Derived algebraic geometry with a view to quantisation I: derived stacks

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Building blocks for AG

- Classical AG built from affine schemes, i.e (commutative rings)^{op}
- Derived AG from $(CDG^{\leqslant 0}A_{\mathbb{Q}})^{op}$:
  - cochain complex $A^0 \overset{\delta}{\leftarrow} A^{-1} \overset{\delta}{\leftarrow} A^{-2} \overset{\delta}{\leftarrow} \ldots$,
  - graded-commutative multiplication $A^i \otimes A^j \to A^{i+j}$, δ a derivation, $1 \in A_0$.
  - Quasi-isomorphisms $f\colon A \to B$ with $\mathrm{H}^*(A) \cong \mathrm{H}^*(B)$.

Also simplicial or E_∞-rings (all equivalent over $\mathbb{Q}$).

DG schemes (Kontsevich)

Pairs $(X^0, \mathscr{O}_X^\bullet)$ with:

- X^0 a scheme,
- $\mathscr{O}_X^{\leqslant 0}$ a sheaf of CDGAs on X^0,
  - $\mathscr{O}_X^n$ quasi-coherent,
  - $\mathscr{O}_X^0 = \mathscr{O}_{X^0}$.
- X quasi-isomorphic to completion along $\pi^0 X = \mathbf{Spec}_{(X^0)} \mathrm{H}^0 \mathscr{O}_X$ if Noetherian.
- Too few objects, far too few morphisms. (Ambient scheme X^0 not really intrinsic.)

Functor of points

- Want to associate functors to derived affines.
- How about $\mathrm{Hom}(R, -)\colon \mathrm{CDGA} \to \mathrm{Set}$? Not quasi-IM invariant.
- What about taking homotopy classes $[A, -]$? Not left-exact; won't glue.
- Have to take derived Hom to topological spaces/simplicial sets $\mathrm{map}(R, -)\colon \mathrm{CDGA} \to s\mathrm{Set}$.

Example

$\mathbb{A}^1 = \operatorname{Spec} k[x]$

- $\mathrm{Hom}(k[x], A) = A^0$, not homotopy-invariant.
- Homotopy classes of maps $k[x] \to A$ given by $\mathrm{H}^0 A$, not left-exact.
- $\mathrm{map}(k[x], A)$ a space with $\pi_i \mathrm{map}(k[x], A) = \mathrm{H}^{-i} A$.

From local to global

- Any geometry has simple building blocks:
  - Convex opens (DG)
  - Affine schemes Aff (AG)
- Have to glue/quotient in larger category $\mathcal{A}$ to get global objects (manifolds, orbifolds, schemes, algebraic stacks).
- Conventionally in DAG, take $\mathcal{A}$ to be simplicial presheaves. We'll see smaller alternative.

- Schemes $\subset$ ringed spaces
- Algebraic stacks $\subset$ functors on Aff
- Quasi-coherent sheaves $\subset$ $\mathscr{O}_X$-modules (enough injectives)

Nice categories, with many nasty objects.

- Who cares about arbitrary sheaves on the big affine site?
- Or about arbitrary $\mathscr{O}_X$-modules?

Are there smaller ambient categories?

Čech nerves

Manifold X, covered by nice open subspaces $U_1, \ldots U_k$. For $U := \coprod U_i$, the Čech nerve

$$\check{X}_n \;:=\; \underbrace{U \times_X U \times_X \ldots \times_X U}_{n+1}$$

$$= \coprod_{i_0,\ldots,i_n} U_{i_0} \cap \ldots \cap U_{i_n}.$$

$$\check{X}_0 \leftleftarrows \check{X}_1 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \check{X}_2 \;\substack{\longleftarrow\\ \vdots\\ \longleftarrow}\; \check{X}_3 \;\substack{\longleftarrow\\ \vdots\\ \longleftarrow}\; \cdots$$

recovers X, $\mathrm{H}^*(X, \mathbb{R})$ from simplicial diagram of nice opens.

Algebraic analogue

Quasi-compact, semi-separated scheme X, affine cover $\{U_i\}_i$, same expression gives simplicial diagram $\check{X}$ of affine schemes, recovering X, $\mathrm{H}^*(X, \mathscr{O}_X)$.

