QUANTISATION OF DERIVED POISSON STRUCTURES

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Abstract. We prove that every 0-shifted Poisson structure on a derived Artin n-stack admits a curved A_∞ quantisation whenever the stack has perfect cotangent complex; in particular, this applies to LCI schemes. Where the Kontsevich–Tamarkin approach to quantisation hinges on invariance of the Hochschild complex under affine transformations, we instead exploit the observation that it carries an anti-involution, and that such anti-involutive deformations of the complex of polyvectors are essentially unique.

Introduction

For smooth algebraic varieties in characteristic 0, Kontsevich and Yekutieli showed in [Kon2, Yek1] that all Poisson structures admit DQ algebroid quantisations. Via local choices of connections, the question reduced to constructing quantisations of affine space. These could then be handled as in [Tam, Kon1, Yek2, VdB]: formality of the E_2 operad associates to the Hochschild complex a deformation of the P_2-algebra of multiderivations, and invariance under affine transformations ensures that it is the unique deformation.

We now consider generalisations of this question to singular varieties and to derived stacks, considering quantisations of 0-shifted Poisson structures in the sense of [Pri3, CPT$^+$]. For positively shifted structures, the analogous question is a formality, following from the equivalence $E_{n+1} \simeq P_{n+1}$ of operads. Quantisations for non-degenerate 0-shifted Poisson structures were established in [Pri4], and we now consider degenerate quantisations, addressing the remaining unsolved case of [Toë, Conjecture 5.3].

The construction of non-degenerate quantisations in [Pri4, Pri2] only relied on the fact that the Hochschild complex is an anti-involutive deformation of the complex of multiderivations. Our strategy in this paper is closer to [Tam, Kon1] in that we establish an equivalence between the two complexes. As in [Pri4, Pri2], the key observation is still that the Hochschild complex of a CDGA carries an anti-involution corresponding to the endofunctor on deformations sending an algebra to its opposite. For a suitable choice of formality quasi-isomorphism for the E_2 operad, which can be deduced from the action of the Grothendieck–Teichmüller group (§2.2), the Hochschild complex becomes an anti-involutive deformation of the P_2-algebra of multiderivations.

We show (Theorem 1.18) that such deformations are essentially unique whenever the complexes of polyvectors and of multiderivations are quasi-isomorphic. This is satisfied when the CDGA has perfect cotangent complex, so gives an equivalence between polyvectors and the Hochschild complex (Theorem 2.10). For derived Deligne–Mumford stacks with perfect cotangent complex, this yields quantisations of 0-shifted Poisson structures (Corollary 2.12), which take the form of curved A_∞ deformations of the étale structure sheaf.

In Section 3, we extend these results to derived Artin n-stacks. This follows by essentially the same argument, but is much more technically complicated because of the subtleties in formulating polyvectors and Hochschild complexes for Artin stacks. The key ingredient is to regard the Hochschild complex as an E_2-algebra in a certain coloured dg operad with multi-operations defined in terms of differential operators (Definition 3.10); this dg operad is chosen to ensure that the homotopy P_k-algebra derivations of interest correspond to the correct notion of shifted polyvectors for stacks (Corollary 3.15). For stacky thickenings of derived affine schemes, this leads to an equivalence between polyvectors and the Hochschild complex (Theorem 3.25), yielding quantisations of 0-shifted Poisson structures on derived Artin stacks with perfect cotangent complex(Corollary 2.12).

Notation and terminology. We write CDGAs (commutative differential graded algebras) and DGAAs (differential graded associative algebras) as chain complexes (homological grading), and denote the differential on a chain complex by δ. Our conventions for shifts of chain and cochain complexes are that $(V_{[n]})_i := V_{n+i}$ and $(V^{[n]})^i := V^{n+i}$.

Given a DGAA A, and A-modules M, N in chain complexes, we write $\underline{\mathrm{Hom}}_A(M, N)$ for the chain complex given by

$$\underline{\mathrm{Hom}}_A(M, N)_i = \mathrm{Hom}_{A_\#}(M_\#, N_{\#[i]}),$$

with differential $\delta f = \delta_N \circ f \pm f \circ \delta_M$, where $V_\#$ denotes the graded vector space underlying a chain complex V.

Contents

1. Involutively filtered deformations of Poisson algebras

In this section, we characterise a certain class of filtered deformations of Poisson algebras equipped with anti-involutions. We initially work in a very general setting, in order to obtain results we can immediately apply to quantisation problems for diagrams, and with a view to possible future generalisations. Of the results in the first two subsections, readers might find it easier initially just to concentrate on Definition 1.6, Lemma 1.11 and Remark 1.12, which suffice for applications to derived affine schemes.

We will assume that all filtrations are increasing and exhaustive, unless stated otherwise. Given a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-vector space V, we write $\mathcal{W}_i V$ for the summand of weight i, and define a weight filtration W by setting $W_n V := \bigoplus_{i \leqslant n} \mathcal{W}_i V$. For $\mathbb{G}_m$-equivariant complexes U, V, we write $\mathcal{W}_i \underline{\mathrm{Hom}}(U, V)$ for the complex $\prod_j \underline{\mathrm{Hom}}(\mathcal{W}_j U, \mathcal{W}_{i+j} V)$ of homomorphisms of weight i, with similar conventions for complexes of derivations etc.

1.1. Involutively filtered deformations of $\mathcal{P}$-algebras.

Definition 1.1. We say that a vector space V is involutively filtered if it is equipped with a filtration W and an involution e which preserves W and acts on $\mathrm{gr}_i^W V$ as multiplication by $(-1)^i$.

Observe that if V is involutively filtered, then the involution gives an eigenspace decomposition $V = V^{e=1} \oplus V^{e=-1}$, and because $\mathrm{gr}^W_{i-1} V^{e=(-1)^i} = 0$, we have $W_{2j+1} V^{e=1} = W_{2j} V^{e=1}$ and $W_{2j} V^{e=-1} = W_{2j-1} V^{e=-1}$.

Definition 1.2. Define a $\mathbb{G}_m$-equivariant dg Hopf algebra $\mathbb{Q}[\eth]$ over $\mathbb{Q}$ by taking $\eth$ of homological degree -1 and of weight -2 for the $\mathbb{G}_m$ action, commutativity forcing $\eth^2 = 0$. The comultiplication $\mathbb{Q}[\eth] \to Q[\eth] \otimes_{\mathbb{Q}} \mathbb{Q}[\eth]$ is defined by $\eth \mapsto \eth \otimes 1 + 1 \otimes \eth$.

This is slightly different from the dg Hopf algebra $\mathbb{Q}[\rlap{-}d]$ of [Pri2, §1.1.1], in which $\rlap{-}d$ has weight -1; the difference corresponds to the data of an involution, and the results below hold without involutions if we replace $\eth$ with $\rlap{-}d$. Also beware that [Pri2] considered decreasing filtrations, so its weights have the opposite signs to ours.

Definition 1.3. For a complete involutively filtered chain complex $(V, W_i V, e)$ over $\mathbb{Q}$, we define a $\mathbb{G}_m$-equivariant $\mathbb{Q}[\eth]$-module $\mathfrak{gr}^{W,e} V$ to be given in weight i by

$$\begin{aligned} \mathfrak{gr}_i^{W,e} V &:= \{v \in \mathrm{cone}(W_{i-1} V \to W_i V) \ : \ e(v) = (-1)^i v\}, \\ &= \{v \in \mathrm{cone}(W_{i-2} V \to W_i V) \ : \ e(v) = (-1)^i v\} \end{aligned}$$

with $\eth \colon \mathfrak{gr}_i^W V \to \mathfrak{gr}_{i-2}^W V_{[-1]}$ given by the identity on $W_{i-2} V_{[-1]}$ (and necessarily 0 elsewhere).

This gives an equivalence of ∞-categories from the category of complete filtered $\mathbb{Q}$-chain complexes localised at filtered quasi-isomorphisms to the category of $\mathbb{G}_m$-equivariant $\mathbb{Q}[\eth]$-modules in chain complexes localised at quasi-isomorphisms. The homotopy inverse functor is given by

$$\bigoplus_i \mathcal{W}_i E \mapsto (\bigoplus_{i>0} \mathcal{W}_i E \oplus \prod_{i \leqslant 0} \mathcal{W}_i E, \delta \pm \eth),$$

equipped with the complete exhaustive filtration

$$W_i := (\prod_{j \leqslant i} \mathcal{W}_j E, \delta \pm \eth)$$

and involution $e(\sum_i v_i) = \sum_i (-1)^i v_i$ for $v_i \in \mathcal{W}_i E$.

Defining a symmetric monoidal structure on $\mathbb{Q}[\eth]$-modules by $\otimes_{\mathbb{Q}}$, with $\eth$ acting on $M \otimes_{\mathbb{Q}} N$ via the comultiplication on $\mathbb{Q}[\eth]$, the functor $\mathfrak{gr}$ and its homotopy inverse above are both lax monoidal.

Definition 1.4. Given a $\mathbb{G}_m$-equivariant dg operad $\mathcal{P}$ over $\mathbb{Q}$ and a dg Hopf algebra B over $\mathbb{Q}$, define the operad $\mathcal{P} \circ B$ by $(\mathcal{P} \circ B)(n) := \mathcal{P}(n) \otimes B^{\otimes n}$, with operad structure defined by the distributive law $B \otimes \mathcal{P}(n) \to \mathcal{P}(n) \otimes B^{\otimes n}$ given by the comultiplication on B.

If we define a filtration W on a $\mathbb{G}_m$-equivariant dg operad $\mathcal{P}$ by setting $W_i\mathcal{P}$ to be spanned by terms of weight $\leqslant i$ for the $\mathbb{G}_m$-action, and an involution e given by the action of $-1 \in \mathbb{G}_m$, then the functor $\mathfrak{gr}^{W,e}$ above gives an equivalence from the ∞-category of $(\mathcal{P}, W, e)$-algebras A in complete filtered $\mathbb{Q}$-chain complexes to the ∞-category of $\mathcal{P} \circ \mathbb{Q}[\eth]$-algebras in $\mathbb{G}_m$-equivariant $\mathbb{Q}$-chain complexes. If we forget the $\eth$-action, note that $\mathfrak{gr}^{W,e}A$ is quasi-isomorphic to the $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebra $\bigoplus_i \mathrm{gr}_i^W A$.

We thus make the following definition:

Definition 1.5. Given a $\mathbb{G}_m$-equivariant coloured dg operad (i.e. a dg multicategory) $\mathcal{A}$ over $\mathbb{Q}$ and a $\mathbb{G}_m$-equivariant operad $\mathcal{P}$ over $\mathbb{Q}$, define the space of complete anti-involutively filtered derived $(\mathcal{P}, W, e)$-algebras in $\mathcal{A}$ to be the space

$$\mathrm{map}(\mathcal{P} \circ \mathbb{Q}[\eth], \mathcal{A})$$

of maps in the ∞-category of $\mathbb{G}_m$-equivariant coloured dg operads over $\mathbb{Q}$.

Here, the ∞-category of dg operads is defined by localising at morphisms $f\colon \mathcal{A} \to \mathcal{B}$ which induce quasi-isomorphisms $\mathcal{A}(x_1, \ldots, x_n; y) \to \mathcal{A}(fx_1, \ldots, fx_n; fy)$ and equivalences on the underlying homotopy categories (objects are colours, with morphisms $\mathrm{H}_0\mathcal{A}(x, y)$).

In practice, we will mostly apply Definition 1.5 to multicategories underlying symmetric monoidal dg categories $\mathcal{C}$, in which case a morphism $\mathcal{P} \circ \mathbb{Q}[\eth] \to \mathcal{A}$ consists of an object $C \in \mathcal{C}$ together with a morphism

$$\{\mathcal{P}(n) \otimes \mathbb{Q}[\eth]^{\otimes n}\}_n \to \{\underline{\mathrm{Hom}}_{\mathcal{C}}(C^{\otimes n}, C)\}_n$$

of dg operads.

1.2. **Almost commutative Poisson algebras.** We now consider non-unital P_k-algebras (i.e. $(k-1)$-shifted Poisson algebras); these are non-unital CDGAs equipped with a Lie bracket of chain degree $k-1$ acting as a biderivation. They are governed by an operad P_k which can be written as $\mathrm{Com} \circ (s^{1-k}\mathrm{Lie})$ via a distributive law (cf. [LV, §8.6]), for the operads $\mathrm{Com}, \mathrm{Lie}$ governing non-unital commutative algebras and Lie algebras, where $(s\mathcal{P})(n) := \mathcal{P}(n)_{[n-1]}$.

Definition 1.6. Define the $\mathbb{G}_m$-equivariant dg operad P_k^{ac} to be the dg operad $\mathrm{Com} \circ s^{1-k}\hbar\mathrm{Lie}$, where $(\hbar\mathcal{P})(i) := \hbar^{i-1}\mathcal{P}(i)$ for any operad $\mathcal{P}$ and $\hbar$ has degree 0 and weight 1 for the $\mathbb{G}_m$-action.

Define an almost commutative anti-involutive P_k-algebra over a CDGA R to be an $(R \otimes P_k, W, e)$-algebra A in involutively complete filtered R-chain complexes, where W_iP_k is spanned by terms of weight $\leqslant i$ for the $\mathbb{G}_m$-action on P_k^{ac}.

Thus a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra is a P_k-algebra equipped with a $\mathbb{G}_m$-action for which multiplication has weight 0 and the Lie bracket has weight -1. An almost commutative anti-involutive P_k-algebra is a P_k-algebra equipped with a complete filtration W satisfying $W_i \cdot W_j \subset W_{i+j}$ and $[W_i, W_j] \subset W_{i+j-1}$, together with an involution $*$ preserving the filtration, satisfying $(a \cdot b)^* = a^* \cdot b^*$ and $[a, b]^* = -[a^*, b^*]$, and acting as $(-1)^i$ on gr_i^W. Such algebras automatically give rise to complete filtered derived

(P_k^{ac}, W, e)-algebras in the sense of Definition 1.5, which we thus refer to as almost commutative anti-involutive derived P_k-algebras.

Definition 1.7. Given a differential graded Lie algebra (DGLA) L with homological grading, define the the Maurer–Cartan set by

$$\mathrm{MC}(L) := \{\omega \in L_{-1} \mid \delta\omega + \frac{1}{2}[\omega,\omega] = 0 \in \bigoplus_n L_{-2}\}.$$

Following [Hin2], define the Maurer–Cartan space $\underline{\mathrm{MC}}(L)$ (a simplicial set) of a nilpotent DGLA L by

$$\underline{\mathrm{MC}}(L)_n := \mathrm{MC}(L \otimes_{\mathbb{Q}} \Omega^\bullet(\Delta^n)),$$

where

$$\Omega^\bullet(\Delta^n) = \mathbb{Q}[t_0, t_1, \dots, t_n, \delta t_0, \delta t_1, \dots, \delta t_n]/(\sum t_i - 1, \sum \delta t_i)$$

is the commutative dg algebra of de Rham polynomial forms on the n-simplex, with the t_i of degree 0.

