QUANTISATION OF DERIVED LAGRANGIANS

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Abstract. We investigate quantisations of line bundles $\mathscr{L}$ on derived Lagrangians X over 0-shifted symplectic derived Artin N-stacks Y. In our derived setting, a deformation quantisation consists of a curved A_∞ deformation of the structure sheaf $\mathscr{O}_Y$, equipped with a curved A_∞ morphism to the ring of differential operators on $\mathscr{L}$; for line bundles on smooth Lagrangian subvarieties of smooth symplectic algebraic varieties, this simplifies to deforming $(\mathscr{L}, \mathscr{O}_Y)$ to a DQ module over a DQ algebroid.

For each choice of formality isomorphism between the E_2 and P_2 operads, we construct a map from the space of non-degenerate quantisations to power series with coefficients in relative cohomology groups of the respective de Rham complexes. When $\mathscr{L}$ is a square root of the dualising line bundle, this leads to an equivalence between even power series and certain anti-involutive quantisations, ensuring that the deformation quantisations always exist for such line bundles. This gives rise to a likely candidate for the new type of Fukaya category, of algebraic Lagrangians, envisaged by Behrend and Fantechi. We also sketch a generalisation of these quantisation results to Lagrangians on higher n-shifted symplectic derived stacks.

Introduction

A major source of motivation for the study of shifted symplectic and Poisson structures in derived geometry is the desire to develop and understand quantisations. For $n > 0$, existence of quantisations of n-shifted Poisson structures is automatic, following from the formality equivalence $E_{n+1} \simeq P_{n+1}$ of operads. Quantisations of positively shifted symplectic structures thus follow immediately from the equivalence in [Pri4, CPT$^+$] between symplectic and non-degenerate Poisson structures. For lower values of n, quantisation is a much harder problem to tackle, or even formulate, but [Pri5, Pri7] established the existence of quantisations for 0-shifted and (-1)-shifted symplectic structures on derived Artin N-stacks.

The purpose of this paper is simultaneously to generalise the results of [Pri5, Pri7] by formulating and studying quantisations of Lagrangian morphisms $(X, \lambda) \to (Y, \omega)$ over 0-shifted symplectic derived stacks (Y, ω), in the sense of [PTVV]; then [Pri5] corresponds to the case where X is empty, and [Pri7] to the case where Y is a point, forcing the the Lagrangian structure on X to be (-1)-shifted symplectic.

Based on the principle that n-shifted quantisations broadly correspond to E_{n+1}-algebras, a deformation quantisation of an n-shifted Lagrangian structure on $(X \to Y)$ should roughly consist of an E_{n+1}-algebra deformation $\tilde{\mathscr{O}}_Y$ of the structure sheaf $\mathscr{O}_Y$, together with an $\mathscr{O}_Y$-module $\tilde{\mathscr{O}}_Y$ in E_n-algebras deforming the structure sheaf $\mathscr{O}_X$. In the 0-shifted setting, this would mean seeking an associative deformation $\tilde{\mathscr{O}}_Y$ of the structure sheaf $\mathscr{O}_Y$, together with an $\tilde{\mathscr{O}}_Y$-module deformation of the structure sheaf $\mathscr{O}_X$.

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However, the case $n = 0$ which we are considering is where the pattern starts to degenerate, and even for Deligne–Mumford stacks the situation is a little more complicated. We look to deform line bundles $\mathscr{L}$ instead of $\mathscr{O}_X$, the deformation must be given by differential operators, and everything in sight has to incorporate curvature. This last condition is inescapable in the derived setting, because "Lagrangian" is now a structure, not just a property; considering the extreme case of (-1)-shifted symplectic structures as in [Pri7], it is a curvature term (in the form of a BV operator) which quantises the Poisson structure. For quantisations of (underived) non-singular varieties, this curvature manifests itself in the form of DQ algebroids and DQ modules. For our precise formulations of quantised co-isotropic structures, see Definition 2.13, Remark 2.14, Definition 4.7 and Remark 4.9.

Our main result is Theorem 4.20, which implies that deformation quantisations exist for any 0-shifted Lagrangian morphism $X \to Y$ of derived Artin stacks, and any line bundle $\mathscr{L}$ which is a square root the dualising complex K_X. The quantisations thus constructed have a self-duality property, generalising the property $b \star_\hbar a = a \star_{-\hbar} b$ often satisfied by star products. In fact, Theorem 4.20 gives a complete classification of self-dual quantisations of a given Lagrangian structure, parametrising them in terms of de Rham cohomology as a torsor for the group $\hbar^2\mathrm{H}^1(\mathrm{cone}(\mathrm{DR}(Y) \to \mathrm{DR}(X)))[\![\hbar^2]\!]$, — when working over $\mathbb{C}$, this is just the relative cohomology group

$$\hbar^2\mathrm{H}^1(Y(\mathbb{C}), X(\mathbb{C}); \mathbb{C}[\![\hbar^2]\!])$$

of the associated topological spaces $X(\mathbb{C}), Y(\mathbb{C})$ (or to be precise, simplicial spaces when X, Y are stacks), with coefficients in $\mathbb{C}[\![\hbar^2]\!]$.

When Y is a smooth variety, and X a smooth Lagrangian subvariety, this theorem and Proposition 4.12 recover the classification in [BGKP] of quantisations of the pair (Y, X), as explained in Remark 4.13, but our derived Lagrangians X can also be derived enhancements of singular schemes or stacks even when Y is a smooth variety.

Much study of derived quantisation questions has been motivated by the desire to associate a dg category to 0-shifted symplectic derived schemes (Y, ω) in algebraic and complex analytic settings with similar properties to the Fukaya category, as outlined in [BF, Joy, BBD$^+$]. The driving philosophy is that the derived intersection (or rather, homotopy fibre product) of 0-shifted Lagrangians is a (-1)-shifted symplectic derived stack, so carries a perverse sheaf of vanishing cycles, and that there should be a dg category whose Hom-complexes are given by appropriate shifts of derived global sections of these sheaves. The motivation is explained in some detail in [BBD$^+$, Remark 6.15], with vanishing cycles resembling an analogue of Lagrangian Floer cohomology. However, there are serious difficulties in trying to upgrade these complexes to a dg category. Our quantisation results allow us to solve this problem in §5 by attacking it from the opposite direction. Fixing a suitable quantisation $\tilde{\mathscr{O}}_Y$ of (Y, ω), Theorem 4.20 guarantees that compatible quantisations $(\tilde{\mathscr{O}}_Y, \tilde{\mathscr{L}})$ exist for oriented Lagrangians $(X, \mathscr{L})$ over Y, leading to a natural dg category of the associated $\tilde{\mathscr{O}}_Y$-modules in Definition 5.11. We then show in Corollary 5.20 that the Hom-complexes indeed come from vanishing cycles after inverting $\hbar$.

Our approach to proving Theorem 4.20 will be familiar from [Pri4, Pri7, Pri5]. For each quantisation Δ, we define a map μ from generalised Lagrangian structures, defined in terms of power series in de Rham cohomology $\mathrm{cone}(\mathrm{DR}(Y) \to \mathrm{DR}(X))$, to a quantised form of relative Poisson cohomology, giving a filtered quasi-isomorphism when Δ is non-degenerate. To each non-degenerate quantised co-isotropic structure Δ, there is an associated element $\hbar^2\frac{\partial\Delta}{\partial\hbar}$, and hence a power series $\mu^{-1}(\hbar^2\frac{\partial\Delta}{\partial\hbar})$ whose constant term is a Lagrangian structure. Obstruction calculus shows that this induces an equivalence between self-dual quantisations and even power series.

Our main new technical ingredient in this paper is in defining the map μ, where we consider the natural morphism $\mathrm{CC}^\bullet(\mathscr{O}_Y) \to \mathrm{CC}^\bullet(\mathscr{D}_{X/Y}(\mathscr{L}))$ of E_2-algebras induced by the action of $\mathrm{CC}^\bullet(\mathscr{O}_Y)$ on the ring of differential $\mathscr{D}_{X/Y}(\mathscr{L})$. Via formality, we may regard these E_2-algebras as P_2-algebras, and then each quantisation defines a commutative diagram

$$\begin{array}{ccc} \mathrm{DR}(Y) & \longrightarrow & \mathrm{DR}(X) \\ \downarrow & & \downarrow \\ \mathrm{CC}^\bullet(\mathscr{O}_Y) \to \mathrm{CC}^\bullet(\mathscr{D}_{X/Y}(\mathscr{L})) & & \end{array}$$

from the de Rham complexes, with the left-hand side recovering the compatibility map from [Pri5]. The morphism $\mu(-,\Delta)$ is then given by composing with the natural map $\mathrm{CC}^\bullet(\mathscr{D}_{X/Y}(\mathscr{L})) \to \mathscr{D}_{X/Y}(\mathscr{L})$ and taking cones to give a map from de Rham cohomology to a form of quantised relative Poisson cohomology.

The structure of the paper is as follows.

In Section 1, we establish some technical background results on Hochschild complexes. Under the well-known principle (see for instance [GM, Kon1, Man, Pri1]) that deformation problems in characteristic 0 are governed by differential graded Lie algebras (DGLAs), the DGLAs in [Pri5, Pri7] governing 0-shifted and (-1)-shifted quantisations were constructed from Hochschild complexes and rings of differential operators, respectively. Our perspective for quantisations of 0-shifted co-isotropic structures on a morphism $X \to Y$ is that the governing DGLA comes from Hochschild complex $\mathrm{CC}^\bullet(\mathscr{O}_Y)$ acting on $\mathscr{D}_{X/Y}(\mathscr{L})$ via the quasi-isomorphism $\mathscr{D}_{X/Y} \to \mathrm{CC}^\bullet(\mathscr{O}_Y, \mathscr{D}_X)$.

When equipped with a PBW filtration degenerating to Poisson cohomology, we first show that Hochschild complexes of almost commutative algebras become almost commutative brace algebras in a suitable sense (§1.2.1). This allows us to construct suitable semidirect products of Hochschild complexes from morphisms of almost commutative algebras in §1.2.2. Section 2 then applies these constructions to Hochschild complexes acting on rings of differential operators, allowing us to construct a form of quantised relative Poisson cohomology (Definition 2.16), leading to a space $Q\mathcal{P}(A, B; 0)$ of quantisations associated to a morphism $A \to B$ of commutative bidifferential bigraded algebras (i.e. a map $\operatorname{Spec} B \to \operatorname{Spec} A$ of stacky derived affines in the sense of [Pri4]), and more generally a space $Q\mathcal{P}(A, M; 0)$ for each line bundle M over B (Definition 2.13).

Section 3 contains the key technical construction (Definition 3.7) of the compatibility map μ between generalised Lagrangians and quantised co-isotropic structures. The main results of this section are Proposition 3.16, giving a map from non-degenerate quantisations to generalised Lagrangians, and Proposition 3.17, which gives an equivalence between Lagrangians and non-degenerate co-isotropic structures. Proposition 3.18 then shows that the obstruction to quantising a co-isotropic structure is first order.

In Section 4, these constructions are globalised via the method introduced in [Pri4]. §4.3 then introduces the notion of self-duality, enabling us to eliminate the first order obstruction and thus lead to Theorem 4.20, the main comparison result. In §4.4, we then explain how the methods and results of the paper should adapt to Lagrangians on positively shifted symplectic stacks.

Section 5 describes algebraic (and complex analytic) analogues of the Fukaya category based on self-dual quantisations of line bundles on derived Lagrangians, and establishes a few key properties. The main definition is given in Definition 5.11 and the relation with vanishing cycles in Proposition 5.13 and Corollary 5.20, with Proposition 5.14 establishing functoriality with respect to Lagrangian correspondences. Many of these structural results rely on additivity properties established in §5.1 investigating the interaction of quantisation with intersection and Hom, which may be of independent interest.

Contents

Notation and terminology. Throughout the paper, we will usually denote chain differentials by δ. The graded vector space underlying a chain (resp. cochain) complex V is denoted by $V_{\#}$ (resp. $V^{\#}$). Since we often have to work with chain and cochain structures separately, we denote shifts as subscripts and superscripts, respectively, so $(V_{[i]})_n := V_{i+n}$ and $(V^{[i]})^n := V^{i+n}$.

Given an associative algebra A in chain complexes, and A-modules M, N in chain complexes, we write $\underline{\mathrm{Hom}}_A(M,N)$ for the cochain complex given by

$$\underline{\mathrm{Hom}}_A(M,N)^i = \mathrm{Hom}_{A_{\#}}(M_{\#[i]}, N_{\#}),$$

with differential $f \mapsto \delta_N \circ f \pm f \circ \delta_M$.

We refer to associative algebras in chain complexes as DGAAs (i.e. differential graded associative algebra), and commutative algebras in chain complexes as CDGAs (i.e. commutative differential graded algebras); these are assumed unital unless stated otherwise. We will also refer to coassociative coalgebras in chain complexes over a CDGA R as DGACs over R; these are co-unital unless stated otherwise. From Section 2 onwards, we will be working with double complexes $V^\bullet_\bullet$ combining both chain and cochain gradings, where the chain and cochain differentials are denoted by δ and ∂ respectively. We refer to unital commutative (resp. associative) algebras in double complexes as stacky CDGAs (resp. stacky DGAAs), regarding the cochain differential ∂ as stacky structure and the chain differential δ as derived structure.

Definition 0.1. Given a chain cochain complex V, define the cochain complex $\hat{\mathrm{Tot}}\, V \subset \mathrm{Tot}^{\Pi} V$ as a subset of the product total complex by

$$(\hat{\mathrm{Tot}}\, V)^m := (\bigoplus_{i<0} V^i_{i-m}) \oplus (\prod_{i\geq 0} V^i_{i-m})$$

with differential $\partial \pm \delta$. This is sometimes referred to as the Tate realisation.

Definition 0.2. Given a stacky DGAA A and A-modules M, N in chain cochain complexes, we define internal Homs $\mathcal{H}om_A(M,N)$ by

$$\mathcal{H}om_A(M,N)^i_j = \mathrm{Hom}_{A^\#_\#}(M^\#_\#, N^{\#[i]}_{\#[j]}),$$

with differentials $\partial f := \partial_N \circ f \pm f \circ \partial_M$ and $\delta f := \delta_N \circ f \pm f \circ \delta_M$, where $V^\#_\#$ denotes the bigraded vector space underlying a chain cochain complex V.

We then define the Hom complex $\underline{\hat{\mathrm{Hom}}}_A(M,N)$ by

$$\underline{\hat{\mathrm{Hom}}}_A(M,N) := \hat{\mathrm{Tot}}\, \mathcal{H}om_A(M,N).$$

Note that there is a multiplication $\underline{\hat{\mathrm{Hom}}}_A(M,N) \otimes \underline{\hat{\mathrm{Hom}}}_A(N,P) \to \underline{\hat{\mathrm{Hom}}}_A(M,P)$; beware that the same is not true for the product total complexes $\mathrm{Tot}^{\Pi}\mathcal{H}om_A(M,N)$ in general.

When we need to compare chain and cochain complexes, we make use of the equivalence u from chain complexes to cochain complexes given by $(uV)^i := V_{-i}$, and refer to this as rewriting the chain complex as a cochain complex (or vice versa). On suspensions, this has the effect that $u(V_{[n]}) = (uV)^{[-n]}$.

We will denote symmetric and cosymmetric powers by $S^p_B(M) = \mathrm{Symm}^p_B(M) := (M^{\otimes_B p})_{\Sigma_p}$ and $\mathrm{CoS}^p_B(M) = \mathrm{CoSymm}^p_B(M) := (M^{\otimes_B p})^{\Sigma_p}$, respectively given by coinvariants and invariants of the symmetric group action. We also write $\mathrm{Symm}_B(M) = \bigoplus_{p\geq 0} S^p_B(M)$ and $\mathrm{CoSymm}_B(M) = \bigoplus_{p\geq 0} \mathrm{CoS}^p_B(M)$.

1. The centre of an almost commutative algebra

The purpose of this section is to show that the Hochschild complex of an almost commutative algebra is almost commutative as a brace algebra, and to study the resulting almost commutative brace algebra constructions. The primary motivation is to ensure that formality equivalences $E_2 \simeq P_2$ then turn these Hochschild complexes into filtered P_2-algebras (i.e. Gerstenhaber algebras) for which the Lie bracket has weight -1.

1.1. **Almost commutative algebras.**

1.1.1. *Homological algebra of complete filtrations.* We now introduce a formalism for working with complete filtered complexes. Although we make little explicit use of these characterisations in the rest of the paper, they implicitly feature in the reasoning for complete filtered functors to have given properties.

Definition 1.1. Given a vector space V with a decreasing filtration F, the Rees module $\xi(V,F)$ is given by $\xi(V,F) := \bigoplus_p F^p V\hbar^{-p} \subset V[\hbar,\hbar^{-1}]$. This has the structure of a $\mathbb{G}_m$-equivariant (i.e. graded) $\mathbb{Z}[\hbar]$-module, setting $\hbar$ to be of weight -1 for the $\mathbb{G}_m$-action.

The functor ξ gives an equivalence between exhaustively filtered vector spaces and flat $\mathbb{G}_m$-equivariant $\mathbb{Z}[\hbar]$-modules — see [Pri3, Lemma 2.1] for instance.

We will be interested in filtrations which are complete, in the sense that $V = \varprojlim_i V/F^i$. Via the Rees constructions, this amounts to looking at the inverse limit over k of the categories of $\mathbb{G}_m$-equivariant $\mathbb{Z}[\hbar]/\hbar^k$-modules. However, Koszul duality provides a much more efficient characterisation. The Koszul dual of $\mathbb{Z}[\hbar]$ is the dg algebra $\mathbb{Z}[\text{đ}] \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[\hbar]}(\mathbb{Z},\mathbb{Z})$ for đ of chain degree -1 and weight 1 with $\text{đ}^2 = 0$. Weak equivalences of $\mathbb{Z}[\text{đ}]$-modules in graded chain complexes are quasi-isomorphisms of the underlying chain complexes, forgetting đ, and these correspond to filtered quasi-isomorphisms of the associated complete filtered complexes.

Definition 1.2. For a filtered chain complex (V,F), the corresponding $\mathbb{G}_m$-equivariant $\mathbb{Z}[\text{đ}]$-module $\mathfrak{gr}_F V$ is given in weight i by

$$\mathfrak{gr}_F^i V := \mathrm{cone}(F^{i+1}V \to F^i V),$$

with đ: $\mathfrak{gr}_F^i V \to \mathfrak{gr}_F^{i+1} V_{[-1]}$ given by the identity on $F^{i+1}V$ (and necessarily 0 elsewhere).

There is an obvious quasi-isomorphism from $\mathfrak{gr}_F V$ to the associated graded $\mathrm{gr}_F V$, but the latter does not have a natural đ-action.

The homotopy inverse functor to $\mathfrak{gr}$ can be realised explicitly as follows:

Definition 1.3. Given a $\mathbb{Z}[\text{đ}]$-module E in $\mathbb{G}_m$-equivariant chain complexes, define the chain complex $\mathfrak{f}(E)$ to be the semi-infinite total complex

$$\mathfrak{f}(E) := (\bigoplus_{i<0} E(i) \oplus \prod_{i\geq 0} E(i), \delta \pm \text{đ}),$$

equipped with the complete exhaustive filtration

$$F^p\mathfrak{f}(E) := (\prod_{i\geq p} E(i), \delta \pm \text{đ}).$$

This clearly maps weak equivalences to filtered quasi-isomorphisms.

One way of thinking of the category of $\mathbb{Z}[\text{đ}]$-modules is that we are allowed to split the filtration on a filtered complex, but only at the expense of having a component đ of the differential which does not respect the grading. The associated graded complex is then simply given by forgetting the action of đ.

Another way of understanding this equivalence is to observe that a cofibrant resolution of $\mathbb{Z}[\text{đ}]$ as an associative algebra in chain complexes is given by the free algebra $\mathbb{Z}\langle \text{đ}_1, \text{đ}_2, \ldots\rangle$ with đ_m of chain degree -1 and weight m, with differential δ given by $\delta\text{đ}_m = -\sum_{i+j=m} \text{đ}_i\text{đ}_j$. Thus the structure of a $\mathbb{Z}\langle \text{đ}_1, \text{đ}_2, \ldots\rangle$-module on a chain complex E is the same as a differential $\delta + \sum \text{đ}_i$ on $\bigoplus_{i<0} E(i) \oplus \prod_{i\geq 0} E(i)$ respecting the filtration and agreeing with δ on the associated graded.

Definition 1.4. Given a ring k, a linear algebraic group G over k, and a G-equivariant commutative algebra R in chain complexes over k, define the category $dg\mathrm{Mod}_G(R)$ to consist of G-equivariant R-modules in chain complexes.

Thus the Rees construction $\xi(V,M)$ of a filtered R-module M lies in $dg\mathrm{Mod}_{\mathbb{G}_m}(R[\hbar])$, while $\mathfrak{gr}_F M \in dg\mathrm{Mod}_{\mathbb{G}_m}(R[\hbar\!\!\!/])$. When G is linearly reductive, standard arguments show that there is a cofibrantly generated model structure on $dg\mathrm{Mod}_G(R)$ in which fibrations are surjections and weak equivalences are quasi-isomorphisms of the underlying chain complexes.

The dg algebra $R[\hbar\!\!\!/]$ has the natural structure of a dg Hopf R-algebra, by setting $\hbar\!\!\!/$ to be primitive, so the comultiplication $R[\hbar\!\!\!/] \to R[\hbar\!\!\!/] \otimes_R R[\hbar\!\!\!/]$ sends $\hbar\!\!\!/$ to $\hbar\!\!\!/ \otimes 1 + 1 \otimes \hbar\!\!\!/$.

Definition 1.5. We define a closed symmetric monoidal structure $\otimes_R$ on the category $dg\mathrm{Mod}_{\mathbb{G}_m}(R[\hbar\!\!\!/])$ by giving the chain complex $M \otimes_R N$ an $R[\hbar\!\!\!/]$-module structure via the comultiplication on the Hopf algebra $R[\hbar\!\!\!/]$.

With respect to this structure, the functors $\mathfrak{gr}$ and $\mathfrak{f}$ are both lax monoidal. By way of comparison, note that for the usual tensor product of filtered complexes over k, we have $\mathrm{gr}_F(U \otimes_k V) = \mathrm{gr}_F(U) \otimes_k \mathrm{gr}_F(V)$.

1.1.2. *Koszul duality for almost commutative rings.* From now on, we fix a commutative algebra R in chain complexes over $\mathbb{Q}$. We refer to associative algebras in chain complexes as DGAAs, and commutative algebras in chain complexes as CDGAs. We will also refer to to coassociative coalgebras in chain complexes over R as DGACs over R.

Definition 1.6. We say that a complete filtered DGAA (A, F) is almost commutative if $\mathrm{gr}_F A$ is a CDGA. Similarly, a filtered DGAC (C, F) is said to be almost cocommutative if the comultiplication on $\mathrm{gr}_F C$ is cocommutative.

Remark 1.7. An almost commutative DGAA (A, F) can be regarded as an algebra in filtered complexes for the filtered operad given by the PBW filtration on the associative operad Ass, which is given by powers of the augmentation ideal of $T(V) \to \mathrm{Symm}(V)$. The Rees construction $\xi(A, F)$ is thus automatically an algebra for the $\mathbb{BD}_1$-operad over $[\mathbb{A}^1/\mathbb{G}_m]$ as described in [CPT+, §3.5.1] (or [CG, §2.4.2] for its completion). Since we only wish to consider complete filtrations, we are effectively studying algebras $\mathfrak{gr}(A, F)$ over the operad $\mathfrak{gr}(BD_1)$ in $dg\mathrm{Mod}_{\mathbb{G}_m}(\mathbb{Q}[\hbar\!\!\!/])$, where we write BD_1 for the complete filtered operad associated to $\mathbb{BD}_1$.

Definition 1.8. We write B for the bar construction from possibly non-unital DGAAs over R to ind-conilpotent DGACs over R. Explicitly, this is given by taking the tensor coalgebra

$$\mathrm{B}A := T(A_{[-1]}) = \bigoplus_{i\geq 0} (A_{[-1]})^{\otimes_R i},$$

with chain differential given on cogenerators $A_{[-1]}$ by combining the chain differential and multiplication on A. Write B_+A for the subcomplex $T_+(A_{[-1]}) = \bigoplus_{i>0} A_{[-1]}^{\otimes_R i}$.

Let Ω_+ be the left adjoint to B_+, given by the tensor algebra

$$\Omega_+ C := \bigoplus_{j>0} (C_{[1]})^{\otimes_R i},$$

with chain differential given on generators $C_{[1]}$ by combining the chain differential and comultiplication on C. We then define $\Omega C := R \oplus \Omega_+ C$ by formally adding a unit.

Definition 1.9. Given an almost commutative DGAA (A, F), we define the filtration βF on $\mathrm{B}A$ by convolution with the Poincaré–Birkhoff–Witt filtration β. Explicitly, there is a shuffle multiplication ∇ on $(\mathrm{B}A)_{\#}$ given on cogenerators by the identity maps

$(A\otimes R)\oplus(R\otimes A)\to A$, making $(\mathrm{B}A)_{\#}$ into a Hopf algebra. Writing F as an increasing filtration, we then set $\beta^j\mathrm{B}A$ to be the image of the j-fold shuffle product $(\mathrm{B}_+A)^{\otimes j}\to\mathrm{B}A$ (i.e. $b_1\otimes\ldots\otimes b_j\mapsto b_1\nabla b_2\nabla\ldots\nabla b_j$), and

$$(\beta F)_i\mathrm{B}A:=\sum_j F_{i+j}\cap\beta^j\mathrm{B}A.$$

Lemma 1.10. *The filtration βF makes $\mathrm{B}A$ into an almost cocommutative DGAC.*

Proof. The filtration β automatically behaves with respect to the comultiplication, making $(\mathrm{B}A)_{\#}$ a filtered coalgebra, and so (βF) also gives a filtered coalgebra structure. To see that $\mathrm{B}A$ is a filtered DGAC, it only remains to show that the spaces $(\beta F)_i\mathrm{B}A$ are closed under the chain differential. Since the latter is a coderivation, it suffices to check that it induces a filtered map on cogenerators.

The filtration induced on cogenerators by β is just $A_{[-1]}=\mathrm{gr}^{\beta}_1A_{[-1]}$, so $(\beta F)_iA_{[-1]}=F_{i+1}A_{[-1]}$. We also get $\beta^1(A^{\otimes 2}_{[-1]})=A^{\otimes 2}_{[-1]}$, $\beta^2(A^{\otimes 2}_{[-1]})=\Lambda^2A_{[-1]}$, and $\beta^3(A^{\otimes 2}_{[-1]})=0$, so

$$(\beta F)_i(A^{\otimes 2}_{[-1]})=F_{i+1}(A^{\otimes 2}_{[-1]})+F_{i+2}(\Lambda^2A)_{[-2]}.$$

Multiplication and the chain differential on A automatically preserve F, so the only remaining condition is that multiplication sends $F_{i+2}(\Lambda^2A)$ to $F_{i+1}A$ — this is precisely the condition that gr_FA be commutative.

Finally, observe that on associated gradeds, the multiplication map $\mathrm{gr}^{\beta F}_i(A\otimes A)\to\mathrm{gr}^{\beta F}_i(A)$ is the map

$$\mathrm{gr}^F_{i+1}\mathrm{Symm}^2(A)\oplus\mathrm{gr}^F_{i+2}\Lambda^2A\to\mathrm{gr}^F_{i+1}A$$

given by multiplication on the first factor and Lie bracket on the second. Thus

$$\mathrm{gr}^{\beta F}\mathrm{B}A_{\#}$$

is the free Poisson coalgebra $\mathrm{CoSymm}_R(\mathrm{gr}^F_{*+1}\mathrm{CoLie}_RA)_{\#}$, the chain differential involving both product and Lie bracket on gr^FA. In particular, the comultiplication on $\mathrm{B}A$ is cocommutative. ☐

In fact, observe that we can characterise βF as the smallest almost cocommutative filtration on $\mathrm{B}A$ for which the induced filtration on cogenerators is $(\beta F)_iA_{[-1]}=F_{i+1}A_{[-1]}$.

Definition 1.11. Given an almost cocommutative DGAC (C,F) over R, define the filtration β^*F on ΩC and Ω_+C by convolution with the PBW filtration. Explicitly, define a comultiplication Δ on $T(C_{[1]})$ to be the algebra morphism sending $c\in C_{[1]}$ to $c\otimes 1+1\otimes c$, and let $\beta^*_r:=\ker(\Delta^{(r+1)}\colon T(C_{[1]})\to T_+(C_{[1]})^{\otimes r+1})$ be the kernel of the iterated comultiplication. We then set

$$(\beta^*F)_i\Omega C:=\sum_j F_{i-j}\cap\beta^*_j\Omega C,$$

and similarly for Ω_+C. We then define $\hat{\Omega}_+C$ to be the completion with respect to β^*.

Lemma 1.12. *The filtration β^*F makes $\hat{\Omega}A$ into an almost commutative DGAA.*

Proof. The constructions (B,β) and (Ω,β^*) are dual to each other, so the proof of Lemma 1.10 adapts after taking shifts and duals. ☐

Definition 1.13. Define the functors B_{BD_1} and Ω_{BD_1} by $\mathrm{B}_{BD_1}(A,F) := (\mathrm{B}A, \beta F)$ and $\Omega_{BD_1}(C,F) := (\hat{\Omega}C, \beta^* F)$; define $\mathrm{B}_{BD_1,+}$ and $\Omega_{BD_1,+}$ similarly.

