Derived algebraic geometry with a view to quantisation II: symplectic and Poisson structures

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de Rham complexes

- $A = (A, \delta)$ a CDGA over R.
- Kähler differentials $\Omega^1_{A/R}$ (a complex).
- Exterior powers $\Omega^p_{A/R}$.
- de Rham differential $d\colon\ \Omega^p_{A/R} \to \Omega^{p+1}_{A/R}$.
- de Rham complex

$$\mathrm{DR}(A/R) = (\prod_p \Omega^p_{A/R}[-p], d \pm \delta).$$

- Hodge filtration $F^p\mathrm{DR}(A) = \prod \Omega_A^{\geqslant p}$.

Derived de Rham cohomology

- Derived de Rham $\mathbf{L}\mathrm{DR}(A/R) := \mathrm{DR}(\tilde{A}/R)$, for $\tilde{A} \to A$ a cofibrant ($\approx$ quasi-free) resolution.
- $\mathbf{L}F^p\mathrm{DR}(A/R) := F^p\mathrm{DR}(\tilde{A}/R)$.
- Also write $\mathbf{L}\Omega^p(A) := \Omega^p(\tilde{A})$.
- $\equiv$ Pieces of cyclic homology (Feigin–Tsygan, HKR).

n-shifted pre-symplectic structures

- $\omega \in \mathrm{Z}^{n+2}\mathbf{L}F^2\mathrm{DR}(A/R)$ [$\equiv$ PTVV].
- Explicitly, $\omega = \sum_{p\geqslant 2} \omega_p$, with

$$\delta\omega_2 = 0, \quad d\omega_p = \delta\omega_{p+1}.$$

- Morphisms given by chain homotopies.
- Sheafify for global, so for hgpd X, $\omega \in \prod_i F^2\mathbf{L}\mathrm{DR}(O(X_i)/R)^{n+2-i}$.
- Symplectic if non-degenerate:

$$\omega_2^\sharp \colon (\mathbf{L}\Omega^1_X)^\vee \xrightarrow{\sim} \mathbf{L}\Omega^1_X[n].$$

n-shifted polyvectors

- $\widehat{\mathrm{Pol}}(A/R, n)$:

$$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathrm{CoSymm}_A(\mathbf{L}\Omega^1_{A/R}[n+1]), A),$$

$$(\approx \widehat{\mathrm{Symm}}_A(T_{A/R}[n+1])).$$

- Filtration $F^p\widehat{\mathrm{Pol}}(A/R, n)$:

$$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathrm{CoSymm}_A^{\geqslant p}(\mathbf{L}\Omega^1_{A/R}[n+1]), A).$$

- Commutative product $F^p \cdot F^q \subset F^{p+q}$.
- Schouten–Nijenhuis Lie bracket on $\widehat{\mathrm{Pol}}(A/R, n)[n+1]$, $[F^p, F^q] \subset F^{p+q-1}$.

n-Poisson structures (affine case)

- $\pi = \sum_{p \geqslant 2} \pi_p \in F^2 \widehat{\mathrm{Pol}}(A/R, n)^{n+2}$,
$$\delta\pi + \frac{1}{2}[\pi, \pi] = 0 \text{ (Maurer–Cartan).}$$
- Morphisms from Thom–Sullivan homotopies give space $\mathcal{P}(A/R)$ of n-Poisson structures.
- Non-degenerate if
$$\pi_2^\sharp \colon \mathbf{L}\Omega^1_A[n] \xrightarrow{\sim} (\mathbf{L}\Omega^1_A)^\vee.$$

n-Poisson algebras

- P_{n+1}-algebra is CDGA with compatible Lie bracket of degree $-n$.
- $\pi \in \mathcal{P}(A)$ gives L_∞-structure on $A[n]$.
- n-Poisson structures $\Longleftrightarrow$ homotopy P_{n+1}-algebra structures on A [Melani].

n-Poisson structures on DM stacks

- X a derived DM stack, resolved by hgpd $\check{X}_\bullet$ (simplicial derived affine).
- $\mathcal{P}(X/R, n)$ the space of cosimplicial homotopy P_{n+1}-algebra structures on cosimplicial CDGA $O(\check{X}_\bullet)$.
- Invariance under trivial DM hgpds $\implies$ well-defined up to coherent homotopy (contractible space of choices).
- Similarly ∞-functorial for étale maps.

