrm{H}^0\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F, \mathbb{Q}) \to \mathcal{D}(\mathbb{Q})$$

associated to an embedding $\sigma\colon F \to \mathbb{C}$, giving a Hopf algebra $\mathcal{H}_{\mathrm{mot}}(F, \sigma) \in \mathcal{D}(\mathbb{Q})$. Replacing $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F, \mathbb{Q})$ with $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F, \mathbb{Q})$ gives a bialgebra $\mathcal{H}^{\mathrm{eff}}_{\mathrm{mot}}(F, \sigma) \in \mathcal{D}(\mathbb{Q})$.

From Remark 2.11, it follows that $\mathcal{H}_{\mathrm{mot}}(F, \sigma)$ and $\mathcal{H}^{\mathrm{eff}}_{\mathrm{mot}}(F, \sigma)$ are just the homotopy classes of our dg coalgebras C, C^{eff} above, equipped with their natural multiplications and antipodes in the homotopy category coming from the rigid monoidal structure of $\mathrm{Ho}(E)$.

A variant of the construction above is given by considering generators of $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}(\mathbb{Q})$ given by $\mathrm{cone}(\mathbb{Q}(Y) \to \mathbb{Q}(X))[i]$, for Nori's good pairs (X, Y, i) as in [HMS, Definition 1.1]. These have the property that their Betti realisations are cohomologically concentrated in degree 0. Writing $\mathcal{A}_{\mathrm{Nori}}$ for the full dg subcategory on these generators, we have $C^{\mathrm{eff}} \simeq C_{E'}(\mathcal{A}_{\mathrm{Nori}})$.

We also have $C_{E'}(\mathrm{Z}^0\mathcal{A}_{\mathrm{Nori}}) \simeq C_{\mathrm{H}^0E'}(\mathrm{Z}^0\mathcal{A}_{\mathrm{Nori}})$ and $\mathrm{H}^0C_{\mathrm{H}^0E'}(\mathrm{Z}^0\mathcal{A}_{\mathrm{Nori}}) \cong \mathrm{H}^0C_{\mathrm{H}^0E'}(\mathrm{H}^0\mathcal{A}_{\mathrm{Nori}})$, and (by [JS, Theorem 7.3]) comodules of the latter are precisely Nori's abelian category $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ of effective mixed motives as in [HMS, Definition 1.3], since the diagram D^{eff} of good pairs generates $\mathrm{Z}^0\mathcal{A}$. Then $\mathrm{Spec}\,\mathrm{H}^0C_{E'}(\mathrm{Z}^0\mathcal{A}_{\mathrm{Nori}})$ is a pro-algebraic monoid whose group of units is Nori's Galois group G_{Nori}. The inclusion $\mathrm{H}^0C_{E'}(\mathrm{Z}^0\mathcal{A}_{\mathrm{Nori}}) \hookrightarrow \mathrm{H}^0C^{\mathrm{eff}}$ thus induces a surjection $\mathrm{Spec}\,\mathrm{H}^0C \twoheadrightarrow G_{\mathrm{Nori}}$.

Contrast this with [Iwa, Remark 5.20], highlighting that a comparison between Nori's and Voevodsky's motives is beyond the reach of Iwanari's Tannakian formulation.

We now introduce alternative simplifications of the dg coalgebra in special cases.

Remark 2.18. When S is the spectrum of a field, the comparison of [Ayo2, Appendix B] combines with the results of [VSF] or [CD2, Corollary 4.4.3] to show that a set of generators of $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}$ is given by the motives of the form $M_k(X)(r)$ for X smooth and projective over S, and $r \in \mathbb{Z}$. Thus the set of motives of the form $M_{k,r}(X) := M_k(X)(r)[2r]$ is another generating set. For X of dimension d over S, the dual of $M_{k,r}(X)$ is $M_{k,d-r}(X)$, so this set of generators is closed under duals, and

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(M_{k,r}(X), M_{k,s}(Y)) \simeq \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(M_k(S), M_{k,d+s-r}(X \times_S Y)).$$

When S is the spectrum of a perfect field, this implies that

$$\mathrm{H}^i\mathcal{M}_{\mathbb{A}^1}(M_{k,r}(X), M_{k,s}(Y)) \cong \mathrm{CH}^{d+s-r}(X \times_S Y, -i) \otimes_{\mathbb{Z}} k,$$

so these generators have no positive Ext groups between them, and we can replace the full dg category on these generators with its good truncation $\mathcal{B}$ in non-positive degrees, given by

$$\mathcal{B}(M_{k,r}(X), M_{k,s}(Y)) := \tau^{\leq 0}\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(M_{k,r}(X), M_{k,s}(Y));$$

we then have $\mathcal{D}_{\mathrm{dg}}(\mathcal{B}) \simeq \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}$.

The mixed Weil cohomology theory E when restricted to $\mathcal{B}$ thus admits a good truncation filtration, whose associated graded is quasi-isomorphic to

$$\mathrm{H}^*E\colon\ \mathrm{H}^0\mathcal{B}^{\mathrm{opp}} \to \mathcal{C}_{\mathrm{dg}}(k);$$

think of this as a formal Weil cohomology theory. Note that this is a strong monoidal functor determined by the Chern character $\mathrm{CH}^s(Y) \to \mathrm{H}^{2s}(Y, E(s))$.

Since H^*E is finite-dimensional, we can then form the dg coalgebra $C_{\mathrm{H}^*E}(\mathcal{B})$, without needing to take a cofibrant replacement of $\mathcal{B}$. Explicitly, this is given by the total complex of

$$n \mapsto \bigoplus_{X_0,\ldots X_n, r_0,\ldots,r_n} \mathrm{H}^{*+2r_0}(X_0, E(r_0)) \otimes_k \mathcal{B}(M_{k,r_0}(X_0), M_{k,r_1}(X_1)) \otimes_k \ldots$$
$$\ldots \otimes_k \mathcal{B}(M_{k,r_{n-1}}(X_{n-1}), M_{k,r_n}(X_n)) \otimes_k \mathrm{H}^{*+2r_n}(X_n, E(r_n))^{\vee},$$

with the dg coalgebra structure of Proposition 1.10. We then have

$$\mathcal{D}_{\mathrm{dg}}(C_{\mathrm{H}^*E}(\mathcal{B})) \simeq \mathcal{D}_{\mathrm{dg}}(\mathcal{B})/\ker \mathrm{H}^*E \simeq \mathcal{D}_{\mathrm{dg}}(\mathcal{B})/\ker E \simeq \mathcal{M}_{\mathrm{dg},\mathbb{A}^1}/\ker E.$$

Remark 2.19. If we write $\mathcal{Z}(X,\bullet)$ for the k-linearisation of Bloch's cycle complex as in [Blo, Proposition 1.3], then we can follow [Han] in defining $\mathcal{Z}(X\times Y,\bullet)\hat{\otimes}\mathcal{Z}(Y\times Z,\bullet)\subset \mathcal{Z}(X\times Y,\bullet)\otimes_k \mathcal{Z}(Y\times Z,\bullet)$ to be the quasi-isomorphic subcomplex of cycles intersecting transversely. We then have a bicomplex

$$n\mapsto \bigoplus_{X_0,\ldots X_n,r_0,\ldots,r_n} \mathrm{H}^{*+2r_0}(X_0,E(r_0))\otimes_k \mathcal{Z}^{d_0+r_1-r_0}(X_0\times X_1,-*)\hat{\otimes}\ldots$$
$$\ldots\hat{\otimes}\mathcal{Z}^{d_{n-1}+r_n-r_{n-1}}(X_{n-1}\times X_n,-*)\otimes_k \mathrm{H}^{*+2r_n}(X_n,E(r_n))^\vee,$$

for X_i of dimension d_i, and with differentials as in Definition 1.9.

The formulae of Proposition 1.10 make this into a dg coalgebra, which should be Morita equivalent to the dg coalgebra $C_{\mathrm{H}^*E}(\mathcal{B})$ of Remark 2.18.

2.6. **Monoidal comparisons.** We now consider the case where $(\mathcal{A},\boxtimes)$ is a monoidal dg category and $\omega\colon\ \mathcal{A}\to \mathrm{per}_{\mathrm{dg}}(k)$ a strong monoidal dg functor. We also assume that D is a universal bialgebra in the sense of §1.3.3.

Note that since C is a dg bialgebra by Lemma 1.36, the dg category $\mathcal{C}_{\mathrm{dg}}(C)$ has a monoidal structure $\otimes_k$, where the coaction on $N\otimes N'$ is the composition

$$N\otimes N'\to (N\otimes C)\otimes(N'\otimes C)\cong (N\otimes N')\otimes(C\otimes C)\to (N\otimes N')\otimes C$$

of the co-actions with the multiplication on C.

Lemma 2.20. *For $M,M'\in\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ and $N,N'\in\mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}})$, there is a natural transformation*

$$(M\otimes_{\mathcal{A}}N)\otimes_k(M'\otimes_{\mathcal{A}}N')\to(M\boxtimes M')\otimes_{\mathcal{A}}(N\boxtimes N').$$

Proof. When $M,M'=h_X,h_Y$ and $N,N'=h_Y,h'_Y$ for $X,X',Y,Y'\in\mathcal{A}$, this is just the map

$$\mathcal{A}(X,Y)\otimes_k\mathcal{A}(X',Y')\to\mathcal{A}(X\boxtimes X',Y\boxtimes Y')$$

given by the bilinearity of $\boxtimes$. This extends uniquely to complexes and direct summands. □

Proposition 2.21. *The dg functor $(-\otimes_{\mathcal{A}}P)\colon\ \mathcal{D}_{\mathrm{dg}}(\mathcal{A})\to\mathcal{C}_{\mathrm{dg}}(C)$ is lax monoidal, with the transformations*

$$(M\otimes_{\mathcal{A}}P)\otimes_k(M'\otimes_{\mathcal{A}}P)\to(M\boxtimes M')\otimes_{\mathcal{A}}P$$

being quasi-isomorphisms.

Proof. Lemma 2.20 gives the required transformations

$$(M\otimes_{\mathcal{A}}P)\otimes_k(M'\otimes_{\mathcal{A}}P)\to(M\boxtimes M')\otimes_{\mathcal{A}}(P\boxtimes P)\to(M\boxtimes M')\otimes_{\mathcal{A}}P.$$

The quasi-isomorphism $P\to\omega$ then maps these transformations quasi-isomorphically to

$$(M\otimes_{\mathcal{A}}\omega)\otimes_k(M'\otimes_{\mathcal{A}}\omega)\to(M\boxtimes M')\otimes_{\mathcal{A}}\omega,$$

i.e.

$$\omega(M)\otimes_k\omega(M)\to\omega(M\boxtimes M'),$$

which is an isomorphism because ω is required to be a strong monoidal dg functor. □

Example 2.22 (Motives). The model category $\mathcal{M}^{\mathrm{eff}}$ of k-linear presheaves from §2.5 is monoidal, as is its localisation $\mathcal{M}^{\mathrm{eff}}_{\mathbb{A}^1}$. However, the tensor product does not preserve fibrant objects, so the dg categories $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}, \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}$ of fibrant cofibrant objects are not monoidal. [At best, they are multicategories (a.k.a. coloured operads), with $\underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}}(X_1,\ldots,X_n;Y) := \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{eff}}}(X_1\otimes\ldots\otimes X_n;Y)$.]

However, if we take the dg category $\mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg}}$ of cofibrant objects in $\mathcal{M}$, with $\mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg},c}$ the full subcategory of compact objects, then $\mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg}}$ and $\mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg},c}$ are monoidal dg categories. The dg categories $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}, \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},c}$ are quasi-equivalent to the dg quotients of $\mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg}}, \mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg},c}$ by the class of weak equivalences in $\mathcal{M}^{\mathrm{eff}}$. By [CD1, Theorem 1], a mixed Weil cohomology theory then gives rise to a contravariant monoidal dg functor E from $\mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg}}$ to cohomologically finite complexes, by setting $E := \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{eff}}}(-,\mathcal{E})$ for the associated presheaf $\mathcal{E}$ of DGAs.

Since symmetric monoidal dg categories do not form a model category, we cannot then mimic the construction of §2.5 and replace E with a monoidal dg functor from a cofibrant replacement of $\mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg}}$ to finite-dimensional complexes. However, we can apply Proposition 2.21 if we can find a Weil cohomology theory taking values in finite-dimensional complexes.

Objects of $\mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg}}$ are formal k-linear complexes of smooth varieties over S. When S is a field admitting resolution of singularities, we can instead consider the model category $\mathcal{N}^{\mathrm{eff}}$ of presheaves on the category of pairs $j\colon U\to X$, where X is smooth and projective over S, with U the complement of a normal crossings divisor. Then for any mixed Weil cohomology theory E, there is an associated formal theory

$$E_f(j\colon U\to X) := (\bigoplus_{a,b}\mathrm{H}^a(X,\mathbf{R}^b j_*\mathcal{E}_U, d_2),$$

where d_2 is the differential on the second page of the Leray spectral sequence. Alternatively, this can be rewritten (as in [Del2, 3.2.4]) in terms of $\mathrm{H}^a(\tilde{D}^b, E(-b))$ and Gysin maps, where $\tilde{D}^n$ consists of local disjoint unions of n-fold intersections in D.

The constructions of §2.5 all adapt from $\mathcal{M}$ to $\mathcal{N}$, and the restriction of E_f to $\mathcal{N}^{\mathrm{eff}'}_{\mathrm{dg},c}$ takes values in finite-dimensional complexes, so we have a dg bialgebra $C := C_{E_f}(\mathcal{N}^{\mathrm{eff}'}_{\mathrm{dg},c})$, and Proposition 2.21 gives a monoidal dg functor

$$\mathcal{N}^{\mathrm{eff}'}_{\mathrm{dg}} \to \mathcal{C}_{\mathrm{dg}}(C),$$

inducing an equivalence $\mathcal{N}^{\mathrm{eff}'}_{\mathrm{dg}}/\ker E \simeq \mathcal{D}_{\mathrm{dg}}(C)$. With some work (see Appendix A.1.3), it follows that $\mathcal{N}^{\mathrm{eff}'}_{\mathrm{dg}}/\ker E \simeq \mathcal{M}^{\mathrm{eff}'}_{\mathrm{dg}}/\ker E$ for all known Weil cohomology theories, and (for Betti cohomology) that E_f is quasi-isomorphic to E.

Corollaries 2.15 and 2.16 then ensure that C is essentially equivalent to the dg coalgebra of §2.5, so the constructions above give a strong compatibility result for the comparisons of §2.5 with respect to the monoidal structures.

By Proposition 1.23, the multiplication on C comes from the fibred dg functor $\boxtimes\colon (\mathcal{N}^{\mathrm{eff}'}\otimes\mathcal{N}^{\mathrm{eff}'},\omega\otimes\omega)\to(\mathcal{N}^{\mathrm{eff}'},\omega)$. Applying [Ayo3, Corollary 1.14] to these fibred dg categories gives bialgebra structures on the coalgebras $[C\otimes C],[C]\in\mathcal{D}(k)$. By Ayoub's weak universal property [Ayo3, Proposition 1.55], the functors $\boxtimes$ and $\mathbb{1}$ on derived categories induce morphisms $[C\otimes C]\to[C]$ and $[k]\to[C]$ of commutative bialgebras; the relations between these morphisms force them to be Ayoub's multiplication and unit maps. Functoriality of the comparison of universal constructions in Remark

2.11 then ensures that this weak bialgebra structure must come from our dg bialgebra structure.

3. Dense subcategories and semisimplicity

The beauty of Theorem 2.9 is that it describes the derived category $\mathcal{D}(\mathcal{A})$ in terms of a fibre functor on $\mathcal{A}$, so is invariant under Morita equivalences. In particular, for any derived Morita equivalence $\mathcal{B} \to \mathcal{A}$, we have a quasi-equivalence $\mathcal{D}_{\mathrm{dg}}(\mathcal{B}) \to \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$. This becomes particularly important when we can find a Morita equivalent dg category $\mathcal{B}$ for which the category $\mathrm{Z}^0\mathcal{B}$ is semisimple, since the representing coalgebra then admits a particularly simple description.

In Remark 2.10, it was observed that different choices of universal coalgebra will give dg coalgebras which are derived Morita equivalent. A quasi-isomorphism $C \to C'$ of dg coalgebras need not be a derived Morita equivalence, in general. However, for the Tannakian dg coalgebras constructed in this section, it turns out that quasi-isomorphisms will be derived Morita equivalences (see §3.2 below).

3.1. Non-negatively graded dg categories.

Definition 3.1. Let $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$ denote the category of dg k-coalgebras C in non-negative cochain degrees, satisfying the additional property that the map $\mathrm{H}^0C \to C$ of coalgebras is ind-conilpotent. This means that we can write C as a nested union $C = \varinjlim_\alpha C_\alpha$ of dg coalgebras with $\mathrm{H}^0C = \mathrm{H}^0C_\alpha$ for all α and C_α/H^0C conilpotent in the sense that the comultiplication

$$C_\alpha/\mathrm{H}^0C \to (C_\alpha/\mathrm{H}^0C)^{\otimes m}$$

is 0 for some $m \geq 2$.

For any $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$, the maximal cosemisimple subcoalgebra ([Pos, 4.3]) $C_{\mathrm{red}} := (\mathrm{H}^0C)_{\mathrm{red}} \subset \mathrm{H}^0C$ thus gives an ind-conilpotent map $C_{\mathrm{red}} \to C$. Since C_{red} is cosemisimple, the ind-conilpotent morphism $C_{\mathrm{red}} \to C_0$ admits a retraction, so C is of the form $C = C_{\mathrm{red}} \oplus N$, for N an ind-conilpotent dg coalgebra with a compatible C_{red}-bicomodule structure.

Proposition 3.2. *Take a k-linear dg category $\mathcal{A}$ with $\mathcal{A}(X,Y)$ concentrated in non-negative degrees, $d\mathcal{A}^0(X,Y) = 0$ for all X, Y, and with $\mathcal{A}^0$ a semisimple abelian category. Assume that we have a k-linear functor $\omega\colon \mathcal{A}^0 \to \mathrm{FDVect}_k$. Then there is a model for the coalgebra $C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega$ of §1.2.3 with $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$.*

Proof. For $i\colon \mathcal{A}^0 \hookrightarrow \mathcal{A}$, we set D to be the direct sum total complex $N\mathrm{CC}(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ of the normalisation $N\mathrm{CC}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ of the simplicial cochain complex $\underline{\mathrm{CC}}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ given by

$$\underline{\mathrm{CC}}_n(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i) := \mathcal{A}(-,-) \otimes_{\mathcal{A}^0} \underbrace{\mathcal{A}(-,-) \otimes_{\mathcal{A}^0} \ldots \otimes_{\mathcal{A}^0} \mathcal{A}(-,-)}_{n} \otimes_{\mathcal{A}^0} \mathcal{A}(-,-).$$

Equivalently, $N\mathrm{CC}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ is the total complex of

$$n \mapsto \mathcal{A}(-,-) \otimes_{\mathcal{A}^0} \underbrace{\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{A}^0} \ldots \otimes_{\mathcal{A}^0} \mathcal{A}^{>0}(-,-)}_{n} \otimes_{\mathcal{A}^0} \mathcal{A}(-,-).$$

The comultiplication and counit are given by the formulae of Proposition 1.10, so we need only show that the counit $D \to \mathrm{id}_{\mathcal{A}}$ is a quasi-isomorphism and that D is a cofibrant module.

The identity $\mathrm{id}_X \in \mathcal{A}(X,X)$ gives a contracting homotopy of the complex $D(X,Y) \to \mathcal{A}(X,Y)$, which ensures that the counit is a quasi-isomorphism. To see that D is cofibrant, we just note that

$$\mathcal{A}(-,iX) \otimes_k \underbrace{\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{A}^0} \ldots \otimes_{\mathcal{A}^0} \mathcal{A}^{>0}(-,-)}_{n} \otimes_k \mathcal{A}(iY,-)$$

is a cofibrant module for all $X, Y \in \mathcal{A}^0$, and that semisimplicity of $\mathcal{A}^0$ ensures that taking $\mathcal{A}^0$-coinvariants is an exact functor, so preserves cofibrancy.

