

# AN OUTLINE OF SHIFTED POISSON STRUCTURES AND DEFORMATION QUANTISATION IN DERIVED DIFFERENTIAL GEOMETRY

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ABSTRACT. We explain how to translate several recent results in derived algebraic geometry to derived differential geometry. These concern shifted Poisson structures on NQ-manifolds, Lie groupoids, smooth stacks and derived generalisations, and include existence and classification of various deformation quantisations.

## INTRODUCTION

In recent years, there have been many developments in the study of shifted Poisson structures and deformation quantisations in derived algebraic geometry, beginning with the systematic study of shifted symplectic structures in [PTVV]. Translating these results into a differential geometric setting is fairly straightforward, but in most cases this has not been done explicitly, a notable exception being [PS]. This situation has arisen partly because the most suitable setting for derived differential geometry in which to write down these results is that built on dg  $\mathcal{C}^\infty$ -rings, for which the foundations have only recently been written down in [CR, Nui].

The aim of this manuscript is to explain how to formulate shifted Poisson structures and various deformation quantisations in differential geometric settings, and to indicate how to adapt existing algebro-geometric proofs, in most cases with a summary of the argument. In places we have imposed unnecessarily strong hypotheses for the purposes of exposition, with pointers which we hope will enable readers who need more general statements to recover them from the cited results in the literature.

The first section is concerned with shifted symplectic structures. These should be familiar as natural generalisations of the homotopy symplectic structures of [KV, Bru]. We start defining these for NQ-manifolds, which should be the most familiar of the objects we will consider. We then consider dg manifolds with differentials going in the opposite direction, set up as the analogue of the algebraic dg manifolds of [CFK]; derived critical loci in the form of classical BV complexes give rise to examples of such dg manifolds. The obvious difference between the formulation of NQ-manifolds and of dg manifolds is in the direction of the differentials  $Q$  and  $\delta$ , but the more important distinction is that we use  $\delta$  to define equivalences via homology isomorphisms. Homological considerations then lead to major differences in the behaviour of  $Q$  and  $\delta$  (see Remarks 1.13). We then formulate shifted symplectic structures for dg NQ-manifolds and super dg NQ-manifolds, where the main difficulty is in keeping track of all the different gradings.

In Section 2, we introduce shifted Poisson structures on all these objects, and establish the equivalence between shifted symplectic structures and non-degenerate shifted Poisson structures (Theorems 2.16 and 2.21). On an NQ-manifold, a shifted Poisson structure is essentially just a shifted  $L_\infty$ -algebra structure on the dg algebra of smooth

functions, with each operation acting as a smooth multiderivation. The description for dg manifolds is similar, while for dg NQ-manifolds the formulation has some subtleties arising from the multiple gradings.

Section 3 then discusses deformation quantisation of  $n$ -shifted Poisson structures. We focus our attention on the cases  $n = 0$  (Theorems 3.7 and 3.9),  $n = -1$  (Theorem 3.15). These quantisations respectively correspond to curved  $A_\infty$  and  $BV_\infty$  deformations of the dg algebra of smooth functions. We then briefly sketch the deformation quantisation of 0-shifted Lagrangians (Theorem 3.18). We also look at the case  $n = -2$  (Theorem 3.24), in which setting quantisations are solutions of a quantum master equation.

The final section then explains how these results translate to Lie groupoids, including higher and derived Lie groupoids. For smooth Artin stacks, including higher stacks, the corresponding stacky CDGAs of [Pri6] or graded mixed cdgas of [CPT<sup>+</sup>] are just given by NQ-manifolds, and the formulation of shifted Poisson structures for such stacks comes down to establishing a simplicial resolution of a higher Lie groupoid by NQ-manifolds in which all the maps induce quasi-isomorphisms on cotangent complexes. We begin by outlining the subtleties of functoriality in §4.1. In §§4.2, 4.3, we then show how to resolve Lie groupoids, higher Lie groupoids and derived higher Lie groupoids by suitable simplicial dg NQ-manifolds, and thus to extend all our constructions to these objects.

I would like to thank Ping Xu and Ted Voronov for helpful comments; unfortunately, I was unable to implement all their suggested changes to terminology without generating clashes elsewhere.

## CONTENTS

Introduction	1
Notation and terminology	2
1. Symplectic structures on stacky and derived enhancements of supermanifolds	4
1.1. Super NQ-manifolds	4
1.2. Differential graded supermanifolds	6
1.3. Derived NQ-manifolds	10
2. Shifted Poisson structures on super derived NQ-manifolds	13
2.1. Shifted Poisson structures on super NQ-manifolds	13
2.2. Shifted Poisson structures on super dg NQ-manifolds	18
3. Deformation quantisation	20
3.1. Quantisation of 0-shifted Poisson structures	20
3.2. Quantisation of $(-1)$ -shifted symplectic structures on dg NQ-manifolds	24
3.3. Quantisation of shifted Lagrangians	28
3.4. Quantisation of $(-2)$ -shifted symplectic structures on dg NQ-manifolds	29
4. Functoriality, derived and higher Lie groupoids	31
4.1. Étale functoriality for dg NQ-manifolds	31
4.2. Lie groupoids	34
4.3. Higher Lie groupoids	37
References	40

**Notation and terminology.** From the outset, we will be working with differential graded superalgebras. Thus our objects are initially  $\mathbb{Z} \times \mathbb{Z}/2$ -graded, and we later encounter objects which are  $\mathbb{Z}^2 \times \mathbb{Z}/2$ - or even  $\mathbb{Z}^3 \times \mathbb{Z}/2$ -graded. However, our indexing

conventions differ from those usually found in supermathematics (e.g. in [CR, 4.6] or [Vor2]), in that for us the parity of an element is the mod 2 sum of its indices. We accordingly denote our copy of  $\mathbb{Z}/2$  by  $\{=, \neq\}$ , so the parity of  $(m, =)$  is  $m \bmod 2$  and parity of  $(m, \neq)$  is  $m + 1 \bmod 2$ . We refer to the  $\mathbb{Z}$ -gradings as degrees (chain or cochain denoted by subscripts and superscripts, respectively) rather than weights, and to the indices  $\{=, \neq\}$  as equal and unequal parity.

In particular, a *super chain complex* will be a  $\mathbb{Z} \times \{=, \neq\}$ -graded vector space, equipped with a square-zero operation  $\delta$  of degree  $-1$  and odd (hence equal) parity. For a super chain complex  $(V, \delta)$ , the subcomplexes of equal and unequal parity are thus given by

$$\begin{aligned} V_{\bullet,=} &= (\dots \xrightarrow{\delta} V_3^{\text{odd}} \xrightarrow{\delta} V_2^{\text{even}} \xrightarrow{\delta} V_1^{\text{odd}} \xrightarrow{\delta} V_0^{\text{even}} \xrightarrow{\delta} \dots), \\ V_{\bullet,\neq} &= (\dots \xrightarrow{\delta} V_3^{\text{even}} \xrightarrow{\delta} V_2^{\text{odd}} \xrightarrow{\delta} V_1^{\text{even}} \xrightarrow{\delta} V_0^{\text{odd}} \xrightarrow{\delta} \dots). \end{aligned}$$

Similarly, a *super cochain complex* will be a  $\mathbb{Z} \times \{=, \neq\}$ -graded vector space equipped with a square-zero operation  $Q$  (corresponding to the  $\partial$  of [Pri6]) or  $d$  of degree  $+1$  and odd (hence equal) parity. For a super cochain complex  $(V, Q)$ , we thus have subcomplexes

$$\begin{aligned} V_{=}^\bullet &= (\dots \xrightarrow{Q} V^{0,\text{odd}} \xrightarrow{Q} V^{1,\text{even}} \xrightarrow{Q} V^{2,\text{odd}} \xrightarrow{Q} V^{3,\text{even}} \xrightarrow{Q} \dots), \\ V_{\neq}^\bullet &= (\dots \xrightarrow{Q} V^{0,\text{even}} \xrightarrow{Q} V^{1,\text{odd}} \xrightarrow{Q} V^{2,\text{even}} \xrightarrow{Q} V^{3,\text{odd}} \xrightarrow{Q} \dots). \end{aligned}$$

of equal and unequal parity.

For a chain (resp. cochain) complex  $M$ , we write  $M_{[i]}$  (resp.  $M^{[j]}$ ) for the complex  $(M_{[i]})_m = M_{i+m}$  (resp.  $(M^{[j]})^m = M^{j+m}$ ). We also denote the parity reversion operator by  $\Pi$ , so for a super chain complex  $M$ , we have  $(\Pi M)_i^- := M_i^{\neq}$  and  $(\Pi M)_i^{\neq} := M_i^-$ .

We define tensor products to follow the usual super/graded conventions, so for super chain complexes  $M, N$ , we have

$$\begin{aligned} (M \otimes N)_n^- &= \bigoplus_{i+j=n} ((M_i^- \otimes N_j^-) \oplus (M_i^{\neq} \otimes N_j^{\neq})) \\ (M \otimes N)_n^{\neq} &= \bigoplus_{i+j=n} ((M_i^- \otimes N_j^{\neq}) \oplus (M_i^{\neq} \otimes N_j^-)), \end{aligned}$$

and we define symmetric powers  $\text{Sym}^n M$  by passing to  $S_n$ -coinvariants of tensor powers  $M^{\otimes n}$ , where the  $S_n$ -action on  $M^{\otimes n}$  is twisted by the sign of the permutation on terms of odd parity.

Given a commutative algebra  $A$  in super chain complexes, and  $A$ -modules  $M, N$  in super chain complexes, we write  $\underline{\text{Hom}}_A(M, N)$  for the super chain complex given by setting  $\underline{\text{Hom}}_A(M, N)_i = \underline{\text{Hom}}_A(M, N)_i^- \oplus \underline{\text{Hom}}_A(M, N)_i^{\neq}$  to consist of  $A$ -linear morphisms from  $M$  to  $N$  of chain degree  $i$ , with the decomposition corresponding to equal and unequal parity; the differential on  $\underline{\text{Hom}}_A(M, N)$  is given by  $\delta f = \delta_N \circ f \pm f \circ \delta_M$ , where  $V_{\#}$  denotes the graded vector space underlying a chain complex  $V$ . We follow analogous conventions for internal Homs in super cochain complexes, and in super chain cochain complexes.