What about Artin stacks?

- Don't have smooth functoriality.
- So resolve $\mathfrak{X}$ by more exotic objects.

- Example
  - Y acted on by G, Lie algebra $\mathfrak{g}$.
  - Chart for $[Y/G]$ will be $[Y/\mathfrak{g}]$.
  - $O([Y/\mathfrak{g}])$ Chevalley–Eilenberg complex

$$O(Y) \xrightarrow{\partial} O(Y) \otimes \mathfrak{g}^\vee \xrightarrow{\partial} O(Y) \otimes \Lambda^2 \mathfrak{g}^\vee \xrightarrow{\partial} \dots$$

  - *bigraded* CDGA.

Stacky CDGAs

- commutative algebras in double complexes:

$$
\begin{array}{ccccccc}
A^{0,0} & \xrightarrow{\partial} & A^{1,0} & \xrightarrow{\partial} & A^{2,0} & \xrightarrow{\partial} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
A^{0,-1} & \xrightarrow{\partial} & A^{1,-1} & \xrightarrow{\partial} & A^{2,-1} & \xrightarrow{\partial} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
\vdots & \xrightarrow{\partial} & \vdots & \xrightarrow{\partial} & \vdots & \xrightarrow{\partial} & \ddots
\end{array}
\qquad
\begin{array}{c}
\xrightarrow{\text{stacky}} \\
\Big\downarrow \text{derived}
\end{array}
$$

- Take weak equivalences to have quasi-isomorphic columns.

- Functor D^* from cosimplicial CDGAs to stacky CDGAs.
- $D^* O([Y/G]) = O(Y/\mathfrak{g})$.
- Given derived Artin hgpd $\check{X}$, can form cosimplicial stacky CDGA $D^* O(\check{X}^{\Delta^\bullet})$:

$$D^* O(\check{X}) \rightrightarrows D^* O(\check{X}^{\Delta^1}) \rightrightarrows D^* O(\check{X}^{\Delta^2}) \cdots ,$$

a <u>DM</u> hgpd in stacky derived affines!

- Example: when $\check{X} = B[Y/G]$, this is

$$[Y/\mathfrak{g}] \leftleftarrows [Y \times G/\mathfrak{g}^2] \leftleftarrows [Y \times G^2/\mathfrak{g}^3] \cdots$$

Structures on Artin stacks

- Internal $\mathcal{H}om$ in double complexes.
- Semi-infinite total complex behaves

$$\hat{\mathrm{Tot}}\,(V)^i = (\bigoplus_{n<0} V^{n,i-n} \oplus \prod_{n\geqslant 0} V^{n,i-n}, \delta \pm \partial).$$

- Hence define $\hat{\mathrm{Hom}} = \hat{\mathrm{Tot}}\,\mathcal{H}om$, $\widehat{\mathrm{Pol}}$, ...
- Poisson structures $\mathcal{P}(A) \xrightarrow{\neq} \mathcal{P}(\hat{\mathrm{Tot}}\,A)$, for stacky CDGA A.

Symplectic versus Poisson

- Classical case: 2-form ω is symplectic iff inverse π is Poisson.
- Standard proof uses Darboux theorem (cotangent bundle).
- Shifted Darboux theorems [B-BBBJ], [BG] give local comparison for shifted structures.

A more direct approach

- ω_2 homotopy inverse to π_2.
- Higher components??
- Look to generalise

$$\pi^\sharp \circ \omega^\sharp \circ \pi^\sharp = \pi^\sharp.$$

- Then globalise (via hypergroupoids).