Lemma 1.14. *The functor $\Omega_{BD_1,+}$ is left adjoint to the functor $\mathrm{B}_{BD_1,+}$ from complete non-unital almost commutative DGAAs A over R to non-counital almost cocommutative DGACs C over R.*

Proof. Given A and C, the sets $\mathrm{Hom}_{DGAA}(\Omega_+ C, A)$ and $\mathrm{Hom}(C, \mathrm{B}A)$ can both be identified with the set

$$\{f \in F_1 \underline{\mathrm{Hom}}_R(C,A)^1 \ : \ [\delta, f] + f \smile f = 0\},$$

where the product $\smile$ combines multiplication on A with comultiplication on C. ☐

Observe that the product $\smile$ makes the complex $\underline{\mathrm{Hom}}_R(C,A)$ into an almost commutative DGAA, so $F_1\underline{\mathrm{Hom}}_R(C,A)$ is closed under the commutator, hence a DGLA.

Lemma 1.15. *If A is a complete filtered non-unital almost commutative DGAA with $\mathrm{gr}_F A$ flat over R, then the co-unit $\varepsilon_A \colon \Omega_{BD_1,+}\mathrm{B}_{BD_1,+}A \to A$ of the adjunction is a filtered quasi-isomorphism.*

Proof. It suffices to show that ε gives quasi-isomorphisms on the graded algebras associated to the filtrations. The functors $\mathrm{gr}_\beta \mathrm{B}_{BD_1,+}$ and $\mathrm{gr}_{\beta^*}\Omega_{BD_1,+}$ are then just the bar and cobar functors for the Poisson operad, equipped with a $\mathbb{G}_m$-action setting the commutative multiplication to be of weight 0 and the Lie bracket of weight -1. For $\hbar$ a formal variable of weight -1, the graded Poisson operad can be written as $\mathrm{Com} \circ \hbar\mathrm{Lie}$, where $(\hbar\mathcal{P})(i) := \hbar^{i-1}\mathcal{P}(i)$ for any operad $\mathcal{P}$. The $\mathbb{G}_m$-equivariant Koszul dual of the graded Poisson operad is then $(\mathrm{Com}\circ\hbar\mathrm{Lie})^! = (\hbar^{-1}\mathrm{Com})\circ\mathrm{Lie} = \hbar^{-1}(\mathrm{Com}\circ\hbar\mathrm{Lie})$, so it is self-dual after a shift in filtrations. This shift is precisely the difference between PBW and lower central series, so $\mathrm{gr}\varepsilon$ is a graded quasi-isomorphism by Koszul duality for the Poisson operad. ☐

1.2. Hochschild complexes.

Recall that we are fixing a CDGA R over $\mathbb{Q}$.

Definition 1.16. For an almost commutative DGAA (A,F) over R and a filtered (A,F)-bimodule (M,F) in chain complexes for which the left and right $\mathrm{gr}^F A$-module structures on $\mathrm{gr}^F M$ agree, we define the filtered chain complex

$$\mathrm{CC}_{R,BD_1}(A,M)$$

to be the completion of the cohomological Hochschild complex $\mathrm{CC}_R(A,M)$ (rewritten as a chain complex) with respect to the filtration γF defined as follows. We may identify $\mathrm{CC}_R(A,M)$ with the subcomplex of

$$\underline{\mathrm{Hom}}_R(\mathrm{B}A, \mathrm{B}(A\oplus M_{[1]}))$$

consisting of coderivations extending the zero coderivation on $\mathrm{B}A$. The hypotheses on M ensure that $A\oplus M$ is almost commutative (regarding M as a square-zero ideal), so we have filtrations βF on $\mathrm{B}A$ and $\mathrm{B}(A\oplus M_{[1]})$. We then define $(\gamma F)_i$ to consist of coderivations sending $(\beta F)_j \mathrm{B}A$ to $(\beta F)_{i+j-1}\mathrm{B}(A\oplus M)$.

Since a coderivation is determined by its value on cogenerators, and the cogenerators of the bar construction have weight 1 with respect to the PBW filtration β, we may regard $(\gamma F)_i \mathrm{CC}_R(A,M)_{\#}$ as the subspace of $\underline{\mathrm{Hom}}_R(\mathrm{B}A, M)^{\#}$ consisting of maps sending $(\beta F)_j\mathrm{B}A$ to $F_{i+j}M$.

We also define the subcomplex $\mathrm{CC}_{R,BD_1,+}(A,M)$ to be the kernel of $\mathrm{CC}_{R,BD_1}(A,M)\to M$, or equivalently $\underline{\mathrm{Hom}}_R(\mathrm{B}_+A, M)^{\#}$.

Remark 1.17. When the filtrations F are trivial in the sense that $A = \mathrm{gr}^F_0 A$, $M = \mathrm{gr}^F_0 M$, we simply write $\gamma := \gamma F$, and observe that $\gamma_0 \mathrm{CC}_R(A, M) = M$, while $\gamma_1 \mathrm{CC}_R(A, M)$ is just the Harrison cohomology complex. When A is moreover cofibrant as a CDGA, observe that the HKR isomorphism gives a filtered levelwise quasi-isomorphism $(\mathrm{CC}_R(A, M), \tau^{\mathrm{HH}}) \to (\mathrm{CC}_{R,BD_1}(A, M), \gamma)$, where τ^{HH} denotes good truncation in the Hochschild direction as featured in [Pri5, Definition 1.13].

Lemma 1.18. *If $\phi\colon (A, F) \to (D, F)$ is a morphism of almost commutative DGAAs over R, then $\mathrm{CC}_{R,BD_1}(A, D)$ is an almost commutative DGAA under the cup product, and $\mathrm{CC}_{R,BD_1}(A, D) \to D$ is a morphism of almost commutative DGAAs.*

Proof. This just follows because $\mathrm{gr}^{\gamma F}\mathrm{CC}_R(A, D)^{\#} = \underline{\mathrm{Hom}}(\mathrm{gr}^{\beta F}\mathrm{B}A, \mathrm{gr}^F D)^{\#}$, with $\mathrm{gr}^{\beta F}\mathrm{B}A$ cocommutative and $\mathrm{gr}^F D$ commutative. □

1.2.1. *Brace algebra structures.* Recall that a brace algebra B over R is an R-cochain complex equipped with a cup product in the form of a chain map

$$B \otimes B \xrightarrow{\smile} B,$$

and braces in the form of maps

$$\{-\}\{-, \ldots, -\}_r \colon B \otimes B^{\otimes r} \to B^{[-r]}$$

satisfying the conditions of [Vor, §3.2] with respect to the differential. There is a brace operad Br in cochain complexes, whose algebras are brace algebras. The commutator of the brace $\{-\}\{-\}_1$ is a Lie bracket, so for any brace algebra B, there is a natural DGLA structure on $B^{[1]}$. The brace operad is weakly equivalent to the rationalisation of the little discs operad, so brace algebras re a model for E_2-algebras in cochain complexes.

Definition 1.19. Define an decreasing filtration γ on the brace operad Br by putting the cup product in γ^0 and the braces $\{-\}\{-, \ldots, -\}_r$ in γ^r.

Thus a (brace, γ)-algebra (A, F) in filtered complexes is a brace algebra for which the cup product respects the filtration, and the r-braces send F_i to F_{i-r}. We refer to (brace, γ)-algebras as almost commutative brace algebras.

Beware that the filtration γ is not the same as that featuring in [Saf1, Definition 5.3], since we assign higher weights to higher braces.

In an almost commutative brace algebra A, the brace $\{-\}\{-\}_1$ is of weight -1; since it gives a homotopy between the cup product and its opposite, it follows that the commutator of the cup product is of weight -1, so A is almost commutative as a DGAA. Moreover, a brace algebra structure on A induces a dg bialgebra structure on $\mathrm{B}A$, as in [Vor, §3.2], and because $\beta^r\mathrm{B}A \subset (A_{[-1]})^{\otimes \geq r}$, the multiplication on $\mathrm{B}A$ given by braces preserves the filtration βF on $\mathrm{B}_{BD_1}A$, so it is a filtered bialgebra (with almost cocommutative comultiplication).

Lemma 1.20. *For any almost commutative DGAA A over R, there is a natural almost commutative brace algebra structure on $\mathrm{CC}_{R,BD_1}(A)$ over R. In particular, $\mathrm{CC}_{R,BD_1}(A)_{[-1]}$ is a filtered DGLA over R, and its associated graded DGLA is abelian.*

Proof. The formulae of [Vor, §3] define a brace algebra structure on $\mathrm{CC}_R(A)$. By Lemma 1.18, we know that $(\mathrm{CC}_R(A), \gamma F)$ is an almost commutative DGAA, so it suffices to show that the brace operations have the required weights.

Given $f \in (\gamma F)_p\underline{\mathrm{Hom}}(\mathrm{B}A, A)$ and $g_i \in (\gamma F)_{q_i}\underline{\mathrm{Hom}}(\mathrm{B}A, A)$, each g_i corresponds to a coalgebra coderivation $\tilde{g}_i$ on $\mathrm{B}A$ sending $(\beta F)_j\mathrm{B}A$ to $(\beta F)_{j+q_i-1}\mathrm{B}A$.

The element $\{f\}\{g_1,\ldots,g_m\} \in \underline{\mathrm{Hom}}(\mathrm{B}A,A)$ is the composition

$$\mathrm{B}A \xrightarrow{\Delta^{(m)}} (\mathrm{B}A)^{\otimes m} \xrightarrow{\tilde{g}_1\otimes\ldots\otimes\tilde{g}_m} (\mathrm{B}A)^{\otimes m} \xrightarrow{\nabla} \mathrm{B}A \xrightarrow{f} A,$$

where $\Delta^{(m)}$ is the iterated coproduct, and ∇ the shuffle product. The definition of β ensures that ∇ preserves the filtration βF, so we have

$$\{f\}\{g_1,\ldots,g_m\} \in (\gamma F)_{(p+q_1+\ldots+q_m-m)}\underline{\mathrm{Hom}}(\mathrm{B}A,A).$$

□

Definition 1.21. Given a brace algebra C, define the opposite brace algebra C^{opp} to have the same elements as C, but multiplication $b^{\mathrm{opp}} \smile c^{\mathrm{opp}} := (-1)^{\deg b \deg c}(c \smile b)^{\mathrm{opp}}$ and brace operations given by the multiplication $(\mathrm{B}C^{\mathrm{opp}}) \otimes (\mathrm{B}C^{\mathrm{opp}}) \to \mathrm{B}C^{\mathrm{opp}}$ induced by the isomorphism $(\mathrm{B}C^{\mathrm{opp}}) \cong (\mathrm{B}C)^{\mathrm{opp}}$. Explicitly,

$$\{b^{\mathrm{opp}}\}\{c_1^{\mathrm{opp}},\ldots,c_m^{\mathrm{opp}}\} := \pm\{b\}\{c_m,\ldots,c_1\}^{\mathrm{opp}},$$

where $\pm = (-1)^{m(m+1)/2+(\deg f-m)(\sum_i \deg c_i - m)+\sum_{i<j}\deg c_i \deg c_j}$.

Observe that when a filtered brace algebra C is almost commutative, then so is C^{opp}.

Lemma 1.22. *Given DGAAs A,D over R, there is an anti-involution*

$$-i\colon\ \mathrm{CC}_R(A,D)^{\mathrm{opp}} \to \mathrm{CC}_R(A^{\mathrm{opp}},D^{\mathrm{opp}})$$

of DGAAs given by

$$i(f)(a_1,\ldots,a_m) = -(-1)^{\sum_{i<j}\deg a_i \deg a_j}(-1)^{m(m+1)/2} f(a_m^{\mathrm{opp}},\ldots,a_1^{\mathrm{opp}})^{\mathrm{opp}}.$$

When $A=D$, the anti-involution $-i$ is a morphism of brace algebras, and in particular $i\colon \mathrm{CC}_R(A)_{[-1]} \to \mathrm{CC}_R(A)_{[-1]}$ is a morphism of DGLAs. Whenever A is a cofibrant CDGA over R, the map i corresponds under the HKR isomorphism to the involution which acts on $\underline{\mathrm{Hom}}_A(\Omega^p_{A/R},A)$ as scalar multiplication by $(-1)^{p-1}$.

Proof. This is effectively [Bra, §2.1], adapted along the lines of [Pri5, Lemma 1.15], together with the observation that $-i$ acts on braces in the prescribed manner. □

1.2.2. *Semidirect products.*

Lemma 1.23. *Given a morphism $\phi\colon A \to D$ of almost commutative filtered DGAAs over R, the almost commutative brace algebra $\mathrm{CC}_{R,BD_1}(A)$ of Hochschild cochains acts on the almost commutative DGAA $\mathrm{CC}_{R,BD_1}(A,D)$ in the form of a morphism*

$$\mathrm{B}_{BD_1,+}\mathrm{CC}_{R,BD_1}(A) \to \mathrm{B}_{BD_1,+}\mathrm{CC}_{R,BD_1}(\mathrm{CC}_{R,BD_1}(A,D))$$

of almost cocommutative bialgebras.

Proof. Given $g_1,\ldots,g_m \in \mathrm{CC}_{R,BD_1}(A)$ and $f \in \mathrm{CC}_{R,BD_1}(A,D)$, the brace operation $\{f\}\{g_1,\ldots,g_m\}$ is well-defined as an element of $\mathrm{CC}_{R,BD_1}(A,D)$. Reasoning as in [Vor, §3.2], this combines with the morphism $\phi_*\colon \mathrm{CC}_{R,BD_1}(A) \to \mathrm{CC}_{R,BD_1}(A,D)$ to give an action

$$M_{\bullet,\bullet}\colon\ \mathrm{B}_{BD_1}\mathrm{CC}_{R,BD_1}(A,D) \otimes_R \mathrm{B}_{BD_1}\mathrm{CC}_{R,BD_1}(A) \to \mathrm{B}_{BD_1}\mathrm{CC}_{R,BD_1}(A,D)$$

of almost cocommutative dg coalgebras, associative with respect to the brace multiplication of [Vor]. This respects the filtrations for the same reason that the multiplication does on the bar construction of an almost commutative brace algebra (Definition 1.19).

Indeed, $\mathrm{CC}_{R,BD_1}(A,D)$ is a brace $\mathrm{CC}_{R,BD_1}(A)$-module in the sense of [Saf1, Definition 3.2]. On restricting to cogenerators, the multiplication above gives a map

$$\begin{aligned}\mathrm{B}_{BD_1}\mathrm{CC}_{R,BD_1}(A,D) \to& \underline{\mathrm{Hom}}(\mathrm{B}_{BD_1}\mathrm{CC}_{R,BD_1}(A), \mathrm{CC}_{R,BD_1}(A,D))\\ &\cong \mathrm{CC}_{R,BD_1}(\mathrm{CC}_{R,BD_1}(A,D)),\end{aligned}$$

and as in [Saf1, Proposition 4.2], this induces a morphism

$$\mathrm{B}_{BD_1,+}\mathrm{CC}_{R,BD_1}(A) \to \mathrm{B}_{BD_1,+}\mathrm{CC}_{R,BD_1}(\mathrm{CC}_{R,BD_1}(A,D))$$

of almost cocommutative bialgebras, compatibility with the filtrations being automatic from the description above. □

For an E_2-algebra C to act on an E_1-algebra E is the same as a morphism from C to the Hochschild complex of E. This is what we now construct for Hochschild complexes in the almost commutative setting, so that we will have an almost commutative brace algebra acting on an almost commutative DGAA (or equivalently a BD_2-algebra acting on a BD_1-algebra). Proposition 1.15 then combines with the adjunction property to give morphisms

$$\mathrm{CC}_{R,BD_1}(A) \xleftarrow{\sim} \Omega_{BD_1,+}\mathrm{B}_{BD_1,+}\mathrm{CC}_{R,BD_1}(A) \to \mathrm{CC}_{R,BD_1}(\mathrm{CC}_{R,BD_1}(A,D)),$$

of almost commutative DGAAs, and we need to enhance this to keep track of the brace algebra structures:

Lemma 1.24. *If A is a complete filtered non-unital almost commutative brace algebra over R, then there is a natural almost commutative brace algebra structure on the DGAA $\Omega_{BD_1,+}\mathrm{B}_{BD_1,+}A$. If $\mathrm{gr}_F A$ is moreover flat over R, then there is a zigzag of filtered quasi-isomorphisms of almost commutative brace algebras between A and $\Omega_{BD_1,+}\mathrm{B}_{BD_1,+}A$.*

Proof. As in [Kad], there is a natural brace algebra structure on Ω_+C for any bialgebra C; we now show that when C is almost cocommutative, the resulting brace algebra structure on $\Omega_{BD_1,+}C$ is almost commutative. For $c \in C$, the brace operation

$$\{c\}\{-\}\colon \Omega(C) \to \Omega(C)$$

is defined by first taking the element $\sum_r \Delta^{(r)}c \in TC$, then applying the multiplication from C internally within each subspace $C^{\otimes r}$. Since Δ is almost cocommutative and ΩC almost commutative, it follows that when $c \in F_pC$, we get $\{c\}\{(\beta^*F)_i\Omega C\} \subset (\beta^*F)_{i+p}\Omega C$. Equivalently, for $y \in (\beta^*F)_i\Omega C$, the map $\{-\}\{y\}$ sends $(\beta^*F)_pC = F_{p-1}C$ to $(\beta^*F)_{i+p-1}\Omega C$.

We automatically have $\{c\}\{\}_0 = c$, and the higher braces $\{c\}\{-\}_n\colon \Omega(C)^{\otimes n} \to \Omega(C)$ are then set to be 0 for $c \in C$, and extended to the whole of ΩC via the identities

$$\{xz\}\{y_1, \ldots, y_n\} = \sum_{i=0}^{n} \pm x\{y_1, \ldots, y_i\}z\{y_{i+1}, \ldots, y_n\}.$$

In particular, this means that $\{-\}\{y\}$ is a derivation, so must map $(\beta^*F)_p\Omega C$ to $(\beta^*F)_{i+p-1}\Omega C$, since it does so on generators. We can then describe higher braces $\{-\}\{y_1, \ldots, y_n\}$ as the composition

$$\Omega(C) \xrightarrow{\Delta^{(n)}} \Omega(C)^{\otimes n} \xrightarrow{\{-\}\{y_1\}\otimes \ldots \otimes\{-\}\{y_n\}} \Omega(C)^{\otimes n} \to \Omega(C),$$

the final map being given by multiplication. By the construction of β^*, the map $\Delta^{(n)}$ preserves the filtration (β^*F), so for $y_i \in (\beta^*F)_{q_i}\Omega C$, we have

$$\{-\}\{y_1, \ldots, y_n\}\colon (\beta^*F)_p\Omega(C) \to (\beta^*F)_{(p+q_1+\ldots+q_n-n)}\Omega C,$$

making $\Omega_{BD_1,+}C$ almost commutative

Taking $C = \mathrm{B}_{BD_1,+}A$ gives an almost commutative brace algebra $\Omega_{BD_1,+}\mathrm{B}_{BD_1,+}A$ and an almost commutative DGAA quasi-isomorphism $\Omega_{BD_1,+}\mathrm{B}_{BD_1,+}A \to A$ by Lemma 1.15, but this is not a brace algebra morphism in general. If we let $\Omega_{\mathrm{Br},+}$ be the left adjoint to B_{BD_1} as a functor from almost commutative brace algebras to almost cocommutative bialgebras, then it suffices to establish a filtered brace algebra quasi-isomorphism $\Omega_{BD_1,+}\mathrm{B}_{BD_1,+}A \to \Omega_{\mathrm{Br},+}\mathrm{B}_{BD_1,+}A$. If we disregard the filtrations, this is the main result of [You], and the filtered case follows by observing that the homotopy of [You, Theorem 3.3] preserves the respective filtrations. □

Combining Lemmas 1.23 and 1.24 gives:

Proposition 1.25. *For any morphism $\phi\colon A \to D$ of almost commutative filtered DGAAs over R, there is a canonical zigzag*

$$\mathrm{CC}_{R,BD_1}(A) \xleftarrow{\sim} \tilde{C} \to \mathrm{CC}_{R,BD_1}(\mathrm{CC}_{R,BD_1}(A,D))$$

of almost commutative brace algebras over R, where the first map is a quasi-isomorphism.

Definition 1.26. Given an almost commutative brace algebra C over R, and an almost commutative DGAA E over R which is a left brace C-module compatibly with the filtrations, define the semidirect product $E_{[1]} \rtimes C$ to be the almost commutative non-unital brace algebra given by the homotopy fibre product of the diagram

$$\tilde{C} \to \mathrm{CC}_{R,BD_1}(E) \leftarrow \mathrm{CC}_{R,BD_1,+}(E),$$

for the brace algebra resolution $\tilde{C}$ of C mapping to $\mathrm{CC}_{R,BD_1}(E)$ via Lemma 1.24 and the proof of Lemma 1.23.

Remark 1.27. Observe that we have a natural morphism $E_{[1]} \rtimes C \to C$ of non-unital brace algebras, with homotopy fibre given by the homotopy kernel of $\mathrm{CC}_{R,BD_1,+}(E) \to \mathrm{CC}_{R,BD_1}(E)$. As a complex, this kernel is just $E_{[1]}$, and the underlying DGLA is just the DGLA underlying the DGAA E. For more discussion of the map $\mathrm{CC}_{R,+}(E) \to \mathrm{CC}_R(E)$ of E_2-algebras, see [Kon2, §2.7].

2. Defining quantisations for derived co-isotropic structures

In this section, we develop a precise notion of quantisation for derived co-isotropic structures in a stacky affine setting. Recall that we are fixing a CDGA R over $\mathbb{Q}$.

2.1. **Stacky thickenings of derived affines.**

We now recall some definitions and lemmas from [Pri4, §3], as summarised in [Pri7, §3.1]. By default, we will regard the CDGAs in derived algebraic geometry as chain complexes $\ldots \xrightarrow{\delta} A_1 \xrightarrow{\delta} A_0 \xrightarrow{\delta} \ldots$ rather than cochain complexes — this will enable us to distinguish easily between derived (chain) and stacky (cochain) structures.

Definition 2.1. A stacky CDGA is a chain cochain complex $A_\bullet^\bullet$ equipped with a commutative product $A \otimes A \to A$ and unit $\mathbb{Q} \to A$. Given a chain CDGA R, a stacky CDGA over R is then a morphism $R \to A$ of stacky CDGAs. We write $DGdg\mathrm{CAlg}(R)$ for the category of stacky CDGAs over R, and $DG^+dg\mathrm{CAlg}(R)$ for the full subcategory consisting of objects A concentrated in non-negative cochain degrees.

When working with chain cochain complexes $V_\bullet^\bullet$, we will usually denote the chain differential by $\delta\colon V^i_j \to V^i_{j-1}$, and the cochain differential by $\partial\colon V^i_j \to V^{i+1}_j$. On a first reading, readers interested primarily in DM (as opposed to Artin) stacks may ignore the stacky part of the structure and consider only CDGAs $A_\bullet = A^0_\bullet$ throughout this section.

Example 2.2. We now recall an important example of a class of stacky CDGAs from [Pri4, Example 3.6]. Given a Lie algebra $\mathfrak{g}$ of finite rank acting as derivations on a derived affine scheme Y, we write $O([Y/\mathfrak{g}])$ for the stacky CDGA given by the Chevalley–Eilenberg double complex

$$O(Y) \xrightarrow{\partial} O(Y)\otimes \mathfrak{g}^\vee \xrightarrow{\partial} O(Y)\otimes \Lambda^2\mathfrak{g}^\vee \xrightarrow{\partial} \ldots$$

of $\mathfrak{g}$ with coefficients in the chain $\mathfrak{g}$-module $O(Y)$. We think of this as a form of derived Lie algebroid.

Definition 2.3. Say that a morphism $U \to V$ of chain cochain complexes is a levelwise quasi-isomorphism if $U^i \to V^i$ is a quasi-isomorphism for all $i \in \mathbb{Z}$. Say that a morphism of stacky CDGAs is a levelwise quasi-isomorphism if the underlying morphism of chain cochain complexes is so.

There is a model structure on chain cochain complexes over R in which weak equivalences are levelwise quasi-isomorphisms and fibrations are surjections — this follows by identifying chain cochain complexes with the category $dg\mathrm{Mod}_{\mathbb{G}_m}(R[\partial]/\partial^2)$ of §1.1.1, for instance, for ∂ of chain degree 0 and weight 1, with $\partial^2 = 0$.

The following is [Pri4, Lemma 3.4]:

Lemma 2.4. *There is a cofibrantly generated model structure on stacky CDGAs over R in which fibrations are surjections and weak equivalences are levelwise quasi-isomorphisms.*

There is a denormalisation functor D from non-negatively graded CDGAs to cosimplicial algebras, combining Dold–Kan denormalisation of a cochain complex with the Eilenberg–Zilber shuffle product (for an explicit description, see [Pri1, Definition 4.20]; it has a left adjoint D^*, described explicitly in [Pri6, Definition 4.14]. Given a cosimplicial CDGA A, D^*A is then a stacky CDGA in non-negative cochain degrees. By [Pri4, Lemma 3.5], D^* is a left Quillen functor from the Reedy model structure on cosimplicial CDGAs to the model structure of Lemma 2.4.

Since DA is a pro-nilpotent extension of A^0, when $\mathrm{H}_{<0}(A) = 0$ we think of the simplicial hypersheaf $\mathbf{R}\mathrm{Spec}\, DA$ as a stacky derived thickening of the derived affine scheme $\mathbf{R}\mathrm{Spec}\, A^0$. Stacky CDGAs arise as formal completions of derived Artin N-stacks along affine atlases, as in [Pri4, §3.1]. When X is a 1-geometric derived Artin stack (i.e. has affine diagonal), the formal completion of a smooth affine 1-atlas $U \to X$ is given by the relative de Rham complex

$$O(U) \xrightarrow{\partial} \Omega^1_{U/X} \xrightarrow{\partial} \Omega^2_{U/X} \xrightarrow{\partial} \ldots,$$

which arises by applying the functor D^* to the Čech nerve of U over X. The construction of Example 2.2 is the special case of this construction corresponding to the atlas $Y \to [Y/G]$ when G is an algebraic group with Lie algebra $\mathfrak{g}$.

Definition 2.5. A morphism $A \to B$ in $DG^+dg\mathrm{CAlg}(R)$ is said to be homotopy formally étale when the map

$$\{\mathrm{Tot}\, \sigma^{\leq q}(\mathbf{L}\Omega^1_A \otimes^{\mathbf{L}}_A B^0)\}_q \to \{\mathrm{Tot}\, \sigma^{\leq q}(\mathbf{L}\Omega^1_B \otimes^{\mathbf{L}}_B B^0)\}_q$$

on the systems of brutal cotruncations is a pro-quasi-isomorphism (i.e. an essentially levelwise quasi-isomorphism in the sense of [Isa, §2.1]), where $\sigma^{\leq q}$ denotes the brutal cotruncation

$$(\sigma^{\leq q} M)^i := \begin{cases} M^i & i \geq q, \\ 0 & i < q. \end{cases}$$

Combining [Pri4, Proposition 3.13] with [Pri2, Theorem 4.15 and Corollary 6.35], every strongly quasi-compact derived Artin N-stack over R can be resolved by a derived DM hypergroupoid (a form of homotopy formally étale cosimplicial diagram) in $DG^+dg\mathrm{CAlg}(R)$.

The constructions of §1 all adapt to chain cochain complexes, by just regarding the cochain structure as a $\mathbb{G}_m$-equivariant $\mathbb{Q}[\partial]/\partial^2$-module structure; quasi-isomorphisms are only considered in the chain direction. We refer to associative (resp. brace) algebras in chain cochain complexes as stacky DGAAs (resp. stacky brace algebras), and have the obvious notions of almost commutativity for filtered stacky DGAAs and filtered stacky brace algebras. We define bar constructions B generalising Definition 1.8 by taking shifts exclusively in the chain direction.

Definition 2.6. For a stacky DGAA A over R and an A-bimodule M in chain cochain complexes, we define the internal cohomological Hochschild complex $\mathcal{CC}_R(A, M)$ to be the chain cochain subcomplex of

$$\mathcal{H}om_R(\mathrm{B}A, \mathrm{B}(A \oplus M_{[1]}))$$

consisting of coderivations extending the zero derivation on $\mathrm{B}A$, where the algebra structure on $A \oplus M_{[1]}$ is defined so that $M_{[1]}$ is a square-zero ideal.

Since a coderivation is determined by its value on cogenerators, the complex $\mathcal{CC}_R(A, M)$ is given explicitly by

$$\mathcal{CC}_R(A, M)_{\#} := \prod_n \mathcal{H}om_R(A^{\otimes n}, M)_{[n]},$$

with chain differential $\delta \pm b$, for the Hochschild differential b given by

$$\begin{aligned}(bf)(a_1, \ldots, a_n) =& a_1 f(a_2, \ldots, a_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i f(a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n) \\ &+ (-1)^n f(a_1, \ldots, a_{n-1}) a_n.\end{aligned}$$

We simply write $\mathcal{CC}_R(A)$ for $\mathcal{CC}_R(A, A)$.

When (A, F) is almost commutative and (M, F) is a filtered A-bimodule for which the left and right $\mathrm{gr}^F A$-module structures on $\mathrm{gr}^F M$ agree, we define the filtered chain cochain complex

$$\mathcal{CC}_{R,BD_1}(A, M)$$

by endowing $\mathcal{CC}_R(A, M)$ with the filtration γF of Definition 1.16, and completing with respect to it.

2.2. **Differential operators.** We now fix a stacky CDGA B over a CDGA R, and recall the definitions of differential operators from [Pri7, §3.2].