Now, observe that $C := \omega^\vee \otimes_{\mathcal{A}} D \otimes_{\mathcal{A}} \omega$ is the direct sum total complex of

$$n \mapsto \omega^\vee \otimes_{\mathcal{A}^0} \underbrace{\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{A}^0} \ldots \otimes_{\mathcal{A}^0} \mathcal{A}^{>0}(-,-)}_{n} \otimes_{\mathcal{A}^0} \omega,$$

which has no negative terms, since ω is concentrated in degree 0 and $(\mathcal{A}^{>0})^{\otimes n}$ in degrees $\geq n$.

Finally, observe that the morphism $\omega^\vee \otimes_{\mathcal{A}^0} \omega \to C$ is an ind-conilpotent extension (cofiltering by copowers of $\omega^\vee \otimes_{\mathcal{A}^0} \mathcal{A}^{>0} \otimes_{\mathcal{A}^0} \omega$) and that $\omega^\vee \otimes_{\mathcal{A}^0} \omega \subset \mathrm{H}^0 C$, so the morphism $\mathrm{H}^0 C \to C$ is necessarily also conilpotent. ☐

Remark 3.3. Like the construction of Example 1.14, the universal bialgebra above can be written as a Čech nerve. Set $L := \mathcal{A}(-,-) \otimes_{\mathcal{A}^0} \mathcal{A}(-,-)$, which is a $\otimes_{\mathcal{A}}$-coalgebra in $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{A})$. We then have

$$\underline{\mathrm{CC}}_n(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i) = \underbrace{L \otimes_{\mathcal{A}} L \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} L}_{n+1},$$

giving the Čech nerve of the $\otimes_{\mathcal{A}}$-comonoid L.

Remark 3.4. If k is algebraically closed, then the complex $CC_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i)$ admits a simpler description. Let $\{V_\alpha\}_\alpha$ be a set of irreducible objects of $\mathcal{A}^0$, with one in each isomorphism class. Since k is algebraically closed, $\mathrm{End}_{\mathcal{A}^0}(V_\alpha) \cong k$, and we get

$$\underline{\mathrm{CC}}_n(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i) \cong$$
$$\bigoplus_{\alpha_0,\ldots,\alpha_n} \mathcal{A}(-,V_{\alpha_0}) \otimes_k \mathcal{A}(V_{\alpha_0}, V_{\alpha_1}) \otimes_k \ldots \otimes_k \mathcal{A}(V_{\alpha_{n-1}}, V_{\alpha_n}) \otimes_k \mathcal{A}(V_{\alpha_n}, -).$$

Writing $\mathcal{A}_s \subset \mathcal{A}$ for the full dg subcategory on objects $\{V_\alpha\}_\alpha$, this gives an isomorphism

$$\mathrm{CC}_\bullet(\mathcal{A}_s, i^{\mathrm{opp}} \otimes i) \cong \mathrm{CC}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i).$$

Thus the quasi-isomorphism between $\mathrm{CC}_\bullet(\mathcal{A}/\mathcal{A}^0, i^{\mathrm{opp}} \otimes i) \to \mathrm{CC}_\bullet(\mathcal{A}, h_{\mathcal{A}^{\mathrm{opp}}} \otimes h_{\mathcal{A}})$ is a consequence of the derived Morita equivalence $\mathcal{A}_s \to \mathcal{A}$.

Corollary 3.5. *Take a k-linear dg category $\mathcal{A}$ with $\mathcal{A}(X,Y)$ concentrated in non-negative degrees, and $\mathrm{H}^0\mathcal{A}$ a semisimple abelian category. Assume that we have a k-linear functor $\omega\colon \mathcal{A}^0 \to \mathrm{FDVect}_k$. Then there is a dg coalgebra $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$ with $C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} \omega$, together with quasi-equivalences $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker \omega) \simeq (\ker \omega)^\perp \simeq \mathcal{D}_{\mathrm{dg}}(C)$.*

Proof. Consider the morphism $d\colon \mathcal{A}^0(-,-) \to \mathcal{A}^1(-,-)$ of $\mathrm{H}^0\mathcal{A}$-bimodules. Since $\mathrm{H}^0\mathcal{A}$ is semisimple, there exists a $\mathrm{H}^0\mathcal{A}$-bimodule decomposition

$$\mathcal{A}^1(-,-) = d\mathcal{A}^0(-,-) \oplus \mathcal{B}^1(-,-).$$

We may therefore define a dg subcategory $\mathcal{B} \subset \mathcal{A}$ by

$$\mathcal{B}^n(-,-) := \begin{cases} \mathcal{A}^n(-,-) & n \neq 0,1 \\ \mathcal{B}^1(-,-) & n = 1 \\ \mathrm{H}^0\mathcal{A}(-,-) & n = 0. \end{cases}$$

Then $\mathcal{B} \to \mathcal{A}$ is a quasi-equivalence, and $\mathcal{B}$ satisfies the conditions of Proposition 3.2, giving a dg coalgebra C concentrated in non-negative degrees. We then apply Theorem 2.9. □

Definition 3.6. For a dg coalgebra $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$, define $\mathcal{D}^+_{\mathrm{dg}}(C) \subset \mathcal{D}_{\mathrm{dg}}(C)$ to be the full dg subcategory on cochain complexes V for which $\mathrm{H}^*(V)$ is bounded below. Write $\mathcal{D}^+(C) := \mathrm{H}^0\mathcal{D}^+_{\mathrm{dg}}(C)$

For a k-linear dg category $\mathcal{A}$ with $\mathcal{A}(X,Y)$ concentrated in non-negative degrees, define $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{A}) \subset \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ to be the full dg subcategory consisting of dg functors F for which $\prod_{X\in\mathcal{A}} \mathrm{H}^*F(X)$ is bounded below. Write $\mathcal{D}^+(\mathcal{A}) := \mathrm{H}^0\mathcal{D}^+_{\mathrm{dg}}(\mathcal{A})$.

Proposition 3.7. *Under the conditions of Corollary 3.5, if the functor* $\omega\colon \mathrm{H}^0\mathcal{A} \to \mathrm{FDVect}_k$ *is faithful, then we have a quasi-equivalence* $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{A}) \simeq \mathcal{D}^+_{\mathrm{dg}}(C)$.

Proof. We first replace $\mathcal{A}$ with the dg category $\mathcal{B}$ from the proof of Corollary 3.5, so $\mathcal{B}^0 = \mathrm{H}^0\mathcal{B}$. Since ω is additive and $\mathcal{B}^0$ semisimple, it follows that $\omega|_{\mathcal{B}^0}$ is exact, and hence represented by some $T \in \mathrm{ind}((\mathcal{B}^0)^{\mathrm{opp}})$, with $\omega|_{\mathcal{B}^0} = -\otimes_{\mathcal{B}^0} T$. We may write T as a filtered colimit $T = \varinjlim T_\alpha$ for $T_\alpha \in (\mathcal{B}^0)^{\mathrm{opp}}$, and because $\mathcal{B}^0$ is abelian we may assume that each T_α is a subobject of T. Since $\mathcal{B}^0$ is semisimple, this means that T_α is a direct summand of T.

Because $\omega|_{\mathcal{B}^0}$ is faithful, it follows that the set $\{T_\alpha\}_\alpha$ generates $\mathcal{B}^0$. Now, the dg functor $\omega\colon \mathcal{D}_{\mathrm{dg}}(\mathcal{B}) \to \mathcal{C}_{\mathrm{dg}}$ is $(-\otimes_{\mathcal{B}} \mathcal{B}^0(-,-)\otimes_{\mathcal{B}^0} T)$, so if $\omega(M) \simeq 0$ then

$$M \otimes_{\mathcal{B}} \mathcal{B}^0(-,T_\alpha) = M \otimes_{\mathcal{B}} \mathcal{B}^0(-,-)(T_\alpha) \simeq 0$$

for all α (T_α being a direct summand of T). Since $\{T_\alpha\}_\alpha$ generates $\mathcal{B}^0$, it follows that $M \otimes_{\mathcal{B}} \mathcal{B}^0(-,-)(X) \simeq 0$ for all objects $X \in \mathcal{B}$, which is precisely the same as saying that $M \otimes_{\mathcal{B}} \mathcal{B}^0(-,-) \simeq 0$.

For the natural projection $\pi\colon \mathcal{B} \to \mathcal{B}^0$, this says that $\pi^*M \simeq 0$; for any $\mathcal{B}^0$-module N, we then have $\underline{\mathrm{Hom}}_{\mathcal{B}}(M, \pi_*N) \simeq 0$, so any complex quasi-isomorphic to π_*N lies in $(\ker\omega)^\perp$. Since $(\ker\omega)^\perp$ is closed under extensions and homotopy limits, and any $R \in \mathcal{D}^+_{\mathrm{dg}}(\mathcal{B})$ can be recovered from the $\mathcal{B}^0$-modules H^iR via these operations, it follows that $R \in (\ker\omega)^\perp$.

Thus Corollary 3.5 shows that the dg functor $-\otimes_{\mathcal{B}} P$ gives a quasi-equivalence from $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{B})$ to a full dg subcategory of $\mathcal{D}^+_{\mathrm{dg}}(C)$. It remains to show that for any $N \in \mathcal{D}^+_{\mathrm{dg}}(C)$, we have $\underline{\mathrm{Hom}}_C(P,N) \in \mathcal{D}^+_{\mathrm{dg}}(\mathcal{B})$; without loss of generality we may assume $\mathrm{H}^{<0}N = 0$. Now,

$$\prod_{X\in\mathcal{B}} \underline{\mathrm{Hom}}_C(P,N)(X) = \prod \underline{\mathrm{Hom}}_C(X \otimes_{\mathcal{B}} P, N),$$

and $\mathrm{H}^*(X \otimes_{\mathcal{B}} P) \cong \mathrm{H}^*(\omega X)$, which is concentrated in degree 0. By applying the cobar resolution in the proof of [Pos, Theorem 4.4] to $\tau^{\geq 0}N$, it follows that that N is quasi-isomorphic to a C-comodule N' concentrated in non-negative degrees and fibrant in the coderived model structure. Then [Pos, Theorem 4.3.1] implies that $\underline{\mathrm{Hom}}_C(-,N) \simeq \underline{\mathrm{Hom}}_C(-,N')$, so

$$\underline{\mathrm{Hom}}_C(X \otimes_{\mathcal{B}} P, N) \simeq \underline{\mathrm{Hom}}_C(\mathrm{H}^0(X \otimes_{\mathcal{B}} P), N'),$$

which is concentrated in non-negative degrees. □

Definition 3.8. For any dg coalgebra C, define $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$ to be the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(C)$ on objects K for which the graded module $K^{\#}$ underlying K is injective as a comodule over the graded coalgebra $C^{\#}$ underlying C.

Note that from the properties of the model structure of [Pos, Theorem 8.2], the homotopy category $\mathrm{H}^0\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$ is equivalent to Positselski's coderived category $\mathcal{D}^{\mathrm{co}}(C)$. Weak equivalences with respect to this model structure are morphisms whose cone L is coacyclic in the sense of [Pos, 4.2] — this is a stronger condition than acyclicity, and is equivalent to saying that $\underline{\mathrm{Hom}}_C(L, K)$ is acyclic for all $K \in \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$.

Proposition 3.9. *The equivalence of Proposition 3.7 induces a quasi-equivalence between* $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ *(see Definition 1.12) and the full dg subcategory* $\mathcal{F}_{\mathrm{dg}}(C)$ *of* $\mathcal{D}_{\mathrm{dg}}(C)$ *on fibrant replacements of C-comodules in finite-dimensional cochain complexes. This gives a quasi-equivalence from* $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ *to* $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$*.*

Proof. Observe that when filtered colimits exist in $\mathcal{D}^{+}_{\mathrm{dg}}(\mathcal{A})$, they are homotopy colimits, and that $\mathrm{Hom}_{\mathcal{A}}(K, -)$ commutes with such limits for all $K \in \mathrm{per}_{\mathrm{dg}}(\mathcal{A})$. Since every object of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ can be written as a filtered colimit of perfect complexes, it follows that $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$ consists of the homotopy-compact objects of $\mathcal{D}^{+}_{\mathrm{dg}}(\mathcal{A})$ (i.e. the objects K for which $\mathrm{Hom}_{\mathcal{D}^*}(\mathcal{A})(K, -)$ commutes with filtered homotopy colimits, when they exist). They therefore correspond under Proposition 3.7 to the homotopy-compact objects of $\mathcal{D}^{+}_{\mathrm{dg}}(C)$.

By [Pos, Theorem 4.3.1(a)], $\mathcal{D}^{+}_{\mathrm{dg}}(C)$ is quasi-equivalent to the full subcategory of $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$ consisting of cochain complexes which are bounded below. By [Pos, 5.5], C-comodules in finite-dimensional cochain complexes are compact generators of $D^{\mathrm{co}}(C)$, and hence of $\mathcal{D}^{+}(C)$. Thus the essential image of

$$- \otimes_{\mathcal{A}} P\colon\ \mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{dg}}(C)$$

is just $\mathcal{F}_{\mathrm{dg}}(C)$.

Moreover, since the finite-k-dimensional C-comodules generate $\mathcal{D}^{\mathrm{co}}(C)$ and the latter is closed under arbitrary direct sums, we have a quasi-equivalence

$$\varinjlim\colon\ \mathrm{ind}(\mathcal{F}_{\mathrm{dg}}(C)) \to \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C),$$

which combines with the quasi-equivalence $\mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{F}_{\mathrm{dg}}(C)$ above to give a quasi-equivalence $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$ on the associated ind-categories. □

Example 3.10. Observe that if $\ker \omega|_{\mathrm{H}^0(\mathcal{A})} = 0$, we need not have $\ker \omega = 0$ on $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$. We see this by considering an example which is in some respects dual to Example 2.13. For t of degree 1, we can take $k\langle t\rangle$ to be the free non-commutative graded algebra generated by t and let $\mathcal{A}$ be the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(k\langle t\rangle)$ on objects $k\langle t\rangle^n$. Then $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathcal{D}_{\mathrm{dg}}(k\langle t\rangle)$, and the dg fibre functor $\omega(M) := k \otimes_{k\langle t\rangle} M$ is faithful on $\mathcal{D}^{+}(\mathcal{A})$. However, it is not faithful on $\mathcal{D}(\mathcal{A})$, since $k\langle t, t^{-1}\rangle$ lies in the kernel.

The associated dg coalgebra C is Morita equivalent to $k[\epsilon]^{\vee}$, for ϵ of degree 0 with $\epsilon^2 = 0$, so a Corollary 3.5 in this case gives

$$\mathcal{D}_{\mathrm{dg}}(k\langle t\rangle)/(k\langle t, t^{-1}\rangle) \simeq \mathcal{D}_{\mathrm{dg}}(k[\epsilon]),$$

while Proposition 3.9 gives an equivalence between $\mathrm{per}_{\mathrm{dg}}(k\langle t\rangle)$ and the dg derived category of finite $k[\epsilon]$-modules. The difference between derived and coderived categories in this case can also be seen by noting that the $k[\epsilon]$-module k is not perfect, but is compact in the coderived category.

3.2. **Koszul duality.** The correspondence of Proposition 3.9 is a manifestation of Koszul duality between modules and comodules, and can be regarded as a partial generalisation of [Pos, Theorem 6.3.a]. In particular, $\mathcal{A} \rightsquigarrow C$ is a cobar construction and $-\otimes_{\mathcal{A}} P$ can be thought of as $\omega \otimes^{\tau,S} C$ for the canonical twisting cochain τ.

Rather than fixing a dg category, we now use Koszul duality to give an equivalence between certain homotopy categories of dg categories and of dg coalgebras. A consequence is that quasi-isomorphisms in the category $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}$ (see Definition 3.1) induce quasi-isomorphisms of the associated categories, so are necessarily derived Morita equivalences.

Fix a cosemisimple coalgebra S, and set $\mathcal{S}$ to be the category of finite-dimensional S-comodules, with ω the forgetful functor to vector spaces. We can then interpret $\mathcal{S}$-bimodules as S-bicomodules, observing that $S = \omega^\vee \otimes_{\mathcal{S}} \omega$. Note that a non-counital coassociative dg $\otimes_{\mathcal{S}}$-coalgebra $\mathcal{B}$ in $\mathcal{S}$-bimodules then corresponds to the non-counital coassociative dg $\otimes^S$-coalgebra $B := \omega^\vee \otimes_{\mathcal{S}} \mathcal{B} \otimes_{\mathcal{S}} \omega$ in S-bicomodules. This is equivalent to a coassociative dg coalgebra structure on $S \oplus B$, for which $S \to S \oplus B \to S$ are morphisms of dg coalgebras.

Definition 3.11. Let $DG^{>0}\mathrm{Cat}(\mathcal{S})$ be the category of dg categories $\mathcal{A}$ in non-negative degrees with $\mathcal{A}^0 = \mathcal{S}$ and $d\mathcal{A}^0 = 0$, as considered in Proposition 3.2.

This is equivalent to the category of associative $\otimes_{\mathcal{S}}$-algebras $\mathcal{A}^{>0}(-,-)$ in cochain complexes of $\mathcal{S}$-bimodules in strictly positive degrees. Applying [Hir, Theorem 11.3.2] to the forgetful functor mapping to $\mathcal{S}$-bimodules, it follows that $DG^{>0}\mathrm{Cat}(\mathcal{S})$ has a cofibrantly generated model structure in which weak equivalences are quasi-isomorphisms and fibrations are surjections.

Definition 3.12. Dually, let $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$ be the category of non-counital ind-conilpotent coassociative dg $\otimes_{\mathcal{S}}$-coalgebras B in complexes of $\mathcal{S}$-bimodules in non-negative cochain degrees.

By analogy with [Hin, Theorem 3.1] and [Pri5, Proposition 1.26], $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$ has a fibrantly cogenerated model structure in which weak equivalences are quasi-isomorphisms and cofibrations are injective in degrees > 0.

Definition 3.13. Write $\beta(\mathcal{A})$ for the cofree ind-conilpotent graded $\otimes_{\mathcal{S}}$-coalgebra on generators $\mathcal{A}^{>0}(-,-)[1]$; thus

$$\beta(\mathcal{A}) = \bigoplus_{n>0} \underbrace{\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{S}} \ldots \otimes_{\mathcal{S}} \mathcal{A}^{>0}(-,-)}_{n}[n].$$

We make this a dg coalgebra by defining the differential on cogenerators to be

$$d_{\beta(\mathcal{A})} = (d_{\mathcal{A}}, \circ)\colon\ (\mathcal{A}^{>0}(-,-)[1]) \oplus (\mathcal{A}^{>0}(-,-) \otimes_{\mathcal{S}} \mathcal{A}^{>0}(-,-)[2]) \to \mathcal{A}^{>0}(-,-)[2];$$

Note that the dg coalgebra C of Proposition 3.2 is just the dg coalgebra $S \oplus (\omega^\vee \otimes_{\mathcal{S}} \beta(\mathcal{A}) \otimes_{\mathcal{S}} \omega)$. This cobar construction defines a functor $\beta\colon DG^{>0}\mathrm{Cat}(\mathcal{S}) \to DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$, with Proposition 3.2 saying that $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathcal{D}_{\mathrm{dg}}(S \oplus (\omega^\vee \otimes_{\mathcal{S}} \beta(\mathcal{A}) \otimes_{\mathcal{S}} \omega))$.

Definition 3.14. Let β^* be the left adjoint to β. This is the bar construction sending C to the tensor algebra

$$\beta^*(C)(X,Y) = \bigoplus_{n\geq 0} \underbrace{C(X,-)\otimes_{\mathcal{S}} \ldots \otimes_{c\mathcal{S}} C(-,Y)}_{n}[-n],$$

with differential defined on generators by $d_C + \Delta_C$.