## 1. SYMPLECTIC STRUCTURES ON STACKY AND DERIVED ENHANCEMENTS OF SUPERMANIFOLDS

In derived algebraic geometry, derived stacks are enhancements of schemes in two different ways. Derived structures give analogues of Kuranishi structures, while stacky structures give analogues of Lie groupoids. Much of the complexity in formulating Poisson structures for derived stacks [Pri6, CPT<sup>+</sup>] arises from considering the derived and stacky structures simultaneously. Since stacky structures in the form of Lie algebroids or NQ-manifolds are more likely to be familiar to readers (cf. [PS]), we start with them (whereas the initial emphasis in [Pri6] was on derived structures).

**1.1. Super NQ-manifolds.** The following definition broadly corresponds to the  $\infty$ -Lie algebroids of [the nlab](#), or to the dg manifolds of [Roy1].

**Definition 1.1.** Define an NQ-manifold  $X$  to be a pair  $(X_0, \mathcal{O}_X)$  where  $X_0$  is a real differentiable manifold and

$$\mathcal{O}_X = (\mathcal{O}_X^0 \xrightarrow{Q} \mathcal{O}_X^1 \xrightarrow{Q} \mathcal{O}_X^2 \xrightarrow{Q} \dots)$$

is a graded-commutative  $\mathbb{R}$ -algebra in cochain complexes of sheaves on  $X_0$ . We require that  $\mathcal{O}_X^0$  be the sheaf of smooth functions on  $X_0$ , that the cochain differential  $Q: \mathcal{O}_X^0 \rightarrow \mathcal{O}_X^1$  be a  $\mathcal{C}^\infty$ -derivation and that  $\mathcal{O}_X$  be locally semi-free in the sense that the underlying graded-commutative sheaf  $\mathcal{O}_X^\#$  is locally of the form  $\mathcal{O}_{X_0} \otimes \text{Symm}_{\mathbb{R}} U$  for some finite-dimensional graded vector space  $U = U^1 \oplus \dots \oplus U^N$ .

Write  $\mathcal{C}^\infty(X) := \Gamma(X_0, \mathcal{O}_X)$  for the cochain complex of global sections of  $\mathcal{O}_X$ .

Beware that we are denoting the underlying manifold  $X_0$  with a subscript 0, although its ring of functions  $\mathcal{O}_X^0$  has a superscript 0. This essentially arises from contravariance, and is part of a general indexing convention in [Pri1, Pri3] inherited from related simplicial and cosimplicial objects.

*Remark 1.2.* For each NQ-manifold, the commutative dg algebra  $\mathcal{C}^\infty(X)$  naturally has the structure of a dg  $\mathcal{C}^\infty$ -ring in the sense of [CR]. However, we have slightly more structure because  $\mathcal{C}^\infty(X)^0$  is a  $\mathcal{C}^\infty$ -ring in the sense of [Dub, MR] and  $Q: \mathcal{C}^\infty(X)^0 \rightarrow \mathcal{C}^\infty(X)^1$  is a  $\mathcal{C}^\infty$ -derivation, while the definition of [CR] would only require that  $H^0 \mathcal{C}^\infty(X)$  be a  $\mathcal{C}^\infty$ -ring.

The following differs slightly from usual conventions, which tend to regard NQ-manifolds and NQ-supermanifolds as synonymous, since we have  $\mathbb{N}_0 \times \mathbb{Z}/2$ -gradings rather than just  $\mathbb{N}_0$ -gradings.

**Definition 1.3.** Define a super NQ-manifold  $X$  to be a pair  $(X_0^=, \mathcal{O}_X)$  where  $X_0^=$  is a real differentiable manifold and  $\mathcal{O}_X = \mathcal{O}_{X_0^=}^{\geq 0}$  is a graded-commutative  $\mathbb{R}$ -algebra in super cochain complexes of sheaves on  $X_0^=$ . We require that  $\mathcal{O}_{X_0^=}^{0,=}$  be the sheaf of smooth functions on  $X_0^=$ , that the cochain differential  $Q: \mathcal{O}_{X_0^=}^{0,=} \rightarrow \mathcal{O}_{X_0^=}^{1,=}$  be a  $\mathcal{C}^\infty$ -derivation and that  $\mathcal{O}_X$  be locally semi-free in the sense that the underlying super graded-commutative sheaf  $\mathcal{O}_X^\#$  is locally of the form  $\mathcal{O}_{X_0^=} \otimes \text{Symm}_{\mathbb{R}} U$  for some finite-dimensional super graded vector space  $U = U^{0,\neq} \oplus U^1 \oplus \dots \oplus U^N$  (with  $U^i = U^{i,=} \oplus U^{i,\neq}$ ).

Write  $\mathcal{C}^\infty(X) := \Gamma(X_0^=, \mathcal{O}_X)$  for the super cochain complex of global sections of  $\mathcal{O}_X$ .

**Definition 1.4.** Given a super NQ-manifold  $X$ , define  $\Omega_X^1$  to be the sheaf of smooth 1-forms of  $\mathcal{O}_X$ . This is a sheaf of  $\mathcal{O}_X$ -modules in super cochain complexes (with basis locally given by  $\{dx^i\}_i$  when  $\mathcal{O}_X$  has local co-ordinates  $\{x^i\}_i$ ), and we write  $\Omega_X^p :=$

$\Lambda_{\mathcal{O}_X}^p \Omega_X^1$ . We also write  $\Omega_{\mathcal{C}^\infty(X)}^p$  for the super cochain complex  $\Gamma(X, \Omega_X^p)$  of global sections of  $\Omega_X^p$ .

Define  $T_X$  to be the sheaf  $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$  of smooth 1-vectors of  $\mathcal{O}_X$ , and write  $T_{\mathcal{C}^\infty(X)}$  for the super cochain complex  $\Gamma(X, T_X)$  of global sections; equivalently, this is the internal Hom space  $\underline{\text{Hom}}_{\mathcal{C}^\infty(X)}(\Omega_{\mathcal{C}^\infty(X)}^1, \mathcal{C}^\infty(X))$ .

*Remark 1.5.* Beware that  $\Omega_{\mathcal{C}^\infty(X)}^1$  is not the module of Kähler differentials of the abstract super dg algebra  $\mathcal{C}^\infty(X)$ , since we have constraints requiring that the derivation  $d: \mathcal{C}^\infty(X) \rightarrow \Omega_{\mathcal{C}^\infty(X)}^1$  restrict to a  $\mathcal{C}^\infty$ -derivation on  $\mathcal{C}^\infty(X_0^-)$ . In the special case when  $X = X_0^-$  is just a manifold, our  $\Omega_{\mathcal{C}^\infty(X)}^1$  is the module  $\Omega_{\mathcal{C}^\infty(X)}$  of [Joy1, §5.3].

Explicitly,  $\Omega_{\mathcal{C}^\infty(X)}^1$  is the  $\mathcal{C}^\infty(X)$ -module in super cochain complexes generated by elements  $da$  (given the same degree and parity as  $a$ ), for homogeneous elements  $a \in \mathcal{C}^\infty(X)$ , subject to the relations

- (1)  $d(ab) = (da)b + (-1)^{\bar{a}}a(db)$ , where  $\bar{a}$  denotes the parity of  $a$ , and
- (2) for  $a_1, \dots, a_n \in \mathcal{C}^\infty(X)^{0,=}$  and  $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ , we have

$$d(f(a_1, \dots, a_n)) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a_1, \dots, a_n) da_i.$$

The cochain differential  $Q$  on  $\Omega_{\mathcal{C}^\infty(X)}^1$  is then given by  $Q(da) = d(Qa)$ . The  $\mathcal{C}^\infty(X)$ -module  $T_{\mathcal{C}^\infty(X)}$  is given by derivations on  $\mathcal{C}^\infty(X)$ , with a similar restriction on  $\mathcal{C}^\infty(X)^{0,=}$ .

**Definition 1.6.** Given a super NQ-manifold  $X$ , define the de Rham complex  $\text{DR}(X)$  to be the total super cochain complex of the double complex

$$\mathcal{C}^\infty(X) \xrightarrow{d} \Omega_{\mathcal{C}^\infty(X)}^1 \xrightarrow{d} \Omega_{\mathcal{C}^\infty(X)}^2 \xrightarrow{d} \dots,$$

so  $\text{DR}(X)^m = \prod_{i+j=m} (\Omega_{\mathcal{C}^\infty(X)}^i)^j$  with total differential  $d \pm Q$ .

We define a filtration  $\text{Fil}$  on  $\text{DR}(X)$  by setting  $\text{Fil}^p \text{DR}(X) \subset \text{DR}(X)$  to consist of terms  $\Omega_X^i$  with  $i \geq p$ .

In our formal arguments, the filtration  $\text{Fil}$  will play the same rôle as the Hodge filtration of [Pri6], but beware that it is very different in situations where both are defined (such as on complex manifolds).

The complex  $\text{DR}(X)$  has the natural structure of a commutative DG super algebra, filtered in the sense that  $\text{Fil}^i \text{Fil}^j \subset \text{Fil}^{i+j}$ .

The following definitions are adapted from [PTVV] (where pre-symplectic structures are referred to as closed  $p$ -forms, and all objects have equal parity), although as noted in [BG], precursors exist in the mathematical physics literature (cf. [KV, Definition 2] and [Bru, Definition 5.2.1], which respectively consider even and odd structures for  $\mathbb{Z}/2$ -gradings rather than  $\mathbb{Z} \times \mathbb{Z}/2$ -gradings):

**Definition 1.7.** Define an  $n$ -shifted pre-symplectic structure  $\omega$  on a super NQ-manifold  $X$  to be an element

$$\omega \in \mathbb{Z}^{n+2} \text{Fil}^2 \text{DR}(X)^= = \{\omega \in \text{Fil}^2 \text{DR}(X)^{n+2,=} : (d \pm Q)\omega = 0\}.$$

Define a parity-reversed  $n$ -shifted pre-symplectic structure to be an element

$$\omega \in \mathbb{Z}^{n+2} \text{Fil}^2 \text{DR}(X)^\neq.$$

Two pre-symplectic structures are regarded as equivalent if they induce the same cohomology class in  $\mathbb{H}^{n+2} \text{Fil}^2 \text{DR}(X)^=$  (resp.  $\mathbb{H}^{n+2} \text{Fil}^2 \text{DR}(X)^\neq$ ).