Definition 2.7. Given B-modules M, N in chain cochain complexes, inductively define the filtered chain cochain complex $\mathcal{D}\mathit{iff}(M,N) = \mathcal{D}\mathit{iff}_{B/R}(M,N) \subset \mathcal{H}\mathit{om}_R(M,N)$ of differential operators from M to N by setting

(1) $F_0\mathcal{D}\mathit{iff}(M,N) = \mathcal{H}\mathit{om}_B(M,N)$,
(2) $F_{k+1}\mathcal{D}\mathit{iff}(M,N) = \{u \in \mathcal{H}\mathit{om}_R(M,N) \;:\; [b,u] \in F_k\mathcal{D}\mathit{iff}(M,N)\, \forall b \in B\}$, where $[b,u] = bu - (-1)^{\deg b \deg u} ub$.
(3) $\mathcal{D}\mathit{iff}(M,N) = \varinjlim_k F_k\mathcal{D}\mathit{iff}(M,N)$.

We simply write $\mathcal{D}\mathit{iff}_{B/R}(M) := \mathcal{D}\mathit{iff}_{B/R}(M,M)$.

We then define the filtered cochain complex $\hat{\mathrm{Diff}}(M,N) = \hat{\mathrm{Diff}}_{B/R}(M,N) \subset \underline{\hat{\mathrm{Hom}}}_R(M,N)$ by $\hat{\mathrm{Diff}}(M,N) := \varinjlim_k \hat{\mathrm{Tot}}\, F_k\mathcal{D}\mathit{iff}(M,N)$.

Definition 2.8. Given a B-module M in chain cochain complexes, write $\mathscr{D}(M) = \mathscr{D}_{B/R}(M) := \hat{\mathrm{Diff}}_{B/R}(M,M)$, which we regard as a sub-DGAA of $\underline{\hat{\mathrm{Hom}}}_R(M,M)$. We simply write $\mathscr{D}_B = \mathscr{D}_{B/R}$ for $\mathscr{D}_{B/R}(B,B)$ and $\mathcal{D}\mathit{iff}_{B/R}$ for $\mathcal{D}\mathit{iff}_{B/R}(B,B)$.

The definitions ensure that the associated gradeds $\mathrm{gr}^F_k\mathcal{D}\mathit{iff}_B(M,N)$ have the structure of B-modules. As in [Pri7], there are maps

$$\mathrm{gr}^F_k\mathcal{D}\mathit{iff}(M,N) \to \mathcal{H}\mathit{om}_B(M \otimes_B \mathrm{CoS}^k_B\Omega^1_B, N)$$

for all k, which are isomorphisms when B is cofibrant. [Here, CoS denotes cosymmetric powers, as in the notation section.]

The following is [Pri7, Definition 3.9]:

Definition 2.9. Define a strict line bundle over B to be a B-module M in chain cochain complexes such that $M^{\#}_{\#}$ is a projective module of rank 1 over the bigraded-commutative algebra $B^{\#}_{\#}$ underlying B.

The motivating examples of strict line bundles, and the only ones we will need to consider for our applications in §4.2, are the double complexes B_c defined as follows. Given $c \in \mathrm{Z}^1\mathrm{Z}_0 B$, we just set $B^{\#}_c$ to be the B-module $B^{\#}$ (so the chain differential is still δ), and then we set the cochain differential to be $\partial + c$.

2.3. **Relative quantised polyvectors.**

Definition 2.10. Given a morphism $\phi\colon A \to B$ of cofibrant stacky CDGAs over R and a strict line bundle M over B, we define the DGLA $Q\widehat{\mathrm{Pol}}(A,M;0)^{[1]}$ of 0-shifted relative quantised polyvectors as follows. We first note that Definition 1.26 and Proposition 1.25 adapt to double complexes to give a non-unital almost commutative stacky brace algebra

$$\mathcal{C} := \mathcal{CC}_{R,BD_1}(A, \mathcal{D}\mathit{iff}_{B/R}(M))_{[1]} \rtimes \mathcal{CC}_{R,BD_1}(A),$$

and then form the complex

$$Q\widehat{\mathrm{Pol}}_R(A,M;0) := \prod_{p\geq 0} \hat{\mathrm{Tot}}\,(\gamma F)_p\mathcal{C}\hbar^{p-1},$$

with $Q\widehat{\mathrm{Pol}}_R(A,M;0)^{[1]}$ becoming a DGLA with bracket given by the commutator of the brace $\{-\}\{-\}_1$, closure under this operator following from the definition of the filtration γ in Definition 1.19.

We define filtrations $\tilde{F}$ and G on $Q\widehat{\mathrm{Pol}}_R(A,M;0)$ by

$$\tilde{F}^i Q\widehat{\mathrm{Pol}}_R(A,M;0) := \prod_{p\geq i} \hat{\mathrm{Tot}}\,(\gamma F)_p \mathcal{C}\hbar^{p-1},$$

$$G^j Q\widehat{\mathrm{Pol}}_R(A,M;0) := Q\widehat{\mathrm{Pol}}_R(A,M;0)\hbar^j.$$

Note that almost commutativity of the brace algebra $\mathcal{C}$ implies that $[\tilde{F}^i Q\widehat{\mathrm{Pol}}, \tilde{F}^j Q\widehat{\mathrm{Pol}}] \subset \tilde{F}^{i+j-1} Q\widehat{\mathrm{Pol}}$ and $[G^i Q\widehat{\mathrm{Pol}}, G^j Q\widehat{\mathrm{Pol}}] \subset G^{i+j} Q\widehat{\mathrm{Pol}}$.

Remark 2.11. When $B = 0$, observe that $\mathscr{D}_{B/R} = 0$, so we just have $Q\widehat{\mathrm{Pol}}_R(A,0;0) \simeq \prod_{p\geq 0}(\hat{\mathrm{Tot}}\,\gamma_p \mathcal{CC}_{R,BD_1}(A)\hbar^{p-1})$, which admits a filtered quasi-isomorphism from the complex $Q\widehat{\mathrm{Pol}}_R(A,0)$ of 0-shifted quantised polyvectors from [Pri5, Definition 1.16] as in Remark 1.17.

By Remark 1.27, there is always a projection $Q\widehat{\mathrm{Pol}}_R(A,M;0)[1] \to Q\widehat{\mathrm{Pol}}_R(A,0;0)[1]$, and the homotopy fibre over 0 is equivalent to the filtered L_∞-algebra underlying the DGAA $Q\widehat{\mathrm{Pol}}_A(M,-1) := \prod_{p\geq 0} F_p \mathscr{D}_{B/A}(M)\hbar^{p-1}$ when B is cofibrant over A. The latter follows because the HKR isomorphism for A ensures that $\mathcal{D}\mathit{iff}_{B/A} \to \mathcal{CC}_R(A, \mathcal{D}\mathit{iff}_{B/R})$ is a filtered quasi-isomorphism.

The following is standard:

Definition 2.12. Given a DGLA L, define the the Maurer–Cartan set by

$$\mathrm{MC}(L) := \{\omega \in L^1 \mid d\omega + \frac{1}{2}[\omega,\omega] = 0 \in L^2\}.$$

Following [Hin], define the Maurer–Cartan space $\underline{\mathrm{MC}}(L)$ (a simplicial set) of a nilpotent DGLA L by

$$\underline{\mathrm{MC}}(L)_n := \mathrm{MC}(L \otimes_{\mathbb{Q}} \Omega^\bullet(\Delta^n)),$$

where

$$\Omega^\bullet(\Delta^n) = \mathbb{Q}[t_0, t_1, \ldots, t_n, \delta t_0, \delta t_1, \ldots, \delta t_n]/(\sum t_i - 1, \sum \delta t_i)$$

is the commutative dg algebra of de Rham polynomial forms on the n-simplex, with the t_i of degree 0.

Given a pro-nilpotent DGLA $L = \varprojlim_i L_i$, define $\underline{\mathrm{MC}}(L) := \varprojlim_i \underline{\mathrm{MC}}(L_i)$.

Definition 2.13. Given a morphism $\phi\colon A \to B$ of cofibrant stacky CDGAs over R and a strict line bundle M over B, define the space $Q\mathcal{P}(A,M;0)$ of quantisations of the pair (A,M) to be the space

$$\underline{\mathrm{MC}}(\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,M;0)^{[1]})$$

of Maurer–Cartan elements of the pro-nilpotent DGLA $\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,M;0)^{[1]}$.

Replacing $\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,M;0)$ with its quotient by G^k gives a space $Q\mathcal{P}(A,B;0)/G^k$; we think of $\mathcal{P}(A,B;0) := Q\mathcal{P}(A,M;0)/G^1$ as being the space of 0-shifted co-isotropic structures on $A \to B$, in the sense that A carries a 0-shifted Poisson structure with respect to which B is co-isotropic.

Remark 2.14. Expanding out the definitions, a quantised co-isotropic structure, i.e. an element of $Q\mathcal{P}(A,M;0)$, is a Maurer–Cartan element of the pro-nilpotent DGLA

$$\tilde{F}^2 Q\widehat{\mathrm{Pol}}_R(A,M;0) = \prod_{p\geq 2} \hat{\mathrm{Tot}}\,(\gamma F)_p(\mathcal{CC}_{R,BD_1}(A, \mathcal{D}\mathit{iff}_{B/R}(M))_{[1]} \rtimes \mathcal{CC}_{R,BD_1}(A))_{[-1]}\hbar^{p-1}.$$

The term $\Delta_A \in \hat{\mathrm{Tot}}\,\mathcal{CC}_{R,BD_1}(A))^{[1]}[\![\hbar]\!]$ gives rise to a curved almost commutative A_∞-deformation $\tilde{A}$ of $\hat{\mathrm{Tot}}\,A$ over $R[\![\hbar]\!]$, via the canonical map

$$\hat{\mathrm{Tot}}\,\mathcal{CC}_{R,BD_1}(A))^{[1]} \to \mathrm{CC}_{R,BD_1}(\hat{\mathrm{Tot}}\,A)^{[1]}.$$

The A_∞-structure consists of operations $m_n\colon\ \hat{\mathrm{Tot}}\,A^{\otimes n} \to \hat{\mathrm{Tot}}\,A_{[n-2]}[\![\hbar]\!]$ for all $n \geq 0$ deforming the multiplication on A, with m_1 being the differential and m_0 the curvature; these satisfy higher associativity conditions.

The remaining term $\Delta_M \in \hat{\mathrm{Tot}}\,\mathcal{CC}_{R,BD_1}(A, \mathcal{D}\mathit{iff}_{B/R}(M))$ then gives rise to the data of a curved almost commutative A_∞-morphism $\tilde{A} \to \hat{\mathrm{Tot}}\,\mathcal{D}\mathit{iff}_{B/R}(M)[\![\hbar]\!]$ deforming the map $\hat{\mathrm{Tot}}\,A \to \hat{\mathrm{Tot}}\,\mathcal{D}\mathit{iff}_{B/R}(M)$, via the canonical map

$$\hat{\mathrm{Tot}}\,\mathcal{CC}_{R,BD_1}(A, \mathcal{D}\mathit{iff}_{B/R}(M)) \to \mathrm{CC}_{R,BD_1}(\hat{\mathrm{Tot}}\,A, \hat{\mathrm{Tot}}\,\mathcal{D}\mathit{iff}_{B/R}(M)).$$

The A_∞-morphism Δ_M consists of maps $f_n\colon\ \hat{\mathrm{Tot}}\,\tilde{A}^{\otimes_{R[\![\hbar]\!]} n} \to \hat{\mathrm{Tot}}\,\mathcal{D}\mathit{iff}_{B/R}(M)_{[n-1]}[\![\hbar]\!]$ for all $n \geq 0$ deforming the composite $A \to B \to \mathcal{D}\mathit{iff}_{B/R}(M)$, with f_0 deforming the differential δ. These satisfy compatibility conditions with the respective A_∞-structures; in particular, $(f_0)^2 = \sum_n f_n(m_0, \ldots, m_0)$, so this gives $\tilde{M} := (\hat{\mathrm{Tot}}\,M[\![\hbar]\!], f_0)$ the structure of a curved $\tilde{A}$-module.

However, there are additional restrictions on the resulting deformations: if we filter $\hat{\mathrm{Tot}}\,V$ by setting $\mathrm{Fil}^p\hat{\mathrm{Tot}}\,V := \mathrm{Tot}^{\Pi} V^{\geq p}$, then each component of the A_∞-structure m or A_∞-morphism f must be bounded in the sense that for some integer r, each Fil^p is mapped to Fil^{p+r}. When the stacky CDGAs are bounded in the cochain direction, as occurs when they originate from 1-geometric derived Artin stacks, these boundedness restrictions are vacuous (cf. [Pri5, Example 1.20]), but there are still restrictions arising because $\hat{\mathrm{Tot}}$ does not preserve cofibrant objects.

Examples 2.15. Here is how the descriptions of quantised co-isotropic structures simplify in settings where we need not worry about the subtleties resulting from $\hat{\mathrm{Tot}}$:

(1) If A and B are smooth R-algebras concentrated in degrees $(0,0)$, then objects of $Q\mathcal{P}(A, M; 0)$ just correspond to $R[\![\hbar]\!]$-deformations $\tilde{A}$ of A as an associative algebra, equipped with $R[\![\hbar]\!]$-algebra homomorphisms $\phi\colon\ \tilde{A} \to \prod_{p\geq 0} F_p\mathcal{D}\mathit{iff}_{B/R}(M)\hbar^p =: \hat{\xi}(\mathcal{D}\mathit{iff}_{B/R}(M), F)$ deforming the composite $A \to B = F_0\mathcal{D}\mathit{iff}_{B/R}(M)$.

However, curvature does manifest itself on the level of morphisms, with an isomorphism between two objects $(\tilde{A}, \phi), (\tilde{A}', \phi')$ consisting of an isomorphism $\theta\colon\ \tilde{A} \cong \tilde{A}'$ deforming id_A, together with an element of

$$\begin{aligned} &\exp(\textstyle\prod_{p\geq 1} F_{p+1}\mathcal{D}\mathit{iff}_{B/R}(M)\hbar^p) \\ = &\{g \in 1 + \hbar\mathcal{D}\mathit{iff}_{B/R}(M)[\![\hbar]\!]\ :\ g\hat{\xi}(\mathcal{D}\mathit{iff}_{B/R}(M), F)g^{-1} \subset \hat{\xi}(\mathcal{D}\mathit{iff}_{B/R}(M), F)\}\end{aligned}$$

intertwining $\phi' \circ \theta$ and ϕ.

There are also 2-isomorphisms in the form of elements of $1 + \hbar\tilde{A} = \exp(\hbar\tilde{A})$ intertwining 1-morphisms in the obvious way. The space $Q\mathcal{P}(A, M; 0)$ is then equivalent to the nerve of this 2-groupoid.

(2) In the special case of the previous example where the morphism $A \to B$ is surjective, we can relax a condition by just requiring that $\phi\colon\ \tilde{A} \to \mathrm{End}_R(M)[\![\hbar]\!]$, since almost commutativity of $\tilde{A}$ then combines with surjectivity to guarantee that the image of $\tilde{A}$ is contained in $\hat{\xi}(\mathcal{D}\mathit{iff}_{B/R}(M), F)$.

If in addition the map $\tilde{A} \to \hat{\xi}(\mathcal{D}\mathit{iff}_{B/R}(M), F)$ is surjective, as happens when the underlying Poisson structure is non-degenerate, then we may also relax the condition on the intertwiner g to say that $g \in 1 + \hbar \mathrm{End}_R(M)[\![\hbar]\!]$, since it must automatically then lie in $\exp(\prod_{p\geq 1} F_{p+1}\mathcal{D}\mathit{iff}_{B/R}(M)\hbar^p)$. For such fixed $\tilde{A}$, this means that the space $Q\mathcal{P}(A, M; 0) \times^h_{Q\mathcal{P}(A,0)} \{\tilde{A}\}$ of quantisations lifting $\tilde{A}$ is equivalent to the nerve of the groupoid of $\tilde{A}$-modules deforming the A-module M.

(3) We can generalise (1) to consider the case where A and B are functions on derived affine schemes, so $A = A^0_\bullet$ and $B = B^0_\bullet$ are chain complexes concentrated in non-negative degrees. Again, curvature does not manifest itself in deformations of A, but there can now be a curvature term in the morphism $f\colon \tilde{A} \to \prod_{p\geq 0} F_p\mathcal{D}\mathit{iff}_{B/R}(M)\hbar^p$.

In this setting, we may use the bar-cobar adjunction $\Omega_{BD_1} \dashv \mathrm{B}_{BD_1}$ to replace A_∞-structures with genuinely associative structures. Explicitly, an element of $Q\mathcal{P}(A; 0)$ corresponds to a flat $\hbar$-adically complete DGAA A' over $R[\![\hbar]\!]$ which is commutative modulo $\hbar$ and equipped with a quasi-isomorphism $A'/\hbar \simeq A$. An element of $Q\mathcal{P}(A, M; 0)$ then combines this with a Maurer–Cartan element (effectively a BV operator, cf. [Pri7, Remark 1.14] and [Kra, Definition 9]) $f_0 \in \prod_{p\geq 1} F_p\mathcal{D}\mathit{iff}_{B/R}(M)_{-1}\hbar^{p-1}$ deforming δ, together with an $R[\![\hbar]\!]$-algebra homomorphism $f_+\colon A' \to (\prod_{p\geq 0} F_p\mathcal{D}\mathit{iff}_{B/R}(M)\hbar^p, \mathrm{ad}_{f_0})$.

The spaces of morphisms in the ∞-groupoid $Q\mathcal{P}(A, M; 0)$ then have contributions from intertwiners generalising the underived situation, in addition to the usual higher homotopies given by localising at quasi-isomorphisms. Beware that the DGAA A' must be genuinely commutative modulo $\hbar$ — a quasi-isomorphism $A'/\hbar \simeq A$ alone does not suffice to make A' almost commutative.

(4) Generalising in the opposite direction, we can consider the case where there is stacky structure but no derived structure, so $A = A^\bullet_0$ and $B = B^\bullet_0$ are cochain complexes concentrated in non-negative degrees, with A^0_0, B^0_0 smooth and $A^{\#}_0, B^{\#}_0$ freely generated over them by graded projective modules. Then an element of $Q\mathcal{P}(A; 0)$ can be described as an almost commutative curved A_∞-deformation $\tilde{A}$ of A, which can more precisely be encoded as a deformation $\tilde{C}$ of the coderivation on the $\hbar$-adic completion of the $R[\hbar]$-DGAC $\xi(\mathrm{B}_{BD_1}A)$. An element of $Q\mathcal{P}(A, M; 0)$ combines this with the structure of an almost commutative curved A_∞-morphism $\tilde{A} \to \prod_{p\geq 0} F_p\mathcal{D}\mathit{iff}_{B/R}(M)\hbar^p$, which can more precisely be characterised as a dg coalgebra morphism from $\tilde{C}$ to the $\hbar$-adic completion of $\xi(\mathrm{B}_{BD_1}(\mathcal{D}\mathit{iff}_{B/R}(M), F))$.

In this setting, curvature manifests itself immediately on the level of objects, so the structural differentials m_1 and f_0 lifting δ in our induced deformations of A and M need not square to 0; in particular, this means that f_0 need not define a (-1)-shifted quantisation of the line bundle M on B in the sense of [Pri7]. Also beware that the nature of the model structure we chose in Lemma 2.4 means that these constructions are not invariant under cochain quasi-isomorphisms, in contrast to the situation with chain quasi-isomorphisms in the purely derived setting above.

Definition 2.16. Define the filtered tangent space to relative quantised polyvectors by

$$T Q\widehat{\mathrm{Pol}}(A,M;0) := Q\widehat{\mathrm{Pol}}(A,M;0) \oplus \hbar Q\widehat{\mathrm{Pol}}_R(A,M;0)\epsilon,$$
$$\tilde{F}^j T Q\widehat{\mathrm{Pol}}(A,M;0) := \tilde{F}^j Q\widehat{\mathrm{Pol}}(A,M;0) \oplus \hbar \tilde{F}^j Q\widehat{\mathrm{Pol}}(A,M;0)\epsilon,$$

for ϵ of degree 0 with $\epsilon^2 = 0$. Then $TQ\widehat{\mathrm{Pol}}(A,M;0)^{[1]}$ is a DGLA, with Lie bracket given by $[u + v\epsilon, x + y\epsilon] = [u,x] + [u,y]\epsilon + [v,x]\epsilon$.

Write $TQ\mathcal{P}(A,M;0)$ for the space

$$\underline{\mathrm{MC}}(\tilde{F}^2 TQ\widehat{\mathrm{Pol}}(A,M;0)^{[1]});$$

this is effectively the tangent bundle on the space of quantised co-isotropic structures.

Definition 2.17. Given a Maurer–Cartan element $\Delta \in \mathrm{MC}(Q\widehat{\mathrm{Pol}}_R(A,M;0))$, define $T_\Delta Q\widehat{\mathrm{Pol}}_R(A,M;0)$ to be the non-unital brace algebra

$$(\hbar Q\widehat{\mathrm{Pol}}_R(A,M;0)_{\#}, \delta_{\mathrm{QPol}} + [\Delta,-]).$$

We define filtrations $\tilde{F}$ and G on $T_\Delta Q\widehat{\mathrm{Pol}}_R(A,M;0)$ by

$$\tilde{F}^i T_\Delta Q\widehat{\mathrm{Pol}}_R(A,M;0)_{\#} := \hbar\tilde{F}^i Q\widehat{\mathrm{Pol}}_R(A,M;0)_{\#},$$
$$G^j T_\Delta Q\widehat{\mathrm{Pol}}_R(A,M;0) := \hbar^j T_\Delta Q\widehat{\mathrm{Pol}}_R(A,M;0).$$

Note that $(T_\Delta Q\widehat{\mathrm{Pol}}_R(A,M;0), \tilde{F})$ is a non-unital almost commutative brace algebra over R.

Observe that $T_\Delta Q\mathcal{P}(A,M;0) := \underline{\mathrm{MC}}(\tilde{F}^2 T_\Delta Q\widehat{\mathrm{Pol}}(A,M;0)^{[1]})$ is just the fibre of $TQ\mathcal{P}(A,M;0) \to Q\mathcal{P}(A,M;0)$ over Δ; we think of this as the tangent space at Δ of the space of quantised co-isotropic structures. We regard the cohomology of $T_\Delta Q\widehat{\mathrm{Pol}}_R(A,M;0)$ as a form of relative quantised Poisson cohomology.

Definition 2.18. Given $\Delta \in Q\mathcal{P}(A,M;0)$, define $\sigma(\Delta) \in \mathrm{Z}^2(\tilde{F}^2 T_\Delta Q\widehat{\mathrm{Pol}}(A,M;0))$ to be

$$-\partial_{\hbar^{-1}}\Delta = \hbar^2 \frac{\partial \Delta}{\partial \hbar}.$$

More generally, define the global section $\sigma\colon Q\mathcal{P}(A,M;0) \to TQ\mathcal{P}(A,M;0)$ of the tangent bundle to be the map induced by the morphism $Q\widehat{\mathrm{Pol}}(A,M;0) \to TQ\widehat{\mathrm{Pol}}(A,M;0)$ of filtered DGLAs given by $\Delta \mapsto \Delta - \partial_{\hbar^{-1}}\Delta\epsilon$.

As in [Pri4, §3.3], we will usually consider stacky CDGAs $A \in DG^+dg\mathrm{CAlg}(R)$ satisfying the following properties, since we can resolve derived Artin stacks by stacky CDGAs of this form, which can be thought of models for derived higher Lie algebroids:

Assumption 2.19.
1. for any cofibrant replacement $\tilde{A} \to A$ in the model structure of Lemma 2.4, the morphism $\Omega^1_{\tilde{A}/R} \to \Omega^1_{A/R}$ is a levelwise quasi-isomorphism,
2. the $A^{\#}$-module $(\Omega^1_{A/R})^{\#}$ in graded chain complexes is cofibrant (i.e. it has the left lifting property with respect to all surjections of $A^{\#}$-modules in graded chain complexes),
3. there exists N for which the chain complexes $(\Omega^1_{A/R} \otimes_A A^0)^i$ are acyclic for all $i > N$.

The following lemma breaks down the complex of quantised relative polyvectors into manageable pieces given by powers of tangent complexes.

Lemma 2.20. *If A and B are both cofibrant and satisfy Assumption 2.19, then $\mathrm{gr}^i_G\tilde{F}^pQ\widehat{\mathrm{Pol}}(A,M;0)$ is quasi-isomorphic to the cocone of*

$$\prod_{j\ge p}\underline{\hat{\mathrm{Hom}}}_A(\Omega^{j-i}_{A/R},A)\hbar^{j-1}[i-j]\to\prod_{j\ge p}\underline{\hat{\mathrm{Hom}}}_B(\mathbf{L}\mathrm{CoS}^{j-i}_B\mathbb{L}_{B/A},B)\hbar^{j-1}$$

coming from the connecting homomorphism $S\colon\ \mathbf{L}\Omega^1_{B/A}=\mathbb{L}_{B/A}\to\Omega^1_{A/R}[1]$.

Moreover, $\mathrm{gr}^i_G\tilde{F}^pT_\Delta Q\widehat{\mathrm{Pol}}(A,M;0)$ is quasi-isomorphic to $\hbar\mathrm{gr}^i_G\tilde{F}^pT_\Delta Q\widehat{\mathrm{Pol}}(A,M;0)$.

Proof. By construction, $\mathrm{gr}^i_G\tilde{F}^pQ\widehat{\mathrm{Pol}}$ is the cocone of

$$\prod_{j\ge p}\hat{\mathrm{Tot}}\,\mathrm{gr}^{\gamma}_{j-i}\mathcal{CC}^\bullet_R(A)\hbar^{j-1}\to\prod_{j\ge p}\hat{\mathrm{Tot}}\,\mathrm{gr}^{\gamma F}_{j-i}\mathcal{CC}^\bullet_R(A,\mathcal{D}\mathit{iff}_{B/R})\hbar^{j-1}.$$

Since B is assumed cofibrant, we have isomorphisms

$$\mathrm{gr}^F_k\mathcal{D}\mathit{iff}_{B/R}\to\mathcal{H}\mathit{om}_B(\mathrm{CoS}^k_B\Omega^1_{B/R},B).$$

The bar-cobar resolution for A as a commutative algebra then gives quasi-isomorphisms

$$\mathcal{H}\mathit{om}_A(\Omega^{j-i}_{A/R},A)[i-j]\to\mathrm{gr}^{\gamma}_{j-i}\mathcal{CC}^\bullet_R(A)$$
$$\mathcal{H}\mathit{om}_A(\mathrm{CoS}^{j-1}_B(\mathrm{cocone}(\Omega^1_{B/R}\to\Omega^1_{A/R}\otimes_A B)),B)\to\mathrm{gr}^{\gamma F}_{j-i}\mathcal{CC}^\bullet_R(A,\mathcal{D}\mathit{iff}_{B/R}).$$

Since $\mathrm{cocone}(\Omega^1_{B/R}\to\Omega^1_{A/R}\otimes_A B)$ is a model for the cotangent complex $\mathbb{L}_{B/A}$, the results follow. □

Given an element $\Delta\in Q\mathcal{P}(A,M;0)$, we write Δ_A for the image in $Q\mathcal{P}(A,0)$ and Δ_B for the image in $\hat{\mathrm{Tot}}\,\mathcal{CC}_{R,BD_1}(A,\mathcal{D}\mathit{iff}_{B/R})$. If we write $\Delta=\sum_{j\ge2}\Delta_j\hbar^{j-1}$, then by working modulo $G^1+\tilde{F}^3$, Lemma 2.20 allows us to identify $\Delta_2=(\Delta_{2,A},\Delta_{2,B})$ with a closed element of degree 0 in the cocone of

$$\underline{\hat{\mathrm{Hom}}}_A(\Omega^2_{A/R},A)\to\mathbf{R}\underline{\hat{\mathrm{Hom}}}_B(\mathbf{L}\mathrm{CoS}^2_B\mathbb{L}_{B/A},B)[2].$$

Now $\Delta_{2,A}$ defines a closed element of the first space, and since the composition of this map with

$$\underline{\hat{\mathrm{Hom}}}_B(\mathbf{L}\mathrm{CoS}^2_B\mathbb{L}_{B/A},B)\to\underline{\hat{\mathrm{Hom}}}_B(\Omega^1_{B/R}\otimes^{\mathbf{L}}_B\mathbb{L}_{B/A},B)$$

is homotopic to 0, $\Delta_{2,B}$ defines a closed element of the latter.

We then have a diagram

$$\begin{array}{ccc}\Omega^1_{A/R} & \longrightarrow & \Omega^1_{B/R}\\ {\scriptstyle\Delta^\sharp_{2,A}}\downarrow & & \downarrow{\scriptstyle\Delta^\sharp_{2,B}}\\ \underline{\hat{\mathrm{Hom}}}_A(\Omega^1_{A/R},A) & \xrightarrow{S} & \mathbf{R}\underline{\hat{\mathrm{Hom}}}_B(\mathbb{L}_{B/A},B)[1]\end{array}$$

commuting up to a canonical homotopy coming from $\Delta_{2,B}$.

Definition 2.21. Say that a quantisation Δ of the pair (A,M) is non-degenerate if the maps

$$\Delta^\sharp_{2,A}\colon\ \mathrm{Tot}^{\Pi}(\Omega^1_{A/R}\otimes_A A^0)\to\underline{\hat{\mathrm{Hom}}}_A(\Omega^1_A,A^0)$$
$$\Delta^\sharp_{2,B}\colon\ \mathrm{Tot}^{\Pi}(\Omega^1_{B/R}\otimes_B B^0)\to\mathbf{R}\underline{\hat{\mathrm{Hom}}}_B(\mathbb{L}_{B/A},B^0)^{[1]}$$

are quasi-isomorphisms and $\mathrm{Tot}^{\Pi}(\Omega^1_{A/R}\otimes_A A^0)$ (resp. $\mathrm{Tot}^{\Pi}(\Omega^1_{B/R}\otimes_B B^0)$) is a perfect complex over A^0 (resp. B^0).