More generally, for affine presentation $U \to \mathfrak{X}$ of Artin stack with affine diagonal,

$$\check{\mathfrak{X}}_n := \underbrace{U \times_{\mathfrak{X}} U \times_{\mathfrak{X}} \ldots \times_{\mathfrak{X}} U}_{n+1}$$

Example: $\mathfrak{X} = [U/G]$; $U \leftleftarrows U \times G \leftleftarrows U \times G^2 \cdots$

Simplicial objects

- $|\Delta^n| := \{x \in \mathbb{R}_+^{n+1} \ : \ \sum_{i=0}^n x_i = 1\}$.
- For topological space X, $\mathrm{Sing}(X)_n := \mathrm{Hom}(|\Delta^n|, X)$, so

$$\mathrm{Sing}(X)_0 \underset{\partial_0}{\overset{\partial_1}{\underset{\sigma_0}{\leftleftarrows\!\!\!\rightarrow}}} \mathrm{Sing}(X)_1 \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} \mathrm{Sing}(X)_2 \ldots,$$

relations like $\partial_i \sigma_i = \mathrm{id}$.
- Any diagram of this form is called simplicial.

Higher algebraic stacks

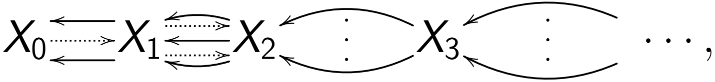
- Motivation: moduli problems.
- 1-stacks keep track of automorphisms.
- n-stacks have higher automorphisms.
- n-groupoids: spaces with $\pi_{>n}X = 0$.
- Example: $\mathscr{E} \in \mathrm{Perf}_X(A)$ has $(i \geqslant 2)$

$$\pi_i(\mathrm{Perf}_X(A), \mathscr{E}) = \mathrm{Ext}^{1-i}_{X \otimes A}(\mathscr{E}, \mathscr{E}).$$

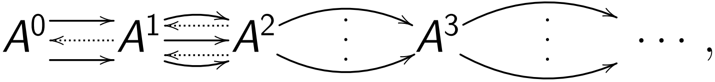
- n-truncated *derived* stack F has $\pi_{>n}F(A) = 0$ for A *underived*.

Technical assumption: from now on, everything is assumed strongly quasi-compact (quasi-compact, quasi-separated . . .)

- Every algebraic *n*-stack can be resolved by a simplicial affine scheme ($s\mathrm{Aff}$).



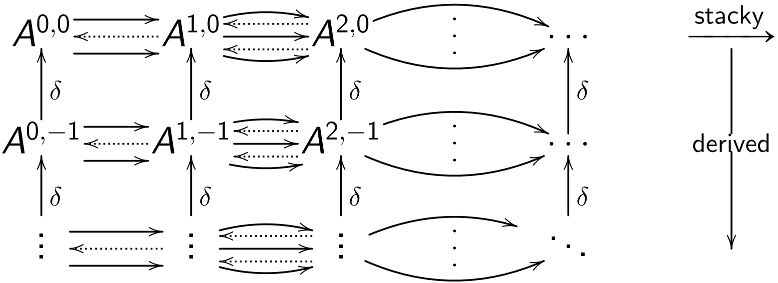
- Equivalently cosimplicial ring



- Which simplicial affines arise this way?
- What about morphisms?

Derived n-stacks

- Every algebraic derived n-stack resolved by a simplicial derived affine (sdAff).
- $d\mathrm{Aff}/\mathbb{Q} \simeq (CDG^{\leqslant 0}A_{\mathbb{Q}})^{\mathrm{opp}}$.
- Equivalently cosimplicial CDGA



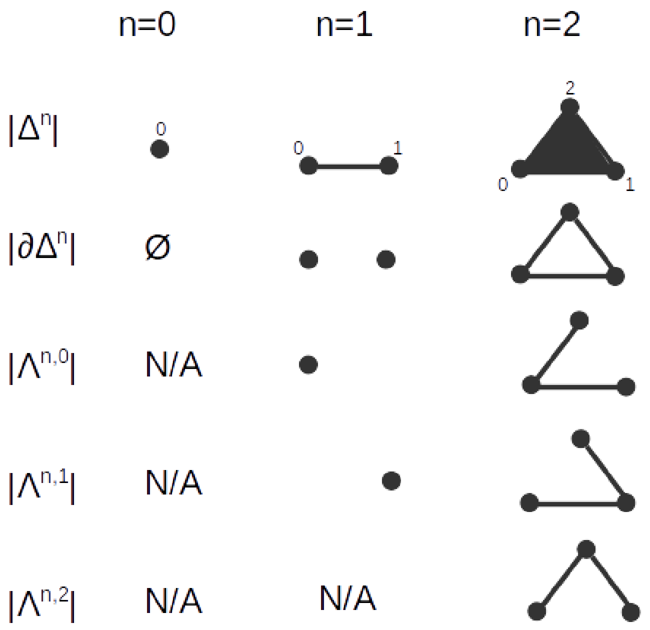
Simplices and horns

- m-simplex Δ^m is simplicial set with

$$\mathrm{Hom}_{s\mathrm{Set}}(\Delta^m, X) = X_m.$$

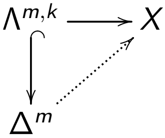
- Boundary $\partial\Delta^m = \bigcup_{i=0}^m \partial^i \Delta^{m-1} \subset \Delta^m$.
- kth horn $\Lambda^{m,k} = \bigcup_{\substack{i=0,\\ i\neq k}}^m \partial^i \Delta^{m-1}$.
- Partial matching objects