The canonical tangent vector

- Tangent space $T_\pi\mathcal{P}$ of Poisson structures at π:

$$\alpha \in F^2\widehat{\mathsf{Pol}}(A/R, n)^{n+2},$$
$$\delta\alpha + [\pi, \alpha] = 0$$

(i.e. $\pi + \alpha\epsilon$ Poisson).

- Differentiating $\mathbb{G}_m$-action gives

$$\sigma(\pi) := \sum_{p \geqslant 2} (p-1)\pi_p \in T_\pi\mathcal{P}.$$

Compatibility (the key)

- Contraction $\mu(-,\pi)$ from de Rham to Poisson cohomology (cf. [K-S M]).
- "Derivative" ν.
- Relates de Rham / Schouten–Nijenhuis:

$$[\pi,\mu(\omega,\pi)] = \mu(d\omega,\pi)+\nu(\omega,\pi;\frac{1}{2}[\pi,\pi]).$$

- $\mu(\omega,\pi) \in T_\pi\mathcal{P}$ for ω pre-symplectic, π Poisson.

In detail

When $\phi = adf_1 \wedge \ldots \wedge df_p$,

$$\mu(\phi, \pi) = a[\pi, f_1] \ldots [\pi, f_p,],$$
$$\nu(\phi, \pi; b) = \sum_i \pm a[\pi, f_1] \ldots [b, f_i] \ldots [\pi, f_p].$$

- Thus $\mu(\omega_2, \pi_2)^\sharp = \pi_2^\sharp \circ \omega_2^\sharp \circ \pi_2^\sharp$.
- $\mu(-, \pi)$ a qu-iso for π non-degenerate.

The affine equivalence

Weak equivalences of ∞-groupoids (i.e. topological spaces):

$$\{\text{Symplectic structures } \omega\}$$
$$\uparrow$$
$$\{\text{Compatible pairs } \mu(\omega, \pi) \simeq \sigma(\pi)\}$$
$$\downarrow$$
$$\{\text{Non-degenerate Poisson structures } \pi\}$$

Thus $\pi \mapsto \mu(-, \pi)^{-1}\sigma(\pi)$ (defined up to coherent homotopy).

Governing DGLAs

For A a stacky CDGA:

- Symplectic: $F^2\mathbf{L}\mathrm{DR}(A/R)[n+1]$ (abelian).
- Poisson: $F^2\widehat{\mathrm{Pol}}(A/R,n)[n+1]$.
- $T\mathcal{P}$: $F^2\widehat{\mathrm{Pol}}(A/R,n)[n+1][\epsilon]$.
- Compatible pairs a homotopy limit.
- Equivalence via obstruction theory.

Obstructions

$\mathrm{gr}_F^p \to F^2\widehat{\mathrm{Pol}}(A)/F^{p+1} \to F^2\widehat{\mathrm{Pol}}(A)/F^p$ central extension of DGLAs.

- Maurer–Cartan obstruction map $\mathcal{P}(A/R, n)/F^p \to \underline{\mathrm{MC}}(\mathrm{gr}_F^p[n+2])$, fibre $\mathcal{P}(A/R, n)/F^{p+1}$.
- Similar obstructions for symplectic structures, compatible pairs.
- Graded pieces equivalent via powers of $(\mathbf{L}\Omega^1_X)^\vee \simeq \mathbf{L}\Omega^1_X[n]$.

Derived Artin N-stacks

- Artin hypergroupoid resolution $\check{X}_\bullet \to \mathfrak{X}$ always exists.
- Thus can look at structures on cosimplicial stacky CDGA $D^*O(\check{X}^{\Delta^\bullet})$.
- Affine equivalences étale functorial (up to coherent homotopy).
- Hence on $\mathfrak{X}$, symplectic $\leftrightarrow$ non-degenerate Poisson.

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