In other words, a non-degenerate quantisation gives an equivalence between the cotangent and tangent complexes of A, and between the cotangent complex of B and the derived normal bundle of B over A.

3. Compatibility of quantisations and isotropic structures

In this section, we introduce generalised isotropic structures, develop the notion of compatibility between a quantisation and a generalised isotropic structure, and give some preliminary existence results for quantisations of Lagrangians.

3.1. Morphisms from the de Rham algebra.

Definition 3.1. Given a stacky CDGA A over R, define the stacky de Rham algebra of A to be the complete filtered stacky CDGA

$$\mathcal{DR}(A/R)^n_i := \prod_{j\geq 0}(\Omega^j_A)^n_{i+j}$$

with filtration $F^p\mathcal{DR}(A/R) = \prod_{j\geq p}(\Omega^j_A)_{[j]}$, cochain differential ∂ and chain differential $\delta \pm d$, where d is the de Rham differential, and the differentials ∂, δ are induced from those on A.

We then write $\mathrm{DR}(A/R) := \hat{\mathrm{Tot}}\, \mathcal{DR}(A/R)$.

In particular, beware that the de Rham differential is absorbed in the chain (derived) structure, not the cochain (stacky) structure.

Lemma 3.2. *Given a morphism $A \to \mathrm{gr}^0_F B$ of stacky CDGAs over R, with A cofibrant and (B, F) a complete filtered stacky CDGA, there is an associated filtered stacky CDGA morphism $\mathcal{DR}(A/R) \to F^0B$ over R, unique up to coherent homotopy.*

Proof. We may assume that A is cofibrant, and then $\mathcal{DR}(A)$ is cofibrant as a complete filtered stacky CDGA, in the sense that it has the left lifting property with respect to surjections of complete filtered stacky CDGAs over R which are levelwise filtered quasi-isomorphisms. For any filtered A-module (M, F), we may regard M as a $\mathcal{DR}(A)$-module via the projection $\mathcal{DR}(A) \to A$. When $M = F^1M$, the double complex $\mathcal{H}om_{\mathcal{DR}(A),\mathrm{Fil}}(\Omega^1_{\mathcal{DR}(A)/R}, M)$ of filtered derivations from $\mathcal{DR}(A)$ to M is then levelwise acyclic, by the construction of $\mathcal{DR}(A)$.

Now, the double complex $\mathcal{H}om_{\mathcal{DR}(A),\mathrm{Fil}}(\Omega^1_{\mathcal{DR}(A)/R}, \mathrm{gr}^r_F B)$ governs the obstruction theory to lifting maps from $\mathcal{DR}(A)$ along the square-zero extension $F^0B/F^{r+1}B \to F^0B/F^rB$. Thus the acyclicity above gives the required equivalence of mapping spaces

$$\mathrm{map}_{\mathrm{Fil}}(\mathcal{DR}(A), B) \simeq \mathrm{map}(A, \mathrm{gr}^0_F B)$$

of filtered stacky CDGAs and of stacky CDGAs, respectively. □

The following is a slight generalisation of [Pri4, Lemma 1.17]:

Lemma 3.3. *Take a cofibrant stacky CDGA A over R, a complete filtered CDGA B over R, and a filtered morphism $\phi\colon \mathrm{DR}(A/R) \to B$. Then for any derivation $\pi \in \mathrm{MC}(F^1\underline{\mathrm{Der}}_R(B))$, there is an associated filtered CDGA morphism*

$$\mu(-,\pi)\colon \mathrm{DR}(A/R) \to (B, \delta + \pi)$$

given by $\mu(a, \pi) = \phi(a)$ and $\mu(df, \pi) = \phi(df) + \pi\phi(f)$ for $a, f \in A$.

Proof. The formulae clearly define a filtered morphism $\mu(-,\pi)\colon \mathrm{DR}(A)^{\#} \to B^{\#}$ of graded algebras, since $\phi\circ d + \pi\circ\phi$ defines a derivation on A with respect to $\phi\colon A \to B$. We therefore need only check that μ is a chain map. We have

$$\begin{aligned} \delta\mu(a,\pi) &= \phi(\delta a) + \phi(da) \\ \pi\mu(a,\pi) &= \pi\phi(a) \\ (\delta+\pi)\mu(a,\pi) &= \mu(\delta a + da, \pi), \end{aligned}$$

and the calculation above applied to $a = f$ and using that $(\delta+\pi)^2 = 0$ gives

$$\begin{aligned} (\delta+\pi)\mu(df,\pi) &= -(\delta+\pi)\mu(\delta f,\pi) \\ &= -(\delta+\pi)\phi(f) \\ &= -\phi(d\delta f) - \pi\phi(\delta f) \\ &= \mu(-d\delta f, \pi) \\ &= \mu((\delta - d)df, \pi), \end{aligned}$$

as required. □

Combining Lemmas 3.2 and 3.3 gives:

Lemma 3.4. *Take a morphism $\phi\colon A \to \mathrm{gr}^0_F B$ of stacky CDGAs over R, with A cofibrant and B a complete filtered stacky CDGA. Then for any $\pi \in \underline{\mathrm{MC}}(\hat{\mathrm{Tot}}\, F^1\mathcal{D}er_R(B))$, there is an associated morphism*

$$\mu(-,\pi)\colon \mathrm{DR}(A/R) \to (\hat{\mathrm{Tot}}\, B, \delta+\pi),$$

of filtered CDGAs, unique up to coherent homotopy.

3.2. **The compatibility map.** We now develop the notion of compatibility between de Rham data and quantisations of a pair $(A \to B)$, generalising the notion of compatibility between generalised 0-shifted pre-symplectic structures and E_1 quantisations from [Pri5]. We begin by recalling some observations from [Pri5, §2.2].

As explained succinctly in [Pet], a choice of Levi decomposition of the Grothendieck–Teichmüller group (equivalently, a Drinfeld 1-associator) over $\mathbb{Q}$ gives a formality quasi-isomorphism $E_2 \simeq P_2$. Writing τ for the good truncation filtration $\tau_{\geq p}$ on a homological operad, a formality quasi-isomorphism automatically gives a filtered quasi-isomorphism $(E_2,\tau) \simeq (P_2,\tau)$. The filtration τ on P_2 gives the commutative multiplication weight 0 and the Lie bracket weight -1, and we refer to (P_2,τ)-algebras in complete filtered complexes as almost commutative P_2-algebras.

Likewise, the map in [Vor] from the E_2 operad to the brace operad Br must preserve the good truncation filtrations. Finally, note that the good truncation filtration is contained in the filtration γ on Br from Definition 1.19, since all operations of homological degree r lie in γ^r, so in particular the closed operations do so. Thus every almost commutative brace algebra can be regarded as an (E_2,τ)-algebra.

Definition 3.5. Given a Levi decomposition $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ of the Grothendieck–Teichmüller group GT over $\mathbb{Q}$, we denote by p_w the resulting ∞-functor from almost commutative brace algebras to almost commutative P_2-algebras over $\mathbb{Q}$, induced by the filtered quasi-isomorphism $(E_2,\tau) \simeq (P_2,\tau)$ as above.

Since the natural morphism from the Lie operad to the E_2 operad is given in each arity by inclusion of the top weight term for the decreasing filtration, it follows that the ∞-functor p_w automatically commutes with the fibre functors $A \mapsto F_1 A$ to the underlying filtered DGLAs,

Definition 3.6. For any of the definitions from §2, we add the subscript w to indicate that we are replacing $\mathcal{CC}_{R,BD_1}(A)$ with $p_w\mathcal{CC}_{R,BD_1}(A)$ in the construction.

Since the DGLAs underlying $\mathcal{CC}_{R,BD_1}(A)$ and $p_w\mathcal{CC}_{R,BD_1}(A)$ are filtered quasi-isomorphic, in particular we have canonical weak equivalences $Q\mathcal{P}_w(A,0) \simeq Q\mathcal{P}(A,0)$. Properties of the filtration $\tilde{F}$ then ensure that the complexes $T_\Delta Q\widehat{\mathrm{Pol}}_w(A,0)$ are filtered (P_2,τ)-algebras.

Definition 3.7. Given a choice $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ of Levi decomposition for GT and an element $\Delta \in Q\mathcal{P}(A,M;0)_w/G^j$, define

$$\mu_w(-,\Delta)\colon \mathrm{cocone}(\mathrm{DR}(A/R) \to \mathrm{DR}(B/R))[\![\hbar]\!]/\hbar^j \to T_\Delta Q\widehat{\mathrm{Pol}}_w(A;B,0)/G^j$$

as follows.

Since $[B, F_i\mathcal{D}\mathit{iff}_{B/A}] \subset F_{i-1}\mathcal{D}\mathit{iff}_{B/A}$, we have a map $B \to \mathrm{gr}^0_{\gamma F}\mathcal{CC}_{R,BD_1}(\mathcal{D}\mathit{iff}_{B/A})$. Combined with the weak equivalence $\mathcal{D}\mathit{iff}_{B/A} \to \mathcal{CC}_{R,BD_1}(A,\mathcal{D}\mathit{iff}_{B/R})$, up to coherent homotopy this gives a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathrm{gr}^0_{\tilde{\gamma}}(p_w\mathcal{CC}_{R,BD_1}(A))[\![\hbar]\!]/\hbar^j) & \longrightarrow & \mathrm{gr}^0_{\widetilde{\gamma F}}(p_w\mathcal{CC}_{R,BD_1}(\mathcal{D}\mathit{iff}_{B/A})[\![\hbar]\!]/\hbar^j) \end{array}$$

where the filtrations on the bottom row are taken to be $(\widetilde{\gamma F})^p := \prod_{i\ge p}(\gamma F)_i\hbar^i$.

Applying Lemma 3.4 to this diagram and the Maurer–Cartan elements on the bottom line induced by Δ yields a diagram

$$\begin{array}{ccc} \mathrm{DR}(A) & \xrightarrow{\mu_w(-,\Delta)} & (\hat{\mathrm{Tot}}\,\tilde{\gamma}^0(p_w\mathcal{CC}_{R,BD_1}(A)[\![\hbar]\!]/\hbar^j), \delta + [\Delta_A,-]) \\ \downarrow & & \downarrow \\ \mathrm{DR}(B) & \xrightarrow{\mu_w(-,\Delta)} & (\hat{\mathrm{Tot}}\,\widetilde{\gamma F}^0(p_w\mathcal{CC}_{R,BD_1}(\mathcal{D}\mathit{iff}_{B/A})[\![\hbar]\!]/\hbar^j), \delta + [\Delta_B,-]) \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & (\hat{\mathrm{Tot}}\,\widetilde{\gamma F}^0(p_w\mathcal{CC}_{R,BD_1,+}(\mathcal{D}\mathit{iff}_{B/A})[\![\hbar]\!]/\hbar^j), \delta + [\Delta_B,-]), \end{array}$$

and taking homotopy limits of the columns gives the desired map.

Remark 3.8. When $B=0$, this recovers the definition of μ_w from [Pri5, Definition 2.11]. When $R=A$, this definition is slightly different from that in [Pri7, Definition 1.32]. The construction there relied on a filtered DGAA resolution $\mathrm{DR}'(B/R)$ of $\mathrm{DR}(B/R)$, with [Pri7, Lemma 1.31] giving a non-commutative analogue of Lemma 3.4.

Instead, Definition 3.7 effectively constructs the map $\mu_w\colon \mathrm{DR}(B/R) \to T_\Delta\mathscr{D}_{B/R}$ in this setting by first taking

$$\mathrm{DR}(B/R) \to p_w\hat{\mathrm{Tot}}\,\mathcal{CC}_{R,BD_1}(\mathcal{D}\mathit{iff}_{B/R})$$

using the commutative structure underlying a P_2-algebra, then applying the projection $\mathcal{CC}_{R,BD_1}(\mathcal{D}\mathit{iff}_{B/R}) \to \mathcal{D}\mathit{iff}_{B/R}$. The map μ_w then converges more quickly than the map μ in [Pri7], but depends on a choice of formality isomorphism.

This raises the question of whether the construction of [Pri7] could be adapted to unshifted symplectic structures, giving equivalences not relying on formality. This would mean establishing an analogue of Lemma 3.2 giving a universal property for $\mathrm{DR}(B/R)$ within a suitable category of filtered E_2-algebras. The filtered DGAA $\mathrm{DR}'(B/R)$ is not almost commutative, but the left and right A-module structures on $\mathrm{gr}_F\mathrm{DR}'(B/R)$ agree. Similarly, $\mathrm{DR}(B/R)$ will not have the desired universal property in BD_2-algebras, but the analogy raises the possibility that it might do so in some larger category.

3.2.1. *Generalised Lagrangians.* We now fix a cofibrant stacky CDGA A over R, and a cofibration $A \to B$ of stacky CDGAs over R.

Definition 3.9. Recall that a 0-shifted pre-symplectic structure ω on A/R is an element
$$\omega \in \mathrm{Z}^2 F^2\mathrm{DR}(A/R).$$
It is called symplectic if $\omega_2 \in \mathrm{Z}^0\mathrm{Tot}^{\Pi}\Omega^2_{A/R}$ induces a quasi-isomorphism
$$\omega_2^\sharp \colon \underline{\hat{\mathrm{Hom}}}_A(\Omega^1_{A/R}, A^0) \to \mathrm{Tot}^{\Pi}(\Omega^1_{A/R} \otimes_A A^0)$$
and $\mathrm{Tot}^{\Pi}(\Omega^1_{A/R} \otimes_A A^0)$ is a perfect complex over A^0.

An isotropic structure on B relative to ω is an element (ω, λ) of
$$\mathrm{Z}^2\mathrm{cocone}(F^2\mathrm{DR}(A/R) \to F^2\mathrm{DR}(B/R))$$
lifting ω. This structure is called Lagrangian if ω is symplectic and the image $\bar{\lambda}_2$ of λ in $\mathrm{Z}^{-1}\mathrm{Tot}^{\Pi}\Omega^1_{B/R} \otimes_B \Omega^1_{B/A}$ induces a quasi-isomorphism
$$\lambda_2^\sharp \colon \underline{\hat{\mathrm{Hom}}}_B(\Omega^1_{B/A}, B^0) \to \mathrm{Tot}^{\Pi}(\Omega^1_{B/R} \otimes_B B^0)^{[-1]}$$
and $\mathrm{Tot}^{\Pi}(\Omega^1_{B/A} \otimes_B B^0)$ is a perfect complex over B^0.

Definition 3.10. Define a decreasing filtration $\tilde{F}$ on $\mathrm{DR}(A/R)[\![\hbar]\!]$ by
$$\tilde{F}^p\mathrm{DR}(A/R) := \prod_{i \geq 0} F^{p-i}\mathrm{DR}(A/R)\hbar^i.$$

Define a further filtration G by $G^k\mathrm{DR}(A/R)[\![\hbar]\!] = \hbar^k\mathrm{DR}(A/R)[\![\hbar]\!]$.

Definition 3.11. Define the space of generalised 0-shifted isotropic structures on the pair (A, B) over R to be the simplicial set
$$G\mathrm{Iso}(A, B; 0) := \underline{\mathrm{MC}}(\tilde{F}^2\mathrm{cone}(\mathrm{DR}(A/R)[\![\hbar]\!] \to \mathrm{DR}(B/R)[\![\hbar]\!])),$$
where we regard the cochain complexes as a DGLA with trivial bracket.

Also write $G\mathrm{Iso}(A, B; 0)/\hbar^k$ for the obvious truncation in terms of $\mathrm{DR}[\hbar]/\hbar^k$, so $G\mathrm{Iso}(A, B; 0) = \varprojlim_k G\mathrm{Iso}(A, B; 0)/\hbar^k$. Write $\mathrm{Iso} = G\mathrm{Iso}/\hbar$.

Set $G\mathrm{Lag}(A, B; 0) \subset G\mathrm{Iso}(A, B; 0)$ to consist of the points whose images in $\mathrm{Iso}(A, B; 0)/\hbar$ are Lagrangians on symplectic structures — this is a union of path-components.

Thus the components of $G\mathrm{Iso}(A, B; 0)$ are just elements in
$$\mathrm{H}^1\mathrm{cone}(\tilde{F}^2\mathrm{DR}(A/R)[\![\hbar]\!] \to \tilde{F}^2\mathrm{DR}(B/R)[\![\hbar]\!]),$$
where $\tilde{F}^2\mathrm{DR}(A/R)[\![\hbar]\!] = F^2\mathrm{DR}(A/R) \oplus \hbar F^1\mathrm{DR}(A/R) \oplus \hbar^2\mathrm{DR}(A/R)[\![\hbar]\!]$ and similarly for $\tilde{F}^2\mathrm{DR}(B/R)[\![\hbar]\!]$, so we can think of these as power series in certain relative cohomology groups. Equivalence classes of n-morphisms in $G\mathrm{Iso}(A, B; 0)$ are then given by elements in H^{1-n} of the same complex.

3.2.2. *Compatible structures.* In addition to our morphism $A \to B$, we now fix a strict line bundle M over B, in the sense of Definition 2.9.

Definition 3.12. We say that a generalised isotropic structure (ω, λ) and a quantisation Δ of the pair (A, M) are w-compatible (or a w-compatible pair) if

$$[\mu_w(\omega, \lambda; \Delta)] = [-\partial_{\hbar^{-1}}(\Delta)] \in \mathrm{H}^1(\tilde{F}^2 T_\Delta Q\widehat{\mathrm{Pol}}_w(A, M; 0)) \cong \mathrm{H}^1(\tilde{F}^2 T_\Delta Q\widehat{\mathrm{Pol}}(A, M; 0)),$$

where $\sigma = -\partial_{\hbar^{-1}}$ is the canonical tangent vector of Definition 2.18.

This definition is chosen to lift the notion of compatibility between Poisson and symplectic structures from [Pri4, §1.3]. As we will see, when Δ is non-degenerate it is fairly straightforward to solve for (ω, λ) in terms of Δ because $\mu_w(-; \Delta)$ is a filtered quasi-isomorphism. The other direction, associating quantised co-isotropic structures to generalised isotropic structures, will require indirect arguments in terms of obstruction theory, as in the unquantised setting.

Definition 3.13. Given a simplicial set Z, an abelian group object A in simplicial sets over Z, a space X over Z and a morphism $s\colon X \to A$ over Z, define the homotopy vanishing locus of s over Z to be the homotopy limit of the diagram

$$X \underset{0}{\overset{s}{\rightrightarrows}} A \longrightarrow Z.$$

Definition 3.14. Define the space $Q\mathrm{Comp}_w(A, M; 0)$ to be the homotopy vanishing locus of

$$(\mu_w - \sigma)\colon\ G\mathrm{Iso}(A, B; 0) \times Q\mathcal{P}_w(A, M; 0) \to TQ\mathcal{P}_w(A, M; 0)$$

over $Q\mathcal{P}_w(A, M; 0)$

We define a cofiltration on this space by setting $Q\mathrm{Comp}_w(A, M; 0)/G^j$ to be the homotopy vanishing locus of

$$(\mu_w - \sigma)\colon\ (G\mathrm{Iso}(A, B; 0)/G^j) \times (Q\mathcal{P}_w(A, M; 0)/G^j) \to TQ\mathcal{P}_w(A, M; 0)/G^j$$

over $Q\mathcal{P}_w(A, M; 0)/G^j$.

Thus an element of $Q\mathrm{Comp}_w(A, M; 0)$ consists of data $(\omega, \lambda, \Delta, \alpha)$, where (ω, λ) is a generalised isotropic structure, Δ a quantisation of (A, M), and α a homotopy between $\mu_w(\omega, \lambda)$ and $\sigma(\Delta)$.

Definition 3.15. Define $Q\mathrm{Comp}_w(A, M; 0)^{\mathrm{nondeg}} \subset Q\mathrm{Comp}_w(A, M; 0)$ to consist of w-compatible quantised pairs (ω, Δ) with Δ non-degenerate. This is a union of path-components, and by [Pri4, Lemma 1.22] any pre-symplectic form compatible with a non-degenerate quantisation is symplectic. The same argument shows that any isotropic pair compatible with a non-degenerate quantisation is Lagrangian, so there is a natural projection

$$Q\mathrm{Comp}_w(A, M; 0)^{\mathrm{nondeg}} \to G\mathrm{Lag}(A, B; 0)$$

as well as the canonical map

$$Q\mathrm{Comp}_w(A, M; 0)^{\mathrm{nondeg}} \to Q\mathcal{P}_w(A, M; 0)^{\mathrm{nondeg}}.$$

3.3. The equivalences.

Proposition 3.16. *For any Levi decomposition w of* GT*, the canonical map*

$$Q\mathrm{Comp}_w(A,M;0)^{\mathrm{nondeg}} \to Q\mathcal{P}_w(A,M;0)^{\mathrm{nondeg}} \simeq Q\mathcal{P}(A,M;0)^{\mathrm{nondeg}}$$

is a weak equivalence. In particular, w gives rise to a morphism

$$Q\mathcal{P}(A,M;0)^{\mathrm{nondeg}} \to G\mathrm{Lag}(A,B;0)$$

(from non-degenerate quantisations to generalised Lagrangians) in the homotopy category of simplicial sets.

Proof. The proof of [Pri4, Proposition 1.26] adapts to this context, along much the same lines as [Pri5, Proposition 2.16]. The essential idea is that non-degeneracy of a quantisation Δ ensures that $\mu_w(-,\Delta)$ is a filtered quasi-isomorphism, so the generalised Lagrangian data (ω,λ) associated to Δ are given by

$$-\mu_w(-,\Delta)^{-1}(\partial_{\hbar^{-1}}\Delta).$$

□

Write $\widehat{\mathrm{Pol}}(A,B;0) := Q\widehat{\mathrm{Pol}}(A,M;0)/G^1$, with a filtration F given by the image of the filtration $\tilde{F}$, then also write $\mathrm{Comp} := Q\mathrm{Comp}_w/G^1$, $\mathcal{P} := Q\mathcal{P}/G^1$, $\mathrm{Lag} := G\mathrm{Lag}/G^1$ and $\mathrm{Iso} := G\mathrm{Iso}/G^1$. In particular, observe that since $\widehat{\mathrm{Pol}}(A,B;0)$ is already a P_2-algebra, the space Comp is independent of the Levi decomposition w of GT.

The following proposition establishes an equivalence between Lagrangians and non-degenerate co-isotropic Poisson structures in the 0-shifted setting:

Proposition 3.17. *The canonical maps*

$$\begin{aligned} \mathrm{Comp}(A,B;0)^{\mathrm{nondeg}} &\to \mathcal{P}(A,B;0)^{\mathrm{nondeg}} \\ \mathrm{Comp}(A,B;0)^{\mathrm{nondeg}} &\to \mathrm{Lag}(A,B;0) \end{aligned}$$

are weak equivalences.

Proof. The first equivalence is given by observing that the equivalences in Proposition 3.16 respect the cofiltration G. For the second equivalence, we adapt the proofs of [Pri4, Corollary 1.36 and Proposition 1.37], establishing the equivalence by induction on the filtration F.

The space $\mathrm{Lag}(A,B;0)/F^3$ is just given by elements (ω,λ) in the cocone of $\hat{\mathrm{Tot}}\,\Omega^2_{A/R} \to \hat{\mathrm{Tot}}\,\Omega^2_{A/R}$ which are non-degenerate in the sense that the induced map $(\omega,\lambda)^\sharp$ induces a quasi-isomorphism

$$\begin{array}{ccc} \underline{\hat{\mathrm{Hom}}}_A(\Omega^1_{A/R},A^0) & \xrightarrow{S} & \underline{\hat{\mathrm{Hom}}}_B(\Omega^1_{B/A},B^0)[1] \\ \omega^\sharp\downarrow & & \downarrow\lambda^\sharp \\ \hat{\mathrm{Tot}}\,(\Omega^1_{A/R}\otimes_A A^0) & \longrightarrow & \hat{\mathrm{Tot}}\,(\Omega^1_{B/R}\otimes_B B^0) \end{array}$$

of diagrams. Since $\mathcal{P}(A,B;0)/F^3$ is given by elements (ϖ,π) in the cocone of $S\colon \underline{\hat{\mathrm{Hom}}}_A(\Omega^2_{A/R},A) \to \underline{\hat{\mathrm{Hom}}}_B(\mathrm{CoS}^2_B\Omega^1_{B/A},B)[2]$, the essentially unique Poisson structure compatible with (ω,λ) is just given by the image of (ω,λ) under the symmetric square of the homotopy inverse of $(\omega,\lambda)^\sharp$, so

$$\mathrm{Comp}(A,B;0)^{\mathrm{nondeg}}/F^3 \xrightarrow{\sim} \mathrm{Lag}(A,B;0)/F^3.$$

Adapting the proof of [Pri4, Corollary 1.36], there is a a commutative diagram

$$\begin{array}{ccc}
(\mathrm{Comp}(A,B;0)^{\mathrm{nondeg}}/F^{p+1})_{(\omega,\lambda,\varpi,\pi)} & \longrightarrow & (\mathrm{Lag}(A,B;0)/F^{p+1})_{(\omega,\lambda)} \\
\downarrow & & \downarrow \\
(\mathrm{Comp}(A,B;0)^{\mathrm{nondeg}}/F^{p})_{(\omega,\lambda,\varpi,\pi)} & \longrightarrow & (\mathrm{Lag}(A,B;0)/F^{p})_{(\omega,\lambda)} \\
\downarrow & & \downarrow \\
\underline{\mathrm{MC}}(M(\omega,\lambda,\varpi,\pi,p)[1]) & \longrightarrow & \underline{\mathrm{MC}}(\hat{\mathrm{Tot}}\,\mathrm{cocone}(\Omega^p_{A/R}\to\Omega^p_{B/R})[2-p])
\end{array}$$

of fibre sequences, where $M(\omega,\lambda,\varpi,\pi,p)$ is defined to be the homotopy limit of the diagram

$$\begin{array}{ccc}
\hat{\mathrm{Tot}}\,\Omega^p_{A/R}[1-p] & \longrightarrow & \hat{\mathrm{Tot}}\,\Omega^p_{B/R}[1-p] \\
{\scriptstyle \Lambda^p(\varpi^\sharp)}\downarrow & & \downarrow{\scriptstyle \Lambda^p(\pi^\sharp)} \\
\hat{\underline{\mathrm{Hom}}}_A(\Omega^p_{A/R},A)[1-p] & \xrightarrow{S} & \hat{\underline{\mathrm{Hom}}}_B(\mathrm{CoS}^p_B\Omega^1_{B/A},B)[1] \\
{\scriptstyle \nu(\omega,\varpi)-(p-1)}\uparrow & & \uparrow{\scriptstyle \nu(\lambda,\pi)-(p-1)} \\
\hat{\underline{\mathrm{Hom}}}_A(\Omega^p_{A/R},A)[1-p] & \xrightarrow{S} & \hat{\underline{\mathrm{Hom}}}_B(\mathrm{CoS}^p_B\Omega^1_{B/A},B)[1].
\end{array}$$

Here $\nu(\omega,\varpi)$ is the tangent map of $\mu(\omega,-)$ at ϖ, given by

$$\mu(\omega,\pi+\rho\epsilon)=\mu(\omega,\pi)+\nu(\omega,\pi)(\rho)\epsilon$$

for $\epsilon^2=0$, with $\nu(\lambda,\pi)$ defined similarly.

Arguing as in [Pri7, Lemma 1.40], $\nu(\omega,\varpi)\simeq p(\varpi^\sharp\circ\omega^\sharp)$ and $\nu(\lambda,\pi)\simeq p(\pi^\sharp\circ\lambda^\sharp)$ in the diagram above. Since we are in the non-degenerate setting, $\varpi^\sharp\circ\omega^\sharp$ and $\pi^\sharp\circ\lambda^\sharp$ are homotopic to the identity maps on their respective spaces, so $\nu(\omega,\varpi)$ and $\nu(\lambda,\pi)$ are homotopic to multiplication by p. Because $p-(p-1)$ is invertible, we then get

$$M(\omega,\lambda,\varpi,\pi,p)\simeq\hat{\mathrm{Tot}}\,\mathrm{cocone}(\Omega^p_{A/R}\to\Omega^p_{B/R})[1-p].$$

Substituting in the diagram of fibre sequences then gives

$$\begin{aligned}
&(\mathrm{Comp}(A,B;0)^{\mathrm{nondeg}}/F^{p+1})\\
&\simeq(\mathrm{Comp}(A,B;0)^{\mathrm{nondeg}}/F^{p})\times^h_{(\mathrm{Lag}(A,B;0)/F^p)}(\mathrm{Lag}(A,B;0)/F^{p+1}),
\end{aligned}$$

from which the desired equivalence $(\mathrm{Comp}(A,B;0)^{\mathrm{nondeg}}/F^{p+1})\simeq(\mathrm{Lag}(A,B;0)/F^{p+1})$ follows by induction. □

Proposition 3.18. *For any Levi decomposition w of* GT, *the maps*

$$\begin{aligned}
&Q\mathcal{P}_w(A,M;0)^{\mathrm{nondeg}}/G^j\\
&\to(Q\mathcal{P}_w(A,M;0)^{\mathrm{nondeg}}/G^2)\times^h_{(G\mathrm{Lag}(A,B;0)/G^2)}(G\mathrm{Lag}(A,B;0)/G^j)\\
&\simeq(Q\mathcal{P}_w(A,M;0)^{\mathrm{nondeg}}/G^2)\times\prod_{2\le i<j}\underline{\mathrm{MC}}(\mathrm{cone}(\mathrm{DR}(A/R)\to\mathrm{DR}(B/R))\hbar^i)
\end{aligned}$$

coming from Proposition 3.16 are weak equivalences for all $j\geq 2$.