Explicitly, this means that  $\omega$  is given by a sum  $\omega = \sum_{i \geq 2} \omega_i$ , with  $\omega_i \in (\Omega_X^i)^{n+2-i}$  (of equal or unequal parity, respectively) and with  $d\omega_i = \pm Q\omega_{i+1}$  and  $Q\omega_2 = 0$ . Thus  $\omega_2$  is  $d$ -closed up to a homotopy given by  $\omega_3$ , and so on.

**Definition 1.8.** Define a (parity-reversed)  $n$ -shifted symplectic structure  $\omega$  on  $X$  to be a (parity-reversed)  $n$ -shifted pre-symplectic structure  $\omega$  for which contraction with the component  $\omega_2 \in Z^n \Omega_{\mathcal{C}^\infty(X)}^2$  induces a quasi-isomorphism

$$\begin{aligned} \omega_2^\sharp: T_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^0 &\rightarrow (\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^0)^{[n]} \quad \text{resp.} \\ \omega_2^\sharp: T_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^0 &\rightarrow \Pi(\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^0)^{[n]}. \end{aligned}$$

*Remarks 1.9.* Note that for super NQ-manifolds, bounds on the generators of  $\Omega_X^1$  mean that  $n$ -shifted symplectic structures can only exist for  $n \in [0, N]$ .

The non-degeneracy condition in Definition 1.8 is more subtle than the one we will see in Definition 1.21 for dg manifolds with differentials in the opposite direction. This is because we need to ensure that  $\omega_2^\sharp$  induces quasi-isomorphisms between tensor powers of tangent complexes and cotangent complexes. Because non-degeneracy is defined as a quasi-isomorphism rather than an isomorphism, we have to treat negatively and positively graded cochain complexes differently, since they interact differently with homological constructions. In technical terms, the  $\mathcal{O}_X$ -modules  $\Omega_X^1$  in cochain complexes are not usually cofibrant in the projective model structure when  $X$  is an NQ-manifold, whereas the corresponding modules for dg manifolds will be.

*Examples 1.10.* Many examples of shifted symplectic structures on NQ-manifolds are given [PS]. Prototypical examples are shifted cotangent bundles  $T^*M[-n]$  of manifolds  $M$  for  $n \geq 0$ , with  $\mathcal{C}^\infty(T^*M[-n])$  given by the free graded-commutative algebra over  $\mathcal{C}^\infty(M)$  generated by the module  $T_{\mathcal{C}^\infty(M)} := \mathcal{C}^\infty(M, TM)_M$  of smooth sections of the tangent bundle placed in cochain degree  $n$ , and with trivial differential  $Q$ . This NQ-manifold carries a natural  $n$ -shifted symplectic structure  $\omega$  given by the canonical closed 2-form  $\omega_2 \in T_{\mathcal{C}^\infty(M)} \otimes_{\mathcal{C}^\infty(M)} \Omega_M^1 \subset (\Omega_{T^*M[-n]}^2)^{[n]}$ .

Similarly, there is a super NQ-manifold  $\Pi T^*M[n]$  with  $\mathcal{C}^\infty(\Pi T^*M[-n])$  freely generated over  $\mathcal{C}^\infty(M)$  by  $\Pi T_{\mathcal{C}^\infty(M)}^{[-n]}$ , and this carries a natural parity-reversed  $n$ -shifted symplectic structure.

For a finite-dimensional real Lie algebra  $\mathfrak{g}$ , we can define an NQ-manifold  $B\mathfrak{g}$  with underlying manifold a point, and functions  $\mathcal{O}(B\mathfrak{g})$  given by the Chevalley–Eilenberg complex

$$\text{CE}(\mathfrak{g}, \mathbb{R}) = (\mathbb{R} \xrightarrow{Q} \mathfrak{g}^* \xrightarrow{Q} \Lambda^2 \mathfrak{g}^* \xrightarrow{Q} \dots).$$

of  $\mathfrak{g}$  with coefficients in  $\mathbb{R}$ . We then have that  $\Omega_{B\mathfrak{g}}^p \cong \text{CE}(\mathfrak{g}, S^p(\mathfrak{g}^*))^{[-p]}$ , so consideration of degrees shows that the space of 2-shifted pre-symplectic structures is just given by  $\mathfrak{g}$ -invariant symmetric bilinear forms  $S^2(\mathfrak{g}^*)^{\mathfrak{g}}$  on  $\mathfrak{g}$ , and that these are symplectic whenever the form is non-degenerate.

**1.2. Differential graded supermanifolds.** We now recall a derived generalisation of the affine  $\mathcal{C}^\infty$ -schemes of [Joy1], in the form of a  $\mathcal{C}^\infty$  analogue of the algebraic dg manifolds of [CFK, §2.5]; these should not be confused with the dg manifolds of [Roy1], which correspond to our NQ manifolds. The homotopy theory of such objects is studied in detail in [CR]. Our dg manifolds correspond to the affine derived manifolds of finite presentation in [Nui]; of the other approaches to derived differential geometry, this formulation is closely related to Joyce’s  $d$ -manifolds [Joy2] (which however discard much

of the derived structure), and is essentially equivalent to the approaches of [BN, Spi] via [Nui, Corollary 2.2.10]. The primary motivation for such objects comes from obstruction theory (notably Kuranishi spaces), and they also allow for phenomena such as well-behaved non-transverse derived intersections.

**Definition 1.11.** Define a  $(\mathcal{C}^\infty)$  dg manifold  $X$  to be a pair  $(X^0, \mathcal{O}_X)$  where  $X^0$  is a real differentiable manifold and

$$\mathcal{O}_X = (\mathcal{O}_{X,0} \xleftarrow{\delta} \mathcal{O}_{X,1} \xleftarrow{\delta} \mathcal{O}_{X,2} \xleftarrow{\delta} \dots)$$

is a graded-commutative  $\mathbb{R}$ -algebra in chain complexes of sheaves on  $X^0$ . We require that  $\mathcal{O}_{X,0}$  be the sheaf of smooth functions on  $X^0$  and that  $\mathcal{O}_X$  be locally semi-free in the sense that the underlying graded-commutative sheaf  $\mathcal{O}_{X,\#}$  is locally of the form  $\mathcal{O}_{X^0} \otimes \text{Sym}_{\mathbb{R}} V$  for some finite-dimensional graded vector space  $V = V_1 \oplus \dots \oplus V_n$ .

Write  $\mathcal{C}^\infty(X) := \Gamma(X^0, \mathcal{O}_X)$  for the chain complex of global sections of  $\mathcal{O}_X$ .

The following notation follows [Pri3]; the corresponding notation in [CFK] is  $\pi_0$ , which we avoid because it usually denotes quotients rather than subspaces, and can lead to confusion when working with stacks or Lie groupoids.

**Definition 1.12.** Given a dg manifold  $X$ , define the truncation  $\pi^0 X$  to be the  $\mathcal{C}^\infty$ -subscheme (in the sense of [Joy1]) of  $X^0$  defined by the ideal  $\delta \mathcal{O}_{X,1}$ . Thus the space underlying  $\pi^0 X$  is the vanishing locus of the vector field  $\delta$ .

*Remarks 1.13.* Because a dg manifold enhances the structure sheaf in the chain direction, it behaves very differently from an NQ-manifold, where the sheaf is enhanced in the cochain direction. The former (derived enhancements) generalise subspaces of a manifold, while the latter (stacky enhancements) generalise quotients; the chain and cochain structures are shadows of simplicial and cosimplicial structures, respectively.

For each dg manifold, the commutative dg algebra  $\mathcal{C}^\infty(X)$  naturally has the structure of a dg  $\mathcal{C}^\infty$ -ring in the sense of [CR] (i.e.  $\mathcal{C}^\infty(X)_0$  is a  $\mathcal{C}^\infty$ -ring in the sense of [Dub, MR]). This gives a full and faithful contravariant functor from dg manifolds to dg  $\mathcal{C}^\infty$ -rings. The image includes all finitely generated cofibrant objects, but is much larger, essentially consisting of finitely generated dg  $\mathcal{C}^\infty$ -rings whose Kähler differentials compute the cotangent complex without needing to pass to a cofibrant replacement.

Where the right notion of equivalence between NQ-manifolds is quite subtle (isomorphism on  $X_0$  and quasi-isomorphism on Kähler differentials), we can regard a morphism of dg manifolds as an equivalence if it induces a quasi-isomorphism (i.e. a homology isomorphism) between complexes of  $\mathcal{C}^\infty$  functions. Many of the singular  $\mathcal{C}^\infty$ -schemes of [Joy1] are then equivalent in this sense to dg manifolds.

Any homotopically meaningful construction then has to be formulated in such a way that it is invariant under equivalences of dg manifolds. In particular, the manifold  $X^0$  is not an invariant of the equivalence class of a dg manifold  $X = (X^0, \mathcal{O}_X)$ , because for any submanifold  $U$  of  $X^0$  containing the vanishing locus  $\pi^0 X$ , the inclusion map

$$(U, \mathcal{O}_X|_U) \rightarrow (X^0, \mathcal{O}_X)$$

is an equivalence. On the other hand, the  $\mathcal{C}^\infty$  scheme  $\pi^0 X$  is an invariant of the equivalence class. Since most quasi-isomorphisms are not strictly invertible, this also means that we cannot usually perform constructions by just taking local charts and gluing.

*Examples 1.14.* Given a manifold  $M$  and a smooth section  $s: M \rightarrow E$  of a vector bundle, there is an associated dg manifold, the derived vanishing locus of  $s$ , given by  $X^0 = M$  and  $\mathcal{O}_X$  the sheaf of sections of  $\text{Symm}_{\mathbb{R}}(E^*_{[-1]})$ , with differential

$$\begin{aligned} \mathcal{C}^\infty(X)_{r+1} &\xrightarrow{\delta} \mathcal{C}^\infty(X)_r \\ \mathcal{C}^\infty(X, \Lambda^{r+1}E^*) &\xrightarrow{s^*} \mathcal{C}^\infty(X, \Lambda^r E^*). \end{aligned}$$

Then  $\pi^0 X$  is the vanishing locus of  $s$ ; a simple example of this form is given by the DGA  $\mathcal{C}^\infty(\mathbb{R}) \xrightarrow{x^2} \mathcal{C}^\infty(\mathbb{R})$  resolving the dual numbers  $\mathbb{R}[x]/x^2$ , but in general  $\mathcal{O}_X$  can have higher homology.