Remark 3.15. A key observation is that the filtration of $\beta(C)$ by powers of $C[-1]$ gives a convergent spectral sequence

$$\mathrm{H}^q(C^{\otimes p}) \implies \mathrm{H}^{p+q}(\beta^* C).$$

Note that convergence of this spectral sequence relies on $\mathrm{H}^{<0}(C)$ vanishing, and allows us to use quasi-isomorphisms for our notion of dg coalgebra weak equivalences where [Hin, Theorem 3.1] used β^* to reflect weak equivalences.

Proposition 3.16. *The functors $\beta^* \dashv \beta$ are a pair of Quillen equivalences between the categories $DG^{>0}\mathrm{Cat}(\mathcal{S}), DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$.*

Proof. We need to show that for any $C \in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}(\mathcal{S})$, the unit $C \to \beta\beta^* C$ of the adjunction is a quasi-isomorphism, and that for any $\mathcal{A} \in DG^{>0}\mathrm{Cat}(\mathcal{S})$, the co-unit $\beta^*\beta\mathcal{A} \to \mathcal{A}$ is a quasi-isomorphism.

We begin by noting that [LV, Proposition 11.4.4] says that for C fibrant, the unit $C \to \beta\beta^*(C)$ gives a quasi-isomorphism on tangent spaces, where $\tan(C) = \ker(\Delta\colon C \to C \otimes C)$. Now, $\tan\beta(\mathcal{A}) = \mathcal{A}^{>0}(-,-)$, so setting $C = \beta(\mathcal{A})$, it follows that the unit $C \to \beta\beta^*(C)$ gives a quasi-isomorphism $\mathcal{A}^{>0} \to \beta^*\beta(\mathcal{A})^{>0}$, from which it follows that the co-unit is a quasi-isomorphism.

Now for C fibrant, filtration by the subspaces $\ker(C \to C^{\otimes n})$ gives a convergent spectral sequence

$$E_1^{pq} = \mathrm{H}^{p+q}((\tan C)^{\otimes -p}) \implies \mathrm{H}^{p+q}(C),$$

so cotangent quasi-isomorphisms of fibrant objects are always quasi-isomorphisms, and in particular the unit $C \to \beta\beta^*(C)$ is a quasi-isomorphism for fibrant C. Since the functor $\beta\beta^*$ preserves quasi-isomorphisms, the unit must be a quasi-isomorphism for all C. □

Remark 3.17. The key step in Proposition 3.16 invokes Koszul duality in the form of [LV, Proposition 11.4.4]. When the field k has characteristic 0, there are therefore analogues for any Koszul-dual pair of operads, with cochain dg $\mathcal{P}$-algebras in strictly positive degrees corresponding to conilpotent $\mathcal{P}^{\text{!`}}$-coalgebras in non-negative degrees.

3.3. **Hearts of t-structures.** For a Morita fibrant dg category $\mathcal{D}$ to admit a compatible t-structure amounts to the existence of a full generating dg subcategory $\mathcal{A}$ with $\mathrm{H}^i\mathcal{A}(X,Y) = 0$ for all $i < 0$ and all X, Y. The objects of $\mathcal{A}$ are given by any choice of generators for the heart $\mathrm{H}^0\mathcal{D}^{\heartsuit}$ of the t-structure, and in particular we can take $\mathcal{A}$ to be the full dg subcategory of $\mathcal{D}$ on the semisimple objects of $\mathrm{H}^0\mathcal{D}^{\heartsuit}$, in which case $\mathrm{H}^0\mathcal{A}$ will be abelian semisimple.

We now show how to extend the results of §3.1 to this generality.

Proposition 3.18. *Take a k-linear dg category $\mathcal{A}$ with the category $\mathrm{H}^0\mathcal{A}$ abelian semisimple and $\mathrm{H}^{<0}\mathcal{A}(X,Y) = 0$ for all objects X, Y. Then $\mathcal{A}$ is quasi-isomorphic to a dg category $\mathcal{B}$ concentrated in non-negative degrees, with $d\mathcal{B}^0(X,Y) = 0$ for all X, Y.*

Proof. First, observe that the good truncation filtration $\tau_n = \tau^{\leq n}$ for $n \geq 0$ gives quasi-isomorphisms

$$\bigoplus_n \mathrm{gr}^\tau_n \mathcal{A}(X,Y) \to \bigoplus_n \mathrm{H}^n\mathcal{A}(X,Y)[-n]$$

for all $X, Y \in \mathcal{A}$, giving $\mathbb{G}_m$-equivariant quasi-isomorphisms of the corresponding dg categories, where gr^τ_n is assigned weight n.

Now, a dg category $\mathcal{B}$ over $\tau_0\mathcal{A}$ on the same objects corresponds to the associative unital $\otimes_{\tau_0\mathcal{A}}$-algebra $\mathcal{B}(-,-)$ in $\mathcal{C}_{\mathrm{dg}}((\tau_0\mathcal{A})^{\mathrm{opp}}\otimes\tau_0\mathcal{A})$. Thus the quasi-isomorphism $\tau_0\mathcal{A} \to \mathrm{H}^0\mathcal{A}$ ensures that $\mathcal{A}$ is quasi-isomorphic to some dg category $\mathcal{A}'$ over $\mathrm{H}^0 A$.

Consider the polynomial ring $k[t]$ with t in degree 0 but equipped with a $\mathbb{G}_m$-action of weight 1. The Rees construction gives us a dg category $\zeta(\mathcal{A},\tau)$ with the same objects as $\mathcal{A}$ and morphisms

$$\zeta(\mathcal{A},\tau)(X,Y) := \bigoplus_n \tau_n\mathcal{A}(X,Y)t^n \cong \bigoplus_n \tau_n\mathcal{A}(X,Y),$$

which then becomes a $\mathbb{G}_m$-equivariant dg category, flat over $k[t]$. This has the properties that $\zeta(\mathcal{A},\tau)/t \cong \mathrm{gr}^\tau\mathcal{A}$ and $\zeta(\mathcal{A},\tau)/(t-1) \cong \mathcal{A}$.

Applying the argument above, we see that the $\mathbb{G}_m$-equivariant dg category $\zeta(\mathcal{A},\tau)$ over $\tau_0\mathcal{A}[t]$ must be quasi-isomorphic to some $\mathbb{G}_m$-equivariant dg category $\mathcal{Z}$ over $\mathrm{H}^0\mathcal{A}[t]$ (flat over $k[t]$). Note that $\mathcal{Z}/t$ and $\mathcal{Z}/(t-1)$ are then quasi-isomorphic to $\mathrm{gr}^\tau\mathcal{A}$ and $\mathcal{A}$, respectively.

Now, set $\mathcal{G}^0 = \mathrm{H}^0\mathcal{A}$, then take a cofibrant replacement $\mathcal{G}^{>0}(-,-)$ of $\mathrm{H}^{>0}\mathcal{A}(-,-)$ as a $\otimes_{\mathcal{G}^0}$-algebra in $\mathcal{G}$-bimodules. This can be constructed canonically as a bar-cobar resolution as in Proposition 3.16, with the output concentrated in strictly positive degrees (this relies on the semisimplicity of $\mathrm{H}^0\mathcal{A}$). Moreover, the $\mathbb{G}_m$-action on $\mathrm{H}^*(\mathcal{A}(X,Y))$ (with H^n of weight n) transfers equivariantly to $\mathcal{G} := \mathcal{G}^0 \oplus \mathcal{G}^{>0}$.

Since $\mathcal{G}$ is cofibrant and $\mathrm{gr}^\tau\mathcal{A}$ is quasi-isomorphic to $\mathrm{H}^*\mathcal{A}$, we may lift the quasi-isomorphism $\mathcal{G} \to \mathrm{H}^*\mathcal{A}$ to give a $\mathbb{G}_m$-equivariant quasi-isomorphism

$$\mathcal{G} \to \mathcal{Z}/t.$$

We now mimic [Pri8, Propositions 14.11,14.12]. Since $\mathcal{G}(-,-)$ is cofibrant as a unital $\otimes_{\mathcal{G}^0}$-algebra, forgetting the differential gives a retract $\mathcal{G}^*$ of a freely generated $\mathbb{G}_m$-equivariant $\otimes_{\mathcal{G}^0}$-algebra. Since $\mathcal{Z} \to \mathcal{Z}/t$ is surjective, we may lift the map $\mathcal{G} \to \mathcal{Z}/t$ to give a $\mathbb{G}_m$-equivariant map $f\colon \mathcal{G}^* \to \mathcal{Z}$ of graded categories, and hence $\mathcal{G}^*[t] \to \mathcal{Z}$.

We then consider possible differentials on $\mathcal{G}^{>0}[t]$ making f into a map of $\otimes_{\mathcal{G}^0}$-algebras. The proof of [Pri7, Proposition 3.7] characterises obstructions to passing from a differential on $\mathcal{G}^{>0}[t]/t^r$ to one on $\mathcal{G}^{>0}[t]/t^{r+1}$ in terms of elements in Hochschild cohomology, which necessarily vanish because the lift $\mathcal{Z}/t^{r+1} \to \mathcal{Z}/t^r$ exists. In the $\mathbb{G}_m$-equivariant category, $\mathcal{G}^*[t]$ is the limit $\varprojlim_r \mathcal{G}^*[t]/t^r$, so a suitable differential δ exists on $\mathcal{G}^*[t]$, set to 0 on $\mathcal{G}^0[t]$.

Writing $\mathcal{R} = (\mathcal{G}^*[t], \delta)$ with its differential, we thus have a $\mathbb{G}_m$-equivariant quasi-isomorphism

$$\mathcal{R} \to \mathcal{Z}$$

over $k[t]$ (deformations of quasi-isomorphisms being quasi-isomorphisms), and hence a quasi-isomorphism

$$\mathcal{R}/(t-1) \to \mathcal{Z}/(t-1) \simeq \mathcal{A},$$

so $\mathcal{B} := \mathcal{R}/(t-1)$ has the required properties. ☐

Corollary 3.19. *Take a k-linear dg category $\mathcal{A}$ with the category $\mathrm{H}^0\mathcal{A}$ abelian semisimple and $\mathrm{H}^{<0}\mathcal{A}(X,Y)=0$ for all objects X,Y. Assume that we have a k-linear dg functor $\omega\colon \mathcal{A}\to\mathcal{C}_{\mathrm{dg}}(k)$ with $\mathrm{H}^*\omega(X)$ finite-dimensional and concentrated in degree 0 for all $X\in\mathcal{A}$. Then there is a dg coalgebra $C\in DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$ with $C\simeq\omega^\vee\otimes^{\mathbf{L}}_{\mathcal{A}}\omega$, together with quasi-equivalences $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})/(\ker\omega)\simeq(\ker\omega)^\perp\simeq\mathcal{D}_{\mathrm{dg}}(C)$.*

Proof. By Proposition 3.18, there is a quasi-isomorphism $q\colon \mathcal{B}\to\mathcal{A}$ with $\mathcal{B}$ satisfying the conditions of Proposition 3.2. It therefore suffices to replace $\omega\circ q\colon \mathcal{B}\to\mathcal{C}_{\mathrm{dg}}(k)$ with a quasi-isomorphic dg functor taking values in FDVect_k. Now, since $\mathcal{B}^0\simeq\mathrm{H}^0\mathcal{A}$ is semisimple, we may decompose the $\mathcal{B}^0$-module $(\omega\circ q)^0$ as $d(\omega\circ q)^{-1}\oplus M^0$ for some $\mathcal{B}^0$-module M^0. Setting $M^i=(\omega\circ q)^i$ for $i>0$ and $M^{<0}=0$ then gives a quasi-isomorphism $M\to\omega\circ q$ of $\mathcal{B}$-modules. Next, choose a decomposition $M^0=\mathrm{Z}^0M\oplus N^0$ of $\mathcal{B}^0$-modules, and set $N^i=M^i$ for $i>0$. Since $\mathrm{H}^iM=0$ for $i\neq 0$, it follows that N is acyclic, so $M\to M/N$ is a quasi-isomorphism, and $M/N\cong\mathrm{H}^0(\omega\circ q)$ takes values in FDVect_k. □

Example 3.20 (The motivic coalgebra and mixed motives). As in §2.5, consider the dg functor $E^\vee\colon \mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}(F,\mathbb{Q})\to\mathcal{C}_{\mathrm{dg}}(\mathbb{Q})$ associated to a mixed Weil homology theory, and let C be the associated dg coalgebra, with quasi-equivalences

$$\mathcal{D}_{\mathrm{dg}}(C)\simeq\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})/(\ker E)\simeq\mathcal{M}_{\mathrm{dg}}(F,\mathbb{Q})/(\ker E).$$

Similarly, let C^{eff} be the dg coalgebra associated to the the restriction of E to effective Beilinson motives, giving

$$\mathcal{D}_{\mathrm{dg}}(C^{\mathrm{eff}})\simeq\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})/(\ker E)\simeq\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}(F,\mathbb{Q})/(\ker E),$$

where $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}(F,\mathbb{Q})$ is the dg category of cofibrant presheaves of $\mathbb{Q}$-complexes on smooth F-schemes.

When E is Betti cohomology, it is shown in [Ayo3, Corollary 2.105] that $\mathrm{H}^{>0}C=0$ and $\mathrm{H}^{>0}C^{\mathrm{eff}}=0$. By [Ayo3, Lemma 2.145], existence of a motivic t-structure would also imply that $\mathrm{H}^{<0}C=0$, so we would have quasi-isomorphisms $C\to\tau^{\geq 0}C\leftarrow\mathrm{H}^0C$ of dg coalgebras, but it is not immediate that these are Morita equivalences.

However, if a motivic t-structure exists, then $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})\subset(\ker E)^\perp$ and applying Corollary 3.19 to the full dg subcategory $\mathcal{A}$ of $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1,c}(F,\mathbb{Q})$ or $\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1,c}(F,\mathbb{Q})$ on semisimple objects in the heart of the t-structure would yield N or N^{eff} in $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$, Morita equivalent to C or C^{eff}, so by Propositions 3.7 and 3.9,

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})\simeq\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(N),\quad \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})\simeq\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(N^{\mathrm{eff}}),$$
$$\mathcal{M}^{+}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})\simeq\mathcal{D}^{+}_{\mathrm{dg}}(N),\quad \mathcal{M}^{\mathrm{eff},+}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})\simeq\mathcal{D}^{+}_{\mathrm{dg}}(N^{\mathrm{eff}}),$$

where $\mathcal{M}^{+}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})$ is the full dg subcategory of $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})$ consisting of objects in $\bigcup_n\mathcal{M}_{\mathbb{A}^1}(F,\mathbb{Q})^{\geq n}$ for the motivic t-structure.

By [Ayo3, Corollary 2.105], the morphisms $\mathrm{H}^0N\to N$ and $\mathrm{H}^0N^{\mathrm{eff}}\to N^{\mathrm{eff}}$ would then be quasi-isomorphisms, hence Morita equivalences by Proposition 3.16, and we would have

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})\simeq\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(\mathrm{H}^0N),\quad \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})\simeq\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(\mathrm{H}^0N^{\mathrm{eff}}).$$

Letting $\mathcal{MM}_F$ and $\mathcal{MM}^{\mathrm{eff}}_F$ be the categories of H^0N- and $\mathrm{H}^0N^{\mathrm{eff}}$-comodules in finite-dimensional vector spaces, and $\mathcal{D}_{\mathrm{dg}}(\mathcal{MM}_F)$ and $\mathcal{D}_{\mathrm{dg}}(\mathcal{MM}^{\mathrm{eff}}_F)$ the dg enhancements of their derived categories, we would then have

$$\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}_{\mathrm{dg}}(\mathcal{MM}_F), \quad \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}_{\mathrm{dg}}(\mathcal{MM}^{\mathrm{eff}}_F),$$

so existence of a motivic t-structure would automatically realise $\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})$ as the dg derived category of an abelian category of mixed motives, implying the $K(\pi,1)$ conjecture of [BK2, (1.2)].

Moreover, the full dg subcategory $\mathcal{M}^{\mathrm{eff},\heartsuit}_{\mathrm{dg}} \subset \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}$ consisting of objects M with $\mathrm{H}^*E(M)$ concentrated in degree 0 contains $\ker E$, so the heart of the t-structure would be

$$\mathcal{M}^{\mathrm{eff}}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q})^{\heartsuit} \simeq \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}(F,\mathbb{Q})^{\heartsuit}/\ker E = \mathcal{M}^{\mathrm{eff}}_{\mathrm{dg}}(F,\mathbb{Q})^{\heartsuit}/(\ker \mathrm{H}^0E).$$

Passing to the associated triangulated categories $\mathcal{M} := \mathrm{H}^0\mathcal{M}_{\mathrm{dg}}$ would give

$$\mathcal{MM}^{\mathrm{eff}}_F \simeq \mathcal{M}^{\mathrm{eff}}_{\mathbb{A}^1}(F,\mathbb{Q})^{\heartsuit} \simeq \mathcal{M}^{\mathrm{eff}}(F,\mathbb{Q})^{\heartsuit}/(\ker \mathrm{H}^0E).$$

Nori's abelian category $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ from Remark 2.17 is defined by forming a diagram D^{eff} of good pairs, and taking the universal $\mathbb{Q}$-linear abelian category under D^{eff} on which H^0E is faithful. The proof of [HMS, Corollary 1.7] gives a functor from D^{eff} to $\mathrm{H}^0\mathcal{G}^{\mathrm{eff}} \subset \mathcal{M}^{\mathrm{eff}}_c(\mathbb{Q},\mathbb{Q})^{\heartsuit}$ and [HMS, Proposition D.3] gives a quasi-inverse to the induced functor

$$\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} \to \mathcal{M}^{\mathrm{eff}}_c(\mathbb{Q},\mathbb{Q})^{\heartsuit}/(\ker \mathrm{H}^0E),$$

so existence of a motivic t-structure would also imply

$$\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} \simeq \mathcal{MM}^{\mathrm{eff}}_{\mathbb{Q}}.$$

Following Example 2.22, all these coalgebras can be promoted to bialgebras, with the equivalences of categories preserving monoidal structures. In particular, H^0C is a naturally a Hopf algebra and $\mathrm{H}^0C^{\mathrm{eff}}$ a bialgebra. Writing $G_{\mathrm{mot}}(F) := \operatorname{Spec}\mathrm{H}^0C$ would then give a motivic Galois group, and letting $\mathcal{MM}_F$ be the abelian tensor category of finite-dimensional $G_{\mathrm{mot}}(F)$-representations, the equivalence

$$\mathrm{H}^0\mathcal{M}_{\mathrm{dg},\mathbb{A}^1}(F,\mathbb{Q}) \simeq \mathcal{D}(\mathcal{MM}_F),$$

would become monoidal, as would the equivalence

$$\mathcal{MM}_{\mathbb{Q}} \simeq \mathcal{MM}_{\mathrm{Nori}}$$

induced from $\mathcal{MM}^{\mathrm{eff}}_{\mathbb{Q}} \simeq \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ by stabilisation. In particular, this would imply that $G_{\mathrm{mot}}(\mathbb{Q})$ is Nori's motivic Galois group from [HMS, Theorem 1.14]. This last result has very recently been proved in [CdS] without assuming existence of a motivic t-structure.

3.4. **Tensor categories.** For the remainder of this section, we require that the field k be of characteristic 0.

3.4.1. *Non-negatively graded dg tensor categories.* Assume that $\mathcal{T}$ is a rigid tensor dg category over k (i.e. a symmetric monoidal dg category with strong duals), with $\mathcal{S} := \mathcal{T}^0$ a rigid tensor subcategory. Define $B \in \mathcal{C}(\mathcal{S}^{\mathrm{opp}})$ by $B(U) := \mathcal{T}(U,\mathbb{1})$. Note that tensor properties ensure that $\mathcal{T}(U,V) = \mathcal{T}(U \boxtimes V^{\vee},\mathbb{1}) = B(U \boxtimes V^{\vee})$.