Given an inverse system $L = \{L_\alpha\}_\alpha$ of nilpotent DGLAs, define

$$\mathrm{MC}(L) := \varprojlim_\alpha \mathrm{MC}(L_\alpha) \quad \underline{\mathrm{MC}}(L) := \varprojlim_\alpha \underline{\mathrm{MC}}(L_\alpha).$$

Note that $\mathrm{MC}(L) = \mathrm{MC}(\varprojlim_\alpha L_\alpha)$, but $\underline{\mathrm{MC}}(L) \neq \underline{\mathrm{MC}}(\varprojlim_\alpha L_\alpha)$.

By [Cav, Theorem 4.22] (following [Hin1, BM]), there is a model structure on $\mathbb{G}_m$-equivariant dg operads over $\mathbb{Q}$ in which all objects are fibrant.

Lemma 1.8. *For the operadic cobar construction Ω of* [LV, §6.5.5], *cofibrant replacements of P_k^{ac} and $P_k^{ac} \circ \mathbb{Q}[\eth]$ are given by*

$$\Omega(s^k \hbar^{-1} P_k^{ac})^\vee \quad \text{and} \quad \Omega(\mathbb{Q}[\hbar^2] \circ s^k \hbar^{-1} P_k^{ac})^\vee.$$

Proof. The shifted analogue of [LV, Theorem 7.4.6] shows that $\Omega\mathcal{P}^{¡}$ is a cofibrant replacement for a Koszul dg operad $\mathcal{P}$, where as in [LV, 7.2.3], the Koszul dual co-operad $\mathcal{P}^{¡}$ is given by $\mathcal{P}^{¡} = (s(\mathcal{P}^!))^\vee$, for the Koszul dual operad $\mathcal{P}^!$. We have $\mathrm{Lie}^! = \mathrm{Com}$, $\mathrm{Com}^! = \mathrm{Lie}$ and $\mathbb{Q}[\eth]^! = \mathbb{Q}[\hbar^2]$, and the proof of [LV, 8.6.11] then adapts to this shifted setting (cf. [Fre, Appendix C]) to give $(P_k^{ac})^!$ as

$$\begin{aligned}(P_k^{ac})^! &= (\mathrm{Com} \circ s^{1-k}\hbar\mathrm{Lie})^! \\ &= (s^{1-k}\hbar\mathrm{Lie})^! \circ \mathrm{Com}^! \\ &\cong (s^{k-1}\hbar^{-1}\mathrm{Com}) \circ \mathrm{Lie} \\ &= s^{k-1}\hbar^{-1}(\mathrm{Com} \circ (s^{1-k}\hbar\mathrm{Lie})) \\ &= s^{k-1}\hbar^{-1}P_k^{ac},\end{aligned}$$

with

$$(P_k^{ac} \circ \mathbb{Q}[\eth])^! = \mathbb{Q}[\eth]^! \circ (P_k^{ac})^! = \mathbb{Q}[\hbar^2] \circ s^{k-1}\hbar^{-1}P_k^{ac}.$$

□

Definition 1.9. Given a dg operad $\mathcal{A}$, define the dg operad $\mathrm{Rep}(\mathbb{G}_m, \mathcal{A})$ of $\mathbb{G}_m$-equivariant objects in $\mathcal{A}$ to have objects (colours) $(\mathrm{Ob}\mathcal{A})^{\mathbb{Z}}$ and $\mathbb{G}_m$-equivariant multimorphism spaces $\mathrm{Rep}(\mathbb{G}_m, \mathcal{A})(M_1, \dots, M_n; N)$ given by

$$\mathcal{W}_i\mathrm{Rep}(\mathbb{G}_m, \mathcal{A})(M_1, \dots, M_n; N) := \prod_{j_1,\dots,j_n} \mathcal{A}(\mathcal{W}_{j_1}M_1, \dots, \mathcal{W}_{j_n}M_n; \mathcal{W}_{i+j_1+\dots+j_n}N)$$

with the obvious composition rules, where $\mathcal{W}_j M \in \mathrm{Ob}\mathcal{A}$ is the jth component of $M \in \mathrm{Ob}\mathrm{Rep}(\mathbb{G}_m, \mathcal{A}) = (\mathrm{Ob}\mathcal{A})^{\mathbb{Z}}$.

Definition 1.10. Take a $\mathbb{G}_m$-equivariant dg operad $\mathcal{P}$ with Koszul dual $\mathcal{P}^!$ (e.g. $\mathcal{P} = P_k^{ac}$, $\mathcal{P}^! = s^{k-1}\hbar^{-1}P_k^{ac}$ or $\mathcal{P} = \mathrm{Com}$, $\mathcal{P}^! = \mathrm{Lie}$). Given a $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebra A in a dg operad $\mathcal{A}$ (i.e. a $\mathbb{G}_m$-equivariant morphism $f\colon \mathcal{P} \to \mathrm{Rep}(\mathbb{G}_m, \mathcal{A})$), and a $\mathbb{G}_m$-equivariant A-module M in $\mathcal{A}$ (i.e. a $\mathcal{P}$-algebra structure on the formal object $A \oplus M\epsilon$, with $\epsilon^2 = 0$), define the $\mathbb{G}_m$-equivariant complex $h\underline{\mathrm{Der}}_{\mathcal{P},\mathcal{A}}(A, M) = \bigoplus_i \mathcal{W}_i h\underline{\mathrm{Der}}_{\mathcal{P},\mathcal{A}}(A, M)$ of homotopy derivations to be the convolution complex

$$(\prod_n (s\mathcal{P}^!)(n) \otimes^{\Sigma_n} \mathcal{A}(\overbrace{A, \ldots, A}^{n}; M), \delta + [f \circ \alpha, -]),$$

where the map $\alpha\colon (s\mathcal{P}^!)^\vee \to \mathcal{P}$ is defined by Koszul duality, and products are taken in the category of $\mathbb{G}_m$-equivariant complexes. When $M = A$, the complex $h\underline{\mathrm{Der}}_{\mathcal{P},\mathcal{A}}(A) := h\underline{\mathrm{Der}}_{\mathcal{P},\mathcal{A}}(A, A)$ is a DGLA, with bracket defined via the convolution product of [LV, 6.4.4] (which is of weight 0 for the $\mathbb{G}_m$-action). When the context is clear, we simply denote $h\underline{\mathrm{Der}}_{\mathcal{P},\mathcal{A}}(A)$ by $h\underline{\mathrm{Der}}_{\mathcal{P}}(A)$.

When $\mathcal{P} = P_k^{ac}$, note that substituting for $(P_k^{ac})^!$ gives a description of $h\underline{\mathrm{Der}}_{\mathcal{P},\mathcal{A}}(A, M)$ as

$$(\prod_n (\hbar^{1-k}(P_k^{ac}(n))_{[k(n-1)]} \otimes^{\Sigma_n} \mathcal{A}(\overbrace{A, \ldots, A}^{n}; M), \delta + [f \circ \alpha, -]).$$

When $\mathcal{A}$ comes from an object A in a symmetric monoidal pre-triangulated dg category containing representations of $\mathbb{G}_m$, then the datum f above defines a P_k^{ac}-algebra structure on A, and the bar construction of [LV, 11.2] gives a $(s^k\hbar^{-1}P_k^{ac})$-coalgebra $\mathrm{B}_\alpha A$, with $\Omega_\alpha \mathrm{B}_\alpha A$ a quasi-free resolution of A. The DGLA $h\underline{\mathrm{Der}}_{P_k^{ac}}(A)$ above then corresponds to the $\mathbb{G}_m$-equivariant complex of $(s^k\hbar^{-1}P_k^{ac})$-coalgebra coderivations on $\mathrm{B}_\alpha A$.

Lemma 1.11. *Given a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra A in a $\mathbb{G}_m$-equivariant dg operad $\mathcal{A}$ (i.e. a $\mathbb{G}_m$-equivariant morphism $P_k^{ac} \to \mathcal{A}$), the space*

$$\mathrm{map}(P_k^{ac} \circ \mathbb{Q}[\eth], \mathcal{A}) \times^h_{\mathrm{map}(P_k^{ac}, \mathcal{A})} \{A\}$$

of almost commutative anti-involutive derived P_k-algebras A' in $\mathcal{A}$ with $\mathrm{gr}^W A' \simeq A$ is given by the Maurer–Cartan space

$$\varprojlim_r \underline{\mathrm{MC}}(\prod_{j=1}^r \mathcal{W}_{-2j} h\underline{\mathrm{Der}}_{P_k^{ac}}(A)).$$

Proof. By Lemma 1.8, we may realise the mapping spaces as spaces of morphisms from the cofibrant replacements $\Omega(s^k\hbar^{-1}P_k^{ac})^\vee$ and $\Omega(\mathbb{Q}[\hbar^2] \circ s^k\hbar^{-1}P_k^{ac})^\vee$ of P_k^{ac} and $P_k^{ac} \circ \mathbb{Q}[\eth]$ to the fibrant simplicial resolution $\Omega^\bullet(\Delta^*) \circ \mathcal{A}$ of $\mathcal{A}$. As in [LV, §6.5], the set of morphisms from the cobar construction is given by the Maurer–Cartan set of the convolution DGLA, realising the desired mapping space as

$$(\varprojlim_r \underline{\mathrm{MC}}(h\underline{\mathrm{Der}}_{P_k^{ac}}(A)[\hbar^2]/\hbar^{2r})^{\mathbb{G}_m}) \times^h_{\underline{\mathrm{MC}}(h\underline{\mathrm{Der}}_{P_k^{ac}}(A))^{\mathbb{G}_m}} \{0\},$$

which simplifies to the expression above, since $\hbar$ has weight 1. $\square$

Remark 1.12. If we were looking at all almost commutative P_k-algebras instead of just the anti-involutive ones, then we would replace $\eth$ with $đ$ and $\hbar^2$ with $\hbar$ in the reasoning above, giving the space $\varprojlim_r \underline{\mathrm{MC}}(\prod_{j=1}^r \mathcal{W}_{-j} h\underline{\mathrm{Der}}_{P_k^{ac}}(A))$. This is unsurprising, since modifying the differential δ_A by a derivation in this space will still preserve the filtration, and not affect the associated graded algebra. The restriction to terms of even weight in Lemma 1.11 ensures that the modified differential $\delta \pm \eth$ on the filtered complex $\bigoplus_{i>0} \mathcal{W}_i A \oplus \prod_{i\leqslant 0} \mathcal{W}_i A$, commutes with the involution $-1 \in \mathbb{G}_m$.

Corollary 1.13. *Given a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra A in a $\mathbb{G}_m$-equivariant dg operad $\mathcal{A}$ for which*

$$\mathrm{H}_i \mathcal{W}_{-2j} h\underline{\mathrm{Der}}_{P_k^{ac}}(A) = 0$$

for all $j \geqslant 1$ and $i \geqslant -2$, the space of almost commutative anti-involutive derived P_k-algebras A' in $\mathcal{A}$ with $\mathrm{gr}^W A' \simeq A$ is contractible.

Proof. By Lemma 1.11, it suffices to show that each map

$$\underline{\mathrm{MC}}(\prod_{j=1}^{r} \mathcal{W}_{-2j} h\underline{\mathrm{Der}}_{P_k^{ac}}(A)) \to \underline{\mathrm{MC}}(\prod_{j=1}^{r-1} \mathcal{W}_{-2j} h\underline{\mathrm{Der}}_{P_k^{ac}}(A))$$

is a weak equivalence, but as in [Pri3, Proposition 1.29], this map can be expressed as the homotopy fibre of a fibration over $\underline{\mathrm{MC}}(\mathcal{W}_{-2r} h\underline{\mathrm{Der}}_{P_k^{ac}}(A))_{[-1]})$, whose ith homotopy group is $\mathrm{H}_{i-2} \mathcal{W}_{-2r} h\underline{\mathrm{Der}}_{P_k^{ac}}(A) = 0$ by hypothesis, making it contractible. $\square$

Remark 1.14. If we take an involutively filtered dg operad $\mathcal{P}$ with an involutive equivalence $\mathrm{gr}^W \mathcal{P} \simeq P_k^{ac}$, then the conditions of Corollary 1.13 also ensure that the space of almost commutative anti-involutive derived $\mathcal{P}$-algebras A' in $\mathcal{A}$ with $\mathrm{gr}^W A' \simeq A$ is contractible. This is because, although the controlling DGLA is defined in terms of $h\underline{\mathrm{Der}}_{\mathcal{P}}$ rather than $h\underline{\mathrm{Der}}_{P_k^{ac}}$, the associated graded pieces are the same.

1.3. **Uniqueness of deformations.** We now fix a CDGA R over $\mathbb{Q}$. There is a model structure on the category of $\mathbb{G}_m$-equivariant R-chain complexes (the projective model structure), in which fibrations are surjections. For a category of the form $[m] = (0 \to 1 \to \ldots \to m)$, we now consider the injective model structure on I-diagrams of $\mathbb{G}_m$-equivariant R-chain complexes, so cofibrations are defined levelwise. This model structure has the property that the tensor products over R is a left Quillen bifunctor.

Writing $\mathrm{CoS}^p_B(M) := \mathrm{CoSymm}^p_B(M) = (M^{\otimes_B p})^{\Sigma_p}$, and expanding out Definition 1.10 in terms of the cotangent complex then gives:

Lemma 1.15. *Given an $[m]$-diagram B of $\mathbb{G}_m$-equivariant R-chain complexes which is fibrant and cofibrant for the injective model structure above (i.e. each $B(i)$ is cofibrant as an R-module and the maps $B(i) \to B(i+1)$ are surjective), and a P_k^{ac}-algebra structure on B, there are canonical quasi-isomorphisms*

$$\mathcal{W}_i h\underline{\mathrm{Der}}_{P_k^{ac},R,I}(B) \simeq (\prod_{p\geqslant 1} \mathbf{R}\mathcal{W}_{i+1-p} \underline{\mathrm{Hom}}_B(\mathbf{L}\mathrm{CoS}^p_B((\mathbf{L}\Omega^1_{B/R})_{[-k]}), B), \delta + [\varpi, -])_{[-k]},$$

giving quasi-isomorphisms of $\mathbb{G}_m$-equivariant DGLAs on taking $\bigoplus_i$, where ϖ denotes the bivector on B corresponding to the Poisson bracket, and the right-hand complex is given the Schouten–Nijenhuis bracket.