This class of examples includes the derived critical locus  $\text{DCrit}(M, f)$  of a function  $f \in \mathcal{C}^\infty(M)$ , by considering the section  $df$  of the cotangent bundle. The  $\mathcal{C}^\infty$ -differential graded algebra  $\mathcal{C}^\infty(\text{DCrit}(M, f))$  is given by the chain complex

$$\mathcal{C}^\infty(M) \xleftarrow{df} T_{\mathcal{C}^\infty(M)} \xleftarrow{df} \Lambda^2_{\mathcal{C}^\infty(M)} T_{\mathcal{C}^\infty(M)} \xleftarrow{df} \dots,$$

so  $H_0 \mathcal{C}^\infty(\text{DCrit}(M, f))$  consists of functions on the critical locus of  $f$ . Explicitly, if  $M = \mathbb{R}^n$ , then  $\mathcal{C}^\infty(\text{DCrit}(M, f))$  is generated by co-ordinates  $x_i, \xi_i$  of chain degrees 0, 1, with  $\delta \xi_i = \frac{\partial f}{\partial x_i}$ . As explained on [the nlab](#),  $\mathcal{O}_{\text{DCrit}(M, f)}$  is the classical BV complex of the function  $f$  on the manifold  $M$ .

Other examples are shifted cotangent bundles  $T^*M[n]$  of manifolds  $M$  for  $n \geq 0$ , with  $\mathcal{C}^\infty(T^*M[n])$  given by the free graded-commutative algebra over  $\mathcal{C}^\infty(M)$  generated by smooth sections  $T_{\mathcal{C}^\infty(M)} := \mathcal{C}^\infty(M, TM)_M$  of the tangent bundle placed in chain degree  $n$ , and with trivial differential  $\delta$ . Note that  $\text{DCrit}(M, 0) = T^*M[1]$ .

**Definition 1.15.** Define a ( $\mathcal{C}^\infty$ ) dg supermanifold  $X$  to be a pair  $(X^{0,=}, \mathcal{O}_X)$  where  $X^{0,=}$  is a real differentiable manifold and  $\mathcal{O}_X = \mathcal{O}_{X, \geq 0}$  is a graded-commutative  $\mathbb{R}$ -algebra in super chain complexes of sheaves on  $X^{0,=}$ . We require that  $\mathcal{O}_{\bar{X}, 0}$  be the sheaf of smooth functions on  $X^{0,=}$ , and that  $\mathcal{O}_X$  be locally semi-free in the sense that the underlying super graded-commutative sheaf  $\mathcal{O}_{X, \#}$  is locally of the form  $\mathcal{O}_{X^{0,=}} \otimes \text{Symm}_{\mathbb{R}} V$  for some finite-dimensional super graded vector bundle  $V = V_0^\neq \oplus V_1 \oplus \dots \oplus V_n$  (with  $V_i = V_i^\neq \oplus V_i^\neq$ ).

Write  $\mathcal{C}^\infty(X) := \Gamma(X^{0,=}, \mathcal{O}_X)$  for the super chain complex of global sections of  $\mathcal{O}_X$ .

*Examples 1.16.* An example of a non-trivial dg supermanifold is given by the shifted cotangent bundle  $T^*M[n]$  of a supermanifold  $M$ , with  $\mathcal{C}^\infty(T^*M[n])$  given by the free graded-commutative algebra over  $\mathcal{C}^\infty(M)$  generated by smooth sections  $T_{\mathcal{C}^\infty(M)}$  of the tangent bundle placed in chain degree  $n$ , and with trivial differential  $\delta$ . Another example is the parity-reversed shifted cotangent bundle  $\Pi T^*M[n]$ , with  $\mathcal{C}^\infty(\Pi T^*M[n])$  given by the free graded-commutative algebra over  $\mathcal{C}^\infty(M)$  generated by the tangent bundle  $T_{\mathcal{C}^\infty(M)}$  placed in chain degree  $n$  and unequal parity, and with trivial differential  $\delta$ .

A more interesting example is given by the derived critical locus  $\text{DCrit}(M, f)$  of an odd function  $f \in \mathcal{C}^\infty(M)^\neq$  on a supermanifold  $M$ . The  $\mathcal{C}^\infty$ -differential graded algebra  $\mathcal{C}^\infty(\text{DCrit}(M, f))$  is given by the chain complex

$$\mathcal{C}^\infty(M) \xleftarrow{df} \Pi T_{\mathcal{C}^\infty(M)} \xleftarrow{df} \Lambda^2_{\mathcal{C}^\infty(M)} \Pi T_{\mathcal{C}^\infty(M)} \xleftarrow{df} \dots$$

Explicitly, if  $M = \mathbb{R}^{m|n}$ , then  $\mathcal{C}^\infty(\text{DCrit}(M, f))$  is generated by co-ordinates  $\{x_i, \xi_i, y_j, \eta_j\}_{1 \leq i \leq m, 1 \leq j \leq n}$  with  $x_i, y_j$  of chain degree 0,  $\xi_i, \eta_j$  of chain degree 1,  $x_i, \eta_j$  of even parity and  $y_i, \xi_i$  of odd parity, with  $\delta \xi_i = \frac{\partial f}{\partial x_i}$  and  $\delta \eta_j = \frac{\partial f}{\partial y_j}$ .



*Remark 1.17.* The finiteness conditions for dg manifolds and supermanifolds in Definitions 1.11, 1.15 have been chosen to be stricter than necessary to ease exposition. In fact, our main constructions and results will apply if we only impose finiteness conditions up to homotopy, as in [Pri6].

Instead of looking at  $\mathcal{C}^\infty(X)$  for dg supermanifolds  $X$ , we could take super chain commutative algebras  $A$  with  $A_0^- = \varinjlim_\alpha \mathcal{C}^\infty(Y_\alpha)$  for inverse systems  $\{Y_\alpha\}_\alpha$  of submersions of manifolds, and  $A_\#$  freely generated as a  $(\mathbb{Z} \times \mathbb{Z}/2)$ -graded-commutative algebra over  $A_0^-$  by projective modules  $P_0^\#, P_1, P_2, \dots$ , possibly of infinite rank. The finiteness condition we would then impose is that the module  $\Omega_A^1$  of  $\mathcal{C}^\infty$ -differential forms (cf. Remark 1.5 or [Joy1, §5.3]) of  $A$  is perfect, meaning that  $\Omega_A^1 \otimes_A H_0 A$  is quasi-isomorphic to a finite complex of projective  $H_0 A$ -modules of finite rank.

This relaxation of finiteness conditions slightly complicates several formulae, giving them the form occurring in [Pri6]. For instance, we would have to replace symmetric powers of tangent complexes with duals of cosymmetric powers of cotangent complexes in the definition of polyvectors.

For the corresponding algebraic notions, see [Pri3, Theorem 6.42], where a slight generalisation of the dg schemes from [CFK] is shown to be equivalent to the notion of derived schemes emerging from higher topoi in [TV].

**Definition 1.18.** Given a dg supermanifold  $X$ , define  $\Omega_X^1$  to be the sheaf of smooth 1-forms of  $\mathcal{O}_X$ . This is a sheaf of super chain complexes, and we write  $\Omega_X^p := \Lambda_{\mathcal{O}_X}^p \Omega_X^1$ . We also write  $\Omega_{\mathcal{C}^\infty(X)}^p$  for the super chain complex  $\Gamma(X, \Omega_X^p)$  of global sections of  $\Omega_X^p$ .

Define  $T_X$  to be the sheaf  $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$  of smooth 1-vectors of  $\mathcal{O}_X$ , and write  $T_{\mathcal{C}^\infty(X)}$  for the super chain complex  $\Gamma(X, T_X)$  of global sections; equivalently, this is the internal Hom complex  $\underline{\text{Hom}}_{\mathcal{C}^\infty(X)}(\Omega_{\mathcal{C}^\infty(X)}^1, \mathcal{C}^\infty(X))$ .

Beware that  $\Omega_{\mathcal{C}^\infty(X)}^1$  is not the module of Kähler differentials of the abstract super dg algebra  $\mathcal{C}^\infty(X)$ , since there are constraints requiring that the derivation  $d: \mathcal{C}^\infty(X) \rightarrow \Omega_{\mathcal{C}^\infty(X)}^1$  restrict to a  $\mathcal{C}^\infty$ -derivation on  $\mathcal{C}^\infty(X^{0,=})$ , similarly to Remark 1.5.

**Definition 1.19.** Given a dg supermanifold  $X$ , define the de Rham complex  $\text{DR}(X)$  to be the product total super cochain complex of the double complex

$$\mathcal{C}^\infty(X) \xrightarrow{d} \Omega_{\mathcal{C}^\infty(X)}^1 \xrightarrow{d} \Omega_{\mathcal{C}^\infty(X)}^2 \xrightarrow{d} \dots,$$

so  $\text{DR}(X)^m = \prod_j (\Omega_{\mathcal{C}^\infty(X)}^{m+j})_j$  with total differential  $d \pm \delta$ .

We define a filtration  $\text{Fil}$  on  $\text{DR}(X)$  by setting  $\text{Fil}^p \text{DR}(X) \subset \text{DR}(X)$  to consist of terms  $\Omega_X^i$  with  $i \geq p$ .

The complex  $\text{DR}(X)$  has the natural structure of a filtered commutative DG super algebra. Note that the quasi-isomorphism class of  $\text{DR}(X)$  only depends on the super  $\mathcal{C}^\infty$ -ringed space  $\pi^0 X$  (cf. [FT, Bha]), so that in particular the map  $\text{DR}(X) \rightarrow \text{DR}(\pi^0 X)$  is a quasi-isomorphism whenever  $\pi^0 X$  is a supermanifold. However, beware that the quasi-isomorphism classes of the filtrations  $\text{Fil}^p \text{DR}(X)$  depend on the full derived structure of  $X$  for  $p > 0$ .