Thus $B\colon \mathcal{S} \to \mathcal{C}$ is a symmetric lax monoidal functor, or equivalently a unital commutative algebra object in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$. This is the same as saying that B is a DGA over $\mathcal{S}$ in the sense of [Pri2, Definition 3.2].

If $\mathcal{S}$ is a semisimple abelian category and $\omega\colon \mathcal{S} \to \mathrm{FDVect}$ a faithful symmetric monoidal functor, then [DMOS, Ch. II] shows that $\mathcal{S}$ is equivalent to the category $\mathrm{Rep}(R)$ of finite-dimensional R-representations for the pro-reductive affine group scheme

$$R := \mathrm{Spec}\, \omega^\vee \otimes_{\mathcal{S}} \omega.$$

Equivalently, this is the category of finite-dimensional $O(R)$-comodules, where $O(R)$ is the Hopf algebra $\omega^\vee \otimes_{\mathcal{S}} \omega$.

As observed in [Pri2, Remark 3.15], the category of DGAs over $\mathcal{S}$ is equivalent to the category of R-equivariant commutative dg algebras. Under this correspondence, B corresponds to $A := B(O(R))$ (regarding the right R-representation $O(R)$ as an object of $\mathrm{ind}(\mathcal{S})$, with the R-action on A coming from the left action on $O(R)$). When this is concentrated in non-negative cochain degrees, note that it defines a schematic homotopy type in the sense of [KPT]. For the inverse construction, we have $B(V) = A \otimes^R V$ and $\mathcal{T}(U, V) = \mathcal{S}(U \otimes A, V) = \mathrm{Hom}_R(V, U \otimes_k A)$.

Definition 3.21. Define $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}_k$ to consist of (commutative but not necessarily cocommutative) dg Hopf algebras C for which the underlying dg k-coalgebra lies in the category $DG^{\geq 0}\mathrm{Co}_n\mathrm{Alg}_k$ of Definition 3.1. In other words, C is concentrated in non-negative cochain degrees, with $\mathrm{H}^0 C \to C$ ind-conilpotent.

Proposition 3.22. *Take a k-linear rigid tensor dg category $\mathcal{T}$ with $\mathcal{T}(X, Y)$ concentrated in non-negative degrees, $d\mathcal{T}^0(X, Y) = 0$ for all X, Y, and $\mathcal{T}^0$ a semisimple rigid tensor subcategory. Assume that we have a strong monoidal k-linear functor $\omega\colon \mathcal{T}^0 \to \mathrm{FDVect}_k$. Then there is a model for the bialgebra $C \simeq \omega^\vee \otimes^{\mathbf{L}}_{\mathcal{T}} \omega$ of §1.3.3 with $C \in DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}_k$.*

Proof. We just take the coalgebra C from the proof of Proposition 3.2, and observe that the formulae of §1.3.1 adapt to define a coproduct Δ and antipode ρ on C, making it into a dg Hopf algebra.

Explicitly, writing $\mathcal{S} = \mathcal{T}^0$, the expression $\mathcal{T}(U, V) = \mathcal{S}(U \otimes A, V)$ above allows us to rewrite

$$D = \bigoplus_{n \geq 0} A^{\otimes n+2} \otimes O(R) \quad C = \bigoplus_{n \geq 0} A^{\otimes n} \otimes O(R),$$

with

$$\Delta(a_1 \otimes \ldots \otimes a_n \otimes 1) = \sum_{0 \leq r \leq n} (a_1 \otimes \ldots \otimes a_r \otimes 1) \otimes (a_{r+1} \otimes \ldots \otimes a_n \otimes 1)$$

and

$$\rho(a_1 \otimes \ldots \otimes a_n \otimes 1) = (-1)^n (a_n \otimes \ldots \otimes a_1 \otimes 1),$$

and with multiplication given by the shuffle product. The coalgebra structure on C is then given as the semidirect tensor product of $\bigoplus_{n \geq 0} A^{\otimes n}$ and $O(R)$. □

Definition 3.23. Given a commutative unital dg algebra A in R-representations, define the dg category $\mathrm{Rep}(R, A)$ to have R-representations in finite-dimensional vector spaces as objects, with morphisms given by

$$\mathrm{Rep}(R, A)(U, V) := A \otimes^R (U \otimes_k V^\vee).$$

Multiplication is induced by multiplication in A, with identities $1_A \otimes \mathrm{id}_V \in A \otimes^R (V \otimes_k V^\vee)$.

Remark 3.24. In the notation of [Pri3, Remark 4.35]), the dg Hopf algebra corresponding to $\mathcal{T}$ is given by $O(R \ltimes \bar{G}(A)) = O(R \ltimes \bar{G}(\mathcal{T}(O(R), \mathbb{1})))$.

Then Propositions 3.7 and 3.9 give quasi-equivalences

$$\begin{aligned} \mathcal{D}^+_{\mathrm{dg}}(\mathrm{Rep}(R, A)) &\simeq \mathcal{D}^+_{\mathrm{dg}}(O(R \ltimes \bar{G}(A))), \\ \mathcal{D}_{\mathrm{dg}}(\mathrm{Rep}(R, A)) &\simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(R \ltimes \bar{G}(A))). \end{aligned}$$

3.4.2. *Koszul duality for tensor categories.* Now fix a pro-reductive affine group scheme R.

Definition 3.25. Let $DG^{>0}\mathrm{Cat}^{\otimes}(R)$ be the category of rigid tensor dg categories $\mathcal{T}$ in non-negative degrees with $\mathcal{T}^0$ the category of finite-dimensional R-representations and $d\mathcal{T}^0 = 0$.

Note that under the discussion of §3.4.1, $DG^{>0}\mathrm{Cat}^{\otimes}(R)$ is equivalent to the category $DG^{>0}\mathrm{Comm}(R)$ of commutative R-equivariant dg algebras A in strictly positive cochain degrees. There is a model structure on $DG^{>0}\mathrm{Cat}^{\otimes}(R)$ in which weak equivalences are quasi-isomorphisms and fibrations are surjections.

Definition 3.26. Dually, let $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$ be the category of R-equivariant ind-conilpotent dg Hopf algebras N in non-negative cochain degrees.

Note that these correspond to Hopf algebras C equipped with maps $O(R) \to C \to O(R)$, such that $O(R) \to C$ is ind-conilpotent. The correspondence sends N to the tensor product $O(R) \otimes N$, with comultiplication given by semidirect product.

Moreover, as in [Qui, Theorem B.4.5], ind-conilpotent dg Hopf algebras N correspond to ind-conilpotent dg Lie coalgebras L, with $L = \tan(N)$. Thus $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$ is equivalent to the category $DG^{\geq 0}\mathrm{Co}_n\mathrm{Lie}(R)$ of ind-conilpotent dg Lie coalgebras L in non-negative cochain degrees. It is thus equivalent to the category $dg\hat{\mathcal{N}}(R)$ of [Pri3, Definition 4.1], so is a model for relative Malcev homotopy types over R.

By analogy with [Hin, Theorem 3.1] and [Pri5, Proposition 1.26], there is then a model structure on $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}_k$ in which weak equivalences are quasi-isomorphisms and cofibrations induce injections on tangent spaces in degrees > 0.

Now, the functor β of Definition 3.13 preserves tensor structures, so induces $\beta\colon DG^{>0}\mathrm{Cat}^{\otimes}(R) \to DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$. Equivalently, we have $\beta_{\otimes}\colon DG^{>0}\mathrm{Comm}(R) \to DG^{\geq 0}\mathrm{Co}_n\mathrm{Lie}(R)$ given by setting $\beta_{\otimes}(A)$ to be the cofree ind-conilpotent graded Lie coalgebra on cogenerators $A[1]$, with differential defined on cogenerators by

$$d_{\beta_{\otimes}(A)} = (d_A, \circ)\colon A[1] \oplus (\mathrm{Symm}^2 A)[2] \to A[2],$$

noting that $(\mathrm{Symm}^2 A)[2] = \bigwedge^2(A[1])$.

Moreover, the commutative and Lie operads are Koszul duals, and β has a left adjoint $\beta^*_{\otimes}$, given by

$$\beta^*_{\otimes}(L) = \bigoplus_{n>0} \mathrm{Symm}^n(L[-1]),$$

with differential given on the generators by $d_L \oplus \Delta\colon L[-1] \to L \oplus (\bigwedge^2 L)[-1]$ where Δ is the Lie cobracket.

Thus Remark 3.17 and the comparisons above allow us to adapt Proposition 3.16 to give:

Proposition 3.27. *The functors $\beta^*_{\otimes} \dashv \beta_{\otimes}$ give a pair of Quillen equivalences between the categories $DG^{>0}\mathrm{Cat}^{\otimes}(R), DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$.*

Using the characterisation of $DG^{>0}\mathrm{Cat}^{\otimes}(R)$ in terms of commutative dg algebras and $DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}(R)$ in terms of dg Lie algebras, this result is effectively one of the equivalences of [Pri3, Theorem 4.41].

3.4.3. *Hearts of tensor t-structures.*

Proposition 3.28. *Take a k-linear rigid tensor dg category $\mathcal{T}$ with the category $\mathrm{H}^0\mathcal{T}$ abelian semisimple and $\mathrm{H}^{<0}\mathcal{T}(X,Y) = 0$ for all objects X, Y. Then $\mathcal{T}$ is quasi-isomorphic to a rigid tensor dg category $\mathcal{B}$ concentrated in non-negative degrees, with $d\mathcal{B}^0(X,Y) = 0$ for all X, Y.*

Proof. If we write $\mathcal{S} := \mathrm{H}^0\mathcal{T}$, we see that the dg tensor categories $\mathcal{S}$ and $\tau^{\leq 0}\mathcal{T}$ are quasi-isomorphic, from which it follows that $\mathcal{T}$ is quasi-isomorphic to some rigid tensor dg category $\mathcal{T}'$ over $\mathcal{S}$. Via the discussion of §3.4.1, this is equivalent to giving a commutative dg algebra A in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$ with $\mathrm{H}^0A = k$.

We now apply the Rees algebra construction $\zeta(A,\tau)$ to the good truncation filtration on A, regarding the Rees algebra as a deformation of $\zeta(A,\tau) = \mathrm{H}^*(A)$. The bar-cobar resolution $\beta^*_{\otimes}\beta_{\otimes}(\mathrm{H}^{>0}A)$ of Proposition 3.27 gives a cofibrant replacement for $\mathrm{H}^{>0}A$ concentrated in strictly positive degrees. The proof of Proposition 3.18 now adapts, substituting André–Quillen cohomology for Hochschild cohomology. □

Corollary 3.29. *Take a k-linear rigid tensor dg category $\mathcal{T}$ with the category $\mathrm{H}^0\mathcal{T}$ abelian semisimple and $\mathrm{H}^{<0}\mathcal{T}(X,Y) = 0$ for all objects X, Y. Assume that we have a lax monoidal k-linear dg functor $\omega\colon \mathcal{T} \to \mathcal{C}_{\mathrm{dg}}(k)$ with $\mathrm{H}^i\omega(X) = 0$ for all $i \neq 0$, $\mathrm{H}^0\omega(X)$ finite-dimensional for all $X \in \mathcal{T}$, and quasi-strong in the sense that the structure maps*

$$\omega(X) \otimes_k \omega(Y) \to \omega(X \otimes Y), \quad k \to \omega(\mathbb{1})$$

quasi-isomorphisms for all $X, Y \in \mathcal{T}$.

Then there is a dg Hopf algebra $C \in DG^{\geq 0}\mathrm{Hopf}_n\mathrm{Alg}_k$ with $C \simeq \omega^{\vee} \otimes^{\mathbf{L}}_{\mathcal{T}} \omega$, together with a tensor dg functor $\mathcal{D}_{\mathrm{dg}}(\mathcal{T}) \to \mathcal{C}_{\mathrm{dg}}(C)$ inducing quasi-equivalences $\mathcal{D}_{\mathrm{dg}}(\mathcal{T})/\ker\omega \simeq (\ker\omega)^{\perp} \simeq \mathcal{D}_{\mathrm{dg}}(C)$. Here, $\mathcal{C}_{\mathrm{dg}}(C)$ and $\mathcal{D}_{\mathrm{dg}}(C)$ are defined using the coalgebra (not the algebra) structure of C.

If the functor $\mathrm{H}^0\omega\colon \mathrm{H}^0\mathcal{T} \to \mathrm{FDVect}_k$ is faithful, then these induce quasi-equivalences $\mathcal{D}^+_{\mathrm{dg}}(\mathcal{T}) \simeq \mathcal{D}^+_{\mathrm{dg}}(C)$ and $\mathcal{D}_{\mathrm{dg}}(\mathcal{T}) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(C)$.

Proof. By Proposition 3.28, there is a tensor quasi-isomorphism $q\colon \mathcal{B} \to \mathcal{A}$ with $\mathcal{B}$ satisfying the conditions of Proposition 3.2. Write $\mathcal{S} := \mathcal{B}^0 \simeq \mathrm{H}^0\mathcal{A}$.

We now show how to replace $\omega\circ q\colon \mathcal{B} \to \mathcal{C}_{\mathrm{dg}}(k)$ with $\mathrm{H}^0(\omega\circ q)$. If A is the commutative dg algebra in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$ corresponding to $\mathcal{B}$, then $\omega \circ q$ corresponds to an A-algebra M in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$. Since $\mathrm{H}^*M(X)$ is concentrated in degree 0 for all X, we have quasi-isomorphisms $M \leftarrow \tau^{\leq 0}M \to \mathrm{H}^0M$ of algebras in $\mathcal{C}(\mathcal{S}^{\mathrm{opp}})$. As A is cofibrant, the map $A \to M$ is homotopic to a map taking values in $\tau^{\leq 0}M$, so M and H^0M are quasi-isomorphic as A-algebras.

Since we now have a monoidal functor $\mathrm{H}^0(\omega\circ q)\colon \mathcal{B}^0 \to \mathrm{FDVect}_k$, the first statement of the corollary follows from Corollary 3.5, with Proposition 2.21 ensuring that the tensor structure is preserved. The second statement is then an immediate consequence of Propositions 3.7 and 3.9. □

3.5. **Comparison with Moriya.**

Definition 3.30. For a dg category $\mathcal{A}$ satisfying the conditions of Corollary 3.5, write $\widehat{\mathrm{H}^0\mathcal{A}} \subset \mathcal{D}(\mathcal{A})$ for the subcategory generated by $\mathrm{H}^0\mathcal{A}$ under finite extensions (but not by suspensions). This is the completion functor of [Mor, §2.2].

Note that all objects of $\widehat{\mathrm{H}^0\mathcal{A}}$ are perfect, so we also have a natural embedding $\mathrm{ind}(\widehat{\mathrm{H}^0\mathcal{A}}) \to \mathcal{D}(\mathcal{A})$. Under the conditions of Proposition 3.7, recall that Proposition 3.9 gives an equivalence between $\mathcal{D}^+(\mathcal{A})$ and $\mathcal{D}^+(C)$, with C concentrated in non-negative degrees, and that $\mathrm{H}^0\mathcal{A}$ is equivalent to the category of semisimple H^0C-comodules in finite-dimensional vector spaces. It then follows that $\widehat{\mathrm{H}^0\mathcal{A}}$ is equivalent to the category of all H^0C-comodules in finite-dimensional vector spaces, and $\mathrm{ind}(\widehat{\mathrm{H}^0\mathcal{A}})$ is equivalent to the category of all H^0C-comodules in vector spaces.

In [Mor, Definition 3.1.1], Moriya gives a notion of Tannakian dg category. Given a rigid dg tensor category $\mathcal{A}$ satisfying the conditions of Corollary 3.5 with a tensor functor $\omega\colon \mathcal{A}^0 \to \mathrm{FDVect}$ such that $\omega|_{\mathrm{H}^0\mathcal{A}}$ is faithful (as in Proposition 3.7), we could form a Tannakian dg category in Moriya's sense by taking $\hat{\mathcal{A}}$ to be the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ on objects $\widehat{\mathrm{H}^0\mathcal{A}}$.

Note that because $\hat{\mathcal{A}}$ is a full generating subcategory of $\mathrm{per}_{\mathrm{dg}}(\mathcal{A})$, it follows that $\mathrm{per}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{per}_{\mathrm{dg}}(\hat{\mathcal{A}})$ is a quasi-equivalence, so Corollary 2.12 gives the same output for both $\mathcal{A}$ and $\hat{\mathcal{A}}$. In topological contexts, this just amounts to saying that semisimple local systems generate all local systems under extension.

In [Mor, §3.3], functors T^{ss} and T are defined from commutative unital dg algebras A in R-representations to dg categories. In fact, $T^{\mathrm{ss}}(R, A) = \mathrm{Rep}(R, A)$ as given in Definition 3.23, and $T(R, A) = \widehat{T^{\mathrm{ss}}(R, A)}$. We have therefore defined Morita equivalences

$$\mathrm{Rep}(R, A) \to T(R, A) \to \mathrm{per}_{\mathrm{dg}}(\mathrm{Rep}(R, A)),$$

so Rep and T give rise to the same theory. However, $T(R, A)$ feels like a halfway house between the minimal choice $\mathrm{Rep}(R, A)$ and the fibrant replacement $\mathrm{per}_{\mathrm{dg}}(\mathrm{Rep}(R, A))$.

Moriya's analogue of the construction in Corollary 2.12 is the construction A_{red} of [Mor, Definition 3.3.3], but this has only limited functoriality, which is a well-known limitation of working with equivariant DGAs. By allowing dg coalgebras to have negative terms, Example 1.35 gives us a completely functorial choice of the dg Hopf algebra $C(\mathcal{T}, \omega)$ corresponding under [Pri3, Theorem 4.41] to Moriya's $A_{\mathrm{red}}(\mathcal{T}, \omega)$.

4. Schematic and Relative Malcev homotopy types

4.1. **de Rham homotopy types.** Take a pointed connected manifold (X, x), and choose a full rigid tensor subcategory $\mathcal{S}$ of the category of real finite-dimensional semisimple local systems on X. Let $\mathcal{T}$ be the real dg tensor category with the same objects as $\mathcal{S}$, but with morphisms

$$\mathcal{T}(U, V) = A^\bullet(X, U \otimes V^\vee),$$

where $A^\bullet(X, -)$ is the de Rham complex. Note that $\mathrm{H}^0\mathcal{T} \simeq \mathcal{S}$.

Now, the basepoint x defines a fibre functor $x^*\colon \mathcal{T}^0 \to \mathrm{FDVect}_{\mathbb{R}}$ sending U to U_x. We are therefore in the setting of Corollary 3.29. Moreover, $x^*\colon \mathcal{S} \to \mathrm{FDVect}$ is faithful because X is connected, so the conditions of Proposition 3.7 are satisfied.

Let R be the real pro-algebraic group $\mathrm{Spec}\,((x^*)^\vee \otimes_{\mathcal{S}} x^*)$; as in §3.4, we have an equivalence $x^*\colon \mathcal{S} \to \mathrm{Rep}(R)$ of tensor categories. Equivalently, we have a Zariski-dense group homomorphism $\rho\colon \pi_1(X, x) \to R(\mathbb{R})$. As in [Pri3], write $\mathbb{O}(R)$ for the local system corresponding to the right R ind-representation $O(R)$. The R-equivariant dg algebra A from §3.4 is then just the dg algebra

$$A^\bullet(X, \mathbb{O}(R))$$

of equivariant cochains from [Pri3, Definition 3.51], so $\mathcal{T}$ is equivalent to $\mathrm{Rep}(R, A^\bullet(X, \mathbb{O}(R)))$.