$$M_{\Lambda^{m,k}} X := \mathrm{Hom}_{s\mathrm{Set}}(\Lambda^{m,k}, X) =$$
$$\{x \in \prod_{\substack{0\leqslant i\leqslant m\\ i\neq k}} X_{m-1} \ : \ \partial_i x_j = \partial_j x_{i+1}, \forall i \geqslant j\}.$$


n=0
n=1
n=2
|Δ^n|
0
0
1
2
0
1
|∂Δ^n|
Ø
|Λ^{n,0}|
N/A
|Λ^{n,1}|
N/A
|Λ^{n,2}|
N/A
N/A

Duskin–Glenn n-hypergroupoids

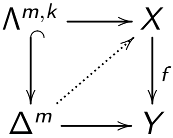
- Horn-fillers



$$X_m \to M_{\Lambda^{m,k}} X$$

are surjective for all m, k, and isomorphisms for $m > n$.

- Relative n-hgpds X/Y:



$$X_m \to M_{\Lambda^{m,k}} X \times_{(M_{\Lambda^{m,k}} Y)} Y_m$$

are surjective for all m, k, and isomorphisms for $m > n$.

- 1-hgpds are nerves of groupoids.
- Relative 0-hgpds are Cartesian:

$$X_m \cong X_0 \times_{Y_0} Y_m.$$

- n-hgpd determined by $X_{\leqslant n+1}$, but have to check conditions at X_{n+2}.
- $n = 1$ case: objects X_0, morphisms X_1, composition $X_{\leqslant 2}$, associativity $X_{\leqslant 3}$.
- Relative $n = 0$ case: $f_0 \colon X_0 \to Y_0$ gives fibres, f_1 gluing data, f_2 cocycle condition.

n-stacks the Grothendieck way

- Can define *n*-hypergroupoids in any category $\mathcal{A}$ with finite limits and a class $\mathbf{C}$ of covering maps.
- Sets and surjections $\rightsquigarrow$ *n*-groupoids.
- Affine schemes and smooth/étale surjections $\rightsquigarrow$ Artin/DM *n*-hgpds.
- In Grothendieck's "Pursuing stacks", apparently.

HAG2 defines n-geometric stacks inductively, but:

Theorem (P)

n-geometric Artin/DM stacks $\leftrightarrow$ hypersheafifications $X^\sharp$ of Artin/DM n-hypergroupoids X.

[HAG2 n-geometric stacks ($\mathfrak{X} \to \mathfrak{X}^{S^{n-1}}$ affine)

$\subset$ Lurie n-stacks ($\mathfrak{X} \simeq \mathfrak{X}^{S^{n+1}}$) $\subset$ $(n+2)$-geom stacks]

Derived n-geometric stacks

- Subtleties: replace isos with quasi-isos,
  - require Reedy fibrant: matching maps $X_m \to M_{\partial\Delta^m} X$ fibrations (i.e. quasi-free)
  - alternatively, use homotopy limits.
- Derived $\substack{\text{Artin} \\ \text{DM}}$ n-hgpd in $sd\mathrm{Aff}$ is Reedy fibrant with $\substack{\text{smooth} \\ \text{étale}}$ horn-fillers.
- (HAG2): $A^\bullet \to B^\bullet$ smooth/étale if $\mathrm{H}^0 A \to \mathrm{H}^0 B$ is so, and $\mathrm{H}^* B \cong \mathrm{H}^* A \otimes_{\mathrm{H}^0 A} \mathrm{H}^0 B$.

Theorem [P] Derived n-geometric $\substack{\text{Artin} \\ \text{DM}}$ stacks $\leftrightarrow X^\sharp$ for derived $\substack{\text{Artin} \\ \text{DM}}$ n-hgpds X.