**Definition 1.20.** Define an  $n$ -shifted pre-symplectic structure  $\omega$  on a dg supermanifold  $X$  to be an element

$$\omega \in Z^{n+2} \text{Fil}^2 \text{DR}(X)^\bullet.$$

Define a parity-reversed  $n$ -shifted pre-symplectic structure to be an element

$$\omega \in Z^{n+2}\mathrm{Fil}^2\mathrm{DR}(X)^\neq.$$

Two pre-symplectic structures are regarded as equivalent if they induce the same cohomology class in  $H^{n+2}\mathrm{Fil}^2\mathrm{DR}(X)^\neq$  (resp.  $H^{n+2}\mathrm{Fil}^2\mathrm{DR}(X)^\neq$ ).

Explicitly, this means that  $\omega$  is given by an infinite sum  $\omega = \sum_{i \geq 2} \omega_i$ , with  $\omega_i \in (\Omega_X^i)_{i-n-2}$  (of equal or unequal parity, respectively) and with  $d\omega_i = \delta\omega_{i+1}$ .

**Definition 1.21.** Define a (parity-reversed)  $n$ -shifted symplectic structure  $\omega$  on a dg supermanifold  $X$  to be a (parity-reversed)  $n$ -shifted pre-symplectic structure  $\omega$  for which contraction with the component  $\omega_2 \in Z^n\Omega_X^2$  induces a quasi-isomorphism

$$\begin{aligned} \omega_2^\sharp: T_X &\rightarrow (\Omega_X^1)_{[-n]} \quad \text{resp.} \\ \omega_2^\sharp: T_X &\rightarrow \Pi(\Omega_X^1)_{[-n]}. \end{aligned}$$

Note that for dg supermanifolds, bounds on the generators mean that  $n$ -shifted symplectic structures can only exist for  $n \leq 0$ .

*Examples 1.22.* The dg manifolds of Examples 1.14 and Examples 1.16 all carry natural shifted symplectic structures. For  $n \geq 0$ , the prototypical example of a  $(-n)$ -shifted symplectic structure is given by the shifted cotangent bundle  $T^*M[n]$ , with symplectic form  $\omega$  given by the canonical closed 2-form  $\omega_2 \in T_{\mathcal{C}^\infty(M)} \otimes_{\mathcal{C}^\infty(M)} \Omega_M^1 \subset (\Omega_{T^*M[n]}^2)_{[n]}$ . A similar expression defines a  $(-1)$ -shifted symplectic structure on the derived critical locus  $\mathrm{DCrit}(M, f)$  of an even function.

The prototypical example of a parity-reversed  $(-n)$ -shifted symplectic structure is given by the parity-reversed shifted cotangent bundle  $\Pi T^*M[n]$ , and the derived critical locus  $\mathrm{DCrit}(M, f)$  of an odd function carries a parity-reversed  $(-1)$ -shifted symplectic structure.

**1.3. Derived NQ-manifolds.** We now consider supermanifolds enhanced in both stacky and derived directions. For technical reasons, we do not combine the structures in a single grading, but instead work with bigraded objects. The objects  $\mathcal{C}^\infty(X)$  we consider are smooth analogues of special cases of the stacky CDGAs of [Pri6] or of the graded mixed cdgas of [CPT<sup>+</sup>]. For simplicity of exposition, we consider more restrictive objects, which broadly correspond to the derived  $\infty$ -Lie algebroids of [the nlab](#), and are global versions of the quasi-smooth functors of [Pri1, Definition 1.33 and Proposition 1.63].

**Definition 1.23.** Define a dg NQ-manifold  $X$  to be a pair  $(X_0^0, \mathcal{O}_X)$  where  $X_0^0$  is a real differentiable manifold and  $\mathcal{O}_X$  is a graded-commutative  $\mathbb{R}$ -algebra

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \downarrow \delta & & \downarrow \delta & & \\ \mathcal{O}_{X,1}^0 & \xrightarrow{Q} & \mathcal{O}_{X,1}^1 & \xrightarrow{Q} & \dots \\ \downarrow \delta & & \downarrow \delta & & \\ \mathcal{O}_{X,0}^0 & \xrightarrow{Q} & \mathcal{O}_{X,0}^1 & \xrightarrow{Q} & \dots \end{array}$$

in chain cochain complexes of sheaves on  $X_0^0$ . We require that  $\mathcal{O}_{X_0^0}^0$  be the sheaf of smooth functions on  $X_0^0$ , that the cochain differential  $Q: \mathcal{O}_{X_0^0}^0 \rightarrow \mathcal{O}_{X_0^0}^1$  be a  $\mathcal{C}^\infty$ -derivation and that  $\mathcal{O}_X$  be locally semi-free in the sense that the underlying bigraded-commutative sheaf  $\mathcal{O}_{X,\#}^\#$  is locally of the form  $\mathcal{O}_{X_0^0} \otimes \text{Symm}_{\mathbb{R}}(U \oplus V)$  for some finite-dimensional bigraded vector spaces  $U = U_0^1 \oplus \dots \oplus U_0^N$  and  $V = V_1^0 \oplus \dots \oplus V_m^0$ .

Write  $\mathcal{C}^\infty(X) := \Gamma(X_0^0, \mathcal{O}_X)$  for the chain cochain complex of global sections of  $\mathcal{O}_X$ .

*Example 1.24.* For a simple example, consider the case where  $X_0^0$  is a point, and  $\mathcal{O}_X$  is freely generated by  $s \in \mathcal{O}_{X_0^0}^2$  and  $t \in \mathcal{O}_{X_0^0}^0$ . Then  $\mathcal{O}_X = \mathbb{R}[s, t]$ , with bigrading given by  $s^i t^j \in \mathcal{O}_{X,2j}^{2i}$ . Homotopy-theoretical considerations mean that the natural associated  $\mathbb{Z}$ -graded object to consider is the product total complex  $\text{Tot}^\Pi \mathcal{O}_X$ . This is given in degree 0 by  $(\text{Tot}^\Pi \mathcal{O}_X)^0 = \mathbb{R}[[st]]$ , the ring of formal power series, which is very far from being a ring of functions on a manifold, demonstrating the utility of bigradings instead of  $\mathbb{Z}$ -gradings for book-keeping purposes.

**Definition 1.25.** Given a dg NQ-manifold  $X$ , define the truncation  $\pi^0 X$  to be the dg-ringed space  $(X_0^0, \mathcal{O}_{X_0^0}/\delta \mathcal{O}_{X,1})$ , which we might think of as a Lie  $\mathcal{C}^\infty$ -subalgebroid of the NQ-manifold  $X^0 = (X_0^0, \mathcal{O}_{X,0})$ .

**Definition 1.26.** Define a super dg NQ-manifold  $X$  to be a pair  $(X_0^{0,=}, \mathcal{O}_X)$  where  $X_0^{0,=}$  is a real differentiable manifold and  $\mathcal{O}_X = \mathcal{O}_{X,\geq 0}^{\geq 0}$  is a graded-commutative  $\mathbb{R}$ -algebra in super chain cochain complexes of sheaves on  $X_0^{0,=}$ . We require that  $\mathcal{O}_{X_0^0}^{0,=}$  be the sheaf of smooth functions on  $X_0^{0,=}$ , that the cochain differential  $Q: \mathcal{O}_{X_0^0}^{0,=} \rightarrow \mathcal{O}_{X_0^0}^{1,=}$  be a  $\mathcal{C}^\infty$ -derivation and that  $\mathcal{O}_X$  be locally semi-free in the sense that the underlying super bigraded-commutative sheaf  $\mathcal{O}_{X,\#}^\#$  is locally of the form  $\mathcal{O}_{X_0^0} \otimes \text{Symm}_{\mathbb{R}}(U \oplus V)$  for some finite-dimensional super bigraded vector spaces  $U = U_0^1 \oplus \dots \oplus U_0^N$  and  $V = V_0^{0,\neq} \oplus V_1^0 \oplus \dots \oplus V_m^0$  (with  $V_i^0 = V_i^{0,=} \oplus V_i^{0,\neq}$  and  $U_0^j = U_0^{j,=} \oplus U_0^{j,\neq}$ ).

Write  $\mathcal{C}^\infty(X) := \Gamma(X_0^{0,=}, \mathcal{O}_X)$  for the super chain cochain complex of global sections of  $\mathcal{O}_X$ .

*Example 1.27.* If  $Y = (Y^{0,=}, \mathcal{O}_Y)$  is a dg supermanifold, with a finite-dimensional Lie algebra  $\mathfrak{g}$  acting on  $\mathcal{O}_Y$ , then there is a dg super NQ-manifold  $[Y/\mathfrak{g}]$  given by setting  $[Y/\mathfrak{g}]_0^{0,=} = Y_0^{0,=}$  and setting  $\mathcal{O}_{[Y/\mathfrak{g}]}$  to be the super cochain chain complex

$$\mathcal{O}_{[Y/\mathfrak{g}]} := (\mathcal{O}_Y \xrightarrow{Q} \mathcal{O}_Y \otimes \mathfrak{g}^* \xrightarrow{Q} \mathcal{O}_Y \otimes \Lambda^2 \mathfrak{g}^* \xrightarrow{Q} \dots),$$

the Chevalley–Eilenberg double complex of  $\mathfrak{g}$  with coefficients in the super chain complex  $\mathcal{O}_Y$ .

*Remark 1.28.* As we saw in Remark 1.17 for dg supermanifolds, the finiteness conditions for dg NQ-manifolds and super dg NQ-manifolds in Definitions 1.23, 1.26 can be relaxed to hold only up to homotopy, as in [Pri6].

Instead of super differential bigraded algebras  $\mathcal{C}^\infty(X)$  for super dg NQ-manifolds  $X$ , we could take super chain cochain commutative algebras  $A$ , with  $A^0$  as in Remark 1.17, and  $A_\#^\#$  freely generated as a graded algebra over  $A_0^0$  by projective modules  $M_j^i$  (possibly of infinite rank) with  $M_0^{0,=} = 0$ . In addition to the finiteness condition on smooth differentials  $\Omega_{A_0^0}^1$  from Remark 1.17, we would have to require that for each  $i$ ,  $(\Omega_A^1 \otimes_A H_0(A^0))^i$  be quasi-isomorphic to a projective  $H_0(A^0)$ -module of finite rank concentrated in chain degree 0, and be quasi-isomorphic to 0 when  $i > N$ . This gives a much more

flexible category of objects with which to work (essentially corresponding to the derived Lie algebroids locally of finite presentation in [Nui]), but several constructions involving duals would have to be rewritten, along the lines of the expressions in [Pri6].