Proof. The proof of [Pri7, Proposition 1.41] and [Pri5, Proposition 2.17] generalises to this setting. For $(\omega, \lambda, \varpi, \pi) \in \mathrm{Comp}(A, B; 0)$, there is a a commutative diagram

$$\begin{array}{ccc} (Q\mathrm{Comp}_w(A, M; 0)/G^{j+1})_{(\omega,\lambda,\varpi,\pi)} & \longrightarrow & (G\mathrm{Iso}(A, B; 0)/G^{j+1})_{(\omega,\lambda)} \\ \downarrow & & \downarrow \\ (Q\mathrm{Comp}_w(A, M; 0)/G^{j})_{(\omega,\lambda,\varpi,\pi)} & \longrightarrow & G\mathrm{Iso}(A, B; 0)/G^{j})_{(\omega,\lambda)} \\ \downarrow & & \downarrow \\ \underline{\mathrm{MC}}(N(\omega, \lambda, \varpi, \pi, j)[1]) & \longrightarrow & \underline{\mathrm{MC}}(\mathrm{cone}(F^{2-j}\mathrm{DR}(A/R) \to F^{2-j}\mathrm{DR}(B/R))\hbar^j) \end{array}$$

of fibre sequences, for a space $N(\omega, \lambda, \varpi, \pi, j)$ defined as follows.

We set $N(\omega, \lambda, \varpi, \pi, j)$ to be the homotopy limit of the diagram

$$\begin{array}{c} \mathrm{cocone}(F^{2-j}\mathrm{DR}(A/R) \to F^{2-j}\mathrm{DR}(B/R))\hbar^j \\ \downarrow{\scriptstyle \mu(-,-,\varpi,\pi)} \\ F^{2-j}T_{(\varpi,\pi)}\widehat{\mathrm{Pol}}(A, B; 0)\hbar^j \\ \uparrow{\scriptstyle \nu(\omega,\lambda,\varpi,\pi)+\partial_{\hbar^{-1}}} \\ (F^{2-j}\widehat{\mathrm{Pol}}(A, B; 0)\hbar^j, \delta_{\varpi,\pi}) = F^{2-j}T_{(\varpi,\pi)}\widehat{\mathrm{Pol}}(A, B; 0)\hbar^{j-1}, \end{array}$$

where $\nu(\omega, \lambda, \varpi, \pi)$ is the tangent map of $\mu(\omega, \lambda, -, -)$ at (ϖ, π), given by

$$\mu(\omega, \lambda, \varpi + \tau\epsilon, \pi + \rho\epsilon) = \mu(\omega, \pi) + \nu(\omega, \pi)(\tau, \rho)\epsilon$$

with $\epsilon^2 = 0$.

On the associated graded pieces, the proof of [Pri5, Proposition 2.17] shows that $\mathrm{gr}^p_F(\nu(\omega, \lambda, \varpi, \pi) + \partial_{\hbar^{-1}})$ is homotopic to $(1 - j)\hbar$. As this is an isomorphism for all $j \geq 2$, the map $N(\omega, \lambda, \varpi, \pi, j) \to \mathrm{cocone}(F^{2-j}\mathrm{DR}(A/R) \to F^{2-j}\mathrm{DR}(B/R))\hbar^j$ is a quasi-isomorphism, which inductively gives the required weak equivalences from the fibre sequences above.

□

Remarks 3.19. Taking the limit over all j, Proposition 3.18 gives an equivalence

$$\begin{aligned} &Q\mathcal{P}_w(A, M; 0)^{\mathrm{nondeg}} \\ &\simeq (Q\mathcal{P}_w(A, M; 0)^{\mathrm{nondeg}}/G^2) \times \prod_{i\geq 2} \underline{\mathrm{MC}}(\mathrm{cone}(\mathrm{DR}(A/R) \to \mathrm{DR}(B/R))\hbar^i); \end{aligned}$$

in particular, this means that there is a canonical map

$$(Q\mathcal{P}(A, M; 0)^{\mathrm{nondeg}}/G^2) \to Q\mathcal{P}(A, M; 0)^{\mathrm{nondeg}},$$

dependent on $w \in \mathrm{Levi}_{\mathrm{GT}}$, corresponding to the distinguished point 0.

Even if π is degenerate, a variant of Proposition 3.18 still holds. Because $\varpi^\sharp \circ \omega^\sharp$ and $\pi^\sharp \circ \lambda^\sharp$ are homotopy idempotent, the map $\mathrm{gr}^p_F\nu(\omega, \lambda, \varpi, \pi)$ has eigenvalues in the interval $[0, p]$, so we just replace $(1 - j)$ with an operator having eigenvalues in the interval $[1 - p - j, 1 - j]$. Since this is still a quasi-isomorphism for $j > 1$, we have

$$\begin{aligned} &Q\mathrm{Comp}_w(A, M; 0) \\ &\simeq (Q\mathrm{Comp}_w(A, M; 0)/G^2) \times \prod_{i\geq 2} \underline{\mathrm{MC}}(\mathrm{cocone}(\mathrm{DR}(A/R) \to \mathrm{DR}(B/R))\hbar^i). \end{aligned}$$

giving a sufficient first-order criterion for degenerate quantisations to exist.

4. Global quantisations

As in [Pri7, §3] and [Pri5, §3], in order to pass from stacky CDGAs to derived Artin stacks, we will exploit a form of étale functoriality. We then introduce the notion of self-duality and thus establish the existence of quantisations for derived Lagrangians.

4.1. Diagrams of quantised pairs.

Definition 4.1. Given a small category I, an I-diagram (A, F) in almost commutative stacky DGAAs over R, and a filtered A-bimodule M in I-diagrams of chain cochain complexes for which the left and right $\mathrm{gr}^F A$-module structures on $\mathrm{gr}^F M$ agree, we define the filtered chain cochain complex

$$\mathcal{CC}_{R,BD_1}(A, M)$$

to be the equaliser of the obvious diagram

$$\prod_{i\in I} \mathcal{CC}^{\bullet}_{R,BD_1}(A(i), M(i)) \implies \prod_{f\colon\, i\to j \text{ in } I} \mathcal{CC}^{\bullet}_{R,BD_1}(A(i), M(j)),$$

for the BD_1 Hochschild complexes of Definition 2.6.

We then write $\mathcal{CC}^{\bullet}_{R,BD_1}(A) := \mathcal{CC}^{\bullet}_{R,BD_1}(A, A)$, which inherits the structure of a stacky brace algebra from each $\mathcal{CC}^{\bullet}_{R,BD_1}(A(i), A(i))$.

Note that if $u\colon I \to J$ is a morphism of small categories and A is a J-diagram of almost commutative stacky DGAAs over R, with $B = A \circ u$, then we have a natural map $\mathcal{CC}^{\bullet}_R(A) \to \mathcal{CC}^{\bullet}_R(B)$.

In order to ensure that $\mathcal{CC}^{\bullet}_R(A, M)$ has the correct homological properties, we now consider categories of the form $[m] = (0 \to 1 \to \ldots \to m)$. Similarly to [Pri5, Lemma 3.2], the construction $\mathcal{CC}^{\bullet}_R(A, M)$ preserves weak equivalences provided we restrict to pairs (A, M) for which each $A(i)$ is cofibrant as an R-module and M is fibrant for the injective model structure (i.e. the maps $M(i) \to M(i + 1)$ are all surjective).

As in [Pri7, §3.4.1], we can do much the same for differential operators:

Definition 4.2. Given a small category I, an I-diagram B of stacky CDGAs over R, and B-modules M, N in chain cochain complexes, define the filtered chain cochain complex $\mathcal{D}\!\mathit{iff}_{B/R}(M, N)$ to be the equaliser of the obvious diagram

$$\prod_{i\in I} \mathcal{D}\!\mathit{iff}_{B(i)/R}(M(i), N(i)) \implies \prod_{f\colon\, i\to j \text{ in } I} \mathcal{D}\!\mathit{iff}_{B(i)/R}(M(i), f_*N(j)),$$

and write $\mathcal{D}\!\mathit{iff}_{B/R}$ for $\mathcal{D}\!\mathit{iff}_{B/R}(B, B)$

If B is an $[m]$-diagram in $DG^+dg\mathrm{CAlg}(R)$ which is cofibrant and fibrant for the injective model structure (i.e. each $B(i)$ is cofibrant in the model structure of Lemma 2.4 and the maps $B(i) \to B(i + 1)$ are surjective), then observe that $\mathrm{gr}^F_k \mathcal{D}\!\mathit{iff}_{B/R}$ is a model for the derived Hom-complex $\mathbf{R}\mathcal{H}om_B(\mathrm{CoS}^k_B \Omega^k_{B/R}, B)$.

The constructions in §2 now all carry over verbatim, generalising from morphisms of cofibrant stacky CDGAs to morphisms $A \to B$ of $[m]$-diagrams of stacky CDGAs which are cofibrant and fibrant for the injective model structure. In particular, for any such morphism and a strict line bundle M over B, we have a DGLA

$$Q\widehat{\mathrm{Pol}}(A, M; 0)^{[1]}$$

of 0-shifted relative quantised polyvectors as in Definition 2.10, and a space

$$Q\mathcal{P}(A, M; 0)$$

of quantisations of the pair (A, M) as in Definition 2.13.

In order to identify $Q\mathcal{P}/G^1$ with $\mathcal{P}$, and for notions such as non-degeneracy to make sense, we have to assume that for our fibrant cofibrant $[m]$-diagrams A, B of stacky CDGAs, each $A(j), B(j)$ satisfies Assumption 2.19, so there exists N for which the chain complexes $(\Omega^1_{A(j)/R} \otimes_{A(j)} A(j)^0)^i$ are acyclic for all $i > N$, and similarly for B.

Definition 4.3. Given a morphism $A \to B$ of fibrant cofibrant $[m]$-diagrams in stacky CDGAs (for the injective model structure) define

$$G\mathrm{Iso}(A, B; 0) := G\mathrm{Iso}(A(0), B(0); 0) = \varprojlim_{i \in [m]} G\mathrm{Iso}(A(i), B(i); 0),$$

for the space GIso of generalised isotropic structures of Definition 3.11, and define the space $G\mathrm{Lag}(A, B; 0)$ of generalised Lagrangians similarly.

Given a choice $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ of Levi decomposition for GT, define

$$\mu_w \colon\ G\mathrm{Iso}(A, B; 0) \times Q\mathcal{P}_w(A, M; 0) \to TQ\mathcal{P}_w(A, M; 0)$$

by setting $\mu_w(\omega, \lambda, \Delta)(i) := \mu_w(\omega(i), \lambda(i), \Delta(i)) \in TQ\mathcal{P}_w(A(i), B(i); 0)$ for $i \in [m]$, and let $Q\mathrm{Comp}_w(A, M; 0)$ be the homotopy vanishing locus of

$$(\mu_w - \sigma) \colon\ G\mathrm{Iso}(A, B; 0) \times Q\mathcal{P}_w(A, M; 0) \to TQ\mathcal{P}_w(A, M; 0).$$

over $Q\mathcal{P}_w(A, M; 0)$.

As in [Pri4, §3.4.2], if we let $(DG^+dg\mathrm{CAlg}(R)^{[1]})^{\text{ét}} \subset DG^+dg\mathrm{CAlg}(R)^{[1]}$ be the wide subcategory of the arrow category with only homotopy formally étale morphisms (see Definition 2.5) between arrows, then for any of the constructions F based on $Q\mathcal{P}$, [Pri4, Definition 2.7] adapts to give an ∞-functor

$$\mathbf{R}F \colon\ \mathbf{L}(DG^+dg\mathrm{CAlg}(R)^{[1]})^{\text{ét}} \to \mathbf{L}s\mathrm{Set}$$

from the ∞-category of stacky CDGAs and homotopy formally étale morphisms to the ∞-category of simplicial sets. This construction has the property that

$$(\mathbf{R}F)(\phi\colon\ A \to B) \simeq F(\phi\colon\ A \to B)$$

for all morphisms ϕ of cofibrant stacky CDGAs A over R.

Immediate consequences of Propositions 3.16 and 3.18 are that for any $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$, the canonical maps

$$\begin{aligned}
&Q\mathrm{Comp}_w(A, M; 0)^{\mathrm{nondeg}} \to Q\mathcal{P}_w(A, M; 0)^{\mathrm{nondeg}} \simeq Q\mathcal{P}(A, M; 0)^{\mathrm{nondeg}};\\
&Q\mathcal{P}_w(A, M; 0)^{\mathrm{nondeg}}/G^j\\
&\to (Q\mathcal{P}_w(A, M; 0)^{\mathrm{nondeg}}/G^2) \times \prod_{2 \le i < j} \underline{\mathrm{MC}}(\mathrm{cocone}(\mathrm{DR}(A/R) \to \mathrm{DR}(B/R))\hbar^i[1])
\end{aligned}$$

are weak equivalences of ∞-functors on the full subcategory of $(\mathbf{L}DG^+dg\mathrm{CAlg}(R)^{[1]})^{\text{ét}}$ consisting of objects satisfying the conditions of Assumption 2.19, for all $j \ge 2$.

4.2. **Descent and line bundles.** In order to translate our constructions from stacky CDGAs to derived Artin stacks, we now follow the approach set out in [Pri4, §3.4.2], adapted to include line bundles as in [Pri7, §3.4.2].

The denormalisation functor $D\colon DG^+dg_+\mathrm{CAlg}(R) \to dg_+\mathrm{CAlg}(R)^\Delta$ from stacky CDGAs to cosimplicial CDGAs (cf. [Pri1, Definition 4.20] allows us to extend simplicial functors F on CDGAs to simplicial functors on stacky CDGAs, given by $B \mapsto \mathrm{ho}\varprojlim_{i\in\Delta} F(D^iB)$.

Definition 4.4. Given a derived Artin N-stack X, and $A \in DG^+dg\mathrm{CAlg}(R)$, we say that an element $f \in \mathrm{ho}\varprojlim_i X(D^iA)$ is homotopy formally étale if the induced morphism

$$N_cf_0^*\mathbb{L}_{X/R} \to \{\mathrm{Tot}\,\sigma^{\le q}\mathbf{L}\Omega^1_{A/R} \otimes^{\mathbf{L}}_A A^0\}_q$$

from [Pri4, §3.2.2] is a pro-quasi-isomorphism.

In this situation, it makes sense to think of A as a derived Lie algebroid locally isomorphic to X.

This allows us to exploit étale functoriality of our constructions on stacky CDGAs, allowing them to descend to derived Artin stacks as follows.

Definition 4.5. Given a morphism $X \to Y$ of derived Artin N-stacks, we write $(dg_+DG\mathrm{Aff}^{[1]}_{\text{ét}} \downarrow X/Y)$ for the arrow ∞-category consisting of morphisms $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$ in the simplicial localisation of $DG^+dg\mathrm{CAlg}(R)^{\mathrm{opp}}$ at levelwise quasi-isomorphisms, equipped with homotopy formally étale elements of $\mathrm{ho}\varprojlim_i X(D^iB) \times^h_{Y(D^iB)} Y(D^iA)$; morphisms in this ∞-category are given by spaces of compatible homotopy formally étale maps $A \to A'$, $B \to B'$.

We now extend the constructions above to line bundles, via $\mathbb{G}_m$-equivariance exactly as in [Pri7, §3.4.2]. When working with CDGAs with no stacky structure, this can be done just by observing that there is a natural $\mathbb{G}_m$-action on $Q\mathcal{P}$ given by conjugation, since the derived stack associated to $B\mathbb{G}_m$ is just the hypersheafification of of the nerve of the functor $B \mapsto \mathbb{G}_m(B_0)$.

However, for stacky CDGAs, we must replace the group $\mathbb{G}_m(B_0)$ with the groupoid

$$\mathrm{TLB}(B) := [\mathrm{Z}^1(\mathrm{Z}_0B)/(\mathrm{Z}_0B^0)^\times]$$

of trivial line bundles, where $f \in (B^0)^\times$ acts on Z^1B by addition of $\partial \log f = f^{-1}\partial f$. Here, an element $c \in \mathrm{Z}^1(\mathrm{Z}_0B)$ corresponds to the strict line bundle $B_c = (B^\#, \partial + c)$, with invertible elements $f \in (\mathrm{Z}_0B^0)$ giving isomorphisms $f\colon B_{c+\partial\log f} \to B_c$. The reason this works is that the nerve of TLB is essentially the smallest functor which hypersheafifies to recover $B \mapsto \mathrm{ho}\varprojlim_{i\in\Delta} B\mathbb{G}_m(D^iB)$.

For any morphism $A \to B$ of cofibrant stacky CDGAs over R, we can then extend $Q\mathcal{P}(A, B; 0)$ to a simplicial representation of the groupoid $\mathrm{TLB}(B)$ above by sending an object $c \in \mathrm{Z}^1(\mathrm{Z}_0B)$ to $Q\mathcal{P}(A, B_c; 0)$, with $(\mathrm{Z}_0B^0)^\times$ acting via functoriality for strict line bundles. Note that the quotient representation $Q\mathcal{P}(-,-;0)/G^1 = \mathcal{P}(-,0)$ is trivial; we also set $G\mathrm{Iso}$ to be a trivial representation $c \mapsto G\mathrm{Iso}(A, B; 0)$.

Definition 4.6. For any of the constructions F of §4.1, let $\mathbf{R}(F/^h\mathbb{G}_m)$ be the ∞-functor on $\mathbf{L}dg\mathrm{CAlg}(R)^{\text{ét}}$ given by applying the construction of [Pri4, §3.4.2] to the right-derived functor of the Grothendieck construction

$$B \mapsto \mathrm{ho}\varinjlim_{c\in\mathrm{TLB}(B)} F(A, B_c),$$

then taking hypersheafification with respect to homotopy formally étale coverings.

Definition 4.7. Given a map $f\colon X \to Y$ of strongly quasi-compact derived Artin N-stacks over R, a line bundle $\mathscr{L}$ on X and any of the functors F above, define $F(Y, \mathscr{L})$ to be the homotopy limit of

$$\mathbf{R}(F/^h\mathbb{G}_m)(A,B) \times^h_{\mathbf{R}(*/^h\mathbb{G}_m)(B)} \{\mathscr{L}|_B\}$$

over objects $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ in the ∞-category $(dg_+DG\mathrm{Aff}^{[1]}_{\text{ét}} \downarrow X/Y)$.

Remark 4.8. In many cases, we can take smaller categories than $(dg_+DG\mathrm{Aff}^{[1]}_{\text{ét}} \downarrow X/Y)$ on which to calculate the homotopy limit. When the $\mathbb{G}_m$-action on F is trivial, we can restrict to compatible hypergroupoid resolutions of X and Y as in [Pri4, §3.4.2], and in general we just need the resolution of X to be compatible with the canonical resolution of $B\mathbb{G}_m$. When X and Y are derived Deligne–Mumford N-stacks, we do not need stacky CDGAs at all, and can just work over $(DG\mathrm{Aff}^{[1]}_{\text{ét}} \downarrow X/Y)$.

When X and Y are 1-geometric derived Artin stacks, we may just consider the ∞-category of commutative diagrams

$$\begin{array}{ccc} U & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ V & \xrightarrow{g} & Y \end{array}$$

with U, V derived affines and the maps f, g being smooth; to this we associate the morphism $\Omega^\bullet_{U/X} \to \Omega^\bullet_{V/Y}$ of stacky CDGAs as in §2.1, giving an object of $(dg_+DG\mathrm{Aff}^{[1]}_{\text{ét}} \downarrow X/Y)$.

Remark 4.9. Following Remark 2.14, we may regard an element of $Q\mathcal{P}(Y, \mathscr{L}; 0)$ as a sheaf on $(DG\mathrm{Aff}^{[1]}_{\text{ét}} \downarrow X/Y)$ deforming the pair $(\mathscr{O}_Y, \mathscr{L})$, by combining a suitable curved A_∞ deformation $\tilde{\mathscr{O}}_Y$ of $\mathscr{O}_Y$ over $R[\![\hbar]\!]$ with an $f^{-1}\tilde{\mathscr{O}}_Y$-module $\tilde{\mathscr{L}}$ deforming $\mathscr{L}$ over $R[\![\hbar]\!]$, the deformation being given by R-linear differential operators with restrictions on their orders.

In fact, there is an $f^{-1}\tilde{\mathscr{O}}_Y - \mathscr{D}_X[\![\hbar]\!]$-bimodule $\mathscr{E}_\hbar := (\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X[\![\hbar]\!], \delta + \Delta_{\tilde{\mathscr{L}}} \cdot -)$, from which we can recover $\tilde{\mathscr{L}}$ as $\tilde{\mathscr{E}} \otimes_{\mathscr{D}_X[\![\hbar]\!]} \mathscr{O}_X[\![\hbar]\!]$. In particular, this gives us a functor from right $\tilde{\mathscr{O}}_Y$-modules $\mathscr{N}$ to right $\mathscr{D}_X[\![\hbar]\!]$-modules $f^{-1}\mathscr{N} \otimes^{\mathbf{L}}_{f^{-1}\tilde{\mathscr{O}}_Y} \mathscr{E}_\hbar$.

Examples 4.10. Here are some cases where the description simplifies:

(1) The simplest case to consider is when X and Y are both smooth (underived) Deligne–Mumford N-stacks, so we can work with algebras instead of stacky CDGAs. Then the description of Example 2.15.(1) implies that $\tilde{\mathscr{O}}_Y$ is locally given by an associative deformation of the sheaf $\mathscr{O}_Y$ on the étale site of Y, but the presence of 2-automorphisms makes $\tilde{\mathscr{O}}_Y$ an algebroid deformation, i.e. an $R[\![\hbar]\!]$-deformation of $\mathscr{O}_Y$ regarded as a 2-sheaf of R-linear categories.

Then $\tilde{\mathscr{L}}$ gives rise to an $R[\![\hbar]\!]$-linear functor from the algebroid $f^{-1}\tilde{\mathscr{O}}_Y$ on the étale site of X to the $R[\![\hbar]\!]$-linear category of right $\mathscr{D}_X[\![\hbar]\!]$-modules on X, together with conditions on orders of differntial operators which are difficult to charactierise directly. However, when f is a closed immersion and $\tilde{\mathscr{O}}_Y$ is non-degenerate, Example 2.15.(2) implies that $\tilde{\mathscr{L}}$ is just a $R[\![\hbar]\!]$-linear functor from the algebroid $f^{-1}\tilde{\mathscr{O}}_Y$ to the category of complete flat $R[\![\hbar]\!]$-modules on

$X_{\text{ét}}$, reducing to the constant functor $f^{-1}\mathscr{O}_Y \mapsto \mathscr{L}$ modulo $\hbar$, with no further conditions necessary.

(2) Generalising to the case where X and Y are both derived Deligne–Mumford N-stacks, the description of Remark 2.15.(3) similarly implies that $\tilde{\mathscr{O}}_Y$ gives rise to an associative $R[\![\hbar]\!]$-deformation $\mathscr{A}$ of $\mathscr{O}_Y$ as a hypersheaf of R-linear dg categories, but this throws away information about almost commutativity, so we cannot recover $\tilde{\mathscr{O}}_Y$ from the algebroid. There is a similar loss of information associating right $\mathscr{D}$-modules to $\tilde{\mathscr{L}}$. Thus each quantisation gives rise to (but cannot be recovered from) an ∞-algebroid $\mathscr{A}$ on Y equipped with an $R[\![\hbar]\!]$-linear ∞-functor from $f^{-1}\mathscr{A}$ to the $R[\![\hbar]\!]$-linear ∞-category of right $\mathscr{D}_X[\![\hbar]\!]$-modules, deforming the constant functor$f^{-1}\mathscr{O}_Y \mapsto \mathscr{L}$.

Adapting [Pri7, Definition 2.21] along the lines of Definition 2.21 gives:

Definition 4.11. Say that a quantisation $\Delta \in Q\mathcal{P}(Y, \mathscr{L}; 0)/G^k$ is non-degenerate if the induced maps from cotangent complexes to tangent complexes

$$\Delta^{\sharp}_{2,Y} \colon \mathbb{L}_{Y/R} \to \mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}(\mathbb{L}_{Y/R}, \mathscr{O}_{Y/R})$$
$$\Delta^{\sharp}_{2,X} \colon \mathbb{L}_{X/R} \to \mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(\mathbb{L}_{X/Y}, \mathscr{O}_X)[1]$$

are quasi-isomorphisms and and $\mathbb{L}_X, \mathbb{L}_Y$ are perfect.

Propositions 3.17 and 3.18 now readily generalise (substituting the relevant results from [Pri4, §3] to pass from local to global), showing that the only obstruction to quantising a non-degenerate co-isotropic structure is first-order:

Proposition 4.12. *For any morphism $X \to Y$ of derived Artin N-stacks, any line bundle $\mathscr{L}$ on X and any $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$, the canonical maps*

$$\mathrm{Comp}(Y, X; 0)^{\mathrm{nondeg}} \to \mathcal{P}(Y, X; 0)^{\mathrm{nondeg}}$$
$$\mathrm{Comp}(Y, X; 0)^{\mathrm{nondeg}} \to \mathrm{Lag}(Y, X; 0)$$
$$Q\mathrm{Comp}_w(Y, \mathscr{L}; 0)^{\mathrm{nondeg}} \to Q\mathcal{P}(Y, \mathscr{L}; 0)^{\mathrm{nondeg}}$$

$$\begin{aligned} Q\mathrm{Comp}_w(Y, \mathscr{L}; 0) \to (Q\mathrm{Comp}_w(Y, \mathscr{L}; 0)/G^2) \times^h_{(G\mathrm{Iso}(Y,X;0)/G^2)} G\mathrm{Iso}(Y, X; 0) \simeq \\ (Q\mathrm{Comp}_w(Y, \mathscr{L}; 0)/G^2) \times \prod_{i \geq 2} \underline{\mathrm{MC}}(\mathrm{cone}(\mathrm{DR}(Y/R) \to \mathrm{DR}(X/R))\hbar^i) \end{aligned}$$

are filtered weak equivalences.

In particular, w gives rise to a morphism in the homotopy category of simplicial sets

$$Q\mathcal{P}(Y, \mathscr{L}; 0)^{\mathrm{nondeg}} \to G\mathrm{Lag}(Y, X; 0)$$

from the space of quantised co-isotropic structures to the space of generalised Lagrangians, which induces a weak equivalence

$$\begin{aligned} Q\mathcal{P}(Y, \mathscr{L}; 0)^{\mathrm{nondeg}} \to (Q\mathcal{P}(Y, \mathscr{L}; 0)^{\mathrm{nondeg}}/G^2) \times^h_{G\mathrm{Lag}(Y,X;0)/G^2} G\mathrm{Lag}(Y, X; 0) \simeq \\ (Q\mathcal{P}(Y, \mathscr{L}; 0)^{\mathrm{nondeg}}/G^2) \times \prod_{i \geq 2} \underline{\mathrm{MC}}(\mathrm{cone}(\mathrm{DR}(Y/R) \to \mathrm{DR}(X/R))\hbar^i). \end{aligned}$$

Remark 4.13. The results of Proposition 4.12 are compatible with those of [BGKP, Theorem 1.1.4], which fixes a sheaf $\tilde{\mathscr{O}}_Y$ of associative algebras quantising a symplectic structure on a smooth variety Y, and describes $\tilde{\mathscr{O}}_Y$-module deformations of line bundles

$\mathscr{L}$ on smooth closed Lagrangians $X \subset Y$. As in Example 4.10.(1), this groupoid corresponds precisely to our space $Q\mathcal{P}(Y,\mathscr{L};0)^{\mathrm{nondeg}} \times^h_{Q\mathcal{P}(Y,0)^{\mathrm{nondeg}}} \{\tilde{\mathscr{O}}_Y\}$ in this specialised setting, although we consider more general quantisations $\tilde{\mathscr{O}}_Y$.

In the generality of Proposition 4.12, the first order deformation problem is a question of lifting $Q\mathcal{P}(Y,\mathscr{L};0)^{\mathrm{nondeg}}/G^2 \to Q\mathcal{P}(Y,0)^{\mathrm{nondeg}}/G^2$ over a Lagrangian structure $\pi \in \mathrm{Lag}(Y,X;0)^{\mathrm{nondeg}}$, so DGLA obstruction theory applied to the complexes of quantised polyvectors allows us to read off the obstruction space as $\mathrm{H}^3(\mathrm{cocone}(F^1T_\pi\widehat{\mathrm{Pol}}(Y,X;0) \to F^1T_\pi\widehat{\mathrm{Pol}}(Y;0)))$, which is isomorphic via the compatibility map $\mu(-,\pi)$ to $\mathrm{H}^2F^1\mathrm{DR}(X)$. By Proposition 4.12, the higher order deformation problem is then simply a case of lifting an element $u \in \hbar^2\mathrm{H}^2\mathrm{DR}(Y/R)[\![\hbar]\!]$ (determined by $\tilde{\mathscr{O}}_Y$) to $\hbar^2\mathrm{H}^1(\mathrm{cone}(\mathrm{DR}(Y) \to \mathrm{DR}(X))[\![\hbar]\!]$, giving the higher order obstruction as the image of u in $\hbar^2\mathrm{H}^2\mathrm{DR}(X/R)[\![\hbar]\!]$.