Proposition 1.16. *If B is a non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebra over a CDGA R for which the map $(\mathcal{W}_1\mathbf{L}\Omega^1_{B/\mathcal{W}_0B})\otimes_{\mathcal{W}_0B} B \to \mathbf{L}\Omega^1_{B/\mathcal{W}_0B}$ is a quasi-isomorphism, with B cofibrant as an R-chain complex, then $\mathcal{W}_i h\underline{\mathrm{Der}}_{P_k^{ac},R}(B) \simeq 0$ for $i \leqslant -2$.*

Proof. Without loss of generality, we may assume that B is cofibrant. We then have an exact triangle

$$\Omega^1_{\mathcal{W}_0B}\otimes_{\mathcal{W}_0B} B \to \Omega^1_B \to \Omega^1_{B/\mathcal{W}_0B} \to \Omega^1_{\mathcal{W}_0B}\otimes_{\mathcal{W}_0B} B_{[-1]},$$

which by hypothesis simplifies to

$$\Omega^1_{\mathcal{W}_0B}\otimes_{\mathcal{W}_0B} B \to \Omega^1_B \to (\mathcal{W}_1\Omega^1_{B/\mathcal{W}_0B})\otimes_{\mathcal{W}_0B} B \to \Omega^1_{\mathcal{W}_0B}\otimes_{\mathcal{W}_0B} B_{[-1]}.$$

Since Ω^1_B is quasi-isomorphic to a B-module freely generated by terms of weights $0,1$, and since the weights of B are non-negative, we thus have that $\mathbf{R}\mathcal{W}_i\underline{\mathrm{Hom}}_B(\mathbf{L}\mathrm{CoS}^p_B((\mathbf{L}\Omega^1_{B/R})_{[-k]}), B)$ is acyclic for $i < -p$. The statement now follows from the description of Lemma 1.15. $\square$

Definition 1.17. We say that a morphism $B \to C$ of $\mathbb{G}_m$-equivariant CDGAs over R is homotopy formally étale if it induces a quasi-isomorphism

$$\mathbf{L}\Omega^1_{B/R}\otimes^{\mathbf{L}}_B C \to \mathbf{L}\Omega^1_{C/R}$$

on cotangent complexes.

Theorem 1.18. *Let B be a non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebra over a CDGA R. If the map $(\mathcal{W}_1\mathbf{L}\Omega^1_{B/\mathcal{W}_0B})\otimes_{\mathcal{W}_0B} B \to \mathbf{L}\Omega^1_{B/\mathcal{W}_0B}$ is a quasi-isomorphism, then the space of almost commutative anti-involutive derived P_k-algebras (B',W) over R with fixed involutive quasi-isomorphism $\mathrm{gr}^W B' \xrightarrow{\sim} B$ is contractible. We thus have an essentially unique quasi-isomorphism $B' \simeq B$ of anti-involutive filtered P_k-algebras for each such (B',W), the anti-involution acting on $\mathcal{W}_i$ as $(-1)^i$.*

Moreover, for any homotopy formally étale morphism $f\colon B \to C$ of non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebras over R, the space of morphisms $g\colon B \to C$ of almost commutative anti-involutive P_k-algebras over R with $\mathrm{gr}^W g \simeq f$ is also contractible.

Proof. The statement is unaffected if we take a cofibrant replacement for B as an R-linear P_k-algebra, so we may in particular assume that B is cofibrant as an R-module in chain complexes. Applying the functor $\mathfrak{gr}$ of Definition 1.3 then associates to B' an element of the space of almost commutative anti-involutive P_k-algebras in the sense of Definition 1.5.

Proposition 1.16 then implies that $\mathcal{W}_{-2j}h\underline{\mathrm{Der}}_{P_k^{ac},R}(B)$ is acyclic for all $j \geqslant 1$, so Corollary 1.13 shows that the space of almost commutative anti-involutive deformations of B is contractible.

If we now write D for the $[1]$-diagram $(B \to C)$, then we may assume that D is fibrant and cofibrant as a diagram of R-chain complexes, and the argument of [Pri3, Lemma 2.3] shows that the restriction map

$$h\underline{\mathrm{Der}}_{P_k^{ac},R,[1]}(D) \to h\underline{\mathrm{Der}}_{P_k^{ac},R}(B)$$

is a $\mathbb{G}_m$-equivariant quasi-isomorphism, so the same argument shows that the space of anti-involutive almost commutative deformations of the diagram is also contractible. $\square$

Remark 1.19. If we were to consider non-involutive deformations instead, then the analogue of Theorem 1.18 would not hold. The non-involutive analogue of Corollary 1.13 involves homology of $h\underline{\mathrm{Der}}_{P_k^{ac},R}(B)$ in all negative weights, and the weight -1 term is given under the hypotheses of Theorem 1.18 by

$$(\prod_{p\geqslant 1}\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{W}_0B}(\mathbf{L}\mathrm{CoS}^p_{\mathcal{W}_0B}((\mathbf{L}\mathcal{W}_1\Omega^1_{B/R})_{[-k]}),\mathcal{W}_0B),\delta+[\varpi,-])_{[-k]}$$

which is seldom acyclic. This is similar to phenomena arising in [Pri6, Pri4, Pri2], where the only obstruction to quantisation is first-order, and can be eliminated by restricting to involutive quantisations.

If ϖ is non-degenerate in the sense that $\mathbf{L}\Omega^1_{\mathcal{W}_0B/R}$ is perfect over $\mathcal{W}_B$ and that the Lie bracket ϖ induces a quasi-isomorphism $(\mathbf{L}\Omega^1_{\mathcal{W}_0B/R})_{[1-k]} \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{W}_0B}(\mathbf{L}\mathcal{W}_1\Omega^1_{B/R},\mathcal{W}_0B)$, then as in [Pri3, Definition 1.16 and Lemma 1.17], there are $\mathbb{G}_m$-equivariant CDGA quasi-isomorphisms

$$\bigoplus_{p\geqslant 0}(\mathbf{L}\Omega^p_{\mathcal{W}_0B/R})_{[p]} \to \bigoplus_{p\geqslant 0}\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{W}_0B}(\mathbf{L}\mathrm{CoS}^p_{\mathcal{W}_0B}((\mathbf{L}\mathcal{W}_1\Omega^1_{B/R})_{[-k]}),\mathcal{W}_0B)$$

given on generators by the identity on B and $df \mapsto [\varpi,f]$. Moreover, this map sends the de Rham differential d to the differential $[\varpi,-]$, so $\mathcal{W}_{-1}h\underline{\mathrm{Der}}_{P_k^{ac},R}(B)$ is then quasi-isomorphic to the truncated de Rham complex $(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0B/R})^{[k]} := (\mathrm{Tot}^{\Pi}\mathbf{L}\Omega^{\geqslant 1}_{\mathcal{W}_0B/R})^{[k]}$. In particular, equivalence classes of deformations of non-involutive deformations are parametrised by $\mathrm{H}^{k+1}(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0B/R})$ under these conditions.

Remark 1.20. Following Remark 1.14, if we take an involutively filtered dg operad $(\mathcal{P},W)$ with an involutive equivalence $\mathrm{gr}^W\mathcal{P}\simeq P_k^{ac}$, then the conditions of Theorem 1.18 also ensure that for B be a non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebra, the space of almost commutative anti-involutive derived $\mathcal{P}$-algebras (B',W) with $\mathrm{gr}^W B'\simeq B$ is contractible. In particular, when $k=1$ we can take $\mathcal{P}$ to be the Beilinson–Drinfeld operad, given by the PBW filtration on the associative operad, to see that there is an essentially unique filtered associative dg algebra (B',W) equipped with a filtered involution $(B')^{\mathrm{opp}}\cong B'$ and an equivalence $\mathrm{gr}^W B'\simeq B$. When B is an algebra of polyvectors, B' will thus be given by the ring of differential operators $\mathscr{D}(\omega^{\frac{1}{2}})$ on a square root $\omega^{\frac{1}{2}}$ of the dualising bundle whenever this exists, and gives a ring of twisted differential operators generalising $\mathscr{D}(\omega^{\frac{1}{2}})$ even when the dualising complex is not a line bundle, or has no square root.

As in Remark 1.19, we could also consider the space of almost commutative derived $\mathcal{P}$-algebras (B',W) with $\mathrm{gr}^W B'\simeq B$, and find this is governed by the abelian DGLA $(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0B/R})^{[k]}$ when ϖ is non-degenerate, so equivalence classes of non-involutive deformations are again a torsor for $\mathrm{H}^{k+1}(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0B/R})$. The same reasoning applies for non-involutively filtered dg operads $(\mathcal{P},W)$ with an involutive equivalence $\mathrm{gr}^W\mathcal{P}\simeq P_k^{ac}$, except that there is then a potentially non-zero obstruction class in $\mathrm{H}^{k+2}(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0B/R})$ to such algebras existing.

2. Quantisations on derived Deligne–Mumford stacks

2.1. **Hochschild complexes.** We now recall some constructions used in [Pri2]. We say that a complete filtered DGAA (A, F) is almost commutative if $\mathrm{gr}_F A$ is a CDGA. The following are abbreviated versions of [Pri2, Definitions 1.7 and 1.15]:

Definition 2.1. We write B for the bar construction from possibly non-unital DGAAs over R to ind-conilpotent differential graded associative coalgebras (DGACs) over R. Explicitly, this is given by taking the tensor coalgebra

$$\mathrm{B}A := T(A_{[-1]}) = \bigoplus_{i \geqslant 0} (A_{[-1]})^{\otimes_R i},$$

with chain differential given on cogenerators $A_{[-1]}$ by combining the chain differential and multiplication on A. Write B_+A for the subcomplex $T_+(A_{[-1]}) = \bigoplus_{i>0} A_{[-1]}^{\otimes_R i}$.

Let Ω_+ be the left adjoint to B_+.

Definition 2.2. For an almost commutative DGAA (A, F) over R and a filtered (A, F)-bimodule (M, F) in chain complexes for which the left and right $\mathrm{gr}^F A$-module structures on $\mathrm{gr}^F M$ agree, we define the filtered chain complex

$$\mathrm{CC}_{R,BD_1}(A, M)$$

to be the completion of the cohomological Hochschild complex $\mathrm{CC}_R(A, M)$ (rewritten as a chain complex) with respect to the filtration γF defined as follows. We may identify $\mathrm{CC}_R(A, M)$ with the subcomplex of

$$\underline{\mathrm{Hom}}_R(\mathrm{B}A, \mathrm{B}(A \oplus M_{[1]}))$$

consisting of coderivations extending the zero coderivation on $\mathrm{B}A$. The hypotheses on M ensure that $A \oplus M$ is almost commutative (regarding M as a square-zero ideal), so we have filtrations βF on $\mathrm{B}A$ and $\mathrm{B}(A \oplus M_{[1]})$, given by convolution with the Poincaré-Birkhoff Witt filtration β (see [Pri2, 1.8] for details). We then define $(\gamma F)_i$ to consist of coderivations sending $(\beta F)_j \mathrm{B}A$ to $(\beta F)_{i+j-1}\mathrm{B}(A \oplus M)$.

We also define the subcomplex $\mathrm{CC}_{R,BD_1,+}(A, M)$ to be the kernel of $\mathrm{CC}_{R,BD_1}(A, M) \to M$, or equivalently $\underline{\mathrm{Hom}}_R(\mathrm{B}_+A, M)^{\#}$. We write $\mathrm{CC}_{R,BD_1}(A) := \mathrm{CC}_{R,BD_1}(A, A)$.

Recall that a brace algebra B is a chain complex equipped with a cup product in the form of a chain map

$$B \otimes B \xrightarrow{\smile} B,$$

and braces in the form of maps

$$\{-\}\{-, \dots, -\}_r \colon B \otimes B^{\otimes r} \to B_{[r]}$$

satisfying the conditions of [Vor, §3.2] with respect to the differential. The commutator of the brace $\{-\}\{-\}_1$ is a Lie bracket, so for any brace algebra B, there is a natural DGLA structure on $B_{[-1]}$.

The following are taken from [Pri2, §1.2.1]:

Definition 2.3. Given a brace algebra B, define the opposite brace algebra B^{opp} to have the same elements as B, but multiplication $b^{\mathrm{opp}} \smile c^{\mathrm{opp}} := (-1)^{\deg b \deg c}(c \smile b)^{\mathrm{opp}}$ and brace operations given by the multiplication $(\mathrm{B}B^{\mathrm{opp}}) \otimes (\mathrm{B}B^{\mathrm{opp}}) \to \mathrm{B}B^{\mathrm{opp}}$ induced by the isomorphism $(\mathrm{B}B^{\mathrm{opp}}) \cong (\mathrm{B}B)^{\mathrm{opp}}$. Explicitly,

$$\{b^{\mathrm{opp}}\}\{c_1^{\mathrm{opp}}, \dots, c_m^{\mathrm{opp}}\} := \pm\{b\}\{c_m, \dots, c_1\}^{\mathrm{opp}},$$

where $\pm = (-1)^{m(m+1)/2+(\deg f-m)(\sum_i \deg c_i - m)+\sum_{i<j}\deg c_i \deg c_j}$.

Observe that when a filtered brace algebra B is almost commutative, then so is B^{opp}.

Definition 2.4. Define a filtration γ on the brace operad Br of [Vor] by putting the cup product in γ_0 and the braces $\{-\}\{-,\ldots,-\}_r$ in γ_{-r}.

Thus a (brace, γ)-algebra (A, F) in filtered complexes is a brace algebra for which the cup product respects the filtration, and the r-braces send F_i to F_{i-r}. We refer to (brace, γ)-algebras as almost commutative brace algebras.

We define an almost commutative anti-involutive brace algebra to be an almost commutative brace algebra (A, F) equipped with an involution $(A, F) \cong (A^{\mathrm{opp}}, F)$ of brace algebras which acts on $\mathrm{gr}^F_i A$ as multiplication by $(-1)^i$.

Lemma 2.5. *Given almost commutative DGAAs A, D over R, following* [Bra, §2.1] *there is an anti-involution*

$$-i\colon \mathrm{CC}_{R,BD_1}(A, D)^{\mathrm{opp}} \to \mathrm{CC}_{R,BD_1}(A^{\mathrm{opp}}, D^{\mathrm{opp}})$$

of DGAAs given by

$$i(f)(a_1,\ldots,a_m) = -(-1)^{\sum_{i<j}\deg a_i \deg a_j}(-1)^{m(m+1)/2} f(a_m^{\mathrm{opp}},\ldots,a_1^{\mathrm{opp}})^{\mathrm{opp}}.$$

When $A = D$ and A is commutative, the anti-involution $-i$ makes $(\mathrm{CC}_{R,BD_1}(A), \gamma F)$ *into an almost commutative anti-involutive brace algebra.*

2.2. **Involutions from the Grothendieck–Teichmüller group.** The good truncation filtration τ on the brace operad is contained in the filtration γ on Br from Definition 2.4, so the quasi-isomorphism in in [Vor] between the brace operad Br and the $\mathbb{Q}$-linear operad $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ of chains on the topological operad E_2 induces filtered quasi-isomorphisms

$$(\mathrm{C}_\bullet(E_2, \mathbb{Q}), \tau) \to (\mathrm{Br}, \tau) \to (\mathrm{Br}, \gamma).$$

As observed in [Pri4, Remark 2.22], the involution of the brace operad in Definition 2.3 corresponds under this quasi-isomorphism to an involution of the E_2 operad, which takes an embedding $[1, k] \times I^2 \to I^2$ of k little squares in a big square, and reverses the order of the labels $[1, k]$ with appropriate signs.