**Definition 1.29.** Given a super dg NQ-manifold  $X$ , define  $\Omega_X^1$  to be the sheaf of smooth 1-forms of  $\mathcal{O}_X$ . This is a sheaf of super chain cochain complexes, and we write  $\Omega_X^p := \Lambda_{\mathcal{O}_X}^p \Omega_X^1$ . We also write  $\Omega_{\mathcal{C}^\infty(X)}^p$  for the super chain cochain complex  $\Gamma(X, \Omega_X^p)$  of global sections of  $\Omega_X^p$ .

Define  $T_X$  to be the sheaf  $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$  of smooth 1-vectors of  $\mathcal{O}_X$ , and write  $T_{\mathcal{C}^\infty(X)}$  for the super chain cochain complex  $\Gamma(X, T_X)$  of global sections; equivalently, this is the internal Hom space  $\underline{\text{Hom}}_{\mathcal{C}^\infty(X)}(\Omega_{\mathcal{C}^\infty(X)}^1, \mathcal{C}^\infty(X))$ .

**Definition 1.30.** Given a super dg NQ-manifold  $X$ , define the de Rham complex  $\text{DR}(X)$  to be the product total super cochain complex of the triple complex

$$\mathcal{C}^\infty(X) \xrightarrow{d} \Omega_{\mathcal{C}^\infty(X)}^1 \xrightarrow{d} \Omega_{\mathcal{C}^\infty(X)}^2 \xrightarrow{d} \dots,$$

so  $\text{DR}(X)^m = \prod_{i+j-k=m} (\Omega_{\mathcal{C}^\infty(X)}^i)_k^j$  with total differential  $d \pm Q \pm \delta$ .

We define a filtration  $\text{Fil}$  on  $\text{DR}(X)$  by setting  $\text{Fil}^p \text{DR}(X) \subset \text{DR}(X)$  to consist of terms  $\Omega_X^i$  with  $i \geq p$ .

The complex  $\text{DR}(X)$  has the natural structure of a filtered commutative DG super algebra.

**Definition 1.31.** Define an  $n$ -shifted pre-symplectic structure  $\omega$  on a super dg NQ-manifold  $X$  to be an element

$$\omega \in Z^{n+2} \text{Fil}^2 \text{DR}(X)^\equiv.$$

Define a parity-reversed  $n$ -shifted pre-symplectic structure to be an element

$$\omega \in Z^{n+2} \text{Fil}^2 \text{DR}(X)^\neq.$$

Two pre-symplectic structures are regarded as equivalent if they induce the same cohomology class in  $H^{n+2} \text{Fil}^2 \text{DR}(X)^\equiv$  (resp.  $H^{n+2} \text{Fil}^2 \text{DR}(X)^\neq$ ).

Explicitly, this means that  $\omega$  is given by an infinite sum  $\omega = \sum_{i \geq 2, j \geq 0} \omega_{i,j}$ , with  $\omega_{i,j} \in (\Omega_X^i)_j^{n+2-i+j}$  (of equal or unequal parity, respectively) and with  $d\omega_{i,j} = \delta\omega_{i+1,j+1} \pm Q\omega_{i+1,j}$ , for the de Rham differential  $d$ .

**Definition 1.32.** Define a (parity-reversed)  $n$ -shifted symplectic structure  $\omega$  on  $X$  to be a (parity-reversed)  $n$ -shifted pre-symplectic structure for which contraction with the component  $\omega_2 \in Z^n \Omega_X^2$  induces a quasi-isomorphism

$$\begin{aligned} \omega_2^\sharp: \text{Tot}(T_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^0) &\rightarrow \text{Tot}(\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^0)^{[n]} \quad \text{resp.} \\ \omega_2^\sharp: \text{Tot}(T_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^0) &\rightarrow \text{II} \text{Tot}(\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^0)^{[n]}. \end{aligned}$$

on total complexes.

Note that if  $X$  carries a 0-shifted symplectic structure, then either it is a supermanifold or it has non-trivial enhancements in both stacky and derived directions, since we need the chain and cochain generators of  $\mathcal{C}^\infty(X)$  to balance each other.

*Examples 1.33.* Shifted cotangent bundles of NQ-manifolds give rise to dg NQ-manifolds with natural shifted symplectic structures. For instance, the NQ-manifold  $B\mathfrak{g}$  (i.e.  $[\ast/\mathfrak{g}]$ ) associated to a Lie algebra  $\mathfrak{g}$  has derived cotangent bundle  $TB\mathfrak{g}$  given by  $[(\mathfrak{g}^\ast[-1])/\mathfrak{g}]$ , which carries a canonical 0-shifted symplectic structure. Explicitly, the manifold underlying  $TB\mathfrak{g}$  is a point, with  $\mathcal{O}_{TB\mathfrak{g}}$  freely generated as a bigraded algebra by  $\mathfrak{g}$  in chain degree 1 and  $\mathfrak{g}^\ast$  in cochain degree 1, with chain differential  $\delta = 0$  and cochain differential  $Q$  given by the Chevalley–Eilenberg complex of  $\mathfrak{g}$  acting on  $\Lambda^\ast\mathfrak{g}$ .

**Definition 1.34.** Given dg NQ-manifolds  $X, Y$ , define the product  $X \times Y$  to be the NQ-manifold with underlying manifold  $(X \times Y)_0^{0,=} = X_0^{0,=} \times Y_0^{0,=}$ , and with structure sheaf given by pullback

$$\mathcal{O}_{X \times Y} := ((\mathrm{pr}_1^{-1}\mathcal{O}_X) \otimes (\mathrm{pr}_2^{-1}\mathcal{O}_Y)) \otimes_{((\mathrm{pr}_1^{-1}\mathcal{O}_{X_0^{0,=}}) \otimes_{\mathbb{R}} (\mathrm{pr}_2^{-1}\mathcal{O}_{Y_0^{0,=}}))} \mathcal{O}_{(X \times Y)_0^{0,=}},$$

for the projection maps  $\mathrm{pr}_1: X_0^{0,=} \times Y_0^{0,=} \rightarrow X_0^{0,=}$  and  $\mathrm{pr}_2: X_0^{0,=} \times Y_0^{0,=} \rightarrow Y_0^{0,=}$ .

## 2. SHIFTED POISSON STRUCTURES ON SUPER DERIVED NQ-MANIFOLDS

This section is adapted from [Pri6], transferring results from the algebraic to the smooth setting. Whereas [Pri6] begins with derived affines (analogous to dg manifolds), we begin with super NQ-manifolds, since these are more likely to be familiar to readers. The two cases behave similarly, both having simplifications compared with the general case (super dg NQ-manifolds), since they do not require us to deal with chain and cochain structures simultaneously. There are however slight differences in the notion of non-degeneracy, which we have to adapt from [Pri6, §3] (rather than [Pri6, §1] as we would for dg manifolds).

**2.1. Shifted Poisson structures on super NQ-manifolds.** For now, we fix a super NQ-manifold  $X = (X_0^=, \mathcal{O}_X)$ .

### 2.1.1. Polyvector fields.

**Definition 2.1.** We define the super cochain complex of  $n$ -shifted polyvector fields on  $X$  by

$$\widehat{\mathrm{Pol}}(X, n) := \prod_i \mathrm{Symm}_{\mathcal{C}^\infty(X)}^i(T_{\mathcal{C}^\infty(X)}^{[-n-1]}),$$

with graded-commutative multiplication following the usual conventions for symmetric powers. Similarly, define the super cochain complex of  $n$ -shifted polyvector fields of reversed parity on  $X$  by

$$\widehat{\mathrm{Pol}}(X, \Pi n) := \prod_i \mathrm{Symm}_{\mathcal{C}^\infty(X)}^i(\Pi T_{\mathcal{C}^\infty(X)}^{[-n-1]})$$

The Lie bracket on  $T_{\mathcal{C}^\infty(X)}$  then extends to give a bracket (the Schouten–Nijenhuis bracket)

$$[-, -]: \widehat{\mathrm{Pol}}(X, n) \times \widehat{\mathrm{Pol}}(X, n) \rightarrow \widehat{\mathrm{Pol}}(X, n)^{[-n-1]},$$

of equal parity determined by the property that it is a bi-derivation with respect to the multiplication operation. Similarly,  $\widehat{\mathrm{Pol}}(X, \Pi n)$  has a Lie bracket bi-derivation of cochain degree  $-n - 1$  and unequal parity.

Thus  $\widehat{\text{Pol}}(X, n)$  has the natural structure of a super  $P_{n+2}$ -algebra (i.e. an  $(n+1)$ -shifted Poisson algebra in super cochain complexes), so  $\widehat{\text{Pol}}(X, n)^{[n+1]}$  is a super differential graded Lie algebra (DGLA) over  $\mathbb{R}$ . In particular, the subcomplexes  $\widehat{\text{Pol}}(X, n)^{[n+1],=}$  and  $\widehat{\text{Pol}}(X, \Pi n)^{[n+1],\neq}$  are DGLAs over  $\mathbb{R}$ .

*Remark 2.2.* Here, we follow the conventions of [Mel] for  $P_k$ -algebras, so they carry a graded-commutative multiplication of degree 0, and a graded Lie bracket of degree  $1-k$ , both of equal parity. In particular, this means that if we commute the  $\mathbb{Z}$ -action to a  $\mathbb{Z}/2$ -action, then a  $P_k$ -algebra is a Poisson algebra for odd  $k$ , and a Gerstenhaber algebra for even  $k$ .

Note that the cochain differential  $Q$  on  $\widehat{\text{Pol}}(X, n)$  (resp.  $\widehat{\text{Pol}}(X, \Pi n)$ ) can be written as  $[Q, -]$ , where  $Q \in \widehat{\text{Pol}}(X, n)^{n+2,=}$  (resp.  $\widehat{\text{Pol}}(X, \Pi n)^{n+2,\neq}$ ) is the element corresponding to the derivation  $Q \in (T_{\mathcal{C}^\infty(X)})^1$ .