Definition 4.1. Write $\mathscr{A}^\bullet_X$ for the sheaf of real $\mathcal{C}^\infty$ differential forms on X, regarded as a sheaf of dg algebras with standard differential $\mathscr{A}^n_X \to \mathscr{A}^{n+1}_X$. Note that $A^\bullet(X, U) = \Gamma(X, U \otimes_{\mathbb{R}} \mathscr{A}^\bullet_X)$.

Definition 4.2. Define the dg category $\mathcal{P}(X)$ to consist of locally perfect $\mathscr{A}^\bullet_X$-modules in complexes of sheaves on X. Define $\mathcal{P}(X, \mathcal{S})$ to be the full dg subcategory of $\mathcal{P}(X)$ generated under shifts and extensions by objects of the form $U \otimes_{\mathbb{R}} \mathscr{A}^\bullet_X$ for $U \in \mathcal{S}$.

Note that because $\mathscr{A}^\bullet_X$ is a flabby resolution of $\mathbb{R}$, the dg category $\mathcal{P}(X)$ is quasi-equivalent to the category of locally constant hypersheaves in real complexes on X.

Lemma 4.3. *When $\mathcal{S}$ consists of all semisimple local systems, we have $\mathcal{P}(X, \mathcal{S}) = \mathcal{P}(X)$.*

Proof. Given $\mathscr{V}^\bullet \in \mathcal{P}(X)$, the sheaf $\bar{\mathscr{V}}^\bullet := \mathscr{V}^\bullet \otimes_{\mathscr{A}^\bullet_X} \mathscr{A}^0_X$ is a finite rank complex of $\mathcal{C}^\infty$-vector bundles on X. We then form the good truncation filtration $\{\tau^{\le n}\bar{\mathscr{V}}^\bullet\}_n$. Now, the morphisms $\mathscr{A}^0_X \to \mathscr{A}^*_X \to \mathscr{A}^0_X$ of sheaves of dg algebras give an isomorphism

$$\mathscr{V}^* \cong \bar{\mathscr{V}}^* \otimes_{\mathscr{A}^0_X} \mathscr{A}^*_X$$

of graded $\mathscr{A}^*_X$-modules (where we write U^* for the graded object underlying a complex $U^\bullet$). We then define an increasing filtration $\{W_n\mathscr{V}^*\}_n$ on $\mathscr{V}^*$ by

$$W_n\mathscr{V}^* = (\tau^{\le n}\bar{\mathscr{V}}^*) \otimes_{\mathscr{A}^0_X} \mathscr{A}^*_X.$$

Writing ∇ for the differential on $\mathscr{V}^\bullet$ and δ for the differential on $\bar{\mathscr{V}}^\bullet$, we can set $\nabla = \delta + D$, for some $D\colon \mathscr{V}^n \to \bigoplus_{i\ge 0} \mathscr{V}^{n-i} \otimes_{\mathscr{A}^0_X} \mathscr{A}^{i+1}_X$. By construction, $\delta W_n \subset W_n$, and we automatically have $DW_n \subset W_n$, so $\{W_n\mathscr{V}^\bullet\}_n$ defines a filtration on $\mathscr{V}^\bullet$.

Let $\mathscr{U}^\bullet$ be the quotient $W_n\mathscr{V}^\bullet/W_{n-1}\mathscr{V}^\bullet$; the only non-zero terms of $\bar{\mathscr{U}}$ are $\bar{\mathscr{U}}^{n-1}, \bar{\mathscr{U}}^n$. For $\nabla = D + \delta$ as above, we have flat connections $D\colon \bar{\mathscr{U}}^i \to \bar{\mathscr{U}}^i \otimes_{\mathscr{A}^0_X} \mathscr{A}^1$ and a map $\delta\colon \mathscr{U}^{n-1} \to \mathscr{U}^n$ commuting with D. Thus there exist local systems U^{n-1}, U^n with $(\bar{\mathscr{U}}^i, D) = (U^i \otimes_{\mathbb{R}} \mathscr{A}^0_X, \mathrm{id}_U \otimes d)$. Because any local system is an extension of semisimple local systems, it follows that the $\mathscr{F}_i := (\bar{\mathscr{U}}^i \otimes_{\mathscr{A}^0_X} \mathscr{A}^\bullet_X, \mathrm{id}\otimes d \pm D \otimes \mathrm{id})$ lie in $\mathcal{P}(X, \mathcal{S})$. Since $\mathscr{U}^\bullet$ is an extension of $\mathscr{F}_{n-1}[1-n]$ by $\mathscr{F}_n[-n]$, we also have $W_n\mathscr{V}^\bullet/W_{n-1}\mathscr{V}^\bullet \in \mathcal{P}(X, \mathcal{S})$. As the filtration is finite ($\bar{\mathscr{V}}^*$ being of finite rank), this means that $\mathscr{V}^\bullet \in \mathcal{P}(X, \mathcal{S})$, as required. □

Lemma 4.4. *The dg categories $\mathrm{per}_{\mathrm{dg}}(\mathcal{T})$ and $\mathcal{P}(X, \mathcal{S})$ are quasi-equivalent.*

Proof. We define a dg functor $\iota\colon \mathcal{T} \to \mathcal{P}(X, \mathcal{S})$ by sending U to $U \otimes_{\mathbb{R}} \mathscr{A}^\bullet_X$. The dg category $\mathcal{P}(X, \mathcal{S})$ is closed under shifts, extensions and direct summands, so $\mathcal{P}(X, \mathcal{S})$ is Morita fibrant — in other words, $\mathcal{P}(X, \mathcal{S}) \to \mathrm{per}_{\mathrm{dg}}(\mathcal{P}(X, \mathcal{S}))$ is a quasi-equivalence.

Now, the dg functor ι is clearly full and faithful, since the maps $\mathcal{T}(U, V) \to \mathcal{P}(X)(\iota U, \iota V)$ are isomorphisms. The definition of $\mathcal{P}(X, \mathcal{S})$ ensures that it is generated by $\iota\mathcal{T}$, so $\mathrm{per}_{\mathrm{dg}}(\mathcal{T}) \to \mathrm{per}_{\mathrm{dg}}(\mathcal{P})$ must be a quasi-equivalence. □

Note that Moriya's category $\hat{\mathcal{T}}$ (see §3.5) embeds in $\mathcal{P}(X)$ as the full dg subcategory on objects $\mathscr{V}^\bullet$ with $\bar{\mathscr{V}}^\bullet$ concentrated in degree 0 — these correspond to flabby resolutions of local systems.

4.2. **Betti homotopy types.** We now let k be a field of characteristic 0.

Definition 4.5. As in [Pri3, Definition 3.11], define the relative Malcev homotopy type $G(X,x)^{R,\mathrm{Mal}}$ of a pointed connected topological space (X,x) with respect to a Zariski-dense representation $\rho\colon \pi_1(X,x) \to R(k)$ as follows. First form the reduced simplicial set $\mathrm{Sing}(X,x)$ of singular simplices based at x, then apply Kan's loop group functor from [Kan] to give a simplicial group $G(X,x) := G(\mathrm{Sing}(X,x))$. Note that $\pi_0 G(X,x) = \pi_1(X,x)$, and apply the relative Malcev completion construction of [Hai] levelwise to $G(X,x) \to R(k)$, obtaining a simplicial affine group scheme

$$G(X,x)^{R,\mathrm{Mal}},$$

with each $G(X,x)^{R,\mathrm{Mal}}_n$ a pro-unipotent extension of R.

In other words, $G(X,x)_n \to (G(X,x)^{R,\mathrm{Mal}})_n(k) \xrightarrow{f(k)} R(k)$ is the universal diagram with f a pro-unipotent extension.

To a relative Malcev homotopy type $G(X,x)^{R,\mathrm{Mal}}$ are associated relative Malcev homotopy groups $\varpi_n(X^{R,\mathrm{Mal}},x) := \pi_{n-1}G(X,x)^{R,\mathrm{Mal}}$. These are affine group schemes, with $\varpi_1(X^{R,\mathrm{Mal}},x) = \pi_1(X,x)^{R,\mathrm{Mal}}$. The higher homotopy groups are pro-finite dimensional vector spaces, and are often just $\pi_n(X,x)\otimes_{\mathbb{Z}} k$ — see [Pri3, Theorem 3.21], [Pri8, Theorem 4.15] and [Pri6, Theorem 3.40].

Examples 4.6. When $\mathcal{S}$ is the category of all semisimple local systems in k-vector spaces on X, we write $G(X,x)^{R,\mathrm{Mal}} = G(X,x)^{\mathrm{alg}}$. Note that [Pri3, Corollary 3.57] shows that $G(X,x)^{\mathrm{alg}}$ is a model for Toën's schematic homotopy types.

When $\mathcal{S}$ is the category of constant local systems on X, note that $R = 1$ and that $G(X,x)^{1,\mathrm{Mal}}$ is the nilpotent k-homotopy type, so Quillen's rational homotopy type from [Qui] when $k = \mathbb{Q}$.

We now specialise to the setting of the previous section, with $k = \mathbb{R}$.

Proposition 4.7. *When X is a manifold and $\mathcal{T} = \mathrm{Rep}(R, A^\bullet(X,\mathbb{O}(R)))$, the dg Hopf algebra $C \simeq (x^*)^\vee \otimes^{\mathbf{L}}_{\mathcal{T}} x^*$ of Corollary 3.29 associated to the fibre functor $x^*\colon \mathcal{T} \to$* FDVect *is a model for the relative Malcev homotopy type $G(X,x)^{R,\mathrm{Mal}}$ of (X,x) under the equivalences of* [Pri3, Theorem 4.41].

Proof. We need to show that the Dold–Kan denormalisation functor D ([Pri3, Definition 4.24]) from dg Hopf algebras to cosimplicial Hopf algebras sends C to a model for the ring of functions on the simplicial group scheme $G(X,x)^{R,\mathrm{Mal}}$. By Remark 3.24, the dg Hopf algebra C is given by $O(R \ltimes \bar{G}(A))$, for $A = A^\bullet(X,\mathbb{O}(R))$. Applying D then gives

$$DC = O(R \ltimes \bar{G}(DA)),$$

where $\bar{G}$ is now the functor on cosimplicial algebras defined in [Pri3, Definition 3.46]. [Equivalently, this is a weak equivalence

$$B\mathrm{Spec}\, DC \simeq [(\mathrm{Spec}\, DA)/R]$$

of affine stacks in the sense of [Toë2], where B is the nerve.]

By [Pri3, Proposition 4.50], the simplicial group scheme $G(X,x)^{R,\mathrm{Mal}}$ is quasi-isomorphic to $R \ltimes \bar{G}(DA)$, so we have shown

$$\mathrm{Spec}\, DC \simeq G(X,x)^{R,\mathrm{Mal}}.$$

□

Remark 4.8. For any reduced simplicial set X and Zariski-dense representation $\rho\colon \pi_1(X) \to R(k)$, there is a relative Malcev homotopy type $G(X)^{R,\mathrm{Mal}}$. By [Pri3, Theorem 3.55], this homotopy type corresponds (via [Pri3, Theorem 4.41]) to the R-equivariant cosimplicial algebra

$$\mathrm{C}^\bullet(X, \rho^{-1}O(R))$$

of equivariant singular cochains with coefficients in the local coefficient system $\rho^{-1}O(R)$ (with right multiplication).

A model for the corresponding R-equivariant dg algebra is given by applying the Thom–Sullivan functor Th. The corresponding dg tensor category $\mathcal{T}$ has finite-dimensional R-representations as objects, and morphisms

$$\mathcal{T}(U,V) = \mathrm{Th}\,\mathrm{C}^\bullet(X, \rho^{-1}(U \otimes V^\vee)).$$

When $R = \pi_1(X)^{\mathrm{red}}$ is the reductive pro-algebraic fundamental group of X, the quasi-isomorphism between cosimplicial and cocubical cochains gives a quasi-isomorphism between $\hat{\mathcal{T}}$ and the dg category $T_{\mathrm{dR}}(X)$ of [Mor, Theorem 1.0.4].

Corollary 4.9. *The dg category $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(G(X,x)^{R,\mathrm{Mal}}))$ of Definition 3.8 is quasi-equivalent to $\mathrm{ind}(\mathcal{P}(X,\mathrm{Rep}(R)))$, for $\mathcal{P}(X,\mathrm{Rep}(R))$ the dg category of derived connections from Definition 4.2. Under this equivalence, $\mathcal{P}(X,\mathrm{Rep}(R))$ corresponds to the full dg subcategory of $\mathcal{D}_{\mathrm{dg}}(O(G(X,x)^{R,\mathrm{Mal}}))$ on fibrant replacements of finite-dimensional comodules. The equivalence respects the tensor structures.*

Proof. Remark 3.24 gives a quasi-equivalence

$$\mathcal{D}_{\mathrm{dg}}(\mathrm{Rep}(R,A)) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(R \ltimes \bar{G}(A))),$$

which is compatible with tensor structures by Corollary 3.29. Proposition 4.7 gives $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(R\ltimes\bar{G}(A))) \simeq \mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(G(X,x)^{R,\mathrm{Mal}}))$, while Lemma 4.4 gives $\mathrm{per}_{\mathrm{dg}}(\mathrm{Rep}(R,A)) \simeq \mathcal{P}(X,\mathrm{Rep}(R))$ and hence $\mathcal{D}_{\mathrm{dg}}(\mathrm{Rep}(R,A)) \simeq \mathrm{ind}(\mathcal{P}(X,\mathrm{Rep}(R)))$. Combining these gives the tensor quasi-equivalence $\mathcal{D}^{\mathrm{co}}_{\mathrm{dg}}(O(G(X,x)^{R,\mathrm{Mal}})) \simeq \mathrm{ind}(\mathcal{P}(X,\mathrm{Rep}(R)))$.

For the characterisation of $\mathcal{P}(X,\mathrm{Rep}(R))$, we appeal to Proposition 3.9. □

Remark 4.10. As in Remark 2.14, we can also consider a finite set T of basepoints. Proposition 4.7 then adapts to show that the dg Hopf algebroid C_T given by $C_T(x,y) \simeq (x^*)^\vee \otimes^{\mathbf{L}}_{\mathcal{T}} y^*$ is a for the unpointed relative Malcev homotopy type $G(X;T)^{R,\mathrm{Mal}}$ of X, where $G(X;T)$ is the restriction of Dwyer and Kan's loop groupoid $G(X)$ (from [DK3]) to the set T of objects.

Because these dg Hopf algebroids are all equivalent as T varies (or equivalently, because the fibre functors are all quasi-isomorphic), taking the colimit over all finite T gives a model for $G(X)^{R,\mathrm{Mal}}$.

Points of X also give a set $\{\bar{x}^*\}$ of fibre functors on the category of all k-linear sheaves, not just on locally constant sheaves. Any such set T of points yields a dg bialgebroid C'_T, but the dg derived category of C'_T-comodules is then just monoidally quasi-equivalent to the dg derived category of k-linear sheaves supported on T. Because the site has enough points, the set of all points gives a jointly faithful set of fibre functors on the category of k-linear sheaves. However, Remark 2.14 only applies to finite sets of fibre functors, so only finitely supported k-linear sheaves arise as comodules of the associated dg bialgebroid $C' = \varinjlim_T C'_T$.

4.3. **The universal Hopf algebra.** An unfortunate feature of relative Malcev homotopy types is that they rely on a choice of basepoint(s). However, the constructions of §1.3.3 give us a universal bialgebra construction $D(X, \mathcal{S})$ associated to a topological space X and a tensor category $\mathcal{S}$ of semisimple local systems. This should be regarded as the ring of functions on the space of algebraic paths generated by $\mathcal{S}$, while $G(X, x)^{R,\mathrm{Mal}}$ is the loop group at a fixed basepoint.

In order to understand $D(X, \mathcal{S})$, we must first understand the category $\mathcal{D}_{\mathrm{dg}}(\mathcal{T} \otimes \mathcal{T}^{\mathrm{opp}})$ in which it lives. When X is a manifold, recall that $\mathcal{T} = \mathrm{Rep}(R, A^\bullet(X, \mathbb{O}(R)))$ and $\mathrm{Rep}(R) \simeq \mathcal{S} = \mathrm{H}^0\mathcal{T}$. By Lemma 4.4, we have a quasi-equivalence

$$\begin{aligned} \mathrm{per}_{\mathrm{dg}}(\mathcal{T} \otimes \mathcal{T}^{\mathrm{opp}}) &\simeq \mathcal{P}(X^2, \mathrm{Rep}(R^2)), \\ (U, V) &\mapsto \iota_{X^2}((\mathrm{pr}_1^{-1} U) \otimes_k (\mathrm{pr}_2^{-1} V^\vee)), \end{aligned}$$

and hence

$$\mathcal{D}_{\mathrm{dg}}(\mathcal{T} \otimes \mathcal{T}^{\mathrm{opp}}) \simeq \mathrm{ind}(\mathcal{P}(X^2, \mathrm{Rep}(R^2))).$$

Understanding $\mathcal{C}_{\mathrm{dg}}(\mathcal{T} \otimes \mathcal{T}^{\mathrm{opp}})$ is harder, but observe that there is a dg functor r from the dg category of $\mathscr{A}^\bullet_{X^2}$-modules to $\mathcal{C}_{\mathrm{dg}}(\mathcal{T} \otimes \mathcal{T}^{\mathrm{opp}})$, given by

$$(rM)(U, V) = \underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^2}}(\iota_{X^2}((\mathrm{pr}_1^{-1} U^\vee) \otimes_k (\mathrm{pr}_2^{-1} V)), M).$$

Now, note that for $K, L \in \mathcal{P}(X)$, we have

$$\underline{\mathrm{Hom}}_{\mathcal{P}(X)}(K, L) \cong \underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^2}}((\mathrm{pr}_1^* K) \otimes_{\mathscr{A}^\bullet_{X^2}} (\mathrm{pr}_2^* L^\vee), \Delta_* \iota_X(k)),$$

where $\Delta \colon X \to X \times X$ is the diagonal morphism. Thus

$$\mathrm{id}_{\mathcal{A}} = r\Delta_* \iota_X(k) \in \mathcal{C}_{\mathrm{dg}}(\mathcal{T} \otimes \mathcal{T}^{\mathrm{opp}}).$$

Likewise,

$$\begin{aligned} &\underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^2}}(\iota_{X^2}((\mathrm{pr}_1^{-1} U) \otimes_k (\mathrm{pr}_2^{-1} V^\vee)), M \otimes_{\mathcal{T}} N) \\ = \; & M(U, -) \otimes_{\mathcal{T}} N(-, V) \\ \cong \; & \underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^3}}(\iota_{X^2}((\mathrm{pr}_1^{-1} U) \otimes_k (\mathrm{pr}_3^{-1} V^\vee)), (\mathrm{pr}_{12}^* M) \otimes_{\mathscr{A}^\bullet_{X^3}} (\mathrm{pr}_{23}^* N)) \\ \cong \; & \underline{\mathrm{Hom}}_{\mathscr{A}^\bullet_{X^2}}(\iota_{X^2}((\mathrm{pr}_1^{-1} U) \otimes_k (\mathrm{pr}_2^{-1} V^\vee)), \mathrm{pr}_{12*}((\mathrm{pr}_{12}^* M) \otimes_{\mathscr{A}^\bullet_{X \times X \times X}} (\mathrm{pr}_{23}^* N)), \end{aligned}$$

so we have

$$M \otimes_{\mathcal{T}} N = r\mathrm{pr}_{13*}((\mathrm{pr}_{12}^* M) \otimes_{\mathscr{A}^\bullet_{X^3}} (\mathrm{pr}_{23}^* N)).$$

Combining these results gives:

Lemma 4.11. *A universal bialgebra $D(X, \mathcal{S})$ corresponds under the equivalence $\mathcal{D}_{\mathrm{dg}}(\mathcal{T} \otimes \mathcal{T}^{\mathrm{opp}}) \simeq \mathrm{ind}(\mathcal{P}(X^2, \mathrm{Rep}(R^2)))$ above to a sheaf $\mathbb{D} \in \mathrm{ind}(\mathcal{P}(X^2, \mathrm{Rep}(R^2)))$ equipped with a commutative unital multiplication*

$$\mathbb{D} \otimes_k \mathbb{D} \to \mathbb{D}$$

and a coassociative $\mathscr{A}^\bullet_{X^2}$-linear comultiplication

$$\mathbb{D}\mathrm{pr}_{13*}((\mathrm{pr}_{12}^* \mathbb{D}) \otimes_{\mathscr{A}^\bullet_{X^3}} (\mathrm{pr}_{23}^* \mathbb{D}))$$

with $\mathscr{A}^\bullet_{X^2}$-linear counit

$$\mathbb{D} \to \Delta_* \iota_X(k).$$

Beware that although the co-unit $\mathbb{D} \to r\Delta_*\iota_X(k)$ is a quasi-isomorphism of sheaves, the induced map $\mathbb{D} \to \Delta_*\iota_X(k)$ is far from being so, with the object on the left locally constant and that on the right supported on the diagonal. In some sense, $\mathbb{D}$ is the universal coalgebra under Δ_*k generated by $\mathcal{S} \otimes \mathcal{S}^{\mathrm{opp}}$. In the same way that a path space in topology is a fibrant replacement for the diagonal, $\mathbb{D}$ is a cofibrant replacement for functions on the diagonal, which is why we think of it as functions on the space of algebraic paths generated by $\mathcal{S}$.

