Derived algebraic geometry with a view to quantisation III: quantisation

J.P.Pridham

Background

- Quantisation of a k-CDGA A is a non-commutative $k[\![\hbar]\!]$-DGA $\tilde{A}$ deforming A.
- On $A = \tilde{A}/\hbar$, have Lie bracket

$$[a, b] := \hbar^{-1}(\tilde{a}\tilde{b} \mp \tilde{b}\tilde{a}) \mod \hbar.$$

$\implies$ A a P_1-algebra.

- When do Poisson structures quantise?
- Analogue for n-shifted Poisson?

Underived quantisations, $n = 0$

- Kontsevich–Tamarkin: $\exists$ quantisations for Poisson structures on smooth k-algebras ($\mathbb{Q} \subset k$).
- Algebroid quantisations for smooth schemes (deforming the category of line bundles).

Positively shifted quantisation

- E_n-algebra A has n (homotopy) compatible associative multiplications.
- KT formality: for $n \geqslant 2$,

$$E_n\text{-algebras} \simeq P_n\text{-algebras}$$

(after choosing Drinfeld associator).

$\Longrightarrow$ Quantisation for P_n-algebras automatic ($n \geqslant 2$).

Sketch of K–T's quantisation I

- Deformations of associative algebra A governed by Hochschild complex.
- $\mathrm{CC}^{\bullet}(A, A)$ an E_2-algebra.
- Formality gives a P_2-algebra P with quasi-iso Lie structure.
- P a deformation of $\widehat{\mathrm{Pol}}(A, 1)$.

Sketch of K–T's quantisation II

- When A smooth underived, $\widehat{\mathrm{Pol}}(A,1)$ doesn't deform, so L_∞ quasi-iso

$$\mathrm{CC}^\bullet(A,A) \simeq \widehat{\mathrm{Pol}}(A,1).$$

- Thus equivalence between quantisations and Poisson structures in smooth underived setting.
- All steps of K–T's quantisation extend to derived stacks except the last.

Derived unshifted quantisation

- In derived setting, can only expect non-degenerate quantisations to exist.
- Unshifted symplectic derived stacks are:
  - either smooth algebraic spaces,
  - or both stacky and derived.
- Reason: tangent complex self dual, so positive (derived) and negative (stacky) degrees balance.

Theorem (P)

Unshifted symplectic structures on derived Artin N-stacks $\mathfrak{X}$ admit (curved, involutive) quantisations, param'd by $\hbar^2 \mathrm{H}^2\mathrm{DR}(\mathfrak{X})[[\hbar^2]]$.

- In particular, get curved deformation of $\mathrm{Perf}_{\mathfrak{X}}$ as a dg category with duality. [*curved* means $\delta^2 \neq 0$ (but close)]

Curved DGAs

Hochschild complex $\rightsquigarrow$ *curved* deformations:

- associative unital graded algebra B^*,
- curvature $\kappa \in B^2$,
- differential $\delta\colon\ B^* \to B^*[1]$,
- $\delta^2 b = [\kappa, b]$, $\delta\kappa = 0$, $\delta(1) = 0$.
- $\approx$ Morita deformations [LvdB].
- Curved dg categories defined similarly.

Strategy of proof I

- Quantisation $\Delta \in \mathrm{MC}(\mathrm{CC}^{\bullet}(A)[\![\hbar]\!])$.
- Formality: regard $\mathrm{CC}^{\bullet}(A)$ as P_2-algebra, so last lecture's formulae define

$$\mu(-,\Delta)\colon\ \mathrm{DR}(A)[\![\hbar]\!] \to T_{\Delta}\mathrm{CC}^{\bullet}(A)[\![\hbar]\!]$$

  (quasi-iso when Δ non-degenerate).
- Essentially unique ω with

$$\mu(\omega,\Delta) \simeq -\hbar^2 \frac{\partial \Delta}{\partial \hbar}.$$

Strategy of proof II

- Given ω, try to solve for Δ.
- Proceed inductively $\mathsf{DR}(A)[\hbar]/\hbar^{k+1} \to \mathsf{DR}(A)[\hbar]/\hbar^{k}$.
- Essentially unique solution at each stage *except* the first.
- So restrict to involutive: $\Delta_{\hbar}^{\mathsf{opp}} \simeq \Delta_{-\hbar}$.
- $\mathsf{DR}(A)[\![\hbar^2]\!] \to T_{\Delta}\mathsf{CC}^{\bullet}(A)[\![\hbar]\!]^{\mathsf{involutive}}$.
- No odd terms, so equivalence.

Quantising (-1)-shifted structures

- E_0-algebra is just module (with unit).
- Have to impose extra restrictions for quantisation.
- Look for deformations $\delta + \Delta$ on $A[\![\hbar]\!]$ given by differential operators.
- $\Delta = \sum \Delta_i \hbar^i$, with Δ_i of order $(i-1)$.
- a.k.a. BV_∞-algebra.
- Notion extends to line bundles $\mathscr{L}$.

Theorem (P)

If $\mathscr{L}^{\otimes 2}$ a right $\mathscr{D}$-module, then any non-degenerate (-1)-shifted Poisson structure on $\mathfrak{X}$ gives self-dual quantisations $\tilde{\mathscr{L}}_\hbar$ of $\mathscr{L}$, parametrised by $\hbar^2 \mathrm{H}^1\mathrm{DR}(\mathfrak{X})[\![\hbar^2]\!]$.