As summarised in the preamble to the theorem in [Pet], the space of homotopy automorphisms of the coloured operad given by rationalisation of E_2 is homotopy equivalent the Grothendieck–Teichmüller group $\mathrm{GT}(\mathbb{Q})$. Thus our involution comes from an element $t \in \mathrm{GT}(\mathbb{Q})$ which lies over $-1 \in \mathbb{G}_m(\mathbb{Q})$.

Definition 2.6. Denote the pro-unipotent radical of the pro-algebraic group GT by GT^1. Write $\mathrm{Levi}_{\mathrm{GT}}$ for the space of Levi decompositions $\mathrm{GT} \cong \mathbb{G}_m \ltimes \mathrm{GT}^1$, or equivalently of sections of the natural map $\mathrm{GT} \to \mathbb{G}_m$. We then define $\mathrm{Levi}^t_{\mathrm{GT}}$ to be the space of sections w of $\mathrm{GT} \to \mathbb{G}_m$ satisfying $w(-1) = t$.

Taking base change to arbitrary commutative $\mathbb{Q}$-algebras A gives sets $\mathrm{Levi}_{\mathrm{GT}}(A), \mathrm{Levi}^t_{\mathrm{GT}}(A)$ of decompositions over A.

Lemma 2.7. *The functor* $\mathrm{Levi}^t_{\mathrm{GT}}$ *is an affine scheme over* $\mathbb{Q}$ *equipped with the structure of a trivial torsor for the subgroup scheme* $(\mathrm{GT}^1)^t \subset \mathrm{GT}^1$ *given by the centraliser of t.*

Proof. We expand the argument from [Pri4, Remark 2.22]. By the general theory [HM] of pro-algebraic groups in characteristic 0, the set $\mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ is non-empty, and for all commutative $\mathbb{Q}$-algebras A, the group $\mathrm{GT}^1(A)$ acts transitively on $\mathrm{Levi}_{\mathrm{GT}}(A)$ via the adjoint action. Because the graded quotients of the lower central series of GT^1 have non-zero weight for the adjoint $\mathbb{G}_m$-action, the centralisers of this action are trivial and $\mathrm{Levi}_{\mathrm{GT}}(A)$ is a torsor for $\mathrm{GT}^1(A)$.

Now, choose any Levi decomposition $w_0 \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ and let $w_0(-1) = tu$ for $u \in \mathrm{GT}^1(\mathbb{Q})$. Since t and $w_0(-1)$ are both of order 2, we have $u = \mathrm{ad}_t(u^{-1})$. Writing $u = \exp(v)$ and $u^{\frac{1}{2}} := \exp(\frac{1}{2}v)$, we have $u^{\frac{1}{2}} = \mathrm{ad}_t(u^{-\frac{1}{2}})$, giving $w := \mathrm{ad}_{u^{-\frac{1}{2}}} \circ w_0 \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$. Thus $\mathrm{Levi}^t_{\mathrm{GT}}$ is non-empty, so $\mathrm{Levi}^t_{\mathrm{GT}} \subset \mathrm{Levi}_{\mathrm{GT}}$ is a torsor for the subgroup $(\mathrm{GT}^1)^t \subset \mathrm{GT}^1$ fixing t under the adjoint action. □

As explained succinctly in [Pet], formality of the operad $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ is a consequence of the observation that the Grothendieck–Teichmüller group is a pro-unipotent extension of $\mathbb{G}_m$. Since GT acts on the operad $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ of chains, any Levi decomposition $w \colon \mathbb{G}_m \to \mathrm{GT}$ gives a weight decomposition (i.e. a $\mathbb{G}_m$-action) of $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ which splits the good truncation filtration τ, so gives an equivalence between $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ and P_2 respecting the natural map from the Lie operad. Since this equivalence necessarily preserves the good truncation filtrations τ, it also gives an equivalence θ_w between $(P_2, \tau) = (\mathrm{H}_*(E_2, \mathbb{Q}), \tau)$ and $(\mathrm{C}_\bullet(E_2, \mathbb{Q}), \tau) \simeq (\mathrm{Br}, \gamma)$.

As explained in [Pri4, §2.2], $\mathrm{Levi}_{\mathrm{GT}}$ is naturally isomorphic to the space of Drinfeld 1-associators, with $\mathrm{Levi}^t_{\mathrm{GT}}$ corresponding to associators which are even.

Definition 2.8. Given a Levi decomposition $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$, we denote by p_w the ∞-functor from almost commutative brace algebras to almost commutative P_2-algebras coming from the equivalence $\theta_w \colon (P_2, \tau) \simeq (\mathrm{Br}, \gamma)$ induced by w. This preserves the underlying filtered L_∞-algebras up to equivalence.

The pro-unipotent radical $\mathrm{GT}^1 \subset \mathrm{GT}$ acts trivially on homology $\mathrm{H}_*(E_2, \mathbb{Q}) \cong P_2$, inducing a $\mathbb{G}_m$-action on P_2. Since the $\mathbb{G}_m$-action on $\mathrm{H}_1(E_2(2), \mathbb{Q})$ has weight 1, this gives a $\mathbb{G}_m$-equivariant isomorphism

$$\mathrm{H}_*(E_2, \mathbb{Q}) \cong P_2^{ac}$$

for the $\mathbb{G}_m$-equivariant operad $P_2^{ac} = \mathrm{Com} \circ s^{-1}\hbar\mathrm{Lie}$ of Definition 1.6.

The element $t \in \mathrm{GT}(\mathbb{Q})$ lies over $-1 \in \mathbb{G}_m(\mathbb{Q})$, so the involution t on E_2 induces the action of $-1 \in \mathbb{G}_m$ on P_2^{ac} under the isomorphism above.

Definition 2.9. When $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$, the equivalence $\theta_w \colon (P_2, \tau) \simeq (\mathrm{Br}, \gamma)$ commutes with the involution t, so it induces an equivalence between almost commutative anti-involutive brace algebras and almost commutative anti-involutive P_2-algebras, which we also denote by p_w.

2.3. **Quantisations on derived Deligne–Mumford stacks.**

Theorem 2.10. *Given a morphism $R \to A$ of CDGAs with perfect cotangent complex $\mathbf{L}\Omega^1_{A/R}$ and $A_\#$ flat over $R_\#$, the filtered DGLA underlying the Hochschild complex*

$$\mathrm{CC}_{R,BD_1}(A)_{[-1]}$$

is quasi-isomorphic to the graded DGLA

$$\mathrm{Pol}(A/R, 0)_{[-1]} := \bigoplus_{p \geqslant 0} \mathbf{L}\Lambda^p_A(T_{A/R})_{[p-1]}$$

of derived polyvectors on A, where $T_{A/R} := \mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{L}\Omega^1_{A/R}, A)$ is the derived tangent space, and the Lie algebra structure is given by the Schouten–Nijenhuis bracket.

This quasi-isomorphism depends only on a choice of even 1*-associator* $w \in \mathrm{Levi}^t_{\mathrm{GT}}$*, and is functorial with respect to homotopy formally étale morphisms.*

Proof. Lemma 2.5 shows that $\mathrm{CC}_{R,BD_1}(A)$ is an almost commutative anti-involutive brace algebra, and the Poincaré–Birkhoff—Witt isomorphism gives $\mathrm{gr}^{\gamma}\mathrm{CC}_{R,BD_1}(A) \simeq \mathrm{Pol}(A/R,0) =: P$. Applying the ∞-functor p_w of Definition 2.9 for some $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$ (or even a point in the space $\mathrm{Levi}^t_{\mathrm{GT}}(R)$) gives an almost commutative anti-involutive P_2-algebra $p_w\mathrm{CC}_{R,BD_1}(A)$ with associated graded P, as a $\mathbb{G}_m$-equivariant P_2^{ac}-algebra.

The $\mathbb{G}_m$-equivariant P_2^{ac}-algebra P over R is non-negatively weighted, and is freely generated over A, by the A-module $(T_{A/R})_{[1]}$, which has weight 1. Thus $\mathbf{L}\Omega^1_{P/A} \simeq P \otimes^{\mathbf{L}}_A (T_{A/R})_{[1]}$ satisfies the conditions of Theorem 1.18, giving an essentially unique equivalence

$$\alpha_{w,A} \colon\ p_w\mathrm{CC}_{R,BD_1}(A) \simeq \mathrm{Pol}(A/R,0)$$

of almost commutative anti-involutive P_2-algebras. In particular, there exists a zigzag of filtered quasi-isomorphisms between the underlying DGLAs.

Finally, if D denotes the [1]-diagram $(A \xrightarrow{f} B)$ with f homotopy formally étale and surjective, then as in [Pri4, §3.1], we have an almost commutative anti-involutive brace algebra $\mathrm{CC}_{R,BD_1}(D)$ with restriction maps

$$\mathrm{CC}_{R,BD_1}(A) \xleftarrow{\sim} \mathrm{CC}_{R,BD_1}(D) \to \mathrm{CC}_{R,BD_1}(B),$$

the left-hand map being a filtered quasi-isomorphism, and the associated graded of the right-hand map being formally étale. In the ∞-category of almost commutative brace algebras, we thus have a map $\phi_f \colon \mathrm{CC}_{R,BD_1}(A) \to \mathrm{CC}_{R,BD_1}(B)$ inducing a formally étale map $\mathrm{gr}^{\gamma}\phi_f \colon \mathrm{Pol}(A/R,0) \to \mathrm{Pol}(B/R,0)$ of $\mathbb{G}_m$-equivariant P_2^{ac}-algebras on the associated gradeds. Theorem 1.18 thus provides an essentially unique homotopy $\alpha_{w,B} \circ \phi_f \circ \alpha_{w,A}^{-1} \simeq \mathrm{gr}_{\gamma}\phi_f$, giving functoriality. □

Remark 2.11. When applied to polynomial rings, the statement of Theorem 1.18 recovers [Kon1, Theorem 4], and for more general smooth algebraic varieties it recovers [VdB, Theorem 1.1]. The preliminary steps are the same, but the arguments for eliminating the potential first-order deformation are very different, as we consider anti-involutive deformations while Tamarkin and Kontsevich looked at invariance under affine transformations.

Corollary 2.12. *Given a derived DM n-stack $\mathfrak{X}$ over R with perfect cotangent complex $\mathbf{L}\Omega^1_{\mathfrak{X}/R}$, the space $Q\mathcal{P}(\mathfrak{X},0)$ of E_1 quantisations of $\mathfrak{X}$ from* [Pri4, Definitions 1.23, 3.9] *is equivalent to the Maurer–Cartan space*

$$\underline{\mathrm{MC}}(\mathbf{R}\Gamma(\mathfrak{X}, (\hbar(\mathscr{O}_{\mathfrak{X}})_{[-1]} \times \hbar T_{\mathfrak{X}/R} \times \prod_{p\geqslant 2} \mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^{p-1})[\![\hbar]\!])).$$

In particular, there exists a quantisation for every Poisson structure

$$\pi \in \mathcal{P}(\mathfrak{X},0) = \underline{\mathrm{MC}}(\mathbf{R}\Gamma(\mathfrak{X}, \prod_{p\geqslant 2} \mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^{p-1})),$$

in the form of a curved A_∞-deformation of $\mathscr{O}_{\mathfrak{X}}$.

Proof. The space $Q\mathcal{P}(\mathfrak{X},0)$ of quantisations is defined to be $\underline{\mathrm{MC}}(\prod_{j\geqslant 2} \mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, \gamma_j\mathrm{CC}_R(\mathscr{O}_{\mathfrak{X}})_{[-1]}\hbar^{j-1})$, and for each $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$, Theorem 2.10 allows us to substitute for γ_j to give an equivalence between this and the space above. Existence of quantisations for a Poisson structure π then follows by observing that extension by zero gives a DGLA morphism

$$\prod_{p\geqslant 2} \mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^{p-1} \to (\hbar(\mathscr{O}_{\mathfrak{X}})_{[-1]} \times \hbar T_{\mathfrak{X}/R} \times \prod_{p\geqslant 2} \mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^{p-1})[\![\hbar]\!].$$

□

Remark 2.13. The hypotheses of Corollary 2.12 are satisfied by any derived Deligne–Mumford stack locally of finite presentation over the CDGA R. When $R = \mathrm{H}_0 R$, this includes underived schemes X which are local complete intersections over R, in which case the cotangent complex $\mathbf{L}\Omega^1_{X/R}$ is concentrated in homological degrees $[0, 1]$. For such underived schemes, a quantisation in the sense of the corollary reduces to the usual notion, namely an associative deformation of $\mathscr{O}_X$ over $R[\![\hbar]\!]$.

Our hypothesis that $\mathfrak{X}$ have perfect cotangent complex cannot be completely removed, since Mathieu's example [Mat] gives an non-quantisable Poisson structure on a non-LCI scheme.

Remark 2.14. Replacing Hochschild complexes with complexes of polydifferential operators allows these results to generalise to derived differential geometry. Complexes of smooth polyvectors will be freely generated as a CDGA over the ring of smooth functions by the shifted smooth tangent complex. This will satisfy the conditions of Theorem 2.10 even though the smooth tangent complex is much smaller than the tangent complex of the ring of derived $\mathcal{C}^\infty$ functions as an abstract CDGA. See [Pri5] for some more details about this approach.

3. Quantisations on derived Artin stacks

3.1. **Double complexes and stacky Hochschild complexes.** When applied to (derived) Artin stacks, the definition of Poisson structures in [Pri3, §3] or [CPT+] and their quantisations in [Pri4, Definitions 1.23, 3.9] is more subtle, since polyvectors and the Hochschild complex are not functorial with respect to smooth morphisms.

This is resolved in [Pri3, §3.1] by observing that the formal completion of a derived Artin stack $\mathfrak{X}$ along an affine atlas $f\colon U \to \mathfrak{X}$ with f smooth can be recovered from a stacky CDGA:

Definition 3.1. A stacky CDGA is a chain cochain complex

$$A^{\bullet}_{\bullet} = (A^0_{\bullet} \xrightarrow{\partial} A^1_{\bullet} \xrightarrow{\partial} A^2_{\bullet} \xrightarrow{\partial} \ldots).$$

equipped with a commutative product $A \otimes A \to A$ and unit $\mathbb{Q} \to A$. Given a chain CDGA R, a stacky CDGA over R is then a morphism $R \to A$ of stacky CDGAs.

As explained in [Pri3, Remark 3.32], these correspond to the "graded mixed cdgas" of [CPT+] (but beware that the latter do not have mixed differentials).

For general derived Artin n-stacks, these formal completions are constructed in [Pri3, §3.1] by forming affine hypercovers as in [Pri1], and then applying the functor D^* (left adjoint to denormalisation) to obtain a stacky CDGA. When $\mathfrak{X}$ is a derived Artin stack whose diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is affine, or even just represented by derived DM stacks, the formal completion of an affine atlas $U \to \mathfrak{X}$ is simply given by the relative de Rham complex

$$O(U) \xrightarrow{\partial} \Omega^1_{U/\mathfrak{X}} \xrightarrow{\partial} \Omega^2_{U/\mathfrak{X}} \xrightarrow{\partial} \ldots,$$

which arises by applying the functor D^* to the Čech nerve of U over $\mathfrak{X}$.