In their restricted setting, [BGKP] indeed show that the potential first order obstruction to quantising $\mathscr{L}$ over $\tilde{\mathscr{O}}_Y$ is given by a class $c_1(\mathscr{L}) - \frac{1}{2}c_1(K_X) - \mathrm{At}(\tilde{\mathscr{O}}_Y,X) \in \mathrm{H}^2F^1\mathrm{DR}(X)$, with higher order obstructions a power series in $\hbar^2\mathrm{H}^2\mathrm{DR}(X)[\![\hbar]\!]$ depending only on $\tilde{\mathscr{O}}_Y$.

When $\mathscr{L}^{\otimes 2}$ has a right $\mathscr{D}$-module structure, the Chern class $c_1(\mathscr{L}) - \frac{1}{2}c_1(K_X)$ vanishes. Moreover, whenever there is an isomorphism $\tilde{\mathscr{O}}_Y \simeq \tilde{\mathscr{O}}_Y^{\mathrm{opp}}$ of quantisations which is semilinear with respect to the transformation $\hbar \mapsto -\hbar$, the calculations of [BGKP, Remark 5.3.4] show that $\mathrm{At}(\tilde{\mathscr{O}}_Y,X) = 0$. Thus their first order obstruction does indeed vanish in the scenario of Theorem 4.20 below, with the higher order obstruction given by Corollary 4.21.

4.3. **Self-duality.** In order to eliminate the potential first order obstruction to quantising a generalised Lagrangian in Proposition 4.12, we now introduce the notion of self-duality, combining the ideas of [Pri7, §4] and [Pri5, §1.6].

We wish to consider line bundles $\mathscr{L}$ on X equipped with an anti-involutive equivalence $(-)^t\colon \mathscr{D}(\mathscr{L}) \simeq \mathscr{D}(\mathscr{L})^{\mathrm{opp}}$. Such an equivalence is the same as a right $\mathscr{D}$-module structure on $\mathscr{L}^{\otimes 2}$. Since a dualising line bundle K_X on X naturally has the structure of a right $\mathscr{D}$-module (see for instance [GR, §2.4] for a proof in the derived setting), we will typically take $\mathscr{L}$ to be a square root of K_X, when this exists. In this case, the equivalence $\mathscr{D}(\mathscr{L}) \simeq \mathscr{D}(\mathscr{L})^{\mathrm{opp}}$ comes from the equivalences $\mathscr{L} \simeq \mathscr{L}^\vee$ and $\mathscr{D}(\mathscr{E})^{\mathrm{opp}} \simeq \mathscr{D}(\mathscr{E}^\vee)$, where $\mathscr{E}^\vee := \mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E},K_X)$.

Definition 4.14. Given a morphism $\phi\colon A \to B$ of cofibrant stacky CDGAs over R and a strict line bundle M over B, equipped with an anti-involution $(-)^t$ of $\mathcal{D}\mathit{iff}_{B/R}(M)$, we define an involution $(-)^*$ on the DGLA $Q\widehat{\mathrm{Pol}}(A,M;0)[1]$ by

$$\Delta^*(\hbar) := i(\Delta)(-\hbar)^t,$$

for the brace algebra anti-involution

$$\begin{aligned} -i\colon\ &(\mathcal{CC}_{R,BD_1}(A,\mathcal{D}\mathit{iff}_{B/R}(M))_{[1]} \rtimes \mathcal{CC}_{R,BD_1}(A))^{\mathrm{opp}} \\ &\to \mathcal{CC}_{R,BD_1}(A,\mathcal{D}\mathit{iff}_{B/R}(M)^{\mathrm{opp}})_{[1]} \rtimes \mathcal{CC}_{R,BD_1}(A) \end{aligned}$$

adapted from Lemma 1.22.

Since $(-)^*$ is a quasi-isomorphism of filtered DGLAs, it gives rise to an involutive weak equivalence

$$(-)^*\colon\ Q\mathcal{P}(A,M;0) \to Q\mathcal{P}(A,M;0)$$

Lemma 4.15. *For the filtration G induced on $\tilde{F}^p Q\widehat{\mathrm{Pol}}(A, M; 0)^{sd}$ by the corresponding filtration on $\tilde{F}^p Q\widehat{\mathrm{Pol}}(A, M; 0)$, we have*

$$\mathrm{gr}^k_G \tilde{F}^p Q\widehat{\mathrm{Pol}}(A, M; 0)^{sd} \simeq \begin{cases} \mathrm{gr}^k_G \tilde{F}^p Q\widehat{\mathrm{Pol}}(A, M; 0) & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}$$

Proof. This combines [Pri7, Lemma 4.5] and [Pri5, Lemma 1.35]. It follows because Lemma 1.22 ensures that the involution acts trivially on $\mathrm{gr}^0_G Q\widehat{\mathrm{Pol}}(A, M; 0)$. It therefore acts as multiplication by $(-1)^k$ on $\mathrm{gr}^k_G Q\widehat{\mathrm{Pol}}(A, M; 0) = \hbar^k \mathrm{gr}^0_G Q\widehat{\mathrm{Pol}}(A, M; 0)$. □

Definition 4.16. For a line bundle $\mathscr{L}$ on X with a right $\mathscr{D}$-module structure on $\mathscr{L}^{\otimes 2}$, we define the space

$$Q\mathcal{P}(Y, \mathscr{L}; 0)^{sd}$$

of self-dual quantisations to be the space of homotopy fixed points of the $\mathbb{Z}/2$-action on $Q\mathcal{P}(Y, \mathscr{L}; 0)$ generated by $(-)^*$.

Remark 4.17. Following Remark 4.9, a self-dual quantisation of $(X \xrightarrow{\phi} Y, \mathscr{L})$ gives rise to a curved A_∞-deformation $\tilde{\mathscr{O}}_Y$ of $\hat{\mathrm{Tot}}\, \mathscr{O}_Y$ over $R[\![\hbar]\!]$, equipped with an anti-involution $*$ which is semilinear under the transformation $\hbar \mapsto -\hbar$, together with a curved anti-involutive A_∞-morphism $\phi^{-1}\tilde{\mathscr{O}}_Y \to \mathscr{D}_{\mathscr{O}_X/R}(\mathscr{L})[\![\hbar]\!]$.

More is true: by [Pri5, Proposition 1.25], a quantisation gives a curved A_∞ deformation of the dg category $\mathrm{per}_{\mathrm{dg}}(\mathscr{O}_Y)$ of perfect complexes on Y, with self-dual quantisations incorporating a semilinear lift of the involution $\mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}(-, \mathscr{O}_Y)$. A self-dual quantisation of the pair $(Y, \mathscr{L})$ thus gives a curved semilinearly involutive A_∞-deformation of the involutive category $\mathrm{per}_{\mathrm{dg}}(\mathscr{O}_Y)$ fibred over $\mathrm{per}_{\mathrm{dg}}(\mathscr{O}_X)$ via the functor

$$(\mathrm{per}_{\mathrm{dg}}(\mathscr{O}_Y), \mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}(-, \mathscr{O}_Y)) \to (\mathrm{per}_{\mathrm{dg}}(\mathscr{O}_X), \mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(-, \mathscr{L}^{\otimes 2}))$$
$$\mathscr{F} \mapsto \phi^*\mathscr{F} \otimes \mathscr{L},$$

with an additional restriction of the curvature of the deformation in terms of differential operators.

Adapting [Pri5, Remark 1.34], we can extend the input data from the space $\mathbf{R}\Gamma(X, B\mathbb{G}_m)$ of line bundles to the space $\mathbf{R}\Gamma(Y, B^2\mathbb{G}_m) \times^h_{\mathbf{R}\Gamma(X, B^2\mathbb{G}_m)} \{1\}$ of pairs $(\mathscr{G}, \mathscr{L})$ with $\mathscr{G}$ a $\mathbb{G}_m$-gerbe on Y, and $\mathscr{L}$ a trivialisation of $\phi^*\mathscr{G}$. There is then a notion of self-dual quantisation for pairs $(\mathscr{G}, \mathscr{L})$ with $\mathscr{G}$ a μ_2-gerbe and $\mathscr{L}$ a trivialisation of the $\mathbb{G}_m$-gerbe associated to $\phi^*\mathscr{G}$, with a right $\mathscr{D}$-module structure on the line bundle $\mathscr{L}^{\otimes 2}$. In particular, we may consider involutive quantisations of $(\mathrm{per}_{\mathrm{dg}}(\mathscr{O}_Y), \mathbf{R}\mathscr{H}om_{\mathscr{O}_Y}(-, \mathscr{M}))$ for any line bundle $\mathscr{M}$, the criterion for self-duality now being that $\mathscr{L}^{\otimes 2} \otimes \phi^*\mathscr{M}$ be a right $\mathscr{D}$-module, so that we consider the involution $\mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(-, \mathscr{L}^{\otimes 2} \otimes \phi^*\mathscr{M})$ on $\mathrm{per}_{\mathrm{dg}}(\mathscr{O}_X)$.

The natural example to take for $\mathscr{M}$ is the dualising line bundle $K_Y = \det \mathbb{L}_Y$ when Y is virtually LCI, but when X is Lagrangian, ϕ^*K_Y will be trivial, so the resulting quantisations are quite similar. In any case, the $\mathbb{G}_m$-actions on our filtered DGLAs are all unipotent, so extend to $\mathbb{G}_m \otimes_{\mathbb{Z}} \mathbb{Q}$-actions. Since $\mu_2 \otimes \mathbb{Q} = 0$, this means there are canonical equivalences between the spaces of self-dual quantisations for varying $(\mathscr{G}, \mathscr{L})$.

Definition 4.18. As in [Pri5, Remark 2.22], write $t \in \mathrm{GT}(\mathbb{Q})$ for the element which induces the anti-involution of Lemma 1.22. We then denote by $\mathrm{Levi}^t_{\mathrm{GT}}$ the space of Levi decompositions w of GT with $w(-1) = t$; these form a torsor for the subgroup

$(\mathrm{GT}^1)^t$ of t-invariants in the pro-unipotent radical GT^1, and correspond to even Drinfeld associators.

Definition 4.19. Define $G\mathrm{Lag}(Y,X;0)^{sd}$ to be the homotopy fixed points of the involution of $G\mathrm{Lag}(Y,X;0)$ given by $\hbar \mapsto -\hbar$. Explicitly, we set $G\mathrm{Iso}(A,B;0)^{sd}$ to be

$$\underline{\mathrm{MC}}(\mathrm{cone}(F^2\mathrm{DR}(A/R) \to F^2\mathrm{DR}(B/R))) \times \prod_{i>0} \underline{\mathrm{MC}}(\mathrm{cone}(\mathrm{DR}(A/R) \to \mathrm{DR}(B/R))\hbar^{2i}),$$

with $G\mathrm{Lag}(A,B;0)^{sd}$ the subspace of non-degenerate elements.

Theorem 4.20. *Take a morphism $X \to Y$ of strongly quasi-compact Artin N-stacks over R, and a line bundle $\mathscr{L}$ on X with a right $\mathscr{D}$-module structure on $\mathscr{L}^{\otimes 2}$ (such as when $\mathscr{L}$ is a square root of K_X). For any even associator $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$, the induced map*

$$Q\mathcal{P}(Y,\mathscr{L};0)^{\mathrm{nondeg},sd} \to G\mathrm{Lag}(Y,X;0)^{sd}$$

(from non-degenerate self-dual quantisations to generalised self-dual Lagrangians) coming from Proposition 4.12 is a weak equivalence.

In particular, w associates a canonical choice of self-dual quantisation of $(Y,\mathscr{L})$ to every Lagrangian structure of X over Y.

Proof. This is much the same as [Pri7, Proposition 4.6]. Lemma 4.15 implies that w gives rise to weak equivalences

$$Q\mathcal{P}(Y,\mathscr{L};0)^{sd}/G^{2i} \to Q\mathcal{P}(Y,\mathscr{L};0)^{sd}/G^{2i-1}$$
$$Q\mathcal{P}(Y,\mathscr{L};0)^{sd}/G^{2i+1} \to (Q\mathcal{P}(Y,\mathscr{L};0)^{sd}/G^{2i}) \times^h_{(Q\mathcal{P}(Y,\mathscr{L};0)/G^{2i})} (Q\mathcal{P}(Y,\mathscr{L};0)/G^{2i+1}).$$

Combined with Proposition 4.12, these give weak equivalences from $Q\mathcal{P}(Y,\mathscr{L};0)^{\mathrm{nondeg},sd}/G^{2i+1}$ to

$$(Q\mathcal{P}(Y,\mathscr{L};0)^{\mathrm{nondeg},sd}/G^{2i}) \times \underline{\mathrm{MC}}(\hbar^{2i}\mathrm{cone}(\mathrm{DR}(Y/R) \to \mathrm{DR}(X/R))$$

for all $i > 0$. Moreover, [Pri5, Remark 2.22] ensures that for our choice of Levi decomposition w, the map μ_w is equivariant under the involutions $*$, so these equivalences are just given by taking homotopy $\mathbb{Z}/2$-invariants. The result then follows by induction, the base case holding because $*$ acts trivially on $Q\mathcal{P}(Y,\mathscr{L};0)/G^1 = \mathcal{P}(Y,X;0)$, so $Q\mathcal{P}(Y,\mathscr{L};0)^{sd}/G^1 \simeq \mathcal{P}(Y,X;0)$. □

Corollary 4.21. *Take a 0-shifted Lagrangian morphism $(X,\lambda) \to (Y,\omega)$ of strongly quasi-compact Artin N-stacks over R, and a line bundle $\mathscr{L}$ on X with a right $\mathscr{D}$-module structure on $\mathscr{L}^{\otimes 2}$ (such as when $\mathscr{L}$ is a square root of K_X). Then for any self-dual E_1-quantisation $\tilde{\mathscr{O}}_Y$ of the symplectic structure, there exist self-dual quantised Lagrangian structures in $Q\mathcal{P}(Y,\mathscr{L};0)^{\mathrm{nondeg},sd}$ lifting $\mathscr{O}_Y \in Q\mathcal{P}(Y,0)^{\mathrm{nondeg},sd}$ if and only if the class*

$$[\mu_w(-,\tilde{\mathscr{O}}_Y)^{-1}\sigma(\tilde{\mathscr{O}}_Y) - \omega] \in \hbar^2\mathrm{H}^2\mathrm{DR}(Y/R)[\![\hbar^2]\!]$$

lies in the kernel of

$$\hbar^2\mathrm{H}^2\mathrm{DR}(Y/R)[\![\hbar^2]\!] \to \hbar^2\mathrm{H}^2\mathrm{DR}(X/R)[\![\hbar^2]\!],$$

where $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$ is an even associator.

Proof. Our space of interest is the homotopy fibre of the canonical map

$$Q\mathcal{P}(Y,\mathscr{L};0)^{\mathrm{nondeg},sd} \to Q\mathcal{P}(Y,0)^{\mathrm{nondeg},sd} \times^h_{\mathrm{Sp}(Y,0)} \mathrm{Lag}(Y,X;0)$$

over $(\tilde{\mathscr{O}}_Y, \omega, \lambda)$, where $\mathrm{Sp}(Y, 0) = \mathrm{Lag}(Y, \emptyset; 0)$ is the space of 0-shifted symplectic structures as in [Pri4, Pri5].

Substituting in Theorem 4.20, this becomes

$$\prod_{i>0} \underline{\mathrm{MC}}(\mathrm{cone}(\mathrm{DR}(Y/R) \to \mathrm{DR}(X/R))\hbar^{2i}) \times_{\underline{\mathrm{MC}}(\mathrm{DR}(Y/R)^{[1]}\hbar^{2i})} \{\mu_w^{-1}\sigma(\tilde{\mathscr{O}}_Y) - \omega\},$$

and exactness of the sequence $\mathrm{H}^1\mathrm{cone}(\mathrm{DR}(Y/R) \to \mathrm{DR}(X/R)) \to \mathrm{H}^2\mathrm{DR}(Y) \to \mathrm{H}^2\mathrm{DR}(X)$ completes the proof. $\square$

4.4. **Quantisations of higher Lagrangians.** Given a Lagrangian (X, λ) with respect to an n-shifted symplectic structure (Y, ω) for $n > 0$, we now discuss how the techniques of this paper should adapt to give a notion of quantised co-isotropic structures and to establish their existence. The broad picture is that we should have an E_{n+1}-algebra deformation of $\mathscr{O}_Y$ acting on an E_n-algebra deformation of $\mathscr{O}_X$.

If we exploit Koszul duality for P_{n+1}-algebras, we may replace the filtered Hochschild complexes of §1 with Poisson coalgebra coderivations on bar complexes to give P_{n+2}-algebras of derived multiderivations acting on P_{n+1}-algebras (instead of brace algebras acting on associative algebras); the details of this construction are worked out in [MS1], via [CW, §3.1]. Proposition 3.17 will then generalise to give a variant proof of the equivalence between n-shifted Lagrangians and non-degenerate n-shifted co-isotropic structures, announced by Costello and Rozenblyum and proved after this paper was first written by Melani and Safronov [MS2] (based on the approach of [Pri4]); [MS2] also established quantisations for n-shifted co-isotropic structures for $n > 1$ via formality of the E_{n+1} operad. We now sketch a parametrisation of quantisations for higher Lagrangians, including the case $n = 1$ not addressed in [MS2].

4.4.1. *Almost commutative E_k-algebras.* We begin with the notion of a BD_k-algebra as a higher analogue of an almost commutative algebra. There is a filtration on the Lie operad given by arity, inducing a filtration on the free Lie algebra generated by any filtered complex. Taking the universal enveloping E_k-algebra of this Lie algebra then gives a filtered E_k-algebra, and this construction corresponds to a filtration on the E_k operad. We can then define the BD_k operad to be the E_k operad equipped with this completed filtration, for $k \geq 1$.

Explicitly, BD_1 is just the operad defined in [CPT$^+$, §3.5.1], whose algebras are almost commutative DGAAs. For $k \geq 2$, the operad BD_k is just given by the re-indexed good truncation filtration $F^p BD_k = \tau_{\geq p(k-1)} E_k$ — this agrees with [CPT$^+$, §3.5.1] for $k = 2$, but differs by the reindexation for higher k. In particular, almost commutative brace algebras are equivalent to BD_2-algebras.

Informally, an n-shifted quantisation of a morphism $A \to B$ of CDGAs consists of a BD_{n+1}-algebra deformation $\tilde{A}$ of A acting on a BD_n-algebra deformation $\tilde{B}$ of B in a sense we now attempt to make precise. An n-shifted quantisation of a morphism $A \to B$ of stacky CDGAs will be an n-shifted quantisation of $\hat{\mathrm{Tot}}\, A \to \hat{\mathrm{Tot}}\, B$ subject to additional boundedness constraints.

4.4.2. *Centres.* From now on, we refer to BD_k-algebras in complete filtered cochain chain complexes (simplicially localised at levelwise filtered quasi-isomorphisms) as stacky BD_k-algebras. Adapting [Lur, Theorem 5.3.1.14] from ∞-operads to the operads BD_k in filtered chain complexes will give a stacky BD_k-algebra

$$\mathbf{R}\mathcal{C}\mathcal{C}_{BD_k,R}(A, D)$$

associated to any morphism $A \to D$ of stacky BD_k-algebras over R, universal with the property that there is a BD_k-algebra morphism $\mathbf{R}\mathcal{CC}_{BD_k,R}(A,D) \otimes^{\mathbf{L}}_R A \to D$ in the associated ∞-category. Explicitly, these centres should be given by the higher order Hochschild complexes of [Gin] equipped with a PBW filtration. The associated graded $\mathrm{gr}\mathbf{R}\mathcal{CC}_{BD_k,R}(A,D)$ is necessarily the centre of the morphism $\mathrm{gr}A \to \mathrm{gr}B$ of graded P_k-algebras, so is given by derived P_k multiderivations from $\mathrm{gr}A$ to $\mathrm{gr}B$.

The universal property implies that $\mathbf{R}\mathcal{CC}_{BD_k,R}(A) := \mathbf{R}\mathcal{CC}_{BD_k,R}(A,A)$ is naturally an E_1-algebra in stacky BD_k-algebras, i.e. a stacky $E_1 \otimes^{\mathbf{L}}_{BV} BD_k$-algebra for the Boardman–Vogt tensor product $\otimes_{BV}$. Moreover, for any morphism $A \to D$, the centre $\mathbf{R}\mathcal{CC}_{BD_k,R}(A,D)$ will then become a $\mathbf{R}\mathcal{CC}_{BD_k}(A)$-module in stacky BD_k-algebras.

For any morphism $A_1 \times A_2 \to D$, the idempotents in the domain give a decomposition $D = D_1 \times D_2$, and by universality for each morphism $A \to D$ we thus have

$$\mathbf{R}\mathcal{CC}_{BD_k,R}(R\times A, R\times D) \simeq \mathbf{R}\mathcal{CC}_{BD_k,R}(R,R)\times\mathbf{R}\mathcal{CC}_{BD_k,R}(A,D) = R\times\mathbf{R}\mathcal{CC}_{BD_k,R}(A,D).$$

The centre of $R \times A \to R \times D$ as in the category of augmented stacky BD_k-algebras over R is just

$$\mathbf{R}\mathcal{CC}_{BD_k,R}(R \times A, R \times D) \times^h_{(R\times D)} R,$$

so the reasoning above shows that

$$\mathbf{R}\mathcal{CC}_{BD_k,R,+}(A,D) := \mathbf{R}\mathcal{CC}_{BD_k,R}(A,D) \times^h_D 0$$

is naturally a non-unital stacky BD_k-algebra, with $\mathbf{R}\mathcal{CC}_{BD_k,R,+}(D)$ a non-unital stacky $E_1 \otimes^{\mathbf{L}}_{BV} BD_k$-algebra.

Adapting Lemma 1.26, we then have:

Definition 4.22. Given a stacky $E_1 \otimes^{\mathbf{L}}_{BV} BD_k$-algebra C over R and a C-module E in stacky BD_k-algebras over R, we define $E_{[1]} \rtimes C$ to be the non-unital stacky $E_1 \otimes^{\mathbf{L}}_{BV} BD_k$-algebra

$$C \times^h_{\mathbf{R}\mathcal{CC}_{BD_k,R}(E)} \mathbf{R}\mathcal{CC}_{BD_k,R,+}(E),$$

the morphism $C \to \mathbf{R}\mathcal{CC}_{BD_k,R}(E)$ existing by universality.

4.4.3. *Quantised n-shifted relative polyvectors for $n > 0$.* Given a morphism $\phi\colon A \to B$ of stacky CDGAs over R, now consider the non-unital $E_1 \otimes^{\mathbf{L}}_{BV} BD_{n+1}$-algebra

$$(\mathcal{C}, F) := \mathbf{R}\mathcal{CC}_{BD_{n+1},R}(A, \mathbf{R}\mathcal{CC}_{BD_n,R}(B))_{[1]} \rtimes \mathbf{R}\mathcal{CC}_{BD_{n+1},R}(A)$$

in complete filtered cochain chain complexes. Definition 2.10 then adapts verbatim to give a complex $Q\widehat{\mathrm{Pol}}(A,B;n)$ equipped with filtrations $\tilde{F}$ and G.

Since we wish $Q\widehat{\mathrm{Pol}}(A,B;n)[n+1]$ to have the structure of a DGLA with $[\tilde{F}^i, \tilde{F}^j] \subset \tilde{F}^{i+j-1}Q$ and $[G^i, G^j] \subset G^{i+j}$, acting as derivations on the bifiltered $E_1 \otimes^{\mathbf{L}}_{BV} BD_{n+1}$-algebra $\hbar Q\widehat{\mathrm{Pol}}(A,B;n)$, we need to know that $\mathbf{R}\mathcal{CC}_{BD_k}(A)$ has the structure of a BD_{k+1}-algebra. The analogous statement for $k = 1$ is the content of Lemma 1.20. In general, the property would follow from the following conjecture:

Conjecture 4.23. *For $k \geq 1$, the additivity isomorphism $E_{k+1} \simeq E_1 \otimes^{\mathbf{L}}_{BV} E_k$ of* [Lur, Theorem 5.1.2.2] *induces a map $BD_{k+1} \simeq E_1 \otimes^{\mathbf{L}}_{BV} BD_k$ of operads in complete filtered chain complexes.*

Here, $\otimes^{\mathbf{L}}_{BV}$ denotes the derived Boardman–Vogt tensor product, so the conjecture amounts to saying that an A_∞-algebra in BD_k-algebras is naturally a BD_{k+1}-algebra. On passing to associated graded complexes, the equivalence would give

$P_{k+1} \to E_1 \otimes^{\mathbf{L}}_{BV} P_k$, which has been proved to be an equivalence by Rozenblyum (unpublished, cf. [CPT$^+$, §3.4]) and independently by Safronov [Saf2]; thus the map in the conjecture is necessarily an equivalence if it exists. A proof of Conjecture 4.23 has also been announced by Rozenblyum (cf. [CPT$^+$, comment after Conjecture 3.5.7]). For $k \geq 2$, the conjecture would follow if additivity is compatible with the action of the Grothendieck–Teichmüller group.

The conjecture would also ensure that the centres $\mathbf{R}\mathcal{CC}_{BD_k,R}(A,D)$ above all exist by appealing directly to [Lur, Theorem 5.3.1.14] for $k \geq 1$, regarding BD_k-algebras as E_{k-1}-algebras in BD_1-algebras.

The definitions of §§2, 3 all then adapt, replacing $Q\widehat{\mathrm{Pol}}(A,M;0)$ with $Q\widehat{\mathrm{Pol}}(A,B;n)$ and taking appropriate shifts. The space $Q\mathcal{P}(A,B;n)$ of n-shifted quantisations of the pair (A,B) is just

$$\underline{\mathrm{MC}}(\tilde{F}^2 Q\widehat{\mathrm{Pol}}(A,B;n)[n+1]),$$

elements of which give rise to curved E_{n+1}-algebra deformations of $\hat{\mathrm{Tot}}\, A$ acting on curved E_n-algebra deformations of B.

The space $G\mathrm{Iso}(A,B;n)$ of n-shifted isotropic structures is

$$\underline{\mathrm{MC}}(\tilde{F}^2\mathrm{cone}(\mathrm{DR}(A/R)[\![\hbar]\!] \to \mathrm{DR}(B/R)[\![\hbar]\!])[n]),$$

and Definition 3.7 then adapts to give a compatibility map

$$\mu_w(-,\Delta)\colon\ \mathrm{cocone}(\mathrm{DR}(A/R) \to \mathrm{DR}(B/R)[\![\hbar]\!]/\hbar^j \to T_\Delta Q\widehat{\mathrm{Pol}}_w(A;B,0)/G^j$$

for each quantisation Δ; Definition 3.14 adapts to give a space $Q\mathrm{Comp}_w(A,B;n)$ for each $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$.

Propositions 3.16, 3.17 and 3.18 will all carry over directly, in particular giving a map $Q\mathcal{P}(A,B;n)^{\mathrm{nondeg}} \to G\mathrm{Lag}(A,B;n)$, the non-degenerate locus in $G\mathrm{Iso}(A,B;n)$. The techniques of §4 then extend these to global constructions for Artin N-stacks.

4.4.4. *Self-duality.* The functor $D \mapsto D^{\mathrm{opp}}$ sending an almost commutative algebra to its opposite gives an involutive endofunctor of the category of BD_1-algebras, and hence of the categories of $E_1 \otimes^{\mathbf{L}}_{BV} BD_k$-algebras. The universal property of centres then gives an anti-involution

$$-i\colon\ \mathbf{R}\mathcal{CC}_{BD_k,R}(A,D)^{\mathrm{opp}} \to \mathbf{R}\mathcal{CC}_{BD_k,R}(A^{\mathrm{opp}},D^{\mathrm{opp}}),$$

which in the $k=1$ case is the anti-involution $-i$ of Lemma 1.22. Defining an anti-involutive $E_1 \otimes^{\mathbf{L}}_{BV} BD_k$-algebra to be a homotopy fixed point of the involutive endofunctor $(-)^{\mathrm{opp}}$, the anti-involution above makes $\mathbf{R}\mathcal{CC}_{BD_k,R}(A,D)$ a stacky anti-involutive BD_k-algebra whenever A and D are stacky anti-involutive BD_k-algebras. In fact, this is necessarily the centre of $A \to D$ in the category of stacky anti-involutive BD_k-algebras — the operad governing anti-involutive BD_k-algebras is $BD_k \circ (0, \mathbb{Q}.(\mathbb{Z}/2), 0, \ldots)$, with distributivity transformation given by the anti-involution.

As in §4.3, we then have an involution $(-)^*$ on the DGLA $Q\widehat{\mathrm{Pol}}(A,D;0)[n+1]$ given by $\Delta^*(\hbar) := i(\Delta)(-\hbar)^t$, and we can define $Q\mathcal{P}(A,B;n)^{sd}$ to be the fixed points of the resulting $\mathbb{Z}/2$-action, so its points give rise to involutive quantisations.

The proof of Theorem 4.20 will then adapt to give:

Theorem 4.24. *Take a morphism $X \to Y$ of strongly quasi-compact Artin N-stacks over R. If Conjecture 4.23 holds, then for any even associator $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$, the induced map*

$$Q\mathcal{P}(Y,X;n)^{\mathrm{nondeg},sd} \to G\mathrm{Lag}(Y,X;n)^{sd}$$

(from non-degenerate self-dual quantisations to generalised self-dual Lagrangians) is a weak equivalence for all $n > 0$.

In particular, w associates a canonical choice of self-dual quantisation of (Y, X) to every n-shifted Lagrangian structure of X over Y.

For $n > 1$, this has been proved without the self-duality conditions by [MS2] since this paper was first written, by using a direct formality argument. Their argument also implies these self-dual statements.

Remark 4.25 (Twisted quantisations). One significant difference between Theorems 4.20 and 4.24 is that the former incorporates the data of a line bundle. Similar input data are not essential for positively shifted quantisations because a commutative algebra is canonically isomorphic to its opposite E_1-algebra, whereas $\mathscr{O}_X$ is not in general a right $\mathscr{D}$-module.