- right $\mathscr{D}$-mod $\approx$ flat conn'n on $\otimes \det T_{\mathfrak{X}}$
- self-dual says right $\mathscr{D}$-mod equivalence $\tilde{\mathscr{L}}_{-\hbar} \simeq \mathscr{H}om_{\mathscr{O}_X[\![\hbar]\!]}(\tilde{\mathscr{L}}, K_{\mathfrak{X}}[\![\hbar]\!])$
- NB (-1)-shifted structures arise as Lagrangian intersections.

Sketch of proof

- Governing DGLA the ring $\mathscr{D}(\mathscr{L})$ of differential operators.
- Similar construction to before gives

$$\mu(-, \Delta)\colon\ \mathsf{DR}(A)[\![\hbar]\!] \to T_\Delta \mathscr{D}(\mathscr{L})[\![\hbar]\!].$$

- Seek $\mu(\omega, \Delta) \simeq -\hbar^2 \frac{\partial \Delta}{\partial \hbar}$;
- again, only obstruction is first order.
- $\mathscr{D}^{\mathsf{opp}}$-mod $\mathscr{L}^{\otimes 2} \Rightarrow \mathscr{D}(\mathscr{L}) \simeq \mathscr{D}(\mathscr{L})^{\mathsf{opp}}$, so can restrict to self-dual.

Lagrangians

[PTVV] Given $f\colon Z \to X$ and an *n*-shifted symplectic structure ω on X, a *Lagrangian* structure on Z is

- a homotopy $\lambda\colon f^*\omega \simeq 0$
- inducing a quasi-iso

$$T_{Z/X} \simeq \Omega^1_Z[n].$$

Explicitly, 1st condition says

$$(\omega, \lambda) \in \mathrm{Z}^{n+1}\mathrm{cone}(F^2\mathrm{DR}(X) \to F^2\mathrm{DR}(Z)).$$

Lagrangian quantisation

Approximately consists of:

- E_1 quantisation $\tilde{\mathscr{O}}_X$ of X
- and an $\tilde{\mathscr{O}}_X$-linear E_0 quantisation of a line bundle $\mathscr{L}$ on Z.

Explicitly, given by curved A_∞-morphism

$$\alpha_\hbar\colon\ f^{-1}\tilde{\mathscr{O}}_X \to \mathscr{D}(\mathscr{L})[\![\hbar]\!].$$

- Self-dual if $\alpha_{-\hbar} \simeq \alpha_\hbar^{\mathrm{opp}}$.

Theorem (P)

Given a 0-shifted derived Lagrangian $Z \to X$, there exist self-dual quantisations of $(\mathscr{O}_X, \mathscr{L})$, parametrised by $\hbar^2 \mathrm{H}^1 \mathrm{cone}(\mathrm{DR}(X) \to \mathrm{DR}(Z))[\![\hbar^2]\!]$, for any line bundle $\mathscr{L}$ on Z with $\mathscr{L}^{\otimes 2}$ a right $\mathscr{D}$-module.

Strategy of proof

- E_2-algebra $\mathrm{CC}^\bullet(\mathscr{O}_X)$ acts on E_1-algebra $\mathrm{CC}^\bullet(\mathscr{O}_X, \mathscr{D}_Z(\mathscr{L}))$.
- Deformations of $(\mathscr{O}_X, \mathscr{L})$ governed by

$$\mathrm{CC}^\bullet(\mathscr{O}_X) \ltimes \mathrm{CC}^\bullet(\mathscr{O}_X, \mathscr{D}_Z(\mathscr{L}))[-1].$$

- E_2 formality gives $\mu(-, \Delta)$:

$$\begin{array}{ccc} \mathrm{DR}(X) & \longrightarrow & T_\Delta \mathrm{CC}^\bullet(\mathscr{O}_X) \\ \downarrow & & \downarrow \\ \mathrm{DR}(Z) & \longrightarrow & T_\Delta \mathrm{CC}^\bullet(\mathscr{O}_X, \mathscr{D}_Z(\mathscr{L})). \end{array}$$

- Take cones, and proceed as before.

What about higher Lagrangians

- For $n \geqslant 2$, quantisations of n-shifted Lagrangians are E_{n+1}-algebras acting on E_n-algebras.
- Not immediate from formality (Swiss cheese).
- Method above will adapt, if PBW filtrations on higher Hochschild complexes behave.
- Equivalently, if Dunn additivity applies to BD-algebras.

References

Maxim Kontsevich.

Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

Maxim Kontsevich.

Operads and motives in deformation quantization. *Lett. Math. Phys.*, 48(1):35–72, 1999. Moshé Flato (1937–1998).

Dmitry E. Tamarkin.

Operadic proof of M. Kontsevich's formality theorem. Thesis (Ph.D.)–The Pennsylvania State University.

Maxim Kontsevich.

Deformation quantization of algebraic varieties. *Lett. Math. Phys.*, 56(3):271–294, 2001. EuroConférence Moshé Flato 2000, Part III (Dijon).

J. P. Pridham.

Deformation quantisation for unshifted symplectic structures on derived Artin stacks. arXiv: 1604.04458v2 [math.AG], 2016.

J. P. Pridham.

Deformation quantisation for (-1)-shifted symplectic structures and vanishing cycles. arXiv:1508.07936v2 [math.AG], 2015.

J. P. Pridham.

Quantisation of derived Lagrangians. arXiv: 1607.01000v1 [math.AG], 2016.

Masaki Kashiwara and Pierre Schapira.

Deformation quantization modules. *Astérisque*, (345):xii+147, 2012.

C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendröi.

Symmetries and stabilization for sheaves of vanishing cycles. *J. Singul.*, 11:85–151, 2015. arXiv:1211.3259 [math.AG]. With an appendix by Jörg Schürmann.

Wendy Lowen and Michel Van den Bergh.

The curvature problem for formal and infinitesimal deformations. arXiv:1505.03698, 2015.