Lemma 3.2. *There is a cofibrantly generated model structure on cochain chain complexes in which fibrations are surjections and weak equivalences are levelwise quasi-isomorphisms in the chain direction. For any chain operad $\mathcal{P}$, this induces a cofibrantly generated model structure on $\mathcal{P}$-algebras in cochain chain complexes, in which fibrations and weak equivalences are those of the underlying cochain chain complexes.*

Proof. This follows as in the proof of [Pri3, Lemma 3.4] $\square$

Hochschild complexes and multiderivations on stacky CDGAs are then defined in terms of the semi-infinite total complex $\hat{\mathrm{Tot}}\, V \subset \mathrm{Tot}^{\Pi} V$ given by

$$(\hat{\mathrm{Tot}}\, V)^m := (\bigoplus_{i<0} V^i_{i-m}) \oplus (\prod_{i\geqslant 0} V^i_{i-m})$$

with differential $\partial \pm \delta$. This functor has the properties that it maps levelwise quasi-isomorphisms in the chain direction to quasi-isomorphisms, and that it is lax monoidal. In particular, this construction is applied to the internal Hom functor $\mathcal{H}om$, to give complexes

$$\hat{\underline{\mathrm{Hom}}}_R(M,N) := \hat{\mathrm{Tot}}\, \mathcal{H}om_R(M,N)$$

for cochain complexes M, N of R-modules in chain complexes, and hence a dg enhancement of the monoidal category of R-cochain chain complexes.

Definition 3.3. For a stacky DGAA A over R and an A-bimodule M in chain cochain complexes, we define the internal cohomological Hochschild complex $\mathcal{CC}_R(A, M)$ as in [Pri2, Definition 2.7], replacing $\underline{\mathrm{Hom}}$ with $\mathcal{H}om$ in Definition 2.2 to give a chain cochain complex.

Definition 3.4. For A a cofibrant stacky CDGA, define the stacky P^{ac}_{n+2}-algebra $\mathcal{P}ol(A/R, n)$ by

$$\mathcal{P}ol(A/R, n) := \bigoplus_{p\geqslant 0} \mathcal{H}om_A(\mathrm{CoS}^p_A((\Omega^1_{A/R})_{[-n-1]}), A),$$

where p is the weight for the $\mathbb{G}_m$-action, and the commutative multiplication and Lie bracket are defined in the usual way for polyvectors.

In particular, note that $\mathcal{P}ol(A/R, 0) = \bigoplus_{p\geqslant 0} \mathcal{H}om_A(\Omega^p_{A/R}, A)_{[p]}$.

Definition 3.5. For a stacky DGAA A, define the strict dg category $DGdg\mathrm{Mod}(A)_{\mathrm{str}}$ of A-modules in cochain chain complexes by setting the complex of morphisms from M to N to be $\mathrm{Z}^0\mathcal{H}om_A(M,N)$. This has a monoidal structure given by bigraded tensor products $\otimes_A$.

Define the Tate dg category $DGdg\mathrm{Mod}(A)_{\mathrm{Tate}}$ of A-modules in cochain chain complexes by setting the complex of morphisms from M to N to be $\hat{\underline{\mathrm{Hom}}}_A(M,N)$. This also has a monoidal structure given by bigraded tensor products $\otimes_A$.

In other words, morphisms in the strict dg category are required to strictly respect the cochain structure, and all shifts are in the chain direction.

By [Pri4, Lemma 1.14], based on [Vor, §3], $\mathcal{CC}_R(A, A)$ is equipped with a brace algebra structure in $DGdg\mathrm{Mod}(R)_{\mathrm{str}}$ (a "stacky brace algebra"). When A is commutative, Lemma 2.5 moreover adapts to make $(\mathcal{CC}_{R,BD_1}(A), \gamma F)$ into a stacky almost commutative anti-involutive brace algebra.

For $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$, Definition 2.9 gives an equivalence p_w between stacky almost commutative anti-involutive brace algebras and stacky almost commutative anti-involutive P_2-algebras, by considering the respective algebras in the strict dg category

$DGdg\mathrm{Mod}(R)_{\mathrm{str}}$ of cochain complexes of R-modules in chain complexes. However, the proof of Theorem 2.10 does not immediately adapt to this setting, because functoriality for stacky Hochschild complexes and stacky polyvectors is much more subtle.

We now consider conditions for a morphism $f\colon P \to Q$ of A-modules to be a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence, i.e. for the homology class $[f] \in \mathrm{H}_0\underline{\hat{\mathrm{Hom}}}_A(P,Q)$ to have an inverse in $\mathrm{H}_0\underline{\hat{\mathrm{Hom}}}_A(Q,P)$:

Lemma 3.6. *If A is a stacky CDGA concentrated in non-negative cochain degrees, and P and Q are cofibrant A-modules, with the chain complexes $(P \otimes_A A^0)^i, (Q \otimes_A A^0)^i$ zero for all $i < r$ and acyclic for all $i > s$, then a morphism $f\colon P \to Q$ is a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence whenever the map $\mathrm{Tot}\,\sigma^{\leqslant s}(P \otimes_A A^0) \to \mathrm{Tot}\,\sigma^{\leqslant s}(Q \otimes_A A^0)$ is a quasi-isomorphism, for the brutal truncation σ. Under this condition, then morphism $f^*\colon \mathcal{H}om_A(Q,A) \to \mathcal{H}om_A(P,A)$ is also a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence.*

Proof. For any A-module M in cochain chain complexes, observe that $\mathcal{H}om_A(P,M)$ admits a filtration with graded pieces $\mathcal{H}om_A(P,M^i)^{[-i]} = \mathcal{H}om_{A^0}(P \otimes_A A^0, M^i)^{[-i]}$. For M bounded below in cochain degrees ($\geqslant t$, say), the boundedness conditions on $P \otimes_A A^0$ then ensure that $\mathcal{H}om_A(P, \sigma^{\leqslant n}M)$ is concentrated in cochain degrees $\leqslant n-r$, and acyclic in degrees below $t-s$, where σ denotes brutal truncation in the cochain direction.

In particular, the boundedness hypotheses on $P \otimes_A A^0$ give us a natural map

$$\sigma^{\leqslant n-r}\mathcal{H}om_A(P,M) \to \sigma^{\leqslant n-r}\mathcal{H}om_A(P,\sigma^{\leqslant n}M) \cong \mathcal{H}om_A(P,\sigma^{\leqslant n}M),$$

while acyclicity of $\mathcal{H}om_A(P,\sigma^{>n}M)$ in degrees $\leqslant n-s$ gives us a levelwise quasi-isomorphism

$$\sigma^{\leqslant n-s}\mathcal{H}om_A(P,M) \xrightarrow{\sim} \sigma^{\leqslant n-s}\mathcal{H}om_A(P,\sigma^{\leqslant n}M),$$

the latter admitting a natural map from $\mathcal{H}om_A(P,\sigma^{\leqslant n}M)$.

Thus the maps

$$\{\mathcal{H}om_A(P,\sigma^{\leqslant n}M)\}_n \to \{\sigma^{\leqslant m}\mathcal{H}om_A(P,\sigma^{\leqslant n}M)\}_{m,n} \leftarrow \{\sigma^{\leqslant n}\mathcal{H}om_A(P,M)\}_n$$

of inverse systems are levelwise quasi-isomorphisms of pro-objects, in the sense that they induce isomorphisms of pro-objects [Gro] on taking homology groups H_i levelwise. Since the maps $\sigma^{\leqslant n} \to \sigma^{\leqslant n-1}$ are all surjective and

$$\mathcal{H}om(P,M) = \varprojlim_n \sigma^{\leqslant n}\mathcal{H}om_A(P,M),$$

passing to homotopy limits then gives us a levelwise quasi-isomorphism $\mathcal{H}om(P,M) \simeq \varprojlim_n \mathcal{H}om_A(P,\sigma^{\leqslant n}M)$.

Moreover, the boundedness conditions above give $\mathcal{H}om_A(P,\sigma^{\leqslant n}M) \simeq \sigma^{\geqslant t-s}\mathcal{H}om(P,\sigma^{\leqslant n}M)$, so passing to inverse limits and applying $\hat{\mathrm{Tot}}$ gives

$$\underline{\hat{\mathrm{Hom}}}(P,M) \simeq \mathrm{Tot}^{\Pi}\sigma^{\geqslant t-s}\mathcal{H}om(P,M).$$

Now, the quasi-isomorphism $\mathrm{Tot}\,\sigma^{\leqslant s}(P \otimes_A A^0) \to \mathrm{Tot}\,\sigma^{\leqslant s}(Q \otimes_A A^0)$ gives quasi-isomorphisms $\mathrm{Tot}^{\Pi}\sigma^{\geqslant -s}\mathcal{H}om_A(Q,M^i) \to \mathrm{Tot}^{\Pi}\sigma^{\geqslant -s}\mathcal{H}om_A(P,M^i)$ for all i. Using the filtrations and homotopy limits described above, this gives a quasi-isomorphism

$$\mathrm{Tot}^{\Pi}\sigma^{\geqslant t-s}\mathcal{H}om_A(Q,M) \to \mathrm{Tot}^{\Pi}\sigma^{\geqslant t-s}\mathcal{H}om_A(P,M).$$

For all M bounded below, we therefore have quasi-isomorphisms

$$f^*\colon \underline{\hat{\mathrm{Hom}}}_A(Q,M) \to \underline{\hat{\mathrm{Hom}}}_A(P,M).$$

Applying this to $M = P$, which is bounded below, gives a class $[g] \in \mathrm{H}_0\underline{\hat{\mathrm{Hom}}}_A(Q, P)$ with $f^*[g] = [\mathrm{id}]$. Thus $[g]$ is inverse to $[f] \in \mathrm{H}_0\underline{\hat{\mathrm{Hom}}}_A(P, Q)$, so f is indeed a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence.

Finally, observe that the contravariant functor $\mathcal{H}om_A(-, A)$ on A-modules is a $\mathcal{H}om_A$-enriched functor, giving natural maps

$$\mathcal{H}om_A(M, N) \to \mathcal{H}om_A(\mathcal{H}om_A(N, A), \mathcal{H}om_A(M, A))$$

for all A-modules M, N in chain cochain complexes, compatible with composition. Applying $\hat{\mathrm{Tot}}$ then gives maps

$$\underline{\hat{\mathrm{Hom}}}_A(M, N) \to \underline{\hat{\mathrm{Hom}}}_A(\mathcal{H}om_A(N, A), \mathcal{H}om_A(M, A))$$

compatible with composition, so $[f]$ and $[g]$ give rise to mutually inverse elements of

$$\mathrm{H}_0\underline{\hat{\mathrm{Hom}}}_A(\mathcal{H}om_A(Q, A), \mathcal{H}om_A(P, A)) \quad \text{and} \quad \mathrm{H}_0\underline{\hat{\mathrm{Hom}}}_A(\mathcal{H}om_A(P, A), \mathcal{H}om_A(Q, A)).$$

□

If D denotes a $[1]$-diagram $(A \xrightarrow{f} B)$ of cofibrant stacky CDGAs with f surjective, then adapting [Pri4, §3.1], gives a stacky almost commutative anti-involutive brace algebra $\mathcal{CC}_{R,BD_1}(D)$ with restriction maps

$$\mathcal{CC}_{R,BD_1}(A) \leftarrow \mathcal{CC}_{R,BD_1}(D) \to \mathcal{CC}_{R,BD_1}(B).$$

Similarly, setting

$$\mathcal{P}ol(A/R, B, n) := \bigoplus_{p \geqslant 0} \mathcal{H}om_A(\mathrm{CoS}^p_A((\Omega^1_{A/R})_{[-n-1]}), B)$$

and

$$\mathcal{P}ol(D/R, n) := \mathcal{P}ol(A/R, n) \times_{\mathcal{P}ol(A/R,B,n)} \mathcal{P}ol(B/R, n)$$

gives a stacky P^{ac}_{n+2}-algebra with restriction maps

$$\mathcal{P}ol(A/R, n) \leftarrow \mathcal{P}ol(D/R, n) \to \mathcal{P}ol(B/R, n).$$

However, when f is homotopy formally étale in the appropriate stacky sense, the left-hand maps need not be levelwise filtered quasi-isomorphisms, although they do become quasi-isomorphisms on applying $\hat{\mathrm{Tot}}$, as shown in [Pri4, §3.1]. Crucially for us, slightly more is true, as we will see in Lemma 3.8 below.

Definition 3.7. Given an $[m]$-diagram A of stacky CDGAs, and A-modules P, Q cochain chain complexes, we define $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_A(P, Q)$ to be any of the quasi-isomorphic complexes $\underline{\hat{\mathrm{Hom}}}_A(\tilde{P}, \bar{Q})$ given by replacing P and Q with cofibrant and fibrant replacements $\tilde{P} \to P$ and $Q \to \bar{Q}$ in the model structure of Lemma 3.2. We say that a map $f\colon P \to Q$ is an $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence if the induced homology class $[f] \in \mathrm{H}_0\mathbf{R}\underline{\hat{\mathrm{Hom}}}_A(P, Q)$ has an inverse in $\mathrm{H}_0\mathbf{R}\underline{\hat{\mathrm{Hom}}}_A(Q, P)$.

Lemma 3.8. *Take a morphism $D = (A \xrightarrow{f} B)$ of cofibrant stacky CDGAs concentrated in non-negative cochain degrees, such that*

1. *there exists $s \geqslant 0$ for which the chain complexes $(\Omega^1_{A/R} \otimes_A A^0)^i$ and $(\Omega^1_{B/R} \otimes_B B^0)^i$ are acyclic for all $i > s$, and*
2. *f is homotopy formally étale in the sense that the map*

$$\mathrm{Tot}\,\sigma^{\leqslant s}(\Omega^1_A \otimes_A B^0) \to \mathrm{Tot}\,\sigma^{\leqslant s}(\Omega^1_B \otimes_B B^0)$$

is a pro-quasi-isomorphism.

Then for the filtration γ, the natural morphisms

$$\mathrm{gr}_i^{\gamma}\mathcal{CC}_{R,BD_1}(D) \to \mathrm{gr}_i^{\gamma}\mathcal{CC}_{R,BD_1}(A), \text{ and}$$
$$\mathcal{P}ol(D/R,n) \to \mathcal{P}ol(A/R,n)$$

are all $\mathbf{R}\hat{\underline{\mathrm{Hom}}}_A$-homotopy equivalences.

Proof. The Poincaré–Birkhoff—Witt isomorphism gives levelwise quasi-isomorphisms $\mathrm{gr}^{\gamma}\mathcal{CC}_{R,BD_1}(A) \simeq \mathcal{P}ol(A/R,0)$ and $\mathrm{gr}^{\gamma}\mathcal{CC}_{R,BD_1}(D) \simeq \mathcal{P}ol(D/R,0)$, so the question reduces to the statements about $\mathcal{P}ol$.