Example 4.12. Note that the construction of Propositions 3.2 and 3.22 gives an efficient choice $\iota_{X^2}N\mathrm{CC}(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i)$ for the dg bialgebra $\mathbb{D}$, in which case it becomes a dg Hopf algebra. Explicitly, we have

$$\iota_{X^2}N\mathrm{CC}(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}}\otimes i) = N\mathrm{CC}(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}})\otimes(\mathrm{pr}_2^{-1}\iota_X i))\otimes_{[(\mathrm{pr}_1^{-1}\mathscr{A}^\bullet_X)\otimes(\mathrm{pr}_2^{-1}\mathscr{A}^\bullet_X)]}\mathscr{A}^\bullet_{X^2},$$

where

$$\begin{aligned}&\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i)) = \\ &(\mathrm{pr}_1^{-1}\iota_X\mathbb{O}(R)) \otimes^R \underbrace{A^\bullet(X, \mathbb{O}(R)) \otimes^R \ldots \otimes^R A^\bullet(X, \mathbb{O}(R))}_{n} \otimes^R(\mathrm{pr}_2^{-1}\iota_X\mathbb{O}(R)).\end{aligned}$$

Thus $\underline{\mathrm{CC}}_0(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) = \iota_{X^2}(\mathbb{O}(R) \otimes^R \mathbb{O}(R))$, which is quasi-isomorphic to the local system given by the $\pi_1(X)^2$-representation $O(R)$, with the two copies of $\pi_1(X)$ acting as left and right multiplication.

Example 4.13. Following Remark 3.4, for $i\colon \mathcal{S} \to \mathcal{T}$ we may describe $N\mathrm{CC}(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i)$ in terms of irreducibles. Let $\{V_\alpha\}_\alpha$ be a set of irreducible objects of complex R-representations, with one in each isomorphism class. Complex conjugation $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on this set, and then we have

$$\begin{aligned}&\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i)) \otimes_{\mathbb{R}} \mathbb{C} \cong \\ &\bigoplus_{\alpha_0,\ldots,\alpha_n} (\mathrm{pr}_1^{-1}\mathscr{A}^\bullet_X(V^\vee_{\alpha_0})) \otimes_{\mathbb{C}} A^\bullet(X, V_{\alpha_0} \otimes_{\mathbb{C}} V^\vee_{\alpha_1}) \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} A^\bullet(X, V_{\alpha_{n-1}} \otimes_{\mathbb{C}} V^\vee_{\alpha_n}) \otimes_{\mathbb{C}} (\mathrm{pr}_2^{-1}\mathscr{A}^\bullet_X(V_{\alpha_n})),\end{aligned}$$

with $\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i))$ given by taking $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants.

When $\mathcal{S}$ is the category of constant local systems, corresponding to the real (nilpotent) homotopy type as in Examples 4.6, this simplifies to

$$\begin{aligned}&\iota_{X^2}\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) \\ &\cong [\mathrm{pr}_1^{-1}\mathscr{A}^\bullet_X \otimes_{\mathbb{R}} (A^\bullet(X, \mathbb{R}))^{\otimes n} \otimes_{\mathbb{R}} \mathrm{pr}_2^{-1}\mathscr{A}^\bullet_X] \otimes_{[(\mathrm{pr}_1^{-1}\mathscr{A}^\bullet_X)\otimes(\mathrm{pr}_2^{-1}\mathscr{A}^\bullet_X)]} \mathscr{A}^\bullet_{X^2} \\ &= (A^\bullet(X, \mathbb{R}))^{\otimes n} \otimes_{\mathbb{R}} \mathscr{A}^\bullet_{X^2}.\end{aligned}$$

In other words,

$$\iota_{X^2}\underline{\mathrm{CC}}_\bullet(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) = \underline{\mathrm{CC}}_\bullet(A^\bullet(X, \mathbb{R}), \mathscr{A}^\bullet_{X^2}).$$

Remark 4.14. Consider the case of a group G acting on a manifold X, with $\mathcal{S}$ a G-equivariant rigid tensor subcategory of semisimple local systems (so $g^*U \in \mathcal{S}$ whenever $U \in \mathcal{S}$ and $g \in G$). Then we have an action of G on $\mathbb{D}$ over $X \times X$, with respect to the diagonal action of G on $X \times X$. This is because G-equivariance of $\mathcal{S}$ gives an action of G on $\mathbb{O}(R)$ over X (i.e. compatible isomorphisms $\mathbb{O}(R) \cong g^*\mathbb{O}(R)$ for all $g \in G$), and hence an action on $A^\bullet(X, \mathbb{O}(R))$. For well-behaved G-actions, this allows us to regard $\mathbb{D}$ as a sheaf of dg algebras on the quotient $(X \times X)/G$. When $x \in X$ is a fixed point for the G-action, note that the dg Hopf algebra $C = (x, x)^*\mathbb{D}$ inherits a G-action from $\mathbb{D}$.

Of course, in order to define a G-action on $\mathbb{D}$, it suffices to have G-actions on $\mathcal{S}$ and on the relative Malcev homotopy type $A^\bullet(X, \mathbb{O}(R))$. When X is a compact Kähler manifold, [Pri8, Theorem 5.17] show that the Tannakian fundamental group $\Pi(\mathrm{MTS})$ of the category of mixed twistor structures acts algebraically on $A^\bullet(X, \mathbb{O}(R))$ for all R, with trivial action on $\mathcal{S}$ (and hence $\mathbb{O}(R)$). This gives an algebraic action of $\Pi(\mathrm{MTS})$ on $\mathbb{D}$, so would allow us to regard $\mathbb{D}$ as an object of the derived category of mixed twistor modules, compatibly with the Hopf algebra structure. When the local systems in $\mathcal{S}$ all underlie variations of Hodge structure, there is also an algebraic circle action on $\mathcal{S}$ and on $A^\bullet(X, \mathbb{O}(R))$, combining with the $\Pi(\mathrm{MTS})$-action to give an action of the Tannakian fundamental group $\Pi(\mathrm{MHS})$ of the category of mixed Hodge structures. Then $\mathbb{D}$ would lie in the derived category of mixed Hodge modules.

Remark 4.15. Note that the cohomology sheaf $\mathscr{H}^0\mathbb{D}$ is the local system $\mathbb{O}(\varpi_1 X^{R,\mathrm{Mal}})$ on $X \times X$ defined in [Pri8, Corollary 8.7] as corresponding to $O(\varpi_1(X^{R,\mathrm{Mal}}, x))$ with its left and right actions by $\pi_1(X, x)$. All of the cohomology sheaves of $\mathbb{D}$ are necessarily local systems.

The pullback $\Delta^*\mathbb{D}$ to the diagonal is a sheaf of Hopf algebras (the ring of functions on the space of algebraic loops generated by $\mathcal{S}$). Then the higher cohomology sheaves of the sheaf of primitive elements of $\Delta^*\mathbb{D}$ are dual to the local systems $\Pi^n(X^{\rho,\mathrm{Mal}})$ of [Pri8, Corollary 8.7] corresponding to the relative Malcev homotopy groups $\varpi_n(X^{R,\mathrm{Mal}}, x)$ with their adjoint actions by $\pi_1(X, x)$.

When $f\colon X \to Y$ is a fibration with section p, choose R so that $\mathrm{Rep}(R)$ contains the semisimplifications of the local systems $\mathbf{R}^n f_*\mathbb{R}$ for all n. Then observe that we have a decomposition

$$p^*\Delta^*\varpi_1 X^{R,\mathrm{Mal}} \cong \pi_1^{\mathrm{dR}}(X/Y, p) \rtimes \Delta^*\varpi_1 Y^{R,\mathrm{Mal}},$$

where $\pi_1^{\mathrm{dR}}(X/Y, p)$ is Lazda's relative fundamental group from [Laz].

4.4. **$\mathbb{Q}_\ell$-homotopy types.** Take a connected algebraic space X and choose a full rigid tensor subcategory $\mathcal{S}$ of the category of semisimple lisse $\mathbb{Q}_\ell$-sheaves on X. Let $\mathcal{T}$ be the cosimplicial tensor category with the same objects as $\mathcal{S}$, but with morphisms

$$\mathcal{T}(U, V) = \mathrm{C}^\bullet(X, U \otimes V^\vee),$$

where $\mathrm{C}^\bullet(X, -)$ is the ℓ-adic Godement resolution of [Pri1, Definition 2.3]. Note that $\mathrm{H}^0\mathcal{T} \simeq \mathcal{S}$, and that

$$\mathrm{H}^i\mathcal{T}(U, V) \cong \mathrm{H}^i_{\text{ét}}(X, U \otimes V^\vee),$$

the ℓ-adic étale cohomology groups. As in Remark 4.8, we may apply the Thom–Sullivan functor Th to obtain a dg category $\mathrm{Th}\,(\mathcal{T})$.

Now, any geometric point $\bar{x}$ defines a fibre functor $\bar{x}^*\colon \mathcal{T}^0 \to \mathrm{FDVect}_{\mathbb{Q}_\ell}$ sending U to $U_{\bar{x}}$. As in §4.1, we may then construct an affine group scheme $R := \mathrm{Spec}\,(\bar{x}^*)^\vee \otimes_{\mathcal{S}} \bar{x}^*$ over $\mathbb{Q}_\ell$, with an equivalence $\bar{x}^*\colon \mathcal{S} \to \mathrm{Rep}(R)$ of tensor categories. Equivalently, we have a Zariski-dense continuous group homomorphism $\rho\colon \pi_1^{\text{ét}}(X, \bar{x}) \to R(\mathbb{Q}_\ell)$. The R-equivariant dg algebra A from §3.4 is then just the dg algebra

$$\mathrm{Th}\,\mathrm{C}^\bullet(X, \mathbb{O}(R))$$

of equivariant cochains from [Pri6, Definition 1.21], so $\mathcal{T}$ is equivalent to $\mathrm{Rep}(R, \mathrm{C}^\bullet(X, \mathbb{O}(R)))$.

Proposition 4.16. *The dg Hopf algebra $C \simeq (\bar{x}^*)^\vee \otimes^{\mathbf{L}}_{\mathrm{Th}\,(\mathcal{T})} \bar{x}^*$ of Corollary 3.29 associated to the fibre functor $\bar{x}^*\colon \mathcal{T} \to \mathrm{FDVect}$ is a model for the relative Malcev homotopy type $G(X, \bar{x})^{R,\mathrm{Mal}}$ of $(X, \bar{x})$ under the equivalences of* [Pri3, Theorem 4.41].

Proof. The proof of Proposition 4.7 carries over, replacing [Pri3, Proposition 4.50] with [Pri6, Theorem 3.30]. □

Now, we may regard $\mathrm{per}_{\mathrm{dg}}(\mathcal{T})$ as the dg subcategory of generated by $\mathcal{S}$ in the dg category of $\mathscr{C}^\bullet_X(\mathbb{Q}_\ell)$-modules in complexes of $\mathbb{Q}_\ell$-sheaves, where $\mathscr{C}^\bullet_X$ is the sheaf version of the Godement resolution. Since $\mathscr{C}^\bullet_X(\mathbb{Q}_\ell)$ is a flabby resolution of $\mathbb{Q}_\ell$, this means that $\mathcal{D}(\mathcal{T})$ is the derived category of $\mathbb{Q}_\ell$-hypersheaves generated by $\mathcal{S}$ under extensions, shifts and direct sums.

If X is defined over a separably closed field $\bar{F}$, then $(X \times_{\bar{F}} X)_{\text{ét}} \cong X_{\text{ét}} \times X_{\text{ét}}$, which means that a universal bialgebra D for $(X, \mathcal{S})$ corresponds to a bialgebra $\mathbb{D}$ in the category of $\mathbb{Q}_\ell$-hypersheaves on $X \times_{\bar{F}} X$.

Remark 4.17. Although we are working with étale homotopy types rather than Betti homotopy types, the argument of Remark 4.14 carries over to say that symmetries of X transfer to the universal bialgebra. In particular, this applies to Galois actions.

Explicitly, take an an algebraic space X_0 over a field F with separable closure $\bar{F}$, and set $X = X_0 \otimes_F \bar{F}$. Assume that $\mathcal{S}$ is generated by pullbacks of lisse sheaves on X_0 — this is equivalent to saying that $\mathcal{S}$ is $\mathrm{Gal}(F)$-equivariant with finite orbits. Then [Pri6, Theorem 3.32] ensures that the relative Malcev homotopy type $G(X)^{R,\mathrm{Mal}}$ carries a continuous Galois action, so we may regard the universal bialgebra $\mathbb{D}$ as a Galois-equivariant $\mathbb{Q}_\ell$-hypersheaf on $X \times_{\bar{F}} X$, or equivalently as a $\mathbb{Q}_\ell$-hypersheaf on $X_0 \times_F X_0$. For any basepoint $x \in X_0(F')$, this gives an action of $\mathrm{Gal}(F')$ on the dg bialgebra $(\bar{x}, \bar{x})^*\mathbb{D}$, but the $\mathrm{Gal}(F)$-action on the universal bialgebra $\mathbb{D}$ does not require $X(F)$ to be non-empty.

Remark 4.18. As in Remarks 2.14 and 4.10, we can consider multiple basepoints instead, and taking the set of all geometric points gives a dg Hopf algebroid C with $C(\bar{x}, \bar{y}) \simeq (\bar{x}^*)^\vee \otimes^{\mathbf{L}}_{\mathrm{Th}(\mathcal{T})} \bar{y}^*$ as a model for the relative Malcev homotopy type $G(X_{\text{ét}})^{R,\mathrm{Mal}}$.

4.5. **Motivic homotopy types.** There is nothing special about the de Rham, Betti and ℓ-adic cohomology theories considered so far in this section. Each construction of pro-algebraic homotopy types has only relied on a suitable sheaf of dg algebras, and a category of projective modules over it. There are thus analogues for any mixed Weil cohomology theory in the sense of [CD1], or if we are willing to replace Hopf algebras with coalgebras, for any stable cohomology theory.

4.5.1. *Nilpotent homotopy types.* We now look at the simplest relative Malcev homotopy types, when $R = 1$, as in Examples 4.6. A mixed Weil cohomology theory E has an associated sheaf E_X of commutative dg algebras on each scheme X over our base field F, and we write $E(X) := \Gamma(X, E_X)$. Set $\mathcal{S} = \mathrm{FDVect}_k$ and $\mathcal{T} = E(X) \otimes \mathcal{S}$, so $\mathcal{T}$ has the same objects as $\mathcal{S}$, but $\mathcal{T}(U, V) = E(X) \otimes \mathcal{S}(U, V)$. We may then embed $\mathcal{D}_{\mathrm{dg}}(\mathcal{T}^{\mathrm{opp}} \otimes \mathcal{T}) = \mathcal{D}_{\mathrm{dg}}(E(X)^{\mathrm{opp}} \otimes E(X))$ into the category of E_{X^2}-modules by setting

$$\iota_{X^2}(U, V) = U \otimes_k E_{X^2} \otimes_k V^\vee,$$

with the left and right actions of $E(X)$ on E_{X^2} coming from the projections $X^2 \to X$.

As in §4.3, we may now construct a universal Hopf algebra $\mathbb{D}$ on $X^2 = X \times X$, and regard it as the ring of functions on the space of nilpotent algebraic paths. For an explicit model, we follow Examples 1.14 and 4.13, setting

$$\mathbb{D} := \iota_{X^2}\mathrm{CC}(\mathcal{T}/\mathcal{S}, i^{\mathrm{opp}} \otimes i) = \mathrm{CC}(E(X), E_{X^2}),$$

for $i\colon \mathcal{S} \to \mathcal{T}$. This is the Hochschild homology complex of the DGA $E(X)$ with coefficients in the $E(X)$-bimodule E_{X^2} in sheaves on X^2.

As before, we have

$$\mathbb{D} = \mathrm{CC}(E(X), (\mathrm{pr}_1^{-1}E_X) \otimes_k (\mathrm{pr}_2^{-1}E_X)) \otimes_{[(\mathrm{pr}_1^{-1}E_X)\otimes_k(\mathrm{pr}_2^{-1}E_X)]} E_{X^2},$$

which is defined in terms of the $(\mathrm{pr}_1^{-1}E_X) \otimes_k (\mathrm{pr}_2^{-1}E_X)$-modules

$$\underline{\mathrm{CC}}_n(\mathcal{T}/\mathcal{S}, (\mathrm{pr}_1^{-1}\iota_X i^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X i)) \cong \mathrm{pr}_1^{-1}E_X \otimes_k E(X)^{\otimes n} \otimes_k \mathrm{pr}_2^{-1}E_X.$$

on X^2. Since E is a mixed Weil cohomology theory, this is quasi-isomorphic to

$$\mathrm{pr}_1^{-1}E_X \otimes_k E(X^n) \otimes_k \mathrm{pr}_2^{-1}E_X.$$

Writing $h(X/Y)$ for the cohomological motive $M_k(X/Y)^{\mathrm{opp}} \in \mathcal{M}_{\mathbb{A}^1}(Y)^{\mathrm{opp}}$ of X over a base scheme Y and $h(X) := h(X/F)$, we see that the sheaf $\mathbb{D}$ comes from applying E to the simplicial motive

$$n \mapsto \mathrm{pr}_1^{-1}h(X/X) \otimes_k h(X^n) \otimes_k \mathrm{pr}_2^{-1}h(X/X)$$

on X^2.

A choice of basepoint $a\colon \operatorname{Spec} F \to X$ gives a fibre functor $a^*\colon E(X) \to k$, and hence $\mathcal{T} \to \mathrm{FDVect}_k$. The associated dg Hopf algebra is

$$C := (a,a)^*\mathbb{D} \cong E(X^\bullet),$$

with the outer boundary maps coming from pulling back by $(a,\mathrm{id})^*, (\mathrm{id},a)^*\colon X^n \to X^{n+1}$.

Thus C comes from applying the cohomology theory E to the simplicial motive

$$\begin{tikzcd} h(F) \arrow[r, dotted, shift left=0.5] & h(X) \arrow[l, shift right=3, "a^*"'] \arrow[l, shift left=3, "a^*"] \arrow[r, dotted, shift left=1.5] \arrow[r, dotted, shift right=1.5] & h(X \times X) \ldots, \arrow[l, shift right=4, "(\mathrm{id},a)^*"'] \arrow[l, "\Delta^*"] \arrow[l, shift left=4, "(a,\mathrm{id})^*"] \end{tikzcd}$$

which is just $h(\mathcal{P}_a(X))$, for Wojtkowiak's cosimplicial loop space $\mathcal{P}_a(X)$ from [Woj]. The motivic fundamental group of [EL] is then essentially just

$$\operatorname{Spec} \mathrm{H}^0 h(\mathcal{P}_a(X)),$$

so becomes a special case of our Hochschild homology construction for Tannakian duals.