**Definition 2.3.** Define decreasing filtrations  $\text{Fil}$  on  $\widehat{\text{Pol}}(X, n)$  and  $\widehat{\text{Pol}}(X, \Pi n)$  by

$$\begin{aligned} \text{Fil}^i \widehat{\text{Pol}}(X, n) &:= \prod_{j \geq i} \text{Symm}_{\mathcal{C}^\infty(X)}^j (T_{\mathcal{C}^\infty(X)}^{[-n-1]}); \\ \text{Fil}^i \widehat{\text{Pol}}(X, \Pi n) &:= \prod_{j \geq i} \text{Symm}_{\mathcal{C}^\infty(X)}^j (\Pi T_{\mathcal{C}^\infty(X)}^{[-n-1]}); \end{aligned}$$

this has the properties that  $\widehat{\text{Pol}}(X, n) = \varprojlim_i \widehat{\text{Pol}}(X, n) / \text{Fil}^i$ , with  $[\text{Fil}^i, \text{Fil}^j] \subset \text{Fil}^{i+j-1}$ ,  $Q\text{Fil}^i \subset \text{Fil}^i$ , and  $\text{Fil}^i \text{Fil}^j \subset \text{Fil}^{i+j}$ , and similarly for  $\widehat{\text{Pol}}(X, \Pi n)$ .

Observe that this filtration makes  $\text{Fil}^2 \widehat{\text{Pol}}(X, n)^{[n+1],=}$  and  $\text{Fil}^2 \widehat{\text{Pol}}(X, \Pi n)^{[n+1],\neq}$  into pro-nilpotent DGLAs.

### 2.1.2. Poisson structures.

**Definition 2.4.** Given a DGLA  $(L, d)$ , define the the Maurer–Cartan set by

$$\text{MC}(L) := \{\omega \in L^1 \mid d\omega + \frac{1}{2}[\omega, \omega] = 0 \in \bigoplus_n L^2\}.$$

**Definition 2.5.** Define an  $n$ -shifted Poisson structure on  $X$  to be an element of

$$\text{MC}(\text{Fil}^2 \widehat{\text{Pol}}(X, n)^{[n+1],=}),$$

and an  $n$ -shifted Poisson structure of reversed parity on  $X$  to be an element of

$$\text{MC}(\text{Fil}^2 \widehat{\text{Pol}}(X, \Pi n)^{[n+1],\neq}).$$

Regard two  $n$ -shifted Poisson structures as equivalent if they are gauge equivalent as Maurer–Cartan elements (cf. [Man]), i.e. if they lie in the same orbit for the gauge action on the Maurer–Cartan set of the formal group  $\exp(\text{Fil}^2 \widehat{\text{Pol}}(X, n)^{n+1})$  corresponding to the pro-nilpotent Lie algebra  $\text{Fil}^2 \widehat{\text{Pol}}(X, n)^{n+1}$ .

*Remark 2.6.* Observe that  $n$ -shifted Poisson structures consist of infinite sums  $\pi = \sum_{i \geq 2} \pi_i$  with polyvectors

$$\pi_i \in \text{Symm}_{\mathcal{C}^\infty(X)}^i (T_{\mathcal{C}^\infty(X)}^{[-n-1]})^{n+2}$$

satisfying  $Q(\pi_i) + \frac{1}{2} \sum_{j+k=i+1} [\pi_j, \pi_k] = 0$ . This is precisely the condition which ensures that  $\pi$  defines an  $L_\infty$ -algebra structure on  $\mathcal{C}^\infty(X)^{[n]}$ . It then makes  $\mathcal{C}^\infty(X)$  into a

$\hat{P}_{n+1}$ -algebra in the sense of [Mel, Definition 2.9]; in [CF] these are referred to as  $P_\infty$ -algebras in the case  $n = 0$ . In our setting, however, we have more than just an abstract  $\hat{P}_{n+1}$ -algebra structure, since we have a  $\mathcal{C}^\infty$ -ring and  $\mathcal{C}^\infty$  derivations.

*Example 2.7.* For  $n = 0$  and  $X$  a manifold, Definition 2.5 recovers the usual notion of a Poisson structure, as we necessarily have  $\pi = \pi_2$  for degree reasons, and the Maurer–Cartan equation reduces to the Jacobi identity.

*Example 2.8.* As in [Pri6, Examples 3.31], for any manifold  $M$  equipped with an action of a finite-dimensional real Lie algebra  $\mathfrak{g}$ , we may consider the NQ-manifold  $[M/\mathfrak{g}] = (M, \text{CE}(\mathfrak{g}, \mathcal{C}^\infty(M)))$ , where  $\text{CE}(\mathfrak{g}, -)$  denotes the Chevalley–Eilenberg complex of a  $\mathfrak{g}$ -representation. Its tangent complex is then given by

$$T_{\mathcal{C}^\infty([M/\mathfrak{g}])} := \text{CE}(\mathfrak{g}, \text{cone}(\mathfrak{g} \otimes \mathcal{C}^\infty(M))) \rightarrow T_{\mathcal{C}^\infty(M)},$$

which is concentrated in cohomological degrees  $\geq -1$ . Degree restrictions thus show that the set of 2-shifted Poisson structures is given by

$$\{\pi \in (S^2\mathfrak{g} \otimes \mathcal{C}^\infty(M))^{\mathfrak{g}} : [\pi, a] = 0 \in \mathfrak{g} \otimes \mathcal{C}^\infty(M) \forall a \in \mathcal{C}^\infty(M)\}.$$

In fact, this set is a model for the space  $\mathcal{P}([M/\mathfrak{g}], 2)$  of Poisson structures in Definition 2.13 below, there being no automorphisms or higher automorphisms since the DGLA has no terms in non-positive degrees.

A generalisation of this example to Lie pairs is given in [BCSX]. Similar expressions hold for Lie algebroids  $A$  on  $M$ , replacing  $S^k\mathfrak{g} \otimes \mathcal{C}^\infty(M)$  with  $\Gamma(M, S^k A)$ . Specialising to the case where  $M$  is a point, the expression above says that 2-shifted Poisson structures on the NQ-manifold  $B\mathfrak{g}$  of Examples 1.10 are given by  $(S^2\mathfrak{g})^{\mathfrak{g}}$ , the set of quadratic Casimir elements.

*Example 2.9.* Meanwhile, 1-shifted Poisson structures on  $[M/\mathfrak{g}]$  are given by pairs  $(\varpi, \phi)$  with

$$\varpi \in (\mathfrak{g} \otimes T_{\mathcal{C}^\infty(M)}) \oplus (\mathfrak{g}^* \otimes \Lambda^2\mathfrak{g} \otimes \mathcal{C}^\infty(M)), \quad \phi \in \Lambda^3\mathfrak{g} \otimes \mathcal{C}^\infty(M)$$

satisfying, for  $Q \in \mathfrak{g}^* \otimes T_{\mathcal{C}^\infty(M)}$  corresponding to the Chevalley–Eilenberg derivative,

$$\{Q, \varpi\} = 0, \quad \frac{1}{2}\{\varpi, \varpi\} + \{Q, \phi\} = 0, \quad [\varpi, \phi] = 0,$$

where  $\{-, -\}$  is the shifted Schouten–Nijenhuis bracket. This characterisation also generalises to Lie algebroids. As explained in [Saf1, Theorem 3.15] (which uses the more involved formulation of shifted Poisson structures from [CPT<sup>+</sup>]), this structure is just the same as a quasi-Lie bialgebroid in the sense of [IPLGX, Definition 4.6] (after [Roy2, §3]), with  $\varpi$  the 2-differential and  $\phi$  its curvature.

Since the DGLA of 1-shifted polyvectors is concentrated in non-negative cohomological degrees, the space  $\mathcal{P}([M/\mathfrak{g}], 1)$  has no higher homotopy groups, but it does have non-trivial fundamental groups at each point. As in [Saf1, Definition 3.4 and Theorem 3.15], we can calculate morphisms in the fundamental groupoid via gauge transformations in the DGLA of polyvectors; these are given by elements of degree 0 in the DGLA, i.e. by twists  $\lambda \in (\Lambda^2\mathfrak{g} \otimes \mathcal{C}^\infty(M))$ , which send  $(\varpi, \phi)$  to  $(\varpi + \{Q, \lambda\}, \phi + \{\lambda, \varpi\} + \frac{1}{2}\{\lambda, \{Q, \lambda\}\})$ .

*Example 2.10.* A special case of the previous example is given by taking  $M$  to be a point. As in [Saf1, Theorem 2.8], a 1-shifted Poisson structure on  $B\mathfrak{g}$  is then the same as a quasi-Lie bialgebra structure on  $\mathfrak{g}$ , with the condition  $Q(\varpi) = 0$  giving compatibility of bracket and cobracket while the condition  $\frac{1}{2}\{\varpi, \varpi\} + Q(\phi) = 0$  says that the trivector  $\phi$  measures the failure of the cobracket  $\varpi$  to satisfy the Jacobi identity.

For constructions involving functoriality, gluing or descent in §4, we will need to keep track of automorphisms of Poisson structures, including higher automorphisms. To this end, we now define a whole simplicial set (or equivalently, a topological space or  $\infty$ -groupoid) of Poisson structures. As for instance in [Wei, §8.1], a simplicial set is a diagram

$$\begin{array}{ccccccc} & & \partial_2 & & \partial_3 & & \\ & \xleftarrow{\partial_1} & \xleftrightarrow{\partial_2} & \xleftarrow{\partial_3} & & & \\ Z_0 & \xleftarrow{\sigma_0} & Z_1 & \xleftrightarrow{\partial_0} & Z_2 & \xleftrightarrow{\partial_0} & Z_3 & \cdots & \cdots, \\ & \xleftarrow{\partial_0} & & & & & & & \end{array}$$

of sets, with various relations between the face maps  $\partial_i$  and the degeneracy maps  $\sigma_i$  such as  $\sigma_i \partial_i = \text{id}$ . The main motivating examples are given by taking  $Z_n$  to be the set of continuous maps to a topological space from the geometric  $n$ -simplex

$$|\Delta^n| := \{x \in \mathbb{R}_+^{n+1} : \sum_{i=0}^n x_i = 1\}.$$

Path components of the simplicial set will correspond to equivalence classes of Poisson structures, fundamental groups to automorphisms of Poisson structures up to homotopy equivalence, and higher homotopy groups to higher automorphisms. Such homotopy groups come from negative degree cohomology in the DGLA of shifted polyvectors, and it is the possible non-triviality of these higher automorphism groups which complicates gluing arguments.