Since $f\colon A \to B$ is surjective and taking symmetric invariants is an exact functor, the question thus reduces to showing that the maps

$$\mathcal{H}om_B((\Omega^1_{B/R})^{\otimes_B p}, B) \to \mathcal{H}om_A((\Omega^1_{A/R})^{\otimes_A p}, B) \cong \mathcal{H}om_B((\Omega^1_{A/R})^{\otimes_A p} \otimes_A B, B)$$

are $\mathbf{R}\hat{\underline{\mathrm{Hom}}}_A$-homotopy equivalences.

The boundedness hypotheses ensure that the chain complexes $((\Omega^1_{A/R})^{\otimes p} \otimes_A A^0)^i$ and $((\Omega^1_{B/R})^{\otimes p} \otimes_B B^0)^i$ are acyclic for $i > sp$. Combined with the homotopy formally étale hypothesis, this ensures that the maps $(\Omega^1_{A/R})^{\otimes p} \otimes_A B \to (\Omega^1_{B/R})^{\otimes p}$ satisfy the conditions of Lemma 3.6, so are $\hat{\underline{\mathrm{Hom}}}_B$-homotopy equivalences, and hence $\mathbf{R}\hat{\underline{\mathrm{Hom}}}_A$-homotopy equivalences *a fortiori*. □

This motivates the next section, where we adapt [Pri3, §3] to develop a notion of almost commutative shifted Poisson structures on a stacky CDGA B, weaker than homotopy P_2-algebra structures on B with respect to the strict dg category structure on cochain chain complexes, but stronger than homotopy P_2-algebra structures on $\hat{\mathrm{Tot}}\, B$.

3.2. Almost commutative Poisson structures on stacky algebras.

3.2.1. *Polydifferential operators.* Recall the definition of stacky differential operators from [Pri6, §3.2]:

Definition 3.9. Given a stacky R-CDGA A, and A-modules M, N in chain cochain complexes, inductively define the filtered chain cochain complex $\mathcal{D}\!\mathit{iff}(M,N) = \mathcal{D}\!\mathit{iff}_{A/R}(M,N) \subset \mathcal{H}om_R(M,N)$ of differential operators from M to N by setting

1. $F_0\mathcal{D}\!\mathit{iff}(M,N) = \mathcal{H}om_A(M,N)$,
2. $F_{k+1}\mathcal{D}\!\mathit{iff}(M,N) = \{u \in \mathcal{H}om_R(M,N) \ : \ [a,u] \in F_k\mathcal{D}\!\mathit{iff}(M,N)\, \forall a \in A\}$, where $[b,u] = bu - (-1)^{\deg b \deg u}ub$.
3. $\mathcal{D}\!\mathit{iff}(M,N) = \varinjlim_k F_k\mathcal{D}\!\mathit{iff}(M,N)$.

We simply write $\mathcal{D}\!\mathit{iff}_{A/R}(M) := \mathcal{D}\!\mathit{iff}_{A/R}(M,M)$.

We then define the filtered cochain complex $\hat{\mathrm{Diff}}(M,N) = \hat{\mathrm{Diff}}_{A/R}(M,N) \subset \hat{\underline{\mathrm{Hom}}}_R(M,N)$ by $\hat{\mathrm{Diff}}(M,N) := \varinjlim_k \hat{\mathrm{Tot}}\, F_k\mathcal{D}\!\mathit{iff}(M,N)$.

Given a small category I, an I-diagram A of stacky R-CDGAs, and A-modules M, N in I-diagrams of cochain chain complexes, we follow [Pri3, §3.4] in writing $\hat{\mathrm{Diff}}_{A/R,I}(M,N)$ for the filtered complex given by the equaliser of the obvious diagram

$$\prod_{i\in I} \hat{\mathrm{Diff}}_{A(i)/R}(M(i),N(i)) \implies \prod_{f\colon\, i\to j \text{ in } I} \hat{\mathrm{Diff}}_{A(i)/R}(M(i), f_*N(j)).$$

For any small category I, this permits the following definitions.

Definition 3.10. Given an I-diagram A of stacky R-CDGAs, we define the coloured dg operad $\mathcal{DIFF}_{A/R}$ to have objects (or colours) consisting of A-modules in I-diagrams of cochain chain complexes, and multi-operations

$$\mathcal{DIFF}_{A/R}(M_1, \ldots, M_n; N) := \hat{\mathrm{Diff}}_{A \otimes R^n}(M_1 \otimes_R M_2 \otimes_R \ldots \otimes_R M_n, N),$$

with compositions and symmetries induced by the inclusion $\hat{\mathrm{Diff}} \subset \underline{\hat{\mathrm{Hom}}}$. Let $\mathcal{DIFF}_{A/R}(M)$ be the dg suboperad on the single object M, so

$$\mathcal{DIFF}_{A/R}(M)(n) = \mathcal{DIFF}_{A/R}(\overbrace{M, \ldots, M}^{n}; M).$$

Definition 3.11. Given an I-diagram A of stacky CDGAs over R, and a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra P in $\mathcal{DIFF}_{A/R}$ with $\mathcal{W}_0 P = A$, define the $\mathbb{G}_m$-equivariant complex $h\underline{\hat{\mathrm{Der}}}_{P_k^{ac},R,I}(P) = \bigoplus_i \mathcal{W}_i h\underline{\hat{\mathrm{Der}}}_{P_k^{ac},R,I}(P)$ by

$$h\underline{\hat{\mathrm{Der}}}_{P_k^{ac},R,I}(P) := h\underline{\mathrm{Der}}_{P_k^{ac},\mathcal{DIFF}_{A/R}}(P),$$

for the complex $h\underline{\mathrm{Der}}$ of graded deformations from Definition 1.10.

The condition $\mathcal{W}_0 P = A$ in Definition 3.11 is not really necessary, but it is satisfied by all the examples we encounter, and removes the need to keep track of A in the notation $h\underline{\hat{\mathrm{Der}}}_{P_k^{ac},R,I}(P)$.

3.2.2. *Reduction to polyvectors.* We now consider an $[m]$-diagram A of stacky R-CDGAs which is fibrant and cofibrant in the injective model structure of Lemma 3.2 (i.e. each $A(i)$ is cofibrant as a stacky R-CDGA and the maps $A(i) \to A(i+1)$ are all surjective). We also take a $\mathbb{G}_m$-equivariant A-module P which is fibrant and cofibrant for the injective model structure on I-diagrams in the model structure of Lemma 3.2 (i.e. each $P(i)$ is cofibrant as a cochain chain P-module and the maps $P(i) \to P(i+1)$ are all surjective).

Definition 3.12. Given a $\mathbb{G}_m$-equivariant commutative algebra C (in the sense of Definition 1.9) in the Tate dg category $DGdg\mathrm{Mod}(A)_{\mathrm{Tate}}$ of A-modules from Definition 3.5, define the $\mathbb{G}_m$-equivariant C-module $h\hat{\Omega}^1_{C/A}$ in $\mathrm{ind}(\mathrm{Rep}(\mathbb{G}_m, DGdg\mathrm{Mod}(A)_{\mathrm{Tate}}))$ to represent the functor

$$h\underline{\mathrm{Der}}_{\mathrm{Comm},DGdg\mathrm{Mod}(A)_{\mathrm{Tate}}}(C, -)$$

from Definition 1.10. We make analogous definitions for $\mathbb{G}_m^2$-equivariant algebras, replacing $\mathbb{G}_m$ with $\mathbb{G}_m^2$ in Definition 1.10.

Explicitly, we can write the ind-object $h\hat{\Omega}^1_{C/A}$ as a direct system $\{\mathrm{Fil}_r h\hat{\Omega}^1_{C/A}\}_r$ in the category of C-modules, with

$$\mathrm{Fil}_r h\hat{\Omega}^1_{C/A} := (\bigoplus_{n=1}^{r} (C \otimes_R \mathrm{Lie}(n)^\vee) \otimes_A^{\Sigma_n} ((C_{[-1]})^{\otimes_A n})_{[1]}, \delta + \mu),$$

where the C-module structure is given by the leftmost copy of C, and the maps $\mu\colon (C \otimes_R \mathrm{Lie}(n)^\vee) \otimes_A^{\Sigma_n} (C_{[-1]}^{\otimes_A n}) \to (C \otimes_R \mathrm{Lie}(n-1)^\vee) \otimes_A^{\Sigma_{n-1}} (C_{[-1]}^{\otimes_A n-1})_{[-1]}$ are defined in terms of the commutative multiplication on C. Substitution in Definition 1.10 then automatically gives the required isomorphisms

$$\mathcal{W}_i h\underline{\mathrm{Der}}_{\mathrm{Comm},DGdg\mathrm{Mod}(A)_{\mathrm{Tate}}}(C, M) \cong \varprojlim_r \mathcal{W}_i \underline{\hat{\mathrm{Hom}}}_{C,I}(\mathrm{Fil}_r h\hat{\Omega}^1_{C/A}, M).$$

Lemma 3.13. *For a fibrant cofibrant $[m]$-diagram A of stacky R-CDGAs and a fibrant cofibrant $\mathbb{G}_m$-equivariant A-module P as above, assume that we are given a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra structure on P as an object of the dg operad $\mathcal{DIFF}_{A/R}$. Then for the order filtration F on polydifferential operators, and for the $\mathbb{G}_m^2$-equivariant commutative A-algebra $C = \bigoplus_{i,j} \mathcal{W}_{i,j}C$ given by $\mathcal{W}_{i,j}C := \mathcal{W}_i P \otimes_A \mathrm{Symm}^j_A \Omega^1_{A/R}$, there are canonical zigzags of quasi-isomorphisms*

$$\mathrm{gr}^F_j \mathcal{W}_i h\underline{\hat{\mathrm{Der}}}_{P_k^{ac},R,[m]}(P) \simeq \prod_{p\geqslant 1} \mathbf{R}\mathcal{W}_{i+1-p,-j}\underline{\hat{\mathrm{Hom}}}_{P,[m]}(\mathrm{CoS}^p_C((h\hat{\Omega}^1_{C/A})_{[-k]}), P).$$

Proof. Because A is cofibrant and fibrant, the order filtration gives isomorphisms

$$\begin{aligned}\mathrm{gr}^F_j \mathcal{W}_i \hat{\mathrm{Diff}}_{A^{\otimes_R n}}(P^{\otimes_R n}, P) &\cong \mathcal{W}_i \underline{\hat{\mathrm{Hom}}}_{A^{\otimes_R n}}(P^{\otimes_R n} \otimes_{A^{\otimes_R n}} \mathrm{Symm}^j_{A^{\otimes_R n}} \Omega^1_{A^{\otimes_R n}/R}, P)\\ &\cong \mathcal{W}_i \underline{\hat{\mathrm{Hom}}}_A(P^{\otimes_A n} \otimes_A \mathrm{Symm}^j_A((\Omega^1_{A/R})^{\oplus n}), P).\end{aligned}$$

We can rewrite this as

$$\mathrm{gr}^F_j \mathcal{W}_i \mathcal{DIFF}_{A/R}(P)(n) \cong \mathcal{W}_{i,-j}\underline{\hat{\mathrm{Hom}}}_A(C^{\otimes_A n}, P),$$

and this is compatible with the respective operad structures under composition.

Because the Poisson structure on P is a bidifferential operator of order 2, its commutator reduces order by 1, so it does not affect the differential on the associated graded $\mathrm{gr}^F h\underline{\hat{\mathrm{Der}}}_{P_k^{ac},R,[m]}(P)$. The isomorphism above thus gives

$$\mathrm{gr}^F h\underline{\hat{\mathrm{Der}}}_{P_k^{ac},R,[m]}(P) \cong h\underline{\mathrm{Der}}_{P_k^{ac},\bigoplus_j \mathcal{W}_{0,j}\underline{\hat{\mathrm{Hom}}}_A,[m]}(C, P),$$

where the stacky CDGA C is given trivial Poisson bracket to regard it as a stacky $\mathbb{G}_m$-equivariant P_k^{ac}-algebra with an additional $\mathbb{G}_m$-action. Substituting Definition 3.12 into Definition 1.10 then gives the desired expression. □

The following lemma is the technical key to this section, and the reason we have to use differential operators instead of $\underline{\hat{\mathrm{Hom}}}_R$ in Definition 3.11 is to ensure that Corollary 3.15 holds. The proof does not adapt to $\underline{\hat{\mathrm{Hom}}}_R$ because infinite direct sums are not coproducts in the Tate dg category.

Lemma 3.14. *For a fibrant cofibrant $[m]$-diagram A of stacky R-CDGAs and fibrant cofibrant $\mathbb{G}_m^2$-equivariant A-module C, assume that $\mathcal{W}_{i,j}C = 0$ for $i < 0$ or $j < 0$, that we are given a commutative algebra structure on C as a $\mathbb{G}_m^2$-equivariant object of the dg operad $\mathcal{DIFF}_{A/R}$, and an isomorphism $\mathcal{W}_{0,0}C \cong A$ of commutative algebras in $\mathcal{DIFF}_{A/R}$.*

Then the ind-object $h\hat{\Omega}^1_{C/A}$ is isomorphic to a $\mathbb{G}_m^2$-equivariant C-module in the Tate dg category. Moreover, if the commutative algebra structure on C comes from a stacky CDGA structure, then $h\hat{\Omega}^1_{C/A}$ is a model for the cotangent complex $\mathbf{L}\Omega^1_{C/A}$ of C over A.

Proof. For fixed i, j, note that the modules $\mathrm{gr}^{\mathrm{Fil}}_n \mathcal{W}_{i,j} h\hat{\Omega}^1_{C/A} = \mathcal{W}_{i,j}(C \otimes_R \mathrm{Lie}(n)^\vee) \otimes^{\Sigma_n}_A (C^{\otimes_A n})$ are only non-zero for finitely many n. This essentially follows because $\mathcal{W}_{0,0}C = A$, so an element in $\mathcal{W}_{0,0}C$ cannot appear more than twice consecutively in a shifted Lie A-algebra construction, with non-negativity of the bi-weights of C then ensuring finiteness. Thus the direct systems $\{\mathrm{Fil}_r \mathcal{W}_{i,j} h\hat{\Omega}^1_{C/A}\}_r$ are all eventually constant, and the ind-object $h\hat{\Omega}^1_{C/A}$ is isomorphic to the $\mathbb{G}_m^2$-equivariant C-module L given by $\mathcal{W}_{i,j}L := \varinjlim_r \mathrm{Fil}_r \mathcal{W}_{i,j} h\hat{\Omega}^1_{C/A}$.

Finally, when C is a stacky CDGA, Koszul duality for the commutative and Lie operads [LV, Proposition 11.4.4] ensures that the module L is a model for the cotangent complex $\mathbf{L}\Omega^1_{C/A}$ of C over A. □

Corollary 3.15. *Under the hypotheses of Lemma 3.13, assume that P is fibrant and cofibrant as a stacky A-CDGA, that $\mathcal{W}_0P = A$ and that the weights of P are non-negative. Then there is a canonical $\mathbb{G}_m$-equivariant quasi-isomorphism*

$$\alpha\colon \bigoplus_i (\prod_{p\geqslant 1} \mathcal{W}_{i+1-p}\underline{\hat{\mathrm{Hom}}}_{P,[m]}(\mathrm{CoS}^p_P((\Omega^1_{P/R})_{[-k]}), P), \delta + [\varpi, -])_{[-k]} \to h\underline{\hat{\mathrm{Der}}}_{P^{ac}_k, R,[m]}(P)$$

of DGLAs, where ϖ denotes the bivector on P corresponding to the Poisson bracket, and the left-hand complex is given the Schouten–Nijenhuis bracket.