However, by generalising Remark 4.17 we still expect a sensible notion of twisted quantisations for n-shifted Lagrangians, fibred over the space $\mathbf{R}\Gamma(Y, B^{n+2}\mathbb{G}_m) \times^h_{\mathbf{R}\Gamma(X, B^{n+2}\mathbb{G}_m)} \{1\}$ of pairs $(\mathscr{G}, \mathscr{L})$ with $\mathscr{G}$ a $B^{n+1}\mathbb{G}_m$-torsor on Y, and $\mathscr{L}$ a trivialisation of $\phi^*\mathscr{G}$ on X. Self-dual (i.e. involutive) quantisations would then be parametrised by $\mathbf{R}\Gamma(Y, B^{n+2}\mu_2) \times^h_{\mathbf{R}\Gamma(X, B^{n+2}\mu_2)} \{1\}$. Adapting [Lur, Theorem 5.3.2.5] from filtered E_{n+2}-algebras to BD_{n+2}-algebras would establish the required actions of $(n+2)$-groupoids $\operatorname{holim}_{\longleftarrow i \in \Delta} B^{n+2} D^i(A)^\times$ generalising TLB from §4.2.

However, since these spaces will come from unipotent group actions on quantised polyvectors, the actions of the torsion groups $B^{n+1}\mu_2(A), B^{n+1}\mu_2(B)$ must be trivial, so the spaces of twisted self-dual quantisations will be canonically equivalent as $(\mathscr{G}, \mathscr{L})$ varies.

5. A "Fukaya category" for algebraic Lagrangians

In [BF, §5.3], Behrend and Fantechi discussed the construction of a dg category whose objects are local systems on Lagrangian submanifolds of a complex symplectic variety. An extensive survey of related results is given in [BBD+, Remark 6.15], where Joyce et al. discuss possible approaches to constructing such a "Fukaya category" with complexes of vanishing cycles as morphisms, with more details spelt out in [Joy]. There are serious difficulties in trying to upgrade complexes of vanishing cycles to a dg category, but on a complex symplectic manifold, [BBD+, Remark 6.15] explains that a likely candidate for the subcategory of smooth Lagrangians is given by the derived category of simple holonomic DQ modules for a DQ algebroid quantisation of the sheaf of analytic functions, by combining the results of Kashiwara and Schapira [KS2] (cf. [Sch, §3.3]) with [DS]. It is this approach which generalises naturally in our setting.

5.1. **Quantised intersections and internal** Homs. Given a BD_1-algebra $\tilde{A}$ with right and left actions on BD_0-algebras $\tilde{B}$ and $\tilde{C}$, respectively, [Saf1, Proposition 5.8 and Theorem 5.10] give a natural BD_0-algebra structure on $\tilde{B} \otimes^{\mathbf{L}}_{\tilde{A}} \tilde{C}$. Since BD_1-algebras acting on BD_0-algebras are a special case of our definition of quantised co-isotropic structures in Definition 2.13, this can be interpreted for these cases as saying that the intersection of quantised 0-shifted co-isotropic structures is a (-1)-shifted quantisation of the intersection.

The purpose of this section is first to generalise this (Proposition 5.5) by working with both stacky CDGAs and non-trivial line bundles. We will then give an analogous result (Proposition 5.8) for Homs instead of tensors.

Lemma 5.1. *Given a stacky CDGA A, regarded as an almost commutative DGAA with trivial filtration, any element $\phi \in \gamma_r \mathcal{CC}_R(A, A)$ is a differential operator of order $\leq r$ with respect to the shuffle multiplication of Definition 1.8, when regarded as a coderivation on the bar construction* $\mathrm{B}A$.

Proof. To say that ϕ is has order $\leq r$ is equivalent to vanishing of the map $[\phi]_r \colon (\mathbb{A})^{\otimes r+2} \to \mathrm{B}A$ given by

$$a_0 \otimes \ldots \otimes a_r \otimes b \mapsto [\ldots[[\phi, a_0], a_1] \ldots, a_r](b), \quad = \sum_{I \subset \{a_0, \ldots, a_r\}} (-1)^{|I|} (\prod_{j \notin I} a_j) \phi((\prod_{i \in I} a_i) b)$$

where $a_i \in \mathrm{B}A$ is regarded as an element of $\mathcal{H}om_R(\mathrm{B}A, \mathbb{A})$ via the shuffle multiplication, $[-,-]$ denotes the commutator in $\mathrm{End}_R(\mathrm{B}A)$, and $\prod$ is defined using the shuffle product with appropriate Koszul signs.

By construction, $[\phi]_r(a_0 \otimes \ldots \otimes a_r \otimes b) = 0$ whenever any $a_i \in R$. Since $\mathrm{B}A = R \oplus \beta^1 \mathrm{B}A$, it follows that $(\mathrm{B}A)^{\otimes(r+2)}$ is the sum of $\beta^{r+1}((\mathrm{B}A)^{\otimes(r+1)}) \otimes \mathrm{B}A$ and a subspace on which $[\phi]_r$ automatically vanishes.

By definition of the filtration γ_r in Definition 1.16, the coderivation ϕ sends $\beta^j \mathrm{B}A$ to $\beta^{j+1-r} \mathrm{B}A$, for the filtration β of Definition 1.9. Since shuffle multiplication preserves the filtration, it follows that $[\phi]_r$ sends $\beta^j((\mathrm{B}A)^{\otimes r+2})$ to $\beta^{j+1-r}\mathrm{B}A$, and in particular the composite of $[\phi]_r \colon \beta^{r+1}(\mathrm{B}A)^{\otimes r+2}) \to \mathrm{B}A$ with the cogenerator map $\mathrm{B}A \to A_{[-1]}$ vanishes.

Combining the last two paragraphs, it follows that $[\phi]_r$ vanishes on cogenerators. If we denote the iterated shuffle multiplication by $\nabla_i \colon \mathrm{B}A^{\otimes i} \to \mathrm{B}A$, then the maps $\nabla_{i+1} \circ (\mathrm{id}^{\otimes i} \otimes \phi \circ \nabla_j)$ are all ∇_{i+j}-coderivations. In particular, this implies that $[\phi]_r$ is a ∇_{r+2}-coderivation, being an alternating sum of such. Since it vanishes on cogenerators, it must therefore be zero. □

Lemma 5.2. *Take a stacky CDGA A, an A-module M, a coderivation $\phi \in \gamma_r \mathcal{CC}_R(A, A)$ as in Lemma 5.1, and an element $\theta \in \gamma_r \mathcal{CC}_R(A, \mathrm{End}_R(M))$ with respect to the trivial filtration on M. Regarding θ as a map $(\mathrm{B}A) \otimes_R M \to M$, the associated ϕ-coconnection $\theta_\phi \colon (\mathrm{B}A) \otimes_R M \to (\mathrm{B}A) \otimes_R M$, given by*

$$(\phi \otimes \mathrm{id}_M) + (\mathrm{id}_{\mathrm{B}A} \otimes \theta) \circ (\mu_{\mathrm{B}A} \otimes \mathrm{id}_M)$$

for the comultiplication $\mu_{\mathrm{B}A} \colon \mathrm{B}A \to (\mathrm{B}A) \otimes (\mathrm{B}A)$, has order $\leq r$ with respect to the shuffle multiplication by elements of $\mathrm{B}A$.

Proof. This proceeds in exactly the same way as the proof of Lemma 5.1. It suffices to establish vanishing of the map $[\theta_\phi]_r \colon (\mathbb{A})^{\otimes r+2} \otimes M \to (\mathrm{B}A) \otimes M$ given by

$$a_0 \otimes \ldots \otimes a_r \otimes m \mapsto [\ldots[[\theta_\phi, a_0], a_1] \ldots, a_r](m).$$

This automatically vanishes whenever any $a_i \in R$, so it suffices to show that it vanishes on $\beta^{r+1}((\mathrm{B}A)^{\otimes(r+1)}) \otimes (\mathrm{B}A) \otimes M$.

The conditions that $\phi, \theta \in \gamma_r$ then imply that the composite of $[\theta_\phi]_r \colon \beta^{r+1}(\mathbb{A})^{\otimes r+2}) \otimes M \to (\mathrm{B}A) \otimes M$ with the cogenerator map $(\mathrm{B}A) \otimes M \to M$ vanishes, so $[\theta_\phi]_r$ vanishes on cogenerators. Since $[\theta_\phi]_r$ is a $(\nabla_{r+2}, [\phi]_r)$-coconnection, it thus vanishes everywhere. □

Lemma 5.3. *Under the conditions of Lemma 5.2, the ϕ^*-connection $\theta^*_\phi \colon \mathcal{H}om_R(\mathrm{B}A, M) \to \mathcal{H}om_R(\mathrm{B}A, M)$, given by*

$$f \mapsto f \circ \phi + \theta \circ (\mathrm{id}_{\mathrm{B}A} \otimes f) \circ \mu_{\mathrm{B}A},$$

has order $\leq r$ with respect to the shuffle multiplication by elements of $\mathrm{B}A$.

Proof. This works in exactly the same way as Lemma 5.2. The question reduces to showing that the similarly defined commutator map $[\theta_\phi]^*_r\colon \mathcal{H}om_R((\mathrm{B}A)^{\otimes r+2}, M) \to \mathcal{H}om_R(\mathrm{B}A, M)$ vanishes, but this follows from the vanishing of $[\theta_\phi]_r$ in the proof of Lemma 5.2.

□

Definition 5.4. Given morphisms $C \leftarrow A \to B$ of stacky CDGAs and strict line bundles M and N over B and C respectively, define the spaces $Q\mathcal{P}(A, M, N; 0)$ and $Q\mathcal{P}(A, M^{\mathrm{opp}}, N; 0)$ to be the homotopy fibre products

$$Q\mathcal{P}(A, M, N; 0) := Q\mathcal{P}(A, M; 0) \times^h_{Q\mathcal{P}(A,0)} Q\mathcal{P}(A, N; 0)$$
$$Q\mathcal{P}(A, M^{\mathrm{opp}}, N; 0) := Q\mathcal{P}(A, M; 0) \times^h_{i, Q\mathcal{P}(A,0)} Q\mathcal{P}(A, N; 0),$$

where i is the involution of Lemma 1.22, which sends a quantisation $\tilde{A}$ of A to the opposite BD_1-algebra $\tilde{A}^{\mathrm{opp}}$.

In other words, elements of $Q\mathcal{P}(A, M, N; 0)$ consist of quantised co-isotropic structures on the pairs (A, M) and (A, N), with the same underlying 0-shifted quantisation $\tilde{A}$ of A. On the other hand, elements of $Q\mathcal{P}(A, M^{\mathrm{opp}}, N; 0)$ consist of quantised co-isotropic structures on the pairs (A, M) and (A, N), but with opposite underlying 0-shifted quantisations $\tilde{A}^{\mathrm{opp}}$ and $\tilde{A}$ of A. Thus in $Q\mathcal{P}(A, M, N; 0)$, both M and N are being deformed as certain left $\tilde{A}$-modules, while in $Q\mathcal{P}(A, M^{\mathrm{opp}}, N; 0)$ we are deforming M as a right $\tilde{A}$-module and N as a left $\tilde{A}$-module.

Proposition 5.5. *Given morphisms $C \leftarrow A \to B$ of stacky CDGAs and strict line bundles M and N over B and C respectively, there is a natural derived tensor product construction*

$$Q\mathcal{P}(A, M^{\mathrm{opp}}, N; 0) \to Q\mathcal{P}(M \otimes^{\mathbf{L}}_A N, -1)$$

to the space of (-1)-shifted quantised Poisson structures on the line bundle $M \otimes^{\mathbf{L}}_A N$ over $B \otimes^{\mathbf{L}}_A C$.

Proof. We adapt the approach of [Saf1, Proposition 5.8 and Theorem 5.10]. We will deform the Hochschild homology complex

$$\mathcal{CC}^R(A, N \otimes_R M)_\# := \bigoplus_n (M \otimes_R A^{\otimes_R n} \otimes_R N)_{[-n]},$$

which has chain differential $\delta \pm b$, for the Hochschild differential

$$\begin{aligned} b(m, a_1, \ldots, a_r, n) = (ma_1, a_2, \ldots, a_r, n) \\ + \sum_{i=1}^{r-1} (-1)^i (a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_r) \\ + (-1)^r (a_1, \ldots, a_{r-1}, a_r n). \end{aligned}$$

Note that we may write $\mathcal{CC}^R(A, N \otimes_R M)_\# = (M \otimes_R \mathrm{B}A \otimes_R N)_\#$ for the bar construction B of Definition 1.8, and regard this as the cofree left $\mathrm{B}A$-comodule cogenerated by $N \otimes_R M$. Then b is the $\mathrm{B}A$-coderivation given on cogenerators by the difference of the multiplication maps $M \otimes_R A \otimes_R N \to M \otimes_R N$.

There is a graded-commutative multiplication on $\mathcal{CC}^R(A, C \otimes_R B)$ given by combining those on B and C with the shuffle multiplication ∇ on $\mathrm{B}A$ from Definition 1.9. This makes $\mathcal{CC}^R(A, C \otimes_R B)$ a model for the stacky CDGA $B \otimes^{\mathbf{L}}_A C$, and similarly the

$\mathcal{CC}^R(A, C\otimes_R B)$-module $\mathcal{CC}^R(A, N\otimes_R M)$ is a strict line bundle, and a model for the $M\otimes^{\mathbf{L}}_A N$.

Since the map $Q\widehat{\mathrm{Pol}}_R(A,N;0)\to Q\widehat{\mathrm{Pol}}_R(A,0;0)$ of filtered DGLAs is surjective, the map $Q\mathcal{P}(A,N;0)\to Q\mathcal{P}(A,0)$ is a fibration, so a model for $Q\mathcal{P}(A,M^{\mathrm{opp}},N;0)$ is given by the fibre product $Q\mathcal{P}(A,M;0)\times_{i,Q\mathcal{P}(A,0)}Q\mathcal{P}(A,N;0)$. Given an element $(\Delta_A,\Delta_M,\Delta_N\in Q\mathcal{P}(A,M^{\mathrm{opp}},N;0)$, we construct an operator $(\Delta_M)_{\Delta_A}\mp(\Delta_N)_{\Delta_A}$ on $\mathcal{CC}^R(A,N\otimes_R M)[\![\hbar]\!]$ by first extending Δ_A to a coderivation on $\mathrm{B}A$ as in Lemma 5.1, then constructing coconnections associated to $\Delta_M,\Delta_N)$ as in Lemma 5.2.

It suffices to show that $(\Delta_M)_{\Delta_A}\mp(\Delta_N)_{\Delta_A}$ is a differential operator in $\prod_{i\geq1}\hbar^iF_{i+1}\mathscr{D}_{\mathcal{CC}^R(A,C\otimes_R B)}(\mathcal{CC}^R(A,N\otimes_R M))$, since it then defines an element of $Q\mathcal{P}(M\otimes^{\mathbf{L}}_A N,-1)$. Equivalently, for arbitrary elements $x_i\in\mathcal{CC}^R(A,C\otimes_R B)$, this says that we want the commutator $[[\ldots[[(\Delta_M)_{\Delta_A}\mp(\Delta_N)_{\Delta_A},x_1],x_2]\ldots,x_r]$ to be divisible by $\hbar^{r-1}$.

Since $\Delta_A\in\prod_{i\geq1}\hbar^i\gamma_{i+1}$, Lemma 5.1 implies that the associated coderivation lies in $\prod_{i\geq1}\hbar^iF_{i+1}\mathscr{D}_{\mathrm{B}A}$. Lemma 5.2 then implies that for all $x_i\in\mathrm{B}A$, the commutator above has the required property, since $\Delta_M,\Delta_N\in\prod_{i\geq1}\hbar^i(\gamma F)_{i+1}\subset\prod_{i\geq1}\hbar^i\gamma_{i+1-r}$.

Finally, for $y\in B\otimes C$, the commutator $[(\Delta_M)_{\Delta_A}\mp(\Delta_N)_{\Delta_A},x]$ is $\mathrm{B}A$-linear. Moreover, for $y_i\in B\otimes C$ we have $[[\ldots[[(\Delta_M)_{\Delta_A}\mp(\Delta_N)_{\Delta_A},y_1],y_2]\ldots,y_r]\in\prod_{i\geq1}\hbar^i(\gamma F)_{i+1-r}\subset\prod_{i\geq r-1}\hbar^i\gamma_{i+1-r}$, so applying Lemma 5.2 with the trivial coderivation shows that for $x_j\in\mathrm{B}A$, the commutator

$$[[\ldots[[\ldots[[(\Delta_M)_{\Delta_A}\mp(\Delta_N)_{\Delta_A},y_1],y_2]\ldots,y_r],x_1],x_2]\ldots,x_s]$$

is divisible by $\hbar^{r+s-1}$, as required. □

Remarks 5.6. It is natural to ask how this intersection construction for quantisations relates to the natural constructions of generalised symplectic structures on Lagrangian intersections. Even for unquantised shifted Poisson intersections in [MS2, §§3 and 4] this is not spelt out, but we expect that it should be possible to formulate compatibility using a map with a target DGAA related to the brace tensor product $\mathrm{CC}(A,\mathscr{D}_B)\otimes^{\mathbf{L}}_{\mathrm{CC}(A)}\mathrm{CC}(A,\mathscr{D}_C)$.

However, the relation of intersections with self-duality will be much more subtle, because the notion of self-duality depends on choices of line bundles with right $\mathscr{D}$-module structures. In virtually LCI settings, where the dualising complex is a line bundle, we expect the intersection of self-dual quantised 0-shifted co-isotropic structures will give a self-dual E_0-quantisation via Proposition 5.5.

Proposition 5.5 has the following generalisation, with much the same proof, just with an additional A_∞-action by $(A_2[\![\hbar]\!],\Delta_{A_2})$ to incorporate:

Proposition 5.7. *Given morphisms $A_1\to B$ and $A_1\otimes A_2\to C$ of stacky CDGAs and strict line bundles M and N over B and C respectively, there is a natural derived tensor product construction*

$$(Q\mathcal{P}(A_1,M;0)\times Q\mathcal{P}(A_2,0))\times_{(i\otimes\mathrm{id}),Q\mathcal{P}(A_1\otimes A_2,0)}Q\mathcal{P}(A_1\otimes A_2,N;0)\to Q\mathcal{P}(A_2,M\otimes^{\mathbf{L}}_{A_1}N;-1)$$

to the space of 0-shifted quantised co-isotropic structures on the line bundle $M\otimes^{\mathbf{L}}_{A_1}N$ on $B\otimes^{\mathbf{L}}_{A_1}C$ over A_2.

Proposition 5.8. *Given morphisms $C\leftarrow A\to B$ of stacky CDGAs and strict line bundles M and N over B and C respectively, there is a natural derived* Hom *construction* *from $Q\mathcal{P}(A,M,N;0)$ to the space of R-linear deformations of $\mathbf{R}\mathcal{H}om_A(M,N)$ given by differential operators*

$$\Delta \in \prod_{i\geq 1} \hbar^i F_{i+1} \hat{\mathrm{Tot}}\, \mathscr{D}_{B\otimes^{\mathbf{L}}_A C}(\mathbf{R}\mathcal{H}om_A(M,N)).$$

In particular, if $\mathbf{R}\mathcal{H}om_A(M,N)$ is an invertible $B\otimes^{\mathbf{L}}_A C$-module, this gives a map

$$Q\mathcal{P}(A,M,N;0) \to Q\mathcal{P}(\mathbf{R}\mathcal{H}om_A(M,N),-1)$$

to the space of (-1)-shifted quantised Poisson structures on the line bundle $\mathbf{R}\mathcal{H}om_A(M,N)$ over $B\otimes^{\mathbf{L}}_A C$.

Proof. The construction arises by sending $(\Delta_A, \Delta_M, \Delta_N \in Q\mathcal{P}(A,M,N;0)$, to the differential operator $((\Delta_M)_{\Delta_A})^* \mp (\Delta_N)^*_{\Delta_A}$ on $\mathcal{H}om_R(M\otimes \mathrm{B}A, N)[\![\hbar]\!]$, defined entirely analogously to the operator of Proposition 5.5, but replacing one instance of Lemma 5.2 with Lemma 5.3. □

5.2. **DQ modules associated to quantised Lagrangians.** Since we are working algebraically rather than analytically, our analogue of a DQ module is simply an $\hbar$-adically complete module over a fixed quantisation $\tilde{\mathscr{O}}_Y$ of $\mathscr{O}_Y$. When Y is a derived DM stack, we can interpret $\tilde{\mathscr{O}}_Y$ as an A_∞-algebroid deformation of $\mathscr{O}_Y$ on the étale site of Y as in Example 4.10.(2), and DQ modules are then objects of its derived dg category $\mathbf{R}\varprojlim_i \mathscr{D}_{dg}(\tilde{\mathscr{O}}_Y/\hbar^i)$. When Y is a derived Artin stack, the deformation $\tilde{\mathscr{O}}_Y$ is defined on a site of stacky CDGAs and may incorporate curvature, so we have to be a little more careful. Essentially, we take a DQ-module to be a module for the curved A_∞-algebra $\tilde{\mathscr{O}}_Y$, but there are boundedness conditions coming from $\hat{\mathrm{Tot}}$ as in Remark 2.14.

For simplicity, we now just describe the full dg subcategory of DQ modules coming from quantised Lagrangians. The idea is that for a line bundle $\mathscr{L}$ on a derived Lagrangian $\phi\colon X\to Y$, each deformation quantisation $(\tilde{\mathscr{O}}_Y, \Delta_{\mathscr{L}})$ of $(\mathscr{O}_Y, \mathscr{L})$ gives rise to such a DQ module as $\mathbf{R}\phi_*\tilde{\mathscr{L}}$, where as in Remark 4.9,

$$\tilde{\mathscr{L}} := (\mathscr{L}[\![\hbar]\!], \delta + \Delta_{\mathscr{L}}\cdot -)$$

is the left $\phi^{-1}\tilde{\mathscr{O}}_Y$-module associated to the quantisation $\Delta_{\mathscr{L}}$.

Definition 5.9. Given an E_1-quantisation $\tilde{\mathscr{O}}_Y \in Q\mathcal{P}(Y,0)$ of a derived Artin n-stack Y over R, we define the $R[\![\hbar]\!]$-linear dg category $Q\mathcal{CI}_{\mathrm{dg}}(\tilde{\mathscr{O}}_Y)$ of quantised co-isotropic structures over $\tilde{\mathscr{O}}_Y$ as follows.

There is an object for each quantised co-isotropic structure $(\tilde{\mathscr{O}}_Y, \tilde{\mathscr{L}}) \in Q\mathcal{P}(Y,\mathscr{L};0)\times^h_{Q\mathcal{P}(Y,0)}\{\tilde{\mathscr{O}}_Y\}$ over $\tilde{\mathscr{O}}_Y$, for each line bundle $\mathscr{L}$ on each derived Artin stack $\phi\colon X\to Y$ representable over Y by quasi-compact quasi-separated derived algebraic spaces.

Given objects $\tilde{\mathscr{L}}_1, \tilde{\mathscr{L}}_2$, we set the associated Hom-complex $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{1*}\tilde{\mathscr{L}}_1, \mathbf{R}\phi_{2*}\tilde{\mathscr{L}}_2)$ to be the homotopy limit, taken over all homotopy formally étale morphisms $f\colon \operatorname{Spec} DA \to Y$ (i.e. $f \in \operatorname{ho}\varprojlim_i X(D^iA)$) from stacky CDGAs A, of the complexes

$$\hat{\mathrm{Tot}}\,(\mathcal{H}om_R(f^*(\mathbf{R}\phi_{1*}\mathscr{L}_1)\otimes \mathrm{B}A, f^*(\mathbf{R}\phi_{2*}\mathscr{L}_2))[\![\hbar]\!], \delta + ((\Delta_{\mathscr{L}_1})_{\Delta_A})^* \mp (\Delta_{\mathscr{L}_2})^*_{\Delta_A})$$

with notation for differentials as in the proofs of Propositions 5.5, and 5.8.

The associative composition law

$$\begin{aligned}&\mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{1*}\tilde{\mathscr{L}}_1, \mathbf{R}\phi_{2*}\tilde{\mathscr{L}}_2) \otimes_R \mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{2*}\tilde{\mathscr{L}}_2, \mathbf{R}\phi_{3*}\tilde{\mathscr{L}}_3)\\ &\to \mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{1*}\tilde{\mathscr{L}}_1, \mathbf{R}\phi_{3*}\tilde{\mathscr{L}}_3)\end{aligned}$$

arises naturally on rewriting each $\mathcal{H}om_R(M \otimes \mathrm{B}A, N)[\![\hbar]\!]$ symmetrically as the double complex $\mathcal{H}om_{\mathrm{B}A}(M \otimes \mathrm{B}A, N \otimes \mathrm{B}A)[\![\hbar]\!]$ of $\mathrm{B}A$-colinear maps.

Remarks 5.10. Similar reasoning to [Pri5, Proposition 1.25] shows that if we reduce modulo $\hbar$, the resulting R-linear dg category $Q\mathcal{CI}_{\mathrm{dg}}(\tilde{\mathscr{O}}_Y)/\hbar$ is quasi-equivalent to a full dg subcategory of the derived category $\mathscr{D}_{dg}(\mathscr{O}_Y)$ of quasi-coherent complexes on Y. Its objects are complexes of the form $\mathbf{R}\phi_*\mathscr{L}$, for line bundles $\mathscr{L}$ on derived stacks over Y for which the data $(\phi\colon X \to Y, \mathscr{L})$ admits a quantised co-isotropic structure lifting $\tilde{\mathscr{O}}_Y$.

The hypothesis in Definition 5.9 that ϕ be representable is stronger than strictly necessary. All we really need is that $\mathbf{R}\phi_*$ preserves quasi-coherence and commutes with derived base change. With that modification, the representability hypothesis on the correspondence in Proposition 5.14 below can be relaxed accordingly.

Definition 5.11. Fix a non-degenerate involutive quantisation $\tilde{\mathscr{O}}_Y \in Q\mathcal{P}(Y,0)^{\mathrm{nondeg},sd}$ quantising a symplectic structure $\omega \in \mathrm{H}^2F^2\mathrm{DR}(Y/R)$, and assume that $\tilde{\mathscr{O}}_Y$ is w-compatible with $\omega \cdot a$ for some $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$ and $a \in \mathrm{H}^0\mathrm{DR}(Y/R)[\![\hbar^2]\!]$.

Now define the dg category $\mathcal{F}(\tilde{\mathscr{O}}_Y)$ to be the full subcategory of $Q\mathcal{CI}_{\mathrm{dg}}(\tilde{\mathscr{O}}_Y)$ consisting of self-dual quantised Lagrangian structures $\tilde{\mathscr{L}}$ on line bundles $\mathscr{L}$ with a given right $\mathscr{D}$-module structure on $\mathscr{L}^{\otimes 2}$.

Remarks 5.12. The self-duality hypotheses and the condition that $\tilde{\mathscr{O}}_Y$ is w-compatible with $\omega \cdot a$ for some $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$ and $a \in \mathrm{H}^0\mathrm{DR}(Y/R)[\![\hbar^2]\!]$ ensure via Corollary 4.21 that $\mathcal{F}(\tilde{\mathscr{O}}_Y)$ has objects for every self-dual line bundle $\mathscr{L}$ on a Lagrangian (X,λ) over (Y,ω). One such quantisation will correspond to the generalised Lagrangian $(\omega\cdot a, \lambda\cdot a)$.

Thus Corollary 4.21 plays an analogous role in our setting to that played classically by [DS], which shows that there exists a simple holonomic DQ-module supported on any smooth closed complex Lagrangian equipped with a square root of the dualising bundle. As explained in [BBD+, Remark 6.15], the DQ-modules of [DS] are expected to provide the objects of the complex Fukaya category in the smooth underived setting.

Since we are permitting all derived Lagrangians to give rise to elements of $\mathcal{F}(\tilde{\mathscr{O}}_Y)$, we cannot expect all morphisms in our dg category $\mathcal{F}(\tilde{\mathscr{O}}_Y)$ to be related to vanishing cycles as in the dg category conjectured in [BBD+, Joy]. However, when Grothendieck–Verdier duality applies (such as for finite virtually LCI morphisms $X_i \to Y$), we will now relate the Hom-complexes to (-1)-shifted quantisations, which in turn relate to vanishing cycles as in [Pri7].

Proposition 5.13. *In the setting of Definition 5.9, if ϕ_1 is proper and virtually LCI of relative dimension d, then the complex $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{1*}\tilde{\mathscr{L}}_1, \mathbf{R}\phi_{2*}\tilde{\mathscr{L}}_2)[d]$ is given by derived global sections of an E_0-deformation quantisation (see* [Pri7]*) of the line bundle*

$$\mathbf{L}\phi_2^*(\mathscr{L}_1^{-1} \otimes^{\mathbf{L}} \det \mathbf{L}\Omega^1_{X_1/Y}) \otimes^{\mathbf{L}} \mathbf{L}\phi_1^*\mathscr{L}_2$$

on the derived intersection $X_1 \times^h_Y X_2$.

Proof (sketch). By definition, $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{1*}\tilde{\mathscr{L}}_1, \mathbf{R}\phi_{2*}\tilde{\mathscr{L}}_2)[d]$ is given by taking the homotopy limit over all homotopy formally étale stacky CDGAs A over Y (i.e.