Example: Reedy fibrant replacement of $\mathbb{A}^1$

- Need $X_\bullet \in sd\mathrm{Aff}$ with (i) $\mathbb{A}^1 \xrightarrow{\sim} X_m$, and (ii) $X_m \to M_{\partial\Delta^m}X$ quasi-free ($m = 1$ is $X_1 \to X_0 \times X_0$).
- Let $NC_\bullet(\Delta^m, k)$ be normalised chains (gen'd by non-degenerate simplices).
- Set $X_m = \operatorname{Spec} k[NC_\bullet(\Delta^m, k)]$.
- (i) $NC_\bullet(\Delta^m, k) \simeq k$, and (ii) $NC_\bullet(\partial\Delta^m, k) \hookrightarrow NC_\bullet(\Delta^m, k)$.

Example: dg schemes

- Semi-separated dg scheme $X = (X^0, \mathscr{O}_X) \rightsquigarrow$ derived Zariski 1-hgpd, by Reedy fibrant replacement of

$$\check{X}_i := \mathrm{Spec}\,\Gamma(\check{X}^0{}_i, \mathscr{O}_X),$$

  for Čech nerve $\check{X}^0$ of X^0.
- $\check{X}_\bullet$ quasi-isomorphic to completion along $\pi^0 X = \mathbf{Spec}_{(X^0)}\,\mathrm{H}^0\mathscr{O}_X$ if Noetherian.

Example: derived schemes

- Derived scheme is derived Artin/DM n-stack $\mathfrak{X}$ with underived truncation $\pi^0\mathfrak{X} \simeq Y$, a scheme.
- No ambient scheme (unlike dg schemes).
- When Y semi-separated, $\mathfrak{X}$ given by CDGAs $\mathscr{A}^{\leqslant 0}$ on $\check{Y}$ with $\mathrm{H}^0\mathscr{A} = \mathscr{O}_Y$ and $\mathrm{H}^*\mathscr{A}$ Cartesian/quasi-coherent. Zariski 1-hgpd is fibrant replacement of

$$\check{X}_i := \mathrm{Spec}\, \Gamma(\check{Y}_i, \mathscr{A}^\bullet).$$

Trivial n-hypergroupoids

To calculate morphisms or sheaves (functor of points), we need to refine atlases.

X is a trivial n-hgpd over Y if the matching maps

$$X_m \to M_{\partial\Delta^m}X \times_{(M_{\partial\Delta^m}Y)} Y_m$$

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^m & \longrightarrow & Y \end{array}$$

are surjective for all m and isos for $m \geqslant n$.

[Thus determined by $X_{<n} \to Y_{<n}$.]

$\substack{\text{Smooth} \\ \text{Étale}}$ surjections $\rightsquigarrow$ trivial $\substack{\text{Artin} \\ \text{DM}}$ n-hgpds.

Main theorem

Theorem (P)

The ∞-category of $(n, \mathbf{P})$-geometric stacks is the localisation of the category of $(n, \mathbf{P})$-hypergroupoids with respect to trivial $(n, \mathbf{P})$-hypergroupoids.

- $\mathbf{P}$ any property like (derived) Artin/DM.

Morphisms

More explicitly, for Y a (derived) Artin n-hgpd, mapping space is

$$\mathrm{map}(X^\sharp, Y^\sharp)_m = \varinjlim_{\alpha} \mathrm{Hom}(X_\alpha \times \Delta^m, Y)$$

for $\{X_\alpha \to X\}_\alpha$ any weakly initial system ($\approx$universal cover/maximal atlas) of trivial (derived) DM n-hgpds.

N.B. $Y^\sharp(A) = \mathrm{map}(\mathrm{Spec}\ A, Y)$.

Sheaves

- Complexes $\mathscr{F}_\bullet$ of $\mathscr{O}_X$-mods on $X^\sharp$ with qu-coh homology are *quasi-Cartesian* complexes of qu-coh sheaves on X.
- Given by complexes $\mathscr{F}_\bullet(X_m)$ of $O(X_m)$-mods and compatible quasi-isos

$$\partial^i \colon \partial_i^* \mathscr{F}_\bullet(X_m) \to \mathscr{F}_\bullet(X_{m+1}).$$

- Same definition works for any sheaves satisfying descent w.r.t. smooth/étale morphisms.

Stacks in other settings

- Zhu's Lie *n*-groupoids are *n*-hgpds in manifolds, w.r.t. surj submersions.
- $\mathbb{C}$-manifolds work just as well.
- $\mathcal{C}^\infty$-rings for singular Lie *n*-groupoids.
- Simplicial/dg $\mathcal{C}^\infty$-rings for derived Lie *n*-groupoids (Borisov–Noel).
- Poisson DGAs for Poisson str. on DM stacks (Artin more complicated).
- Pro-Artinian rings for formal moduli.

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