Proof. Once we note that every multiderivation is necessarily a polydifferential operator, we see that the inclusion of multiderivations in homotopy multiderivations gives us the injective DGLA morphism α.

The filtration $\alpha^{-1}F$ on $\underline{\hat{\mathrm{Hom}}}_P(\mathrm{CoS}^p_P((\Omega^1_{P/R})_{[-k]}), P)$ is then just the one induced by the short exact sequence

$$0 \to \Omega^1_{A/R} \otimes_A P \to \Omega^1_{P/R} \to \Omega^1_{P/A} \to 0,$$

so in particular $\alpha^{-1}F_0$ consists of A-linear multiderivations.

Substituting Lemma 3.13 into Lemma 3.14 gives canonical zigzags of quasi-isomorphisms

$$\mathrm{gr}^F_j \mathcal{W}_i h\underline{\hat{\mathrm{Der}}}_{P^{ac}_k, R,[m]}(P) \simeq \prod_{p\geqslant 1} \mathbf{R}\mathcal{W}_{i+1-p,-j}\underline{\hat{\mathrm{Hom}}}_{C,[m]}(\mathbf{L}\mathrm{CoS}^p_C(\mathbf{L}\Omega^1_{C/A})_{[-k]}), P),$$

and the canonical equivalence

$$\mathbf{L}\Omega^1_{C/A} \otimes^{\mathbf{L}}_C P \simeq \mathbf{L}\Omega^1_{P/A} \oplus \Omega^1_{A/R},$$

then gives quasi-isomorphisms between $\mathrm{gr}^F_j \mathcal{W}_i h\underline{\hat{\mathrm{Der}}}_{P^{ac}_k, R,[m]}(P)$ and

$$\prod_{p\geqslant 1} \mathbf{R}\mathcal{W}_{i+1-p}\underline{\hat{\mathrm{Hom}}}_{P,[m]}((\mathrm{CoS}^j_A(\Omega^1_A)_{[-k]}) \otimes^{\mathbf{L}}_A \mathbf{L}\mathrm{CoS}^{p-j}_P((\mathbf{L}\Omega^1_{P/A})_{[-k]}), P).$$

Thus the associated graded morphisms $\mathrm{gr}^F_j\alpha$ induced by α are quasi-isomorphisms, so α is a $\mathbb{G}_m$-equivariant quasi-isomorphism. □

3.2.3. *Almost commutative shifted Poisson structures.*

Definition 3.16. Take a fibrant cofibrant $[m]$-diagram A of stacky R-CDGAs and a fibrant cofibrant $\mathbb{G}_m$-equivariant A-module P as in Lemma 3.13, and assume that we are given a $\mathbb{G}_m$-equivariant P^{ac}_k-algebra structure on P as an object of the dg operad $\mathcal{DIFF}_{A/R}$. We then define the space $\mathcal{P}(P, n)^{ac}$ of R-linear almost commutative n-shifted Poisson structures on P by

$$\mathcal{P}(P,n)^{ac} := \varprojlim_r \underline{\mathrm{MC}}(\prod_{j=1}^r \mathcal{W}_{-j} h\underline{\hat{\mathrm{Der}}}_{P^{ac}_{n+1}, R,[m]}(P)),$$

and the space of R-linear almost commutative anti-involutive (or self-dual) n-shifted Poisson structures on P by

$$\mathcal{P}(P,n)^{ac,sd} := \varprojlim_r \underline{\mathrm{MC}}(\prod_{j=1}^r \mathcal{W}_{-2j} h\underline{\hat{\mathrm{Der}}}_{P^{ac}_{n+1}, R,[m]}(P)).$$

Remark 3.17. Since the Tate realisation $\hat{\mathrm{Tot}}$ is symmetric monoidal, it gives rise to a morphism $\hat{\mathrm{Tot}}: \mathcal{DIFF}_{A/R} \to dg\mathrm{Mod}(R)$ of dg operads, and hence to maps $\hat{\mathrm{Tot}}$ from $\mathcal{P}(P,n)^{ac}$ (resp. $\mathcal{P}(P,n)^{ac,sd}$) to the space of of almost commutative (resp. almost commutative anti-involutive) derived P_{n+1}-algebras over R with associated graded $\hat{\mathrm{Tot}}\, P$.

Moreover, for any $[m]$-diagram $P' = W_{\geqslant 0}P'$ of almost commutative anti-involutive stacky P_{n+1}-algebras over R with $A := W_0P'$ and $P := \mathrm{gr}^W P'$, there is an associated element $[\mathfrak{gr}P']$ of $\mathcal{P}(P,n)^{ac,sd}$ given by applying the functor $\mathfrak{gr}$ of Definition 1.3. This follows by observing that the Poisson bracket is necessarily a biderivation with respect to A, and hence a polydifferential operator (of order 2).

This pair of constructions is compatible in the sense that for the diagram P' of almost commutative anti-involutive stacky P_{n+1}-algebras, we have a natural equivalence

$$\hat{\mathrm{Tot}}\,[\mathfrak{gr}P'] \simeq [\mathfrak{gr}\hat{\mathrm{Tot}}\,P'] \in \mathrm{map}(P^{ac}_{n+1} \circ \mathbb{Q}[\eth], dg\mathrm{Mod}(R)) \times^h_{\mathrm{map}(P^{ac}_{n+1}, dg\mathrm{Mod}(R))} \{\hat{\mathrm{Tot}}\,P\}$$

in the relevant space of derived P_{n+1}-algebras.

Definition 3.18. Say that a morphism $P \to Q$ of stacky R-CDGAs is weakly formally étale if the natural map

$$\mathbf{L}\Omega^1_{P/R} \otimes^{\mathbf{L}}_P Q \to \mathbf{L}\Omega^1_{Q/R}$$

is an $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_Q$-homotopy equivalence.

Proposition 3.19. *Given a fibrant cofibrant $[1]$-diagram $f\colon P \to Q$ of $\mathbb{G}_m$-equivariant stacky P^{ac}_k-algebras, there are natural $\mathbb{G}_m$-equivariant maps*

$$h\underline{\hat{\mathrm{Der}}}_{P^{ac}_k,R}(P) \xleftarrow{\mathrm{ev}_1} h\underline{\hat{\mathrm{Der}}}_{P^{ac}_k,R,[1]}(P \to Q) \xrightarrow{\mathrm{ev}_2} h\underline{\hat{\mathrm{Der}}}_{P^{ac}_k,R}(Q).$$

(1) *If f is weakly formally étale, and P and Q are non-negatively weighted, then* ev_1 *is a quasi-isomorphism.*
(2) *If the map $P \to Q$ is a $\underline{\hat{\mathrm{Hom}}}_{\mathcal{W}_0P}$-homotopy equivalence with $\mathcal{W}_0P \cong \mathcal{W}_0Q$, then* ev_2 *is a quasi-isomorphism. If P and Q are also non-negatively weighted, then* ev_1 *is also a quasi-isomorphism.*

Proof. If we write D for the diagram $P \to Q$, then $\mathcal{W}_j\mathcal{DIFF}_{\mathcal{W}_0D/R}(D)(n)$ equals

$$\mathcal{W}_j\hat{\mathrm{Diff}}_{(\mathcal{W}_0P)^{\otimes_R n}}(P^{\otimes_R n}, P) \times_{\mathcal{W}_j\hat{\mathrm{Diff}}_{(\mathcal{W}_0P)^{\otimes_R n}}(P^{\otimes_R n}, Q)} \mathcal{W}_j\hat{\mathrm{Diff}}_{(\mathcal{W}_0Q)^{\otimes_R n}}(Q^{\otimes_R n}, Q),$$

giving natural morphisms

$$\mathcal{DIFF}_{\mathcal{W}_0P/R}(P) \leftarrow \mathcal{DIFF}_{\mathcal{W}_0D/R}(P)(D) \to \mathcal{DIFF}_{\mathcal{W}_0Q/R}(P)(Q)$$

of $\mathbb{G}_m$-equivariant dg operads inducing the maps $\mathrm{ev}_1, \mathrm{ev}_2$.

(1) For ev_1 to be a quasi-isomorphism, by Corollary 3.15 it suffices for the maps

$$\mathcal{W}_j\underline{\hat{\mathrm{Hom}}}_{Q,[m]}(\mathrm{CoS}^p_Q((\Omega^1_{Q/R})_{[-k]}), Q) \to \mathcal{W}_j\underline{\hat{\mathrm{Hom}}}_{P,[m]}(\mathrm{CoS}^p_P((\Omega^1_{P/R})_{[-k]}), Q)$$

to be quasi-isomorphisms. Since $\underline{\hat{\mathrm{Hom}}}$-homotopy equivalences behave under tensor products and summands, it suffices for the maps $(\Omega^1_{P/R}) \otimes_P Q \to \Omega^1_{Q/R}$ to be $\underline{\hat{\mathrm{Hom}}}_Q$-homotopy equivalences, but this is just the statement that f is weakly formally étale.

(2) On the other hand, if $\mathcal{W}_0P = \mathcal{W}_0Q$ and $P \to Q$ is a $\underline{\hat{\mathrm{Hom}}}_A$-homotopy equivalence for $A = \mathcal{W}_0P$, then the maps

$$\mathcal{W}_j\hat{\mathrm{Diff}}_{A^{\otimes_R n}}(P^{\otimes_R n}, P) \to \mathcal{W}_j\hat{\mathrm{Diff}}_{A^{\otimes_R n}}(P^{\otimes_R n}, Q)$$

are all quasi-isomorphisms, inducing the quasi-isomorphism ev_2 on substituting into Definition 1.10.

Under these conditions, we also automatically have a $\hat{\underline{\mathrm{Hom}}}_Q$-homotopy equivalence $h\hat{\Omega}^1_P \otimes_P Q \to h\hat{\Omega}^1_Q$, for $h\hat{\Omega}^1$ as in Definition 3.12. When P and Q are non-negatively weighted, Lemma 3.14 shows that $h\hat{\Omega}^1$ is a model for the cotangent complex, and then the previous part applies to show that ev_1 is a quasi-isomorphism. □

Proceeding as in [Pri3, §3.4.2], we may use Proposition 3.19 to establish functoriality for almost commutative n-shifted Poisson structures with respect to weakly formally étale morphisms by sending any such morphism $f\colon P \to Q$ to the span

$$\mathcal{P}(P,n)^{ac} \xleftarrow{\sim} \mathcal{P}(f\colon P \to Q, n)^{ac} \to \mathcal{P}(Q,n)^{ac},$$

regarded as a morphism in the simplicial localisation at weak equivalences of the category of simplicial sets. The same reasoning also applies to $\mathcal{P}(-,n)^{ac,sd}$.

In more detail, our analogue of [Pri3, Properties 2.5] is to combine Proposition 3.19 with the observation that for any fibrant cofibrant $[m]$-diagram $(P(0) \to \ldots \to P(m))$, the natural map from $h\hat{\underline{\mathrm{Der}}}_{P^{ac}_k,R,[m]}(P(0) \to \ldots \to P(m))$ to the homotopy fibre product

$$h\hat{\underline{\mathrm{Der}}}_{P^{ac}_k,R,[1]}(P(0) \to P(1)) \times^h_{h\hat{\underline{\mathrm{Der}}}_{P^{ac}_k,R}(P(1))} h\hat{\underline{\mathrm{Der}}}_{P^{ac}_k,R,[m-1]}(P(1) \to \ldots \to P(m))$$

is a $\mathbb{G}_m$-equivariant quasi-isomorphism, which follows immediately on unwinding the definitions.

Definition 3.20. Define $\mathbf{L}\mathrm{st}P^{ac,+}_k\mathrm{Alg}(R)$ to be the ∞-category (i.e. $(\infty,1)$-category) given by localising the category of non-negatively weighted $\mathbb{G}_m$-equivariant stacky P^{ac}_k-algebras at those morphisms $f\colon P \to Q$ for which the map $\mathcal{W}_0 f\colon \mathcal{W}_0 P \to \mathcal{W}_0 Q$ is a levelwise quasi-isomorphism, and each map $\mathcal{W}_i f\colon \mathcal{W}_i P \to \mathcal{W}_i Q$ is an $\mathbf{R}\hat{\underline{\mathrm{Hom}}}_{\mathcal{W}_0 P}$-homotopy equivalence.

We then define $\mathbf{L}\mathrm{st}P^{ac,+,\text{ét}}_k\mathrm{Alg}_R$ to be the 2-sub-∞-category on weakly étale morphisms.

If we write $\mathbf{L}s\mathrm{Set}$ for the ∞-category given by localising the category of simplicial sets at weak equivalences, then as in [Pri3, Definition 2.6 and §3.4.2], a Grothendieck construction gives us ∞-functors

$$\mathbf{L}\mathcal{P}(-,k-1)^{ac}, \mathbf{L}\mathcal{P}(-,k-1)^{ac,sd}\colon \mathbf{L}\mathrm{st}P^{ac,+,\text{ét}}_k\mathrm{Alg}_R \to \mathbf{L}s\mathrm{Set}$$

with $\mathbf{L}\mathcal{P}(P,k-1)^{ac} \simeq \mathcal{P}(P,k-1)^{ac}$ and $\mathbf{L}\mathcal{P}(-,k-1)^{ac,sd} \simeq \mathcal{P}(P,k-1)^{ac,sd}$ for cofibrant objects P.

Definition 3.21. Define $\mathbf{L}DG^{+,\flat}dg\mathrm{CAlg}_R$ to be the ∞-category given by taking the category of stacky R-CDGAs A concentrated in non-negative cochain degrees with $(\mathbf{L}\Omega^1_{A/R} \otimes_A A^0)^i$ acyclic for $i \gg 0$, and localising at levelwise quasi-isomorphisms.

Write $\mathbf{L}DG^{+,\flat}dg\mathrm{CAlg}^{\text{ét}}_R$ for the 2-sub-∞-category on morphisms which are homotopy formally étale (cf. Lemma 3.8).

The following is an immediate consequence of Lemma 3.8 and Remark 3.17:

Lemma 3.22. *There is a natural ∞-functor*

$$\mathbf{L}\mathcal{P}ol(-/R,n)\colon \mathbf{L}DG^{+,\flat}dg\mathrm{CAlg}^{\text{ét}}_R \to \mathbf{L}\mathrm{st}P^{ac,+,\text{ét}}_{n+2}\mathrm{Alg}_R$$

with $\mathbf{L}\mathcal{P}ol(A/R,n) \simeq \mathcal{P}ol(A/R,n)$ *for* A *cofibrant. For* $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$ *and* p_w *as in Definition 2.9, the Hochschild complex gives rise to a global section* $[p_w\mathcal{CC}_{R,BD_1}]$ *of the* ∞*-functor*

$$\mathbf{L}\mathcal{P}(-,1)^{ac,sd} \circ \mathbf{L}\mathcal{P}ol(-/R,0)\colon\ \mathbf{L}DG^{+,\flat}dg\mathrm{CAlg}^{\text{ét}}_R \to \mathbf{L}s\mathrm{Set}.$$

In other words, $p_w\mathcal{CC}$ gives an ∞-functor from a suitable ∞-category of homotopy formally étale morphisms of stacky CDGAs to the Grothendieck construction $\int \mathbf{L}\mathcal{P}(-,1)^{ac,sd}$, which we can think of as an ∞-category of almost commutative self-dual 1-shifted Poisson structures and weakly étale morphisms.