$f\colon \operatorname{Spec} DA \to Y$) of $\hat{\mathrm{Tot}}$-complexes of the double complexes

$$C_A := (\mathcal{CC}_R(A, \mathcal{H}om_R(f^*(\mathbf{R}\phi_{1*}\mathscr{L}_1), f^*(\mathbf{R}\phi_{2*}\mathscr{L}_2)))[\![\hbar]\!], \delta + ((\Delta_{\mathscr{L}_1})_{\Delta_A})^* \mp (\Delta_{\mathscr{L}_2})^*_{\Delta_A}).$$

We may rewrite $\mathcal{CC}_R(A, \mathcal{H}om_R(f^*(\mathbf{R}\phi_{1*}\mathscr{L}_1), f^*(\mathbf{R}\phi_{2*}\mathscr{L}_2)))$ as $\mathbf{R}\phi_{2*}\mathcal{CC}_R(\phi_2^{-1}A, \mathcal{H}om_R(\phi_2^{-1}f^*\mathbf{R}\phi_{1*}\mathscr{L}_1, f^*\mathscr{L}_2)$, where we use the same notation for a map and its pullbacks. When ϕ_1 is affine (or equivalently finite), the result now follows as a direct consequence of Proposition 5.8 and Verdier duality, but we must work much harder in general.

As in Remark 2.11, inclusion of differential operators in R-linear maps gives us a levelwise quasi-isomorphism

$$\mathcal{CC}_R(\phi_2^{-1}A, \mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}f^*\mathbf{R}\phi_{1*}\mathscr{L}_1, f^*\mathscr{L}_2)) \to \mathcal{CC}_R(\phi_2^{-1}A, \mathcal{H}om_R(\phi_2^{-1}f^*\mathbf{R}\phi_{1*}\mathscr{L}_1, f^*\mathscr{L}_2)).$$

Since the operators $\Delta_{\mathscr{L}_i}$ are differential operators, their action restricts to this, so we have a levelwise quasi-isomorphism

$$(\mathbf{R}\phi_{2*}\mathcal{CC}_R(\phi_2^{-1}A, \mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}f^*\mathbf{R}\phi_{1*}\mathscr{L}_1, f^*\mathscr{L}_2))[\![\hbar]\!], \delta + ((\Delta_{\mathscr{L}_1})_{\Delta_A})^* \mp (\Delta_{\mathscr{L}_2})^*_{\Delta_A}) \to C_A.$$

Now, $\mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}f^*\mathbf{R}\phi_{1*}\mathscr{L}_1, f^*\mathscr{L}_2) \simeq \mathbf{R}\mathcal{H}om_{\phi_2^{-1}A}(\phi_2^{-1}f^*\mathbf{R}\phi_{1*}\mathscr{L}_1, \mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}A, f^*\mathscr{L}_2))$, so

$$\begin{aligned}&\mathbf{R}\phi_{2*}\mathcal{CC}_R(\phi_2^{-1}A, \mathbf{R}\mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}f^*\mathbf{R}\phi_{1*}\mathscr{L}_1, f^*\mathscr{L}_2)) \simeq \\ &\mathcal{CC}_R(A, \mathbf{R}\mathcal{H}om_A(f^*\mathbf{R}\phi_{1*}\mathscr{L}_1, \mathbf{R}\phi_{2*}\mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}A, f^*\mathscr{L}_2)))\end{aligned}$$

We can now use Grothendieck–Verdier duality to rewrite this as

$$\begin{aligned}&\mathcal{CC}_R(A, \mathbf{R}\phi_{1*}\mathbf{R}\mathcal{H}om_{f^*\mathscr{O}_{X_1}}(f^*\mathscr{L}_1, \phi_1^!\mathbf{R}\phi_{2*}\mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}A, f^*\mathscr{L}_2))) \simeq \\ &\mathcal{CC}_R(A, \mathbf{R}\phi_{1*}(\phi_1^!\mathbf{R}\phi_{2*}\mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}A, f^*\mathscr{L}_2) \otimes^{\mathbf{L}}_{\mathscr{O}_{X_1}} \mathscr{L}_1^{-1}));\end{aligned}$$

importantly for us, $\phi^!$ preserves right $\mathscr{D}$-module structures via the constructions of [GR] and [Pri7, Example 4.1], so our coefficients of $\mathcal{CC}_R(A, -)$ above are a right $\mathscr{D}_{X_1}(\mathscr{L}_1)$-module, as well as inheriting a left $\mathscr{D}_{X_2}(\mathscr{L}_2)$-module structure from $\mathcal{D}\mathit{iff}_{\phi_2^{-1}\mathscr{O}_Y}(\phi_2^{-1}\mathscr{O}_Y, \mathscr{L}_2)$. Thus the operator $\Delta_{\mathscr{L}_1}$ acts on this on the right, while $\Delta_{\mathscr{L}_2}$ acts on the left; this step is our reason for having to introduce $\mathcal{D}\mathit{iff}$ in the proof.

Writing $\psi\colon X_1 \times^h_Y X_2 \to Y$ for the canonical map, we may rearrange this double complex to write C_A as

$$(\mathbf{R}\psi_*\mathcal{CC}_R(\psi^{-1}A, (\phi_1^!\mathcal{D}\mathit{iff}_{\phi_2^{-1}A}(\phi_2^{-1}A, f^*\mathscr{L}_2) \otimes_{\phi_2^{-1}\mathscr{O}_{X_1}} \phi_2^{-1}\mathscr{L}_1^{-1}))[\![\hbar]\!], \delta + ((\Delta_{\mathscr{L}_1})_{\Delta_A})^* \mp (\Delta_{\mathscr{L}_2})^*_{\Delta_A}).$$

The argument of Propositions 5.5 and 5.8 adapts to show that this deformation is given by differential operators of the correct orders, and it only remains to show that the complex

$$\mathrm{CC}_R(\psi^{-1}\mathscr{O}_Y, (\phi_1^!\mathcal{D}\mathit{iff}_{\phi_2^{-1}\mathscr{O}_Y}(\phi_2^{-1}\mathscr{O}_Y, \mathscr{L}_2) \otimes^{\mathbf{L}}_{\phi_2^{-1}\mathscr{O}_{X_1}} \phi_2^{-1}\mathscr{L}_1^{-1}))$$

it deforms is quasi-isomorphic to a shift of a line bundle on $X_1 \times^h_Y X_2 \to Y$. We can then reverse some of the equivalences above, replacing differential operators with R-linear maps, to give quasi-isomorphisms

$$\begin{aligned}&\mathrm{CC}_R(\psi^{-1}\mathscr{O}_Y, (\phi_1^!\mathcal{D}\mathit{iff}_{\phi_2^{-1}\mathscr{O}_Y}(\phi_2^{-1}\mathscr{O}_Y, \mathscr{L}_2) \otimes_{\phi_2^{-1}\mathscr{O}_{X_1}} \phi_2^{-1}\mathscr{L}_1^{-1})) \\ &\simeq \mathbf{R}\psi_*\mathbf{R}\mathscr{H}om_{\mathscr{O}_{X_1 \times^h_Y X_2}}(\mathbf{L}\phi_2^*\mathscr{L}_1, \phi_1^!\mathscr{L}_2) \\ &\simeq \mathbf{R}\psi_*(\mathbf{L}\phi_2^*\mathscr{L}_1^{-1} \otimes^{\mathbf{L}} \mathbf{L}\phi_2^* \det \mathbf{L}\Omega^1_{X_1/Y} \otimes^{\mathbf{L}} \mathbf{L}\phi_1^*\mathscr{L}_2)[-d],\end{aligned}$$

so shifting by d indeed gives us a deformation quantisation in
$$Q\mathcal{P}(\mathbf{L}\phi_2^*\mathscr{L}_1^{-1}\otimes^{\mathbf{L}}\det\mathbf{L}\Omega^1_{X_1/Y}\otimes^{\mathbf{L}}\mathbf{L}\phi_1^*\mathscr{L}_2,-1)$$
with the desired global sections.

□

We also have functoriality of the dg categories $Q\mathcal{CI}_{\mathrm{dg}}(\tilde{\mathscr{O}}_Y)$ of quantised co-isotropic structures with respect to co-isotropic correspondences:

Proposition 5.14. *Assume we are given quantisations $\tilde{\mathscr{O}}_Y\in Q\mathcal{P}(Y,0)$ and $\tilde{\mathscr{O}}_Z\in Q\mathcal{P}(Y,0)$ of derived Artin N-stacks Y,Z, a morphism $\psi\colon T\to Y\times Z$ for which $\psi_2\colon T\to Z$ is representable by quasi-compact quasi-separated derived algebraic spaces, and a line bundle $\mathscr{M}$ on T with a quantised co-isotropic structure $(\tilde{\mathscr{O}}_Y^{\mathrm{opp}}\otimes\tilde{\mathscr{O}}_Z,\tilde{\mathscr{M}})\in Q\mathcal{P}(Y\times Z,\mathscr{M};0)$ lifting the quantisation $\tilde{\mathscr{O}}_Y^{\mathrm{opp}}\otimes\tilde{\mathscr{O}}_Z$ of $Y\times Z$.*

Then there is a natural dg functor $Q\mathcal{CI}_{\mathrm{dg}}(\tilde{\mathscr{O}}_Y)\to Q\mathcal{CI}_{\mathrm{dg}}(\tilde{\mathscr{O}}_Z)$ between the respective dg categories of quantised co-isotropic structures, which modulo $\hbar$ is quasi-equivalent to the dg functor
$$(X\xrightarrow{\phi}Y,\mathscr{L})\mapsto(X\times^h_Y T,\mathbf{L}\mathrm{pr}_2^*\mathscr{M}\otimes^{\mathbf{L}}\mathbf{L}\mathrm{pr}_1^*\mathscr{L}).$$

Proof. On objects, the functor is given by applying the derived tensor product construction of Proposition 5.7. In the Deligne–Mumford setting, that means we send a quantised co-isotropic structure $\tilde{\mathscr{L}}$ on X to the quantised line bundle $(\mathrm{pr}_2^{-1}\tilde{\mathscr{M}})\otimes^{\mathbf{L}}_{\chi^{-1}\tilde{\mathscr{O}}_Y}(\mathrm{pr}_1^{-1}\tilde{\mathscr{L}})$ on $X\times^h_Y T$, for the natural map $\chi\colon X\times^h_Y T\to Y$.

In order to consider morphisms, observe that we can rewrite $\mathbf{R}\chi_*(\mathbf{L}\mathrm{pr}_2^*\mathscr{M}\otimes^{\mathbf{L}}\mathbf{L}\mathrm{pr}_1^*\mathscr{L})$ as $\mathbf{R}\psi_*(\mathscr{M}\otimes^{\mathbf{L}}\mathbf{R}\mathrm{pr}_{2*}\mathrm{pr}_1^*\mathscr{L})\simeq\mathbf{R}\psi_*(\mathscr{M}\otimes^{\mathbf{L}}\mathbf{L}\psi_1^*\mathbf{R}\phi_*\mathscr{L})$, where $\psi_1\colon T\to Y$ composes ψ with the projection $Y\times Z\to Y$. That description allows us to substitute into Definition 5.9 to pass from morphisms in $Q\mathcal{CI}_{\mathrm{dg}}(\tilde{\mathscr{O}}_Y)$ (defined in terms of $\mathbf{R}\phi_*\mathscr{L}$) to morphisms in $Q\mathcal{CI}_{\mathrm{dg}}(\tilde{\mathscr{O}}_Z)$ (defined in terms of $\mathbf{R}\chi_*(\mathbf{L}\mathrm{pr}_2^*\mathscr{M}\otimes^{\mathbf{L}}\mathbf{L}\mathrm{pr}_1^*\mathscr{L})$). □

Remark 5.15. It is natural to ask whether there are conditions under which Proposition 5.14 restricts to give a dg functor $\mathcal{F}(\tilde{\mathscr{O}}_Y)\to\mathcal{F}(\tilde{\mathscr{O}}_Z)$. A necessary condition is that the co-isotropic structure on $\psi\colon T\to Y\times Z$ must be Lagrangian, since the correspondence must send quantised Lagrangians to quantised Lagrangians. Additional conditions will be required to ensure that the correspondence preserves self-duality. It seems plausible that self-duality of the given quantisation of T suffices, but it is not clear that self-duality interacts well with additivity statements such as Propositions 5.5 and 5.7 (and indeed Proposition 5.8), although it seems likely.

5.3. **Local quantisations of Lagrangians.** The Fukaya category envisaged in [BF, §5.3] had an object for each local system on a Lagrangian submanifold L. By contrast, the dg category conjectured in [BBD+, Remark 6.15] only had one object for each square root of K_L. Our approach in Definition 5.11 has an object for each self-dual quantisation of a square root of the dualising complex, making it closest in flavour to the dg category of simple DQ modules supported on smooth Lagrangians constructed using [KS2, DS, KS1] and also described explicitly in [BBD+, Remark 6.15].

While Corollary 4.21 ensured that self-dual quantisations of square roots of the dualising complex always exist, we now investigate how unique they are.

Once we have fixed our quantisation $\tilde{\mathscr{O}}_Y$ in $Q\mathcal{P}(Y,0)^{\mathrm{nondeg},sd}$ and a compatible Lagrangian in $(\omega,\lambda)\in\mathrm{Lag}(Y,X;0)$, the homotopy fibre of
$$Q\mathcal{P}(Y,\mathscr{L};0)^{\mathrm{nondeg},sd}\to\mathrm{Lag}(Y,X;0)\times^h_{\mathrm{Lag}(Y,\emptyset;0)}Q\mathcal{P}(Y,0)^{\mathrm{nondeg},sd}$$

over $(\tilde{\mathscr{O}}_Y, \lambda)$ parametrises self-dual $\tilde{\mathscr{O}}_Y$-module quantisations of the line bundle $\mathscr{L}$ on the Lagrangian (X, λ). We now explain how this homotopy fibre can be regarded as a torsor for the group of self-dual rank 1 local systems, so comes close to the intention of [BF].

By Theorem 4.20, components of the homotopy fibre are a torsor for the even de Rham power series

$$\mathrm{H}^1(F^2\mathrm{DR}(X/R))^{\mathrm{nondeg}} \times \prod_{i>0} \mathrm{H}^1\mathrm{DR}(X/R)\hbar^{2i},$$

although the parametrisation depends on $w \in \mathrm{Levi}^t_{\mathrm{GT}}$.

As in [Pri7, Remark 4.4], quantisations $(\mathscr{L}[\![\hbar]\!], \delta + \Delta)$ of $\mathscr{L}$ correspond to deformations $\mathscr{E}_\hbar := (\mathscr{L} \otimes^{\mathbf{L}}_{\mathscr{O}_X} \mathscr{D}_X[\![\hbar]\!], \delta + \Delta \cdot \{-\})$ of $\mathscr{L} \otimes^{\mathbf{L}}_{\mathscr{O}_X} \mathscr{D}_X$ as a right $\mathscr{D}_X$-module. Other deformations of this form can be obtained by tensoring with deformations $\mathscr{O}'_\hbar$ of $\mathscr{O}_X[\![\hbar]\!]$ as a left $\mathscr{D}_X$-module. When $\mathscr{L}^{\otimes 2} = K_X$, the self-duality condition for $\mathscr{E}_\hbar$ is

$$\mathscr{E}_{-\hbar} \simeq \mathbf{R}\mathscr{H}om_{\mathscr{D}_X^{\mathrm{opp}}[\![\hbar]\!]}(\mathscr{E}_\hbar, \mathscr{D}_X[\![\hbar]\!]) \otimes_{\mathscr{O}_X} K_X$$

as right $\mathscr{D}_X[\![\hbar]\!]$-modules. The condition for $\mathscr{O}'_\hbar \otimes \mathscr{E}_{-\hbar}$ to also be self-dual is then

$$\mathscr{O}'_{-\hbar} \simeq \mathbf{R}\mathscr{H}om_{\mathscr{O}_X[\![\hbar]\!]}(\mathscr{O}'_\hbar, \mathscr{O}_X[\![\hbar]\!])$$

as left $\mathscr{D}_X$-modules.

We now show that the parametrisation in terms of de Rham cohomology corresponds to the homotopy fibre above being a torsor for this group of self-dual rank 1 local systems; this was in fact the original motivation behind the construction of μ in [Pri7].

Lemma 5.16. *If* $\mathrm{H}^0\mathrm{DR}(X/R) = R$ *and* $\mathrm{H}^1\mathrm{DR}(X/R) = 0$*, then the non-empty homotopy fibres of*

$$Q\mathcal{P}(Y, \mathscr{L}; 0)^{\mathrm{nondeg},sd} \to \mathrm{Lag}(Y, X; 0) \times^h_{\mathrm{Lag}(Y,\emptyset;0)} Q\mathcal{P}(Y, 0)^{\mathrm{nondeg},sd}$$

are connected, the space of automorphisms of each quantisation Δ *being the discrete group*

$$\{g \in 1 + \hbar R[\![\hbar]\!] \ : \ g(\hbar)^{-1} = g(-\hbar)\} = \exp(\hbar R[\![\hbar^2]\!]),$$

regarded as a subset of $\mathscr{O}_X = F_0\mathscr{D}_X(\mathscr{L})$*.*

Proof. Theorem 4.20 implies that the homotopy fibre is connected, because $\mathrm{H}^1\mathrm{DR}(X/R) = 0$. Morphisms are parametrised by $\prod_{i>0} \mathrm{H}^0\mathrm{DR}(X/R)\hbar^{2i} = \hbar^2 R[\![\hbar^2]\!]$, and we need to understand what these map to via the equivalences between generalised Lagrangians and quantisations.

If we take $\hbar^2 a(\hbar^2) \in \hbar^2 R[\![\hbar^2]\!]$, then linearity gives

$$\mu_w(\hbar^2 a(\hbar^2)) = \hbar^2 a(\hbar^2) \in \tau_{\geq 0} G^2 T_\Delta Q\widehat{\mathrm{Pol}}(\mathscr{L}, -1)^{sd}.$$

The corresponding gauge automorphism in $\tau_{\geq 0} Q\mathcal{P}(\mathscr{L}, -1)^{\mathrm{nondeg},sd}_\pi$ is then an element g with $-\partial_{\hbar^{-1}}(g)g^{-1} = \hbar^2 a(\hbar^2)$, so

$$g = \exp(\int a d\hbar).$$

Gauge elements are thus precisely exponentials of odd power series, giving the group described above. □

We now consider torsors for the group $\exp(\hbar R[\![\hbar^2]\!])$, and their associated rank 1 $R[\![\hbar]\!]$-linear local systems $\mathbb{V}$ coming from the inclusion $\exp(\hbar R[\![\hbar^2]\!]) \subset R[\![\hbar]\!]^\times$. Observe that such local systems will automatically carry an inner product $\mathbb{V} \times \mathbb{V} \to R[\![\hbar]\!]$ which is sesquilinear in the sense that $\langle a(\hbar)u, b(\hbar)v\rangle = a(\hbar)b(-\hbar)\langle u, v\rangle$, and that the associated left $\mathscr{D}$-modules $\mathbb{V} \otimes_R \mathscr{O}_X$ are then self-dual in the sense above.

Proposition 5.17. *If $R = \mathbb{C}$ and the obstruction of Corollary 4.21 vanishes, then the homotopy fibres of*

$$Q\mathcal{P}(Y, \mathscr{L}; 0)^{\mathrm{nondeg},sd} \to \mathrm{Lag}(Y, X; 0) \times^h_{\mathrm{Lag}(Y,\emptyset;0)} Q\mathcal{P}(Y, 0)^{\mathrm{nondeg},sd}$$

are torsors for the 2*-group of* $\exp(\hbar\mathbb{C}[\![\hbar^2]\!])$*-torsors on the analytic site of* $X(\mathbb{C})$*. An* $\exp(\hbar\mathbb{C}[\![\hbar^2]\!])$*-torsor acts by tensoring the quantisation with the self-dual left $\mathscr{D}$-module of the associated local system.*

Proof (sketch). As observed in [Pri8, §4.4], all of our constructions carry over directly to the complex analytic setting. There is a natural analytification functor from algebraic quantisations to analytic quantisations, and the isomorphism from algebraic to analytic de Rham cohomology ensures via Theorem 4.20 that this analytification functor is an equivalence. We may then apply Lemma 5.16 locally in the analytic topology, and the result follows. □

Remark 5.18. Over more general bases than $\mathbb{C}$, the analogue of Proposition 5.17 should still hold, with the homotopy fibre being a torsor for the 2-group of self-dual rank 1 left $\mathscr{D}[\![\hbar]\!]$-modules. One way to approach this would be to work on the crystalline site. Each quantisation gives rise to a left $\mathscr{D}$-module $\mathscr{E}^*_\hbar := (\mathscr{D}_X \otimes^{\mathbf{L}}_{\mathscr{O}_X} \mathscr{L}^{-1}[\![\hbar]\!], \delta + \{-\} \cdot \Delta)$, corresponding to a deformation $\mathscr{F}_\hbar$ of the $\mathscr{O}_{X,\mathrm{cris}}$-module $u^{-1}_{X/R}\mathscr{L}^{-1}$ on the crystalline site, for $u_{X/R} \colon X_{\mathrm{cris}} \to X_{\text{ét}}$ the natural morphism of sites. The compatibility map $\mu_w(-, \Delta) \colon \mathrm{DR}(X) \to (\mathscr{D}_X(\mathscr{L})[\![\hbar]\!], \delta + \mathrm{ad}_\Delta)$ should then just be the image under $\mathbf{R}u_{X/R,*}$ of the natural map $\mathscr{O}_{\mathrm{cris}} \to \mathbf{R}\mathscr{E}nd_{\mathscr{O}_{\mathrm{cris}}[\![\hbar]\!]}(\mathscr{F}_\hbar)$, with a crystalline analogue of Lemma 5.16 yielding the desired $\exp(\hbar\mathscr{O}_{\mathrm{cris}}[\![\hbar^2]\!])$-torsors.

5.4. **Morphisms in terms of vanishing cycles.** The complex Fukaya category envisaged in [Joy] had complexes of morphisms coming from shifts of the perverse sheaf of vanishing cycles, but the required composition law was purely conjectural. Our construction in Definition 5.11 is manifestly a dg category, and we will now show that on inverting $\hbar$, its resulting Hom-complexes indeed come from sheaves of vanishing cycles, so the $R((\hbar))$-linear dg category $\mathcal{F}(\tilde{\mathscr{O}}_Y)[\hbar^{-1}]$ has all the expected properties.

As in Remark 5.6, we expect the E_0-deformation quantisations of Proposition 5.13 to be self-dual, and then Lemma 5.16 would combine with [Pri7, Lemma 4.8] to show that the complexes $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{1*}\tilde{\mathscr{L}}_1, \mathbf{R}\phi_{2*}\tilde{\mathscr{L}}_2)[d]$ are given by derived global sections of complexes on $X_1 \times^h_Y X_2$ which are given by vanishing cycles locally in the analytic topology. The following results will also apply in the event that the quantisation is not self-dual. They are phrased in terms of the total space $Q\mathcal{P}(Z, -1)^{\mathrm{nondeg}}/^h\mathbb{G}_m$ of (-1)-shifted quantisations from [Pri7, Definition 3.12], whose homotopy fibres over $\mathrm{map}(Z, B\mathbb{G}_m)$ are the spaces $Q\mathcal{P}(\mathscr{L}, -1)^{\mathrm{nondeg}}$ of quantisations of line bundles.

Proposition 5.19. *If $R = \mathbb{C}$, then the non-empty homotopy fibres of*

$$Q\mathcal{P}(Z, -1)^{\mathrm{nondeg}}/^h\mathbb{G}_m \to \mathcal{P}(Z, -1)^{\mathrm{nondeg}}$$

are torsors for the 2*-group of* $\mathbb{C}[\![\hbar]\!]^\times$*-torsors on the analytic site of* $Z(\mathbb{C})$*. A* $\mathbb{C}[\![\hbar]\!]^\times$*-torsor acts by tensoring the quantisation with the self-dual left* $\mathscr{D}$*-module of the associated local system.*

Proof (sketch). We begin by considering the first order deformation

$$(Q\mathcal{P}(Z,-1)^{\mathrm{nondeg}}/G^2)/^h\mathbb{G}_m \to \mathcal{P}(Z,-1)^{\mathrm{nondeg}}.$$

Over a (-1)-shifted Poisson structure π, the obstruction towers of [Pri7, §2.2] show that the non-empty fibres are given by derived global sections of the simplicial sheaf $\underline{\mathrm{MC}}(\mathrm{cone}(\mathscr{O}_Z^\times \to F^1T_\pi\widehat{\mathrm{Pol}}(\mathscr{O}_Z,-1)))$, where the morphism from $\mathscr{O}_Z^\times$ is $a \mapsto a^{-1}[\pi,a]$. Via the unquantised compatibility map, this complex is quasi-isomorphic to the log de Rham complex $\mathscr{O}_Z^\times \xrightarrow{d\log} F^1\mathrm{DR}(Z/R)$. Since we are assuming $R=\mathbb{C}$, this complex locally analytically resolves the constant sheaf $\mathbb{C}^\times$, so the simplicial sheaf reduces to $B\mathbb{C}^\times_{\mathrm{an}}$, corresponding locally to the automorphisms $\mathbb{C}^\times \subset \mathscr{D}_X(\mathscr{L})$ of any quantisation.

For the rest of the tower, we can simply appeal to [Pri7, Proposition 2.20], and the identical argument to Lemma 5.16 and Proposition 5.17 shows that the fibres of $Q\mathcal{P}(Z,-1)^{\mathrm{nondeg}} \to (Q\mathcal{P}(Z,-1)^{\mathrm{nondeg}}/G^2)$ are just torsors for the 2-group of $(1+\hbar\mathbb{C}[\![\hbar]\!])$-torsors on the analytic site of $Z(\mathbb{C})$, the action given locally by multiplication on isomorphisms. □

Corollary 5.20. *In the setup of Proposition 5.13, the complex* $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{1*}\tilde{\mathscr{L}}_1,\mathbf{R}\phi_{2*}\tilde{\mathscr{L}}_2)$ *is given by derived global sections of a complex* $\mathbb{H}$ *of sheaves on the analytic site of the fibre product* $X_1\times_Y X_2$*. In any neighbourhood* U *of* $X_1\times^h_Y X_2$ *which is equivalent as a* (-1)*-shifted* $\mathbb{C}$*-analytic symplectic stack to the derived critical locus of a function* $f\colon Z\to\mathbb{A}^1$ *on a smooth* $\mathbb{C}$*-analytic space* Z*, we have*

$$\mathbf{R}p_*\mathbb{H}|_U \simeq (\Omega^*_Z[\![\hbar]\!],\hbar d + df\wedge -)\otimes_{\mathbb{C}[\![\hbar]\!]}\mathbb{V}[\dim Z - d]$$
$$\mathbf{R}p_*\mathbb{H}|_U[\hbar^{-1}] \simeq \bigoplus_{c\in\mathbb{C}}\phi_{f-c}[\dim Z-d-1]\otimes_{\mathbb{C}}\mathbb{V}[\hbar^{-1}]$$

for $\mathbb{V}$ *some rank* 1 $\mathbb{C}[\![\hbar]\!]$*-linear local system,* $p\colon U\to Z$ *the natural projection, and* ϕ *the vanishing cycles complex.*

Proof. By Proposition 5.13, $\mathbf{R}\underline{\hat{\mathrm{Hom}}}_{\tilde{\mathscr{O}}_Y}(\mathbf{R}\phi_{1*}\tilde{\mathscr{L}}_1,\mathbf{R}\phi_{2*}\tilde{\mathscr{L}}_2)[d]$ is a (-1)-shifted quantisation. We know from [Pri7, Lemma 4.8] that the twisted de Rham complex above is an element of $Q\mathcal{P}(p^*\Omega^d_Z,-1)^{\mathrm{nondeg},sd}_{\lambda_f}$ for the canonical (-1)-shifted symplectic structure λ_f. Proposition 5.19 thus implies that on U, our quantisation is a tensor product of the twisted de Rham complex and a rank 1 $\mathbb{C}[\![\hbar]\!]$-linear local system. On inverting $\hbar$, the twisted de Rham complex becomes a vanishing cycles complex, as in [Sab, Theorem 1.1] (see also [Pri7, Proposition 4.9]). □

Remarks 5.21. It seems reasonable to expect that the quantisations of Proposition 5.13 are self-dual, in which case we could strengthen Corollary 5.20, using Proposition 5.17 in place of Proposition 5.19 to conclude that the local system $\mathbb{V}$ must be semi-linearly self-dual with respect to the involution $\hbar\mapsto-\hbar$ of $\mathbb{C}[\![\hbar]\!]$.

One way to interpret this is that on inverting $\hbar$, non-degenerate self-dual (-1)-shifted quantisations give a form of perverse sheaf over the $*$-algebra (i.e. ring with involution) $\mathbb{C}(([\![\hbar]\!]))$, whereas [BBD+] constructed perverse sheaves of vanishing cycles over rings such as $\mathbb{R}$ or $\mathbb{C}$. The choice of orientation in [BBD+] could be regarded as a torsor for the group $\{\pm 1\} = \{a \in \mathbb{C}^\times \ : \ a = a^{-1}\}$, while the self-dual quantisations of Proposition 5.17 depend on a choice of torsor for the group $\pm \exp(\hbar\mathbb{C}[\![\hbar]\!]) = \{a \in \mathbb{C}[\![\hbar]\!]^\times \ : \ a(-\hbar) = a(\hbar)^{-1}\}$.

The vanishing cycles sheaf from [BBD$^+$] was constructed by discarding much of the derived structure, while our constructions here and in [Pri7] depend on the full derived structure. We expect that the $R[\![\hbar]\!]$-linear dg category $\mathcal{F}(\tilde{\mathscr{O}}_Y)$ depends on the derived structure in an essential way, and that the same would have to be true of any variant such as that envisaged in [Joy].

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