In fact, we can say much more. Following [EL, §6], we define a cosimplicial scheme $X^{[0,1]}$ by $(X^{[0,1]})^n = X^{\Delta^1_n} \cong X^{n+2}$. The vertices of Δ^1 give a cosimplicial map from $X^{[0,1]}$ to the constant cosimplicial scheme X^2, with $\mathcal{P}_a(X)$ the fibre over (a,a). Now, observe that the ring of functions $\mathbb{D}$ on the space of nilpotent algebraic paths is just given by applying our chosen cohomology theory to the simplicial cohomological motive $\mathscr{D} := h(X^I/X^2)$ over X^2.

4.5.2. *Relative Malcev homotopy types.* Rather than just looking at nilpotent homotopy types, we could consider more general motivic homotopy types by choosing a set S of rigid cohomological motives over X, the nilpotent case being $S = \{h(X/X)\}$. Taking $\mathcal{T}$ to be the full dg category of E_X-modules on objects $E_X(M)$ for $M \in S$, we find that the universal coalgebra $\mathbb{D}$ (thought of as the sheaf of functions on the space of algebraic paths generated by S) is the normalised total complex of the simplicial diagram given in level n by

$$\underline{\mathrm{CC}}_n(\mathcal{T}, (\mathrm{pr}_1^{-1}\iota_X h_{\mathcal{T}}^{\mathrm{opp}}) \otimes (\mathrm{pr}_2^{-1}\iota_X h_{\mathcal{T}})) \cong$$
$$\bigoplus_{M_0,\ldots,M_n\in S} \mathrm{pr}_1^{-1}E_X(M_0^\vee) \otimes_k E(M_0 \otimes_X M_1^\vee) \otimes_k \ldots \otimes_k E(M_{n-1} \otimes_X M_n^\vee) \otimes_k \mathrm{pr}_2^{-1}E_X(M_n).$$

Here $M^\vee$ denotes the dual motive to M over X, which is just $M(-d)[-2d]$ when $M = h(Y/X)$ is the motive of a smooth and proper morphism $Y \to X$ of relative dimension d. We write $\otimes_X$ for the derived tensor product of motives over X (i.e. with respect to $k_X := h(X/X)$), and we set $E(N) := \Gamma(X, E_X(N))$.

Beware that the duals and tensor products in this expression are only defined up to homotopy, so we have only described $\mathbb{D}$ as a coalgebra in the derived category of E-modules over $X \times X$, with respect to the tensor product

$$(F, G) \mapsto \mathrm{pr}_{13*}((\mathrm{pr}_{12}^* F) \otimes_{E_{X^3}} (\mathrm{pr}_{23}^* G)).$$

Now, $\mathbb{D}$ arises by applying E to the simplicial cohomological motive $\mathscr{D}$ over X^2 given by

$$n \mapsto \bigoplus_{M_0,\ldots,M_n\in S} M_0^\vee \otimes_k (M_0 \otimes_X M_1^\vee) \otimes_k \ldots \otimes_k (M_{n-1} \otimes_X M_n^\vee) \otimes_k M_n,$$

where the $h(X^2)$-module structure comes from the $h(X)$-module structures of $M_0^\vee$ and M_n. When the set S is closed under the tensor product $\otimes_X$, the universal coalgebra $\mathscr{D}$ becomes a $\otimes_X$-bialgebra over X^2.

To understand the relation between $\mathscr{D}$ and the universal coalgebras of §1.2.2, observe that the six functors formalism of [Ayo1] makes $\mathcal{M}_{\mathbb{A}^1}(X)$ a category enriched in $\mathcal{M}_{\mathbb{A}^1}(F)$ and linear over it. The $\otimes_X$-coalgebra $\mathscr{D}$ on X^2 is then a resolution of the enriched Hom functor on objects in S given by $(N, M) \mapsto \mathbf{R}f_*(M \otimes_X N^\vee)$, for $f\colon X \to \operatorname{Spec} F$. This construction is thus the direct generalisation of §1.2.2 to enriched categories.

At a basepoint $a \in X$, the E-Malcev homotopy type of (X, a) relative to S is the dg coalgebra $C := (a, a)^*\mathbb{D}$, which just comes from applying E to the simplicial cohomological F-motive $(a, a)^*\mathscr{D}$ given by

$$n \mapsto \bigoplus_{M_0,\ldots,M_n\in S} (M_0^\vee)_a \otimes_k (M_0 \otimes_X M_1^\vee) \otimes_k \ldots \otimes_k (M_{n-1} \otimes_X M_n^\vee) \otimes_k (M_n)_a.$$

In other words, we should think of $\mathscr{D}$ as the motive of $\mathcal{M}_{\mathbb{A}^1}(F)$-valued functions on the space of algebraic paths generated by S. At any basepoint a, the motive $(a, a)^*\mathscr{D}$ is then the geometric motivic homotopy type of (X, a) relative to S. Note that the arithmetic homotopy type would replace the motive $(M_{i-1} \otimes_X M_i^\vee)$ with its motivic cohomology complex.

As in Example 1.14, the motive $L := \bigoplus_{M\in S} M^\vee \otimes_k M$ is a $\otimes_X$-coalgebra over X^2. Then $\mathscr{D}_n = \underbrace{L \otimes_X L \otimes_X \ldots \otimes_X L}_{n+1}$, so $\mathscr{D}$ is just the Čech nerve of the comonoid L. Setting $S = \{k_X\}$ (the nilpotent case), we get $L = k_{X^2} = h(X^2/X^2)$ and recover the description $\mathscr{D} = h(X^I/X^2)$ of §4.5.1.

For rigid motives $P, Q \in \mathcal{M}_{\mathbb{A}^1}(X)^{\mathrm{opp}}$, we have

$$\begin{aligned}&\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{M}_{\mathbb{A}^1}^{\mathrm{opp}}(X\times X)}(\mathrm{pr}_1^* P^\vee \otimes_{X\times X} \mathrm{pr}_2^* Q, \mathscr{D}) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{M}_{\mathbb{A}^1}^{\mathrm{opp}}(X)}(P, Q)\\ &\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{M}_{\mathbb{A}^1}^{\mathrm{opp}}(X\times X)}(\mathrm{pr}_1^* P^\vee \otimes_{X\times X} \mathrm{pr}_2^* Q, h(X/X^2)),\end{aligned}$$

where the morphism $X \to X^2$ is the diagonal map. Thus the universal coalgebra $\mathscr{D}$ is just the universal motive under $h(X/X^2)$ generated by motives in S. Since duals and tensor products are here only defined up to homotopy, we should perhaps think of $h(X/X^2)$ (or at least its induced dg functor on rigid motives) as the fundamental object.

When S is the set of all rigid motives and we have a basepoint $a \in X$, $a^*\mathscr{D} \in \mathcal{D}_{\mathbb{A}^1}(F)$ is just Ayoub's motivic Hopf algebra from [Ayo4, §2.4].

Appendix A. Formal Weil cohomology theories

A.1. **Quasi-projective pairs and localisation.**

A.1.1. *Quasi-projective pairs.*

Definition A.1. Given a field F admitting resolution of singularities, we let SmQP/F be the category of pairs $j\colon U \to X$, where X is smooth and projective over F, with U the complement of a normal crossings divisor. We say that a morphism $(U,X) \to (U',X')$ in SmQP/F is an equivalence (or in $\mathcal{E}$) if it induces an isomorphism $U \to U'$.

Lemma A.2. *The pairs* $(\mathrm{SmQP}/F, \mathcal{E})$ *and* $(\mathcal{E},\mathcal{E})$ *admit right calculi of fractions in the sense of* [DK1, §7].

Proof. We begin with the case $(\mathrm{SmQP}/F, \mathcal{E})$, noting that $\mathcal{E}$ contains all identities and is closed under composition. For any diagram $(V,Y) \xrightarrow{f} (U,X) \xleftarrow{a} (U,X')$ in SmQP/F (so a is an equivalence), we first need to find a commutative diagram

$$\begin{array}{ccc} (V,Y') & \xrightarrow{f'} & (U,X') \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ (V,Y) & \xrightarrow{f} & (U,X), \end{array}$$

with b an equivalence. To do this, we first form the fibre product $X' \times_X Y$, and observe that the isomorphism $V = U \times_U V$ gives us a map $V \hookrightarrow X' \times_X Y$. Taking Y' to be a resolution of singularities of the closure of V in $X' \times_X Y$ gives the required diagram.

Secondly, we need to show that if any parallel arrows $f,g\colon (V,Y) \to (U,X')$ in SmQP/F satisfy $af = ag$ for some equivalence $a\colon (U,X') \to (U,X)$, then there exists an equivalence $b\colon (V,Y') \to (V,Y)$ with $fb = gb$. The condition $af = ag$ implies that the maps $f,g\colon V \to U$ are equal. There is therefore a diagonal map

$$V \hookrightarrow Y \times_{f,X',g} Y.$$

Taking Y' to be a resolution of singularities of the closure of V in $Y \times_{f,X',g} Y$ then gives the construction required. Thus $(\mathrm{SmQP}/F, \mathcal{E})$ admits a right calculus of fractions.

Finally, note that $\mathcal{E}$ satisfies the two out of three property, so as observed in [DK1, 7.1], it follows that $(\mathcal{E},\mathcal{E})$ admits a right calculus of fractions. □

A.1.2. *Localisation and DG quotients.*

Definition A.3. Given a category $\mathcal{C}$ and a subcategory $\mathcal{W}$, we follow [DK1, §7] in writing $\mathcal{C}[\mathcal{W}^{-1}]$ for the localised category given by formally inverting all morphisms in $\mathcal{W}$.

Definition A.4. Given a category $\mathcal{C}$ and a subcategory $\mathcal{W}$, and an object $Y \in \mathcal{C}$, we write
$$\mathcal{C}\mathcal{W}^{-1}(X,Y)$$
for the category whose objects are spans
$$Y \xleftarrow{u} Y' \xrightarrow{f} X$$
with u in $\mathcal{W}$, and whose morphisms are commutative diagrams
$$\begin{array}{ccccc} Y & \xleftarrow{u_1} & Y_1 & \xrightarrow{f_1} & X \\ \| & & \downarrow v & & \| \\ Y & \xleftarrow{u_2} & Y_2 & \xrightarrow{f_2} & X, \end{array}$$
with v in $\mathcal{W}$.

Note that this category is denoted in [DK2, 5.1] by $N^{-1}\mathcal{C}\mathcal{W}^{-1}(Y,X)$.

Definition A.5. Given a category $\mathcal{C}$, write $k\mathcal{C}$ for the k-linear category with the same objects as $\mathcal{C}$, but with morphisms given by the free k-modules
$$(k\mathcal{C})(X,Y) := k(\mathcal{C}(X,Y)).$$

Definition A.6. Given $\mathscr{F} \in \mathcal{C}_{\mathrm{dg}}(k\mathcal{C})$ (i.e. a contravariant dg functor from $\mathcal{C}$ to cochain complexes over k) and $Y \in \mathcal{C}$, define the cochain complex $\mathscr{F}\mathcal{W}^{-1}(Y)$ by
$$\mathscr{F}\mathcal{W}^{-1}(Y) := \varinjlim_{Y' \in \mathcal{W}\downarrow Y}{}^{h} \mathscr{F}(Y').$$
Explicitly, this can be realised as the direct sum total complex of the simplicial cochain complex
$$\bigoplus_{Y_0' \to Y} \mathscr{F}(Y_0') \Leftarrow \bigoplus_{Y_1' \to Y_0' \to Y} \mathscr{F}(Y_0') \Lleftarrow \ldots.$$
Beware that this construction is not functorial in Y.

Proposition A.7. *Take a small category $\mathcal{C}$ and a subcategory $\mathcal{W}$ such that $(\mathcal{C},\mathcal{W})$ and $(\mathcal{W},\mathcal{W})$ admit right calculi of fractions. Let $\mathcal{D}$ be the localised category $\mathcal{C}[\mathcal{W}^{-1}]$ given by formally inverting all morphisms in $\mathcal{W}$. Then the functor $\lambda\colon \mathcal{C} \to \mathcal{D}$ gives a left Quillen functor*
$$\lambda_!\colon \mathcal{C}_{\mathrm{dg}}(k\mathcal{C}) \to \mathcal{C}_{\mathrm{dg}}(k\mathcal{D}),$$
left adjoint to λ^{-1}, making $\mathcal{C}_{\mathrm{dg}}(k\mathcal{D})$ Quillen-equivalent to the left Bousfield localisation of $\mathcal{C}_{\mathrm{dg}}(k\mathcal{C})$ at the image $k\mathcal{W}$ of $\mathcal{W}$ under the Yoneda embedding $k\colon \mathcal{C} \to \mathcal{C}_{\mathrm{dg}}(k\mathcal{C})$.

Proof. The functor $\lambda_!$ satisfies $\lambda_!(kC) = k\lambda(C)$, which then determines $\lambda_!$ by right Kan extension. We begin by computing this for cofibrant $k\mathcal{C}$-modules.

Combining [DK1, Propositions 7.2 and 7.3], the morphism
$$B\mathcal{C}\mathcal{W}^{-1}(X,Y) \to \mathcal{D}(\lambda X, \lambda Y)$$
is a weak equivalence of simplicial sets for all $X, Y \in \mathcal{C}$. Since $k\mathcal{D}(\lambda X, \lambda Y) = (\lambda_! kX)(\lambda Y)$, and $B\mathcal{C}\mathcal{W}^{-1}(X,Y) = (kX)\mathcal{W}^{-1}(Y)$, this gives a quasi-isomorphism
$$(kX)\mathcal{W}^{-1}(Y) \to (\lambda_! kX)(\lambda Y),$$

functorial in X (but not in Y). Since any cofibrant $k\mathcal{C}$-module $\mathscr{F}$ is a retraction of a filtered colimit of finite complexes of kX's, this gives quasi-isomorphisms

$$\mathscr{F}\mathcal{W}^{-1}(Y) \xrightarrow{\sim} (\lambda_!\mathscr{F})(\lambda Y)$$

for all $Y \in \mathcal{C}$.

Now, the unit $\mathscr{F} \to \lambda^{-1}\lambda_!\mathscr{F}$ of the adjunction gives maps

$$\mathscr{F}(Y) \to (\lambda_!\mathscr{F})(\lambda Y)$$

for all $Y \in \mathcal{C}$, and these factor through the maps above, giving

$$\mathscr{F}(Y) \to \mathscr{F}\mathcal{W}^{-1}(Y) \xrightarrow{\sim} (\lambda_!\mathscr{F})(\lambda Y).$$

The $k\mathcal{C}$-module $\mathscr{F}$ will be $k\mathcal{W}$-local if and only if $\mathscr{F}$ maps morphisms in $\mathcal{W}$ to quasi-isomorphisms. If this is the case, then the map $\mathscr{F}(Y) \to \mathscr{F}\mathcal{W}^{-1}(Y)$ is a quasi-isomorphism, so the unit

$$\mathscr{F}(Y) \to (\lambda_!\mathscr{F})(\lambda Y)$$

is also a quasi-isomorphism.

Because λ is essentially surjective on objects, the functor λ^{-1} reflects quasi-isomorphisms. Thus the co-unit $\mathbf{L}\lambda_!\lambda^{-1}\mathscr{G} \to \mathscr{G}$ of the derived adjunction is a quasi-isomorphism for all $\mathscr{G}$. Since λ maps $\mathcal{W}$ to isomorphisms, any object in the image of λ^{-1} is $k\mathcal{W}$-local. It therefore suffices to show that for any cofibrant $\mathscr{F} \in DG(k\mathcal{C})$, the unit $\mathscr{F}(Y) \to (\lambda_!\mathscr{F})(\lambda Y)$ of the adjunction is a $k\mathcal{W}$-local equivalence. Now, for any $k\mathcal{W}$-local object $\mathscr{G}$

$$\begin{aligned}
\mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{C}}(\mathscr{F},\mathscr{G}) &\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{C}}(\mathscr{F},\lambda^{-1}\mathbf{L}\lambda_!\mathscr{G})\\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{D}}(\lambda_!\mathscr{F},\mathbf{L}\lambda_!\mathscr{G})\\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{D}}(\mathbf{L}\lambda_!\lambda^{-1}\lambda_!\mathscr{F},\mathbf{L}\lambda_!\mathscr{G})\\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{C}}(\lambda^{-1}\lambda_!\mathscr{F},\lambda^{-1}\mathbf{L}\lambda_!\mathscr{G})\\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{k\mathcal{C}}(\lambda^{-1}\lambda_!\mathscr{F},\mathscr{G}),
\end{aligned}$$

as required. □

Corollary A.8. *In the setting of Proposition A.7, the functor $\lambda_!$ gives a quasi-equivalence $(k\mathcal{W})^\perp \to \mathcal{D}_{\mathrm{dg}}(k\mathcal{D})$ of dg categories. Moreover, the map $(\mathcal{W})^\perp \to \mathcal{D}_{\mathrm{dg}}(k\mathcal{C})/\mathcal{D}_{\mathrm{dg}}(k\mathcal{W})$ to the dg quotient is a quasi-equivalence.*

Proof. First observe that $(k\mathcal{W})^\perp \subset \mathcal{D}_{\mathrm{dg}}(k\mathcal{C})$ consists of the fibrant cofibrant objects in the Bousfield model structure, automatically giving the quasi-equivalence $(k\mathcal{W})^\perp \to \mathcal{D}_{\mathrm{dg}}(k\mathcal{D})$.

Fibrant replacement in the Bousfield model structure gives us morphisms $r\colon M \to \hat{M}$ in for each $M \in \mathcal{D}_{\mathrm{dg}}(k\mathcal{C})$, with $\hat{M} \in (k\mathcal{W})^\perp = \mathcal{D}_{\mathrm{dg}}(k\mathcal{W})^\perp$ and $\mathrm{cone}(r) \in \mathcal{D}_{\mathrm{dg}}(k\mathcal{W})$. Thus $\mathcal{D}_{\mathrm{dg}}(k\mathcal{W})$ is right admissible in the sense of [Dri, §12.6], giving the quasi-equivalence $(\mathcal{W})^\perp \to \mathcal{D}_{\mathrm{dg}}(k\mathcal{C})/\mathcal{D}_{\mathrm{dg}}(k\mathcal{W})$. □

Now write Sm/F for the category of smooth schemes over F.

Corollary A.9. *The excision functor $(X,D) \mapsto X\backslash D$ induces quasi-equivalences $\mathcal{D}_{\mathrm{dg}}(\mathrm{SmQP}/F,k)/\mathcal{D}_{\mathrm{dg}}(k\mathcal{E}) \leftarrow (k\mathcal{E})^\perp \to \mathcal{D}_{\mathrm{dg}}(\mathrm{Sm}/F,k)$.*

Proof. This comes from applying Lemma A.2 to Corollary A.8, noting that the excision functor is essentially surjective, so gives an equivalence

$$(\mathrm{SmQP}/F)[\mathcal{E}^{-1}] \simeq \mathrm{Sm}/F.$$

□

A.1.3. *Formal Weil cohomology theories.* As in Example 2.22, for any k-linear mixed Weil cohomology theory E over the field F, we can now define the formal Weil cohomology theory

$$E_f\colon\ (\mathrm{SmQP}/F)^{\mathrm{opp}} \to \mathcal{C}_{\mathrm{dg}}(k)$$

by

$$E_f(U \xrightarrow{j} X) := (\bigoplus_{a,b} \mathrm{H}^a(X, \mathbf{R}^b j_* E_U, d_2),$$

where d_2 is the differential on the second page of the Leray spectral sequence.