3.2.4. *Uniqueness of deformations.*

Theorem 3.23. *Let* P *be a non-negatively weighted* $\mathbb{G}_m$*-equivariant stacky* P^{ac}_{n+1}*-algebra over a CDGA* R*, cofibrant as a stacky* R*-CDGA. If the map*

$$(\mathcal{W}_1\Omega^1_{P/\mathcal{W}_0P}) \otimes_{\mathcal{W}_0P} P \to \Omega^1_{P/\mathcal{W}_0P}$$

is an $\mathbf{R}\hat{\underline{\mathrm{Hom}}}_P$*-homotopy equivalence, then the space* $\mathcal{P}(P,n)^{ac,sd}$ *of* R*-linear almost commutative anti-involutive* n*-shifted Poisson structures on* P *is contractible. For any* $P' \in \mathcal{P}(P,n)^{ac,sd}$*, we thus have a filtered quasi-isomorphism* $P' \simeq P$ *of anti-involutive filtered* P_{n+1}*-algebras in* $\mathcal{DIFF}_{\mathcal{W}_0P/R}$*, unique up to coherent homotopy.*

Moreover, for any weakly formally étale morphism $f\colon P \to Q$ *of such* $\mathbb{G}_m$*-equivariant stacky* P^{ac}_{n+1}*-algebras, the space* $\mathcal{P}(f\colon P \to Q,n)^{ac,sd}$ *of* R*-linear almost commutative anti-involutive* n*-shifted Poisson structures on the diagram* $f\colon P \to Q$ *is also contractible.*

Proof. We adapt the proof of Theorem 1.18. By Corollary 1.13, it suffices to show that

$$\mathrm{H}_i\mathcal{W}_{-2j}h\hat{\underline{\mathrm{Der}}}_{P^{ac}_k,R}(P) = 0 \quad \text{and} \quad \mathrm{H}_i\mathcal{W}_{-2j}h\hat{\underline{\mathrm{Der}}}_{P^{ac}_k,R,[1]}(P \to Q) = 0$$

for all $j \geqslant 1$ and $i \geqslant -2$. If we substitute the condition on $(\mathcal{W}_1\mathbf{L}\Omega^1_{P/\mathcal{W}_0P}) \otimes_{\mathcal{W}_0P} P \to \mathbf{L}\Omega^1_{P/\mathcal{W}_0P}$ into Corollary 3.15, then the argument of Proposition 1.16 adapts verbatim to give $\mathcal{W}_ih\hat{\underline{\mathrm{Der}}}_{P^{ac}_k,R}(P) \simeq 0$ for $i \leqslant -2$, establishing the first vanishing condition. Proposition 3.19 then shows that the restriction map

$$h\hat{\underline{\mathrm{Der}}}_{P^{ac}_k,R,[1]}(P \to Q) \to h\underline{\mathrm{Der}}_{P^{ac}_k,R}(P)$$

is a $\mathbb{G}_m$-equivariant quasi-isomorphism, establishing the second vanishing condition. □

Applying Theorem 3.23 to the construction $\mathbf{L}\mathcal{P}(-,n)^{ac,sd}$ has the following immediate consequence:

Corollary 3.24. *When restricted to the full subcategory of* $\mathbf{L}\mathrm{st}P^{ac,+,\text{ét}}_k\mathrm{Alg}_R$ *on non-negatively weighted objects* P *with*

$$(\mathcal{W}_1\mathbf{L}\Omega^1_{P/\mathcal{W}_0P}) \otimes_{\mathcal{W}_0P} P \to \mathbf{L}\Omega^1_{P/\mathcal{W}_0P}$$

an $\mathbf{R}\hat{\underline{\mathrm{Hom}}}_P$*-homotopy equivalence, the* ∞*-functor* $\mathbf{L}\mathcal{P}(-,n)^{ac,sd}$ *is naturally equivalent to the constant functor* $P \mapsto *$.

3.3. **Quantisations on derived Artin stacks.** We are now in a position to generalise Theorem 2.10 to stacky CDGAs, and hence Corollary 2.12 to derived Artin n-stacks.

Theorem 3.25. *Given a cofibration $R \to A$ of stacky CDGAs concentrated in non-negative cochain degrees, with cotangent complex $(\Omega^1_{A/R})^{\#}$ perfect as an $A^{\#}$-module, the filtered DGLA underlying the Hochschild complex*

$$\mathrm{CC}_{R,\gamma}(A)_{[-1]} := \varinjlim_p (\hat{\mathrm{Tot}}\, \gamma_p \mathcal{CC}_{R,BD_1}(A))_{[-1]}$$

is filtered quasi-isomorphic to the graded DGLA

$$\mathrm{Pol}_R(A,0)_{[-1]} := \bigoplus_{p \geqslant 0} \underline{\hat{\mathrm{Hom}}}_A(\Omega^p_{A/R}, A)_{[p-1]}$$

of derived polyvectors on A, where the Lie algebra structure is given by the Schouten–Nijenhuis bracket.

This quasi-isomorphism depends only on a choice of even 1-associator $w \in \mathrm{Levi}^t_{\mathrm{GT}}$, and is natural with respect to homotopy formally étale morphisms, for the functoriality of Lemma 3.22.

Proof. We adapt the proof of Theorem 2.10. As explained in §3.1, Lemma 2.5 adapts to show that $\mathcal{CC}_{R,BD_1}(A)$ is an almost commutative anti-involutive stacky brace algebra, and the Poincaré–Birkhoff—Witt isomorphism gives $\mathrm{gr}^{\gamma}\mathcal{CC}_{R,BD_1}(A) \simeq \mathcal{P}ol(A/R,0)$, which we will now denote as P. Applying the ∞-functor p_w of Definition 2.9 for some $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$ (or even a point in the space $\mathrm{Levi}^t_{\mathrm{GT}}(R)$) gives an almost commutative anti-involutive stacky P_2-algebra $p_w\mathcal{CC}_{R,BD_1}(A)$ with associated graded P.

The $\mathbb{G}_m$-equivariant stacky P_2^{ac}-algebra $P = \bigoplus_{p \geqslant 0} \mathcal{H}om_A(\Omega^p_{A/R}, A)_{[p]}$ is non-negatively weighted, and since $(\Omega^1_A)^{\#}$ is perfect over $A^{\#}$, the natural maps

$$\Lambda^p_A \mathcal{H}om_A(\Omega^1_{A/R}, A) \to \mathcal{H}om_A(\Omega^p_{A/R}, A)$$

are levelwise quasi-isomorphisms, so the cotangent complex $\mathbf{L}\Omega^1_{P/A}$ is freely generated by the A-module $(\mathcal{H}om_A(\Omega^1_{A/R}, A))_{[1]}$, which has weight 1. Thus P satisfies the conditions of Theorem 3.23, giving an essentially unique filtered quasi-isomorphism

$$\alpha_{w,A} \colon\ p_w\mathcal{CC}_{R,BD_1}(A) \simeq P = \mathcal{P}ol(A/R,0)$$

of anti-involutive filtered P_2-algebras in $\mathcal{DIFF}_{A/R}$. As in Remark 3.17, applying $\hat{\mathrm{Tot}}$ then gives a filtered quasi-isomorphism

$$\alpha_{w,A} \colon\ p_w\mathrm{CC}_{R,\gamma}(A) \simeq \mathrm{Pol}_R(A,0)$$

of almost commutative anti-involutive P_2-algebras. In particular, there exists a zigzag of filtered quasi-isomorphisms between the underlying DGLAs.

Finally, Corollary 3.24 implies that when restricted to the full subcategory of $\mathbf{L}DG^{+,\flat}dg\mathrm{CAlg}^{\mathrm{\acute{e}t}}_R$ on objects with perfect cotangent complex, the ∞-functor $\mathbf{L}\mathcal{P}(-,1)^{ac,sd} \circ \mathbf{L}\mathcal{P}ol(-/R,0)$ is contractible. Thus its sections $[p_w\mathcal{CC}_{R,BD_1}]$ and $[\mathrm{gr}^{\gamma}\mathcal{CC}_{R,BD_1}] = [\mathcal{P}ol(-/R,0)]$ are canonically equivalent, via an essentially unique path α_w, giving the required naturality of the equivalence above. □

Corollary 3.26. *Given a strongly quasi-compact derived Artin n-stack $\mathfrak{X}$ over R with perfect cotangent complex $\mathbf{L}\Omega^1_{\mathfrak{X}/R}$, any even associator $w \in \mathrm{Levi}^t_{\mathrm{GT}}$ gives rise to a map*

$$\mathcal{P}(\mathfrak{X},0) \to Q\mathcal{P}(\mathfrak{X},0)$$

from the space of 0*-shifted Poisson structures on* $\mathfrak{X}$ *to the space of* E_1 *quantisations of* $\mathfrak{X}$ *in the sense of* [Pri4, Definitions 1.23, 3.9]*. These quantisations give rise to curved* A_∞ *deformations* $(\mathrm{per}_{dg}(\mathfrak{X})[\![\hbar]\!], \{m^{(i)}\}_{i\geqslant 0})$ *of the dg category* $\mathrm{per}_{dg}(\mathfrak{X})$ *of perfect* $\mathscr{O}_{\mathfrak{X}}$*-complexes.*

Proof. As in [Pri2, Remark 2.12], for cofibrant stacky CDGAs A there is a filtered brace algebra quasi-isomorphism between the filtered Hochschild complex $(\mathrm{CC}^\bullet_R(A), F)$ from [Pri4, Definition 1.16] and the filtered complex $(\mathrm{CC}_{R,\gamma}(A), \gamma)$ in the notation of Theorem 3.25. Thus when $A^\#$ has perfect cotangent complex, Theorem 3.25 gives a zigzag of filtered DGLA quasi-isomorphisms

$$(\mathrm{CC}^\bullet_R(A)_{[-1]}, F) \simeq \mathrm{Pol}_R(A,0)_{[-1]},$$

functorial with respect to homotopy formally étale morphisms, dependent on a choice of some $w \in \mathrm{Levi}^t_{\mathrm{GT}}$.

Now, the space $Q\mathcal{P}(A,0)$ of 0-shifted quantisations of A is defined in [Pri4, Definition 1.17] to be

$$\varprojlim_i \underline{\mathrm{MC}}(\prod_{2\leqslant j\leqslant i} F_j\mathrm{CC}^\bullet_R(A)_{[-1]}\hbar^{j-1});$$

via the quasi-isomorphism above, this is weakly equivalent to

$$\mathcal{P}_\hbar(A,0) := \underline{\mathrm{MC}}((\hbar(\hat{\mathrm{Tot}}\, A)_{[-1]} \times \hbar\underline{\hat{\mathrm{Hom}}}_A(\Omega^1_{A/R}, A) \times \prod_{p\geqslant 2}\underline{\hat{\mathrm{Hom}}}_A(\Omega^p_{A/R}, A)_{[p-1]}\hbar^{p-1})[\![\hbar]\!])).$$

Moreover, for the space

$$\mathcal{P}(A,0) = \underline{\mathrm{MC}}(\prod_{p\geqslant 2}\underline{\hat{\mathrm{Hom}}}_A(\Omega^p_{A/R}, A)_{[p-1]}\hbar^{p-1}),$$

of 0-shifted Poisson structures on A, there is an obvious natural map $\mathcal{P}(A,0) \to \mathcal{P}_\hbar(A,0)$, and hence the equivalence above gives a map $\mathcal{P}(A,0) \to Q\mathcal{P}(A,0)$ from Poisson structures on A to quantisations of A.

By construction, the map $\mathcal{P}(A,0) \to Q\mathcal{P}(A,0)$ is natural with respect to homotopy formally étale morphisms in A. The global result now follows immediately from the definitions in [Pri4]. In more detail, the spaces $Q\mathcal{P}(\mathfrak{X},0)$ in [Pri4] are defined by first taking an Artin hypergroupoid resolution $X_\bullet$ of $\mathfrak{X}$, then forming a cosimplicial stacky CDGA $j \mapsto D^*O(X^{\Delta^j})$ with homotopy formally étale morphisms (this can be thought of as giving a formally étale simplicial resolution of $\mathfrak{X}$ by derived Lie algebroids), and setting

$$\mathcal{P}(\mathfrak{X},0) := \mathrm{ho}\varprojlim_{j\in\Delta}\mathcal{P}(D^*O(X^{\Delta^j}),0), \quad Q\mathcal{P}(\mathfrak{X},0) := \mathrm{ho}\varprojlim_{j\in\Delta}Q\mathcal{P}(D^*O(X^{\Delta^j}),0)$$

When $\mathfrak{X}$ has perfect cotangent complex, the stacky CDGAs $D^*O(X^{\Delta^j})$ have perfect cotangent complexes, and the structure maps of the cosimplicial stacky CDGA $j \mapsto D^*O(X^{\Delta^j})$ are all homotopy formally étale, so the natural maps above give

$$\mathcal{P}(\mathfrak{X},0) \to Q\mathcal{P}(\mathfrak{X},0)$$

on passing to homotopy limits.

Finally, by [Pri4, Proposition 3.11], E_1 quantisations of $\mathfrak{X}$ give rise to curved A_∞ deformations of $\mathrm{per}_{dg}(\mathfrak{X})$. $\square$

Remark 3.27. The hypotheses of Corollary 2.12 are satisfied by any derived Artin stack locally of finite presentation over the CDGA R. When $R = \mathrm{H}_0 R$, this includes those underived Artin n-stacks $\mathfrak{X}$ which admit n-atlases by underived schemes which are local complete intersections over R, in which case the cotangent complex $\mathbf{L}\Omega^1_{\mathfrak{X}/R}$ is concentrated in homological degrees $[-n, 1]$.

Remark 3.28. The strongly quasi-compact hypothesis in Corollary 3.26 is not really necessary, and just reflects the generality in which Poisson structures and quantisations are defined in [Pri4]. The definitions and results can be extended to more general derived Artin n-stacks by passing to filtered colimits of étale morphisms of strongly quasi-compact derived stacks.

Remark 3.29. Replacing Hochschild complexes with complexes of polydifferential operators should allow the results of this section to generalise to derived differential geometry, in a similar fashion to Remark 2.14, to give quantisations for Lie n-groupoids and their derived and singular analogues. In this setting, note that we would be using differential operators both to construct the complex governing quantisations and to equate it with smooth polyvectors as in Theorem 3.25.

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