Weight considerations or standard results on Gysin maps imply that the Leray spectral sequence degenerates at E_2 (at least for all known Weil theories), so any equivalence $(U, X) \to (U, X')$ in SmQP/F induces a quasi-isomorphism on E_f.

Writing $\mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'} := \mathcal{D}_{\mathrm{dg}}(\mathrm{SmQP}/K, k)$ as in Example 2.22, the functor E_f extends k-linearly, giving

$$E_f^{\vee}\colon\ \mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'}/\mathcal{D}_{\mathrm{dg}}(k\mathcal{E}) \to \mathcal{C}_{\mathrm{dg}}(k),$$

since $\mathcal{E}$ lies in the kernel of E_f. By Corollary A.9, $\mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'}/\mathcal{D}_{\mathrm{dg}}(k\mathcal{E})$ is quasi-equivalent to $\mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}'}$.

Moreover, since the Leray spectral sequence degenerates, we have $\ker E_f = \ker E$ on $\mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'}$, so

$$\mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'}/\ker E_f = \mathcal{N}_{\mathrm{dg}}^{\mathrm{eff}'}/\ker E \simeq \mathcal{M}_{\mathrm{dg}}^{\mathrm{eff}'}/\ker E.$$

A.2. **Mixed Hodge structures on Betti cohomology.**

Definition A.10. For $\Lambda \subset \mathbb{R}$ a subfield, define MHS_Λ to be the tensor category of mixed Hodge structures in finite-dimensional vector spaces over Λ. Explicitly, an object of MHS_Λ consists of a finite-dimensional vector space V over Λ equipped with an increasing (weight) filtration W, and a decreasing (Hodge) filtration F on $V \otimes_\Lambda \mathbb{C}$ (both exhaustive and Hausdorff), such that

$$\mathrm{gr}_F^p \mathrm{gr}_{\bar{F}}^q \mathrm{gr}_n^W V = 0$$

for $p + q \neq n$.

The functor forgetting the filtrations is faithful, so by Tannakian duality there is a corresponding affine group scheme which (following [Ara]) we refer to as the universal Mumford–Tate group MT_Λ; this allows us to identify MHS_Λ with the category of finite-dimensional MT_Λ-representations.

Denote the pro-reductive quotient of MT_Λ by PMT_Λ — representations of this correspond to Hodge structures (i.e. direct sums of pure Hodge structures) over Λ. The assignment of weights to pure Hodge structures defines a homomorphism $\mathbb{G}_{m,\Lambda} \to \mathrm{PMT}_\Lambda$.

Definition A.11. For $\Lambda \subset \mathbb{R}$ a subfield, a Λ-Hodge complex in the sense of [Bei, Definition 3.2] is a tuple $(V_F, V_\Lambda, V_\mathbb{C}, \phi, \psi)$, where (V_Λ, W) is a filtered complex of Λ-modules, $(V_\mathbb{C}, W)$ is a filtered complex of complex vector spaces, (V_F, W, F) a bifiltered complex of complex vector spaces, and

$$\phi\colon\ V_\Lambda \otimes_\Lambda \mathbb{C} \to V_\mathbb{C} \quad \psi\colon\ V_F \to V_\mathbb{C}$$

are W-filtered quasi-isomorphisms; these must also satisfy the conditions that

(1) the cohomology $\bigoplus_i \mathrm{H}^i(V_\Lambda)$ is finite-dimensional over Λ;
(2) for any $n \in \mathbb{Z}$, the differential in the filtered complex $(\mathrm{gr}_n^W V_F, \mathrm{gr}_n^W F)$ is strictly compatible with the filtration, or equivalently the map $\mathrm{H}^*(F^p \mathrm{gr}_n^W V_F) \to \mathrm{H}^*(\mathrm{gr}_n^W V_F)$ is injective;
(3) the induced Hodge filtration together with the isomorphism $\mathrm{H}^i(\mathrm{gr}_n^W V_\Lambda) \otimes_\Lambda \mathbb{C} \to \mathrm{H}^i(\mathrm{gr}_n^W V_F)$ defines a pure Λ-Hodge structure of weight n on $\mathrm{H}^i(\mathrm{gr}_n^W V_\Lambda)$.

Example A.12. For a sheaf $\mathscr{F}$ on $Y(\mathbb{C})$, write $\mathscr{C}_Y^\bullet(\mathscr{F})$ for the Godement resolution of $\mathscr{F}$ — this is a cosimplicial diagram of flabby sheaves. Write $\mathrm{C}^\bullet(X, \mathscr{F})$ for the global sections of $\mathscr{C}_X^\bullet(\mathscr{F})$.

Take a smooth projective complex variety X and a complement $j\colon Y \hookrightarrow X$ of a normal crossings divisor D. Then set $A_\Lambda^\bullet := \mathrm{C}^\bullet(X, j_*\mathscr{C}_Y^\bullet(\Lambda))$, $A_\mathbb{C}^\bullet := \mathrm{C}^\bullet(X, j_*\mathscr{C}_Y^\bullet(N_c^{-1}\Omega_Y^\bullet))$, and $A_F^\bullet := \mathrm{C}^\bullet(X, N_c^{-1}\Omega_X^\bullet\langle D\rangle)$, where N_c^{-1} is the Dold–Kan denormalisation functor from cochain complexes to cosimplicial modules. The filtration W is given by décalage of the good truncation filtration on j_* in each case. Then N_cA is mixed Hodge complex, and on applying the Thom–Sullivan functor Th from cosimplicial DG algebras to DG algebra, we obtain a commutative algebra $\mathrm{Th}\,(A)$ in mixed Hodge complexes.

Definition A.13. For $\Lambda \subset \mathbb{R}$ a subfield, define MHS_Λ to be the category of mixed Hodge structures in finite-dimensional vector spaces over Λ, and write $\Pi(\mathrm{MHS}_\Lambda)$ for the group scheme over Λ corresponding to the forgetful functor from MHS_Λ to Λ-vector spaces.

Definition A.14. Given a cosimplicial vector space $V^\bullet$ and a simplicial set K, define $(V^\bullet)^K$ to be the cosimplicial vector space given by $((V^\bullet)^K)^n = (V^n)^{K_n}$, with operations $\partial^i\colon ((V^\bullet)^K)^n \to ((V^\bullet)^K)^{n+1}$ defined by composing

$$(V^n)^{K_n} \xrightarrow{(\partial^i)^{K_n}} (V^{n+1})^{K_n} \xrightarrow{(V^{n+1})^{\partial_i}} (V^{n+1})^{K_{n+1}},$$

and operations $\sigma^i\colon ((V^\bullet)^K)^n \to ((V^\bullet)^K)^{n-1}$ defined similarly.

In particular, $(V^\bullet)^{\Delta^1}$ is a path object over $V^\bullet$, with the two vertices $\Delta^0 \to \Delta^1$ inducing two maps $(V^\bullet)^{\Delta^1} \to V^\bullet$.

Definition A.15. The Λ-algebra $O(\mathrm{MT}_\Lambda)$ admits both left and right multiplication by MT_Λ. These induce two different ind-mixed Hodge structures on $O(\mathrm{MT}_\Lambda)$, which we refer to as the left and right mixed Hodge structures $(O(\mathrm{MT}_\Lambda), W^l, F_l), (O(\mathrm{MT}_\Lambda), W^r, F_r)$.

Example A.16. Given $A_{\Lambda,\mathcal{H}}^\bullet(X, D) := (A_\Lambda^\bullet, \phi, A_\mathbb{C}^\bullet, \psi, A_F^\bullet)$ as in Example A.12, we can define $A_{\mathrm{MHS}}^\bullet(X, D; \Lambda)$ to be the limit of the diagram

$$\begin{tikzcd}
(W \otimes W^l)_0(A_\Lambda^\bullet \otimes_\Lambda O(\mathrm{MT}_\Lambda)) \arrow[r] & (W \otimes W^l)_0(A_\mathbb{C}^\bullet \otimes_\Lambda O(\mathrm{MT}_\Lambda)) \\
(W \otimes W^l)_0((A_\mathbb{C}^\bullet)^{\Delta^1} \otimes_\Lambda O(\mathrm{MT}_\Lambda)) \arrow[ur] \arrow[r] & (W \otimes W^l)_0(A_\mathbb{C}^\bullet \otimes_\Lambda O(\mathrm{MT}_\Lambda)) \\
(W \otimes W^l)_0(F \otimes F_l)^0(A_F^\bullet \otimes_\Lambda O(\mathrm{MT}_\Lambda)) \arrow[ur] &
\end{tikzcd}$$

giving a cosimplicial algebra. The right Hodge structure on $O(\mathrm{MT}_\Lambda)$ then gives us a cosimplicial algebra

$$(A_{\mathrm{MHS}}^\bullet(X, D; \Lambda), W^r, F_r)$$

in $\mathrm{ind}(\mathrm{MHS}_\Lambda)$.

Proposition A.17. *The Λ-Hodge complex associated to the cosimplicial algebra $A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$ in $\mathrm{ind}(\mathrm{MHS}_\Lambda)$ is canonically quasi-isomorphic to $A^\bullet_{\Lambda,\mathcal{H}}(X,D)$ as a commutative algebra in cosimplicial Λ-Hodge complexes.*

Proof. If we set

$$\begin{aligned} B_\Lambda &:= A^\bullet_\Lambda \times_{A^\bullet_{\mathbb{C}}} (A^\bullet_{\mathbb{C}})^{\Delta^1} \\ B_{\mathbb{C}} &:= (A^\bullet_{\mathbb{C}})^{\Delta^1} \\ B_F &:= (A^\bullet_{\mathbb{C}})^{\Delta^1} \times_{A^\bullet_{\mathbb{C}}} A^\bullet_F, \end{aligned}$$

then $B := (B_\Lambda, B_{\mathbb{C}}, B_F)$ has the natural structure of a Λ-Hodge complex, and the morphism $\sigma^0\colon \Delta^1 \to \Delta^0$ induces a quasi-isomorphism $A^\bullet_{\Lambda,\mathcal{H}}(X,D) \to B$.

There is a map from the Λ-Hodge complex associated to $A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$ to B given by projections. We need to show that these projections preserve the Hodge and weight filtrations, and are (bi)filtered quasi-isomorphisms.

Now, for any mixed Hodge structure V there is a canonical isomorphism $V \cong V \otimes^{\mathrm{MT}_\Lambda} O(\mathrm{MT}_\Lambda) := (V \otimes O(\mathrm{MT}_\Lambda))^{\mathrm{MT}_\Lambda}$, where the Mumford–Tate action combines the action on V with the left action on $O(\mathrm{MT}_\Lambda)$. The mixed Hodge structure on V then corresponds to the Mumford–Tate action on $(V \otimes O(\mathrm{MT}_\Lambda))^{\mathrm{MT}_\Lambda}$ induced by the right action on $O(\mathrm{MT}_\Lambda)$.

Since W_nV is a sub-MHS, it follows that $W_nV \cong V \otimes^{\mathrm{MT}_\Lambda} W^r_n O(\mathrm{MT}_\Lambda)$, and since W_n is an idempotent functor, this is also isomorphic to $(W_nV) \otimes^{\mathrm{MT}_\Lambda} W^r_n O(\mathrm{MT}_\Lambda)$. In particular,

$$V \otimes^{\mathrm{MT}_\Lambda} W^l_m W^r_n O(\mathrm{MT}_\Lambda) \subset (W_nV) \otimes^{\mathrm{MT}_\Lambda} W^l_m O(\mathrm{MT}_\Lambda) \subset \sum_{\substack{i\le n, j\le m,\\ i+j=0}} (W_iV) \otimes (W^l_j O(\mathrm{MT}_\Lambda)),$$

which is 0 for $m+n<0$. Thus $W^l_{-n-1} W^r_n O(\mathrm{MT}_\Lambda) = 0$. A similar argument shows that $F^{1-p}_l F^p_r(\mathscr{O}(\mathrm{MT}_\Lambda) \otimes_\Lambda \mathbb{C}) = 0$.

Now, the weight filtration $W_n A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$ is given by replacing $O(\mathrm{MT}_\Lambda)$ with $W^r_n O(\mathrm{MT}_\Lambda)$ in the definition of $A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$, and the Hodge filtration $F^p A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$ by replacing $O(\mathrm{MT}_\Lambda) \otimes_\Lambda \mathbb{C}$ with $F^p_r(O(\mathrm{MT}_\Lambda) \otimes_\Lambda \mathbb{C})$. Projection onto the first factor gives a map from $W_n A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$ to $\sum_i (W_i A^\bullet_\Lambda) \otimes W^l_{-i} W^r_n O(\mathrm{MT}_\Lambda)$, which by the vanishing above is contained in $(W_n A^\bullet_\Lambda) \otimes O(\mathrm{MT}_\Lambda)$. Using similar arguments for the other factors and composing with the co-unit $O(\mathrm{MT}_\Lambda) \to \Lambda$ gives compatible (bi)filtered morphisms

$$\begin{aligned} A^\bullet_{\mathrm{MHS}}(X,D;\Lambda) &\to B_\Lambda \\ A^\bullet_{\mathrm{MHS}}(X,D;\Lambda) \otimes_\Lambda \mathbb{C} &\to B_{\mathbb{C}} \\ A^\bullet_{\mathrm{MHS}}(X,D;\Lambda) \otimes_\Lambda \mathbb{C} &\to B_F, \end{aligned}$$

and it only remains to establish quasi-isomorphism.

The data $N_c A^\bullet_{\Lambda,\mathcal{H}}(X,D)$ of Example A.12 define a Λ-Hodge complex, so by [Bei, Theorem 3.4], there exists a complex $V^\bullet$ of mixed Hodge structures whose associated Hodge complex is quasi-isomorphic to $N_c A^\bullet_{\Lambda,\mathcal{H}}(X,D)$.

Now, observe that $N_c A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$ is a cocone calculating absolute Hodge cohomology, so

$$\begin{aligned} N_c A^\bullet_{\mathrm{MHS}}(X,D;\Lambda) &\simeq \mathbf{R}\Gamma_{\mathcal{H}}(A^\bullet_{\Lambda,\mathcal{H}}(X,D)\otimes(O(\mathrm{MT}_\Lambda),W^l,F_l))\\ &\simeq \mathbf{R}\Gamma_{\mathcal{H}}(V^\bullet\otimes(O(\mathrm{MT}_\Lambda),W^l,F_l))\\ &\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathrm{MHS},\Lambda}(\Lambda,V^\bullet\otimes(O(\mathrm{MT}_\Lambda),W^l,F_l))\\ &\simeq \underline{\mathrm{Hom}}_{\mathrm{MHS},\Lambda}(\Lambda,V^\bullet\otimes(O(\mathrm{MT}_\Lambda),W^l,F_l))\\ &\cong V^\bullet\otimes^{\mathrm{MT}_\Lambda}O(\mathrm{MT}_\Lambda)\\ &\cong V^\bullet, \end{aligned}$$

with the last two properties following because $V^\bullet\otimes O(\mathrm{MT}_\Lambda)$ is an injective MT_Λ-representation and because $\mathrm{ind}(\mathrm{MHS}_\Lambda)$ is equivalent to the category of $O(\mathrm{MT}_\Lambda)$-comodules in Λ-vector spaces. The quasi-isomorphisms above all respect mixed Hodge structures (via the right action on $O(\mathrm{MT}_\Lambda)$), completing the proof. □

A.3. **Splittings for Betti cohomology.** Since MT_Λ is an affine group scheme, it is an inverse limit of linear algebraic groups, so by [HM], there exists a Levi decomposition $\mathrm{MT}_\Lambda\cong\mathrm{PMT}_\Lambda\ltimes\mathrm{R_u}(\mathrm{MT}_\Lambda)$ of the universal Mumford–Tate group as the semidirect product of its pro-reductive quotient and its pro-unipotent radical. Beware that this decomposition is not canonical; it might be tempting to think that the functor $V\mapsto\mathrm{gr}^W V$ yields the required section by Tannaka duality, but it is not compatible with the fibre functors.

Moreover, Levi decompositions are conjugate under the action of the radical $\mathrm{R_u}(\mathrm{MT}_\Lambda)$, so the set of decompositions is isomorphic to the quotient

$$\mathrm{R_u}(\mathrm{MT}_\Lambda)/\mathrm{R_u}(\mathrm{MT}_\Lambda)^{\mathrm{PMT}_\Lambda}$$

by the centraliser of PMT_Λ. For any element u of $\mathrm{R_u}(\mathrm{MT}_\Lambda)$, we must have $(u-\mathrm{id})W_nV\subset W_{n-1}V$ for all mixed Hodge structures V. However, any element in the centraliser necessarily has weight 0 for the $\mathbb{G}_m$-action, so must be 1. Thus the set of Levi decompositions is a torsor under

$$\mathrm{R_u}(\mathrm{MT}_\Lambda).$$

Proposition A.18. *Each choice of Levi decomposition for the universal Mumford–Tate group* MT_Λ *gives rise to a zigzag of* W*-filtered quasi-isomorphisms between the cosimplicial algebra-valued functors*

$$\begin{aligned} (X,D)&\mapsto A^\bullet_\Lambda(X,D)\\ (X,D)&\mapsto N_c^{-1}(\mathrm{H}^*(X,\mathbf{R}^*j_*\Lambda),d_2), \end{aligned}$$

where d_2 *is the differential on the* E_2 *page of the Leray spectral sequence and* $j\colon X\backslash D\to X$.

Proof. A choice of Levi decomposition is equivalent to a retraction of MHS_Λ onto HS_Λ, and $V\in\mathrm{MHS}_\Lambda$ is canonically isomorphic to $\mathrm{gr}^W V$. Since the weight filtration is a functorial filtration by mixed Hodge substructures, it is necessarily preserved by any such retraction, which thus amounts to giving a functorial W-filtered isomorphism $V\cong\mathrm{gr}^W V$ for all mixed Hodge structures V.

Proposition A.17 gives a zigzag of functorial W-filtered quasi-isomorphisms between the cosimplicial algebra $A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$ and the Betti complex $A^\bullet_\Lambda(X,D)$. A choice of Levi decomposition then gives a W-filtered isomorphism $A^\bullet_{\mathrm{MHS}}(X,D;\Lambda) \cong \mathrm{gr}^W A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$. Applying Proposition A.17 to the associated gradeds then gives a zigzag of filtered quasi-isomorphisms between $\mathrm{gr}^W A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$ and $\mathrm{gr}^W A^\bullet_{\mathrm{MHS}}(X,D;\Lambda)$, which maps quasi-isomorphically to $N_c^{-1}(\mathrm{H}^*(X,\mathbf{R}^*j_*\Lambda),d_2)$. $\square$

Corollary A.19. *If E_B denotes the mixed Weil cohomology theory associated to Betti cohomology, and $E_{B,f}$ its formal analogue as in Examples 2.22 and §A.1.3, then each choice of Levi decomposition for the universal Mumford–Tate group $\mathrm{MT}_{\mathbb{Q}}$ gives a zigzag of quasi-isomorphisms between E_B and $E_{B,f}$.*

Proof. The functor E_B is given by $X \mapsto \mathrm{Th}\,(\mathrm{C}^\bullet(X(\mathbb{C}),\mathbb{Q}))$, so there is a canonical quasi-isomorphism from $\mathrm{Th}\,(A^\bullet_\Lambda(X(\mathbb{C}),D(\mathbb{C})))$ to $E_B(X)$. Proposition A.18 thus gives a zigzag of quasi-isomorphisms from $E_B(X)$ to $\mathrm{Th}\,N_c^{-1}(\mathrm{H}^*(X(\mathbb{C}),\mathbf{R}^*j_*\Lambda),d_2)$, functorial in $(X\backslash D \xrightarrow{j} X)$ in SmQP/F. The functors Th and N_c^{-1} are homotopy inverses, so this is quasi-isomorphic to $(\mathrm{H}^*(X(\mathbb{C}),\mathbf{R}^*j_*\Lambda),d_2)$, which is just $E_{B,f}(X\backslash D \xrightarrow{j} X)$. $\square$

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