**Definition 2.11.** Following [Hin], define the Maurer–Cartan space  $\underline{\text{MC}}(L)$  (a simplicial set) of a nilpotent DGLA  $L$  by

$$\underline{\text{MC}}(L)_n := \text{MC}(L \otimes_{\mathbb{Q}} \Omega^\bullet(\Delta^n)),$$

where

$$\Omega^\bullet(\Delta^n) = \mathbb{Q}[t_0, t_1, \dots, t_n, dt_0, dt_1, \dots, dt_n] / (\sum t_i - 1, \sum dt_i)$$

is the commutative dg algebra of de Rham polynomial forms on the  $n$ -simplex, with the  $t_i$  of degree 0.

**Definition 2.12.** Given an inverse system  $L = \{L_\alpha\}_\alpha$  of nilpotent DGLAs, define

$$\text{MC}(L) := \varprojlim_{\alpha} \text{MC}(L_\alpha) \quad \underline{\text{MC}}(L) := \varprojlim_{\alpha} \underline{\text{MC}}(L_\alpha).$$

Note that  $\text{MC}(L) = \text{MC}(\varprojlim_{\alpha} L_\alpha)$ , but  $\underline{\text{MC}}(L) \neq \underline{\text{MC}}(\varprojlim_{\alpha} L_\alpha)$ .

**Definition 2.13.** Define the space  $\mathcal{P}(X, n)$  of  $n$ -shifted Poisson structures on  $X$  to be given by the simplicial set

$$\mathcal{P}(X, n) := \underline{\text{MC}}(\{\widehat{\text{Fil}}^2 \widehat{\text{Pol}}(X, n)^{[n+1]} / \widehat{\text{Fil}}^{i+2}\}_i).$$

For the space  $\mathcal{P}(X, \Pi n)$  of  $n$ -shifted Poisson structures with reversed parity, replace  $\widehat{\text{Pol}}(X, n)^\equiv$  with  $\widehat{\text{Pol}}(X, \Pi n)^\neq$ .

Thus observe that  $n$ -shifted Poisson structures are elements of  $\mathcal{P}_0(X, n)$ . Since gauge transformations and polynomial de Rham homotopies both give rise to path objects in a suitable model category of pro-nilpotent DGLAs, two Poisson structures define the same class in the set  $\pi_0 \mathcal{P}(X, n)$  of path components if and only if they are equivalent in the sense of Definition 2.5.



**Definition 2.14.** We say that an  $n$ -shifted Poisson structure (resp.  $n$ -shifted Poisson structure of reversed parity)  $\pi = \sum_{i \geq 2} \pi_i$  is non-degenerate if contraction with  $\pi_2 \in \text{Symm}_{\mathcal{C}^\infty(X)}^2(T_{\mathcal{C}^\infty(X)}^{[-n-1]})^{[n+2],=}$  (resp.  $\text{Symm}_{\mathcal{C}^\infty(X)}^2(\Pi T_{\mathcal{C}^\infty(X)}^{[-n-1]})^{[n+2],\neq}$ ) induces a quasi-isomorphism

$$\begin{aligned} \pi_2^\sharp: (\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^0)^{[n]} &\rightarrow T_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^0, \quad \text{resp.} \\ \pi_2^\sharp: (\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^0)^{[n]} &\rightarrow \Pi T_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^0. \end{aligned}$$

Define  $\mathcal{P}(X, n)^{\text{nondeg}} \subset \mathcal{P}(X, n)$  and  $\mathcal{P}(X, \Pi n)^{\text{nondeg}} \subset \mathcal{P}(X, \Pi n)$  to consist of non-degenerate elements — these are unions of path-components.

2.1.3. *Equivalence of non-degenerate Poisson and symplectic structures.* We can regard  $\text{Fil}^2\text{DR}(X)^{[n+1],=}$  and  $\text{Fil}^2\text{DR}(X)^{[n+1],\neq}$  as filtered DGLAs with trivial bracket. Such abelian DGLAs  $A$  have the property that  $\text{MC}(A) = Z^1 A$ . The following definition thus generalises Definition 1.7

**Definition 2.15.** Define the spaces of  $n$ -shifted pre-symplectic structures and of parity-reversed  $n$ -shifted pre-symplectic structures on a super NQ-manifold  $X$  by

$$\begin{aligned} \text{PreSp}(X, n) &:= \underline{\text{MC}}(\{\text{Fil}^2\text{DR}(X)^{[n+1],=} / \text{Fil}^{i+2}\}_i) \\ \text{PreSp}(X, \Pi n) &:= \underline{\text{MC}}(\{\text{Fil}^2\text{DR}(X)^{[n+1],\neq} / \text{Fil}^{i+2}\}_i). \end{aligned}$$

Set  $\text{Sp}(X, n) \subset \text{PreSp}(X, n)$  and  $\text{Sp}(X, \Pi n) \subset \text{PreSp}(X, \Pi n)$  to consist of the symplectic structures — these subspaces are unions of path-components.

Note that the spaces  $\text{PreSp}(X, n)$  and  $\text{PreSp}(X, \Pi n)$  are canonically weakly equivalent to Dold–Kan denormalisations of good truncations of the equal and unequal parity summands of  $\text{Fil}^2\text{DR}$ , so their homotopy groups are just given by

$$\begin{aligned} \pi_i \text{PreSp}(X, n) &\cong H^{n+2-i} \text{Fil}^2\text{DR}(X)^= \\ \pi_i \text{PreSp}(X, \Pi n) &\cong H^{n+2-i} \text{Fil}^2\text{DR}(X)^\neq. \end{aligned}$$

**Theorem 2.16.** *For a super NQ-manifold  $X$ , there are canonical weak equivalences*

$$\text{Sp}(X, n) \simeq \mathcal{P}(X, n)^{\text{nondeg}} \quad \text{Sp}(X, \Pi n) \simeq \mathcal{P}(X, \Pi n)^{\text{nondeg}}$$

*of simplicial sets.*

*In particular, the sets of equivalence classes of (parity-reversed)  $n$ -shifted symplectic structures and of (parity-reversed) non-degenerate  $n$ -shifted Poisson structures on  $X$  are isomorphic.*

*Proof.* The proof of [Pri6, Corollary 1.38] adapts, *mutatis mutandis*. We now outline the main steps. The passage from non-degenerate Poisson structures to symplectic structures proceeds along the lines of the more specific cases considered in [KSM, Proposition 6.4], [KV, Proposition 2 and Theorem 2] and [Bru, Proposition 5.2.2]. Other related constructions can be found in [KSLG, KS].

Each Poisson structure  $\pi \in \mathcal{P}(X, n)$  (resp.  $\pi \in \mathcal{P}(X, \Pi n)$ ) gives rise to a Poisson cohomology complex

$$\widehat{\text{Pol}}_\pi(X, n) \quad (\text{resp.} \quad \widehat{\text{Pol}}_\pi(X, \Pi n)),$$

defined as the super cochain complex given by the derivation  $Q + [\pi, -]$  acting on  $\widehat{\text{Pol}}(X, n)$  (resp.  $\widehat{\text{Pol}}(X, \Pi n)$ ). There is also a canonical element  $\sigma(\pi) \in Z^{n+2} \widehat{\text{Pol}}_\pi(X, n)^=$

(resp.  $Z^{n+2}\widehat{\text{Pol}}_\pi(X, \Pi n)^\neq$ ) given by

$$\sigma(\pi) = \sum_{i \geq 2} (i-1)\pi_i,$$

for  $\pi_i \in \text{Symm}^i T$ .

The key construction is then given by the ‘‘compatibility map’’

$$\begin{aligned} \mu(-, \pi): \text{DR}(X) &\rightarrow \widehat{\text{Pol}}_\pi(X, n) \quad (\text{resp.} \quad \widehat{\text{Pol}}_\pi(X, \Pi n)) \\ \text{adf}_1 \wedge \dots \wedge \text{df}_p &\mapsto a[\pi, f_1] \dots [\pi, f_p] \end{aligned}$$

of filtered super cochain complexes. When  $\pi$  is non-degenerate, this map is necessarily a quasi-isomorphism, and the symplectic structure associated to  $\pi$  is given by

$$\mu(\pi, -)^{-1}\sigma(\pi) \in \text{H}^{n+2}\text{Fil}^2\text{DR}(X)^\neq \quad (\text{resp.} \quad \text{H}^{n+2}\text{Fil}^2\text{DR}(X)^\neq).$$

In fact, [KV] observe that the inverse map  $\mu(\pi, -)^{-1}$  is a Legendre transform.

Establishing that this gives an equivalence between symplectic and Poisson structures relies on obstruction theory associated to filtered DGLAs, building the Poisson form  $\pi = \pi_2 + \pi_3 + \dots$  inductively from the symplectic form  $\omega = \omega_2 + \omega_3 + \dots$  by solving the equation  $\mu(\omega, \pi) \simeq \sigma(\pi)$  up to coherent homotopy; for a readable summary of the argument from [Pri6], see [Saf2, §2.5].  $\square$

**2.2. Shifted Poisson structures on super dg NQ-manifolds.** The formulation of shifted Poisson structures for super dg NQ-manifolds follows along the same lines as the construction for stacky CDGAs in [Pri6, §3]. The main subtlety is to combine the two gradings in an effective way. In this section, we fix a super dg NQ-manifold  $X = (X_0^{0,\neq}, \mathcal{O}_X)$ .

The definition of an  $n$ -shifted Poisson structure on  $X$  is fairly obvious: it is a Lie bracket of total cochain degree  $-n$  on  $\mathcal{O}_X$ , or rather an  $L_\infty$ -structure in the form of a sequence  $[-]_m$  of  $m$ -ary operations of cochain degree  $1 - (n+1)(m-1)$ . However, the precise formulation (Definition 2.18) is quite subtle, involving lower bounds on the cochain degrees of the operations.

**Definition 2.17.** Given a chain cochain complex  $V$ , define the cochain complex  $\widehat{\text{Tot}} V \subset \text{Tot}^\Pi V$  by

$$(\widehat{\text{Tot}} V)^m := \left( \bigoplus_{i < 0} V_{i-m}^i \right) \oplus \left( \prod_{i \geq 0} V_{i-m}^i \right)$$

with differential  $Q \pm \delta$ .

An alternative description of  $\widehat{\text{Tot}} V$  is as the completion of  $\text{Tot} V$  with respect to the filtration  $\{\text{Tot} \sigma^{\geq m} V\}_m$ , where  $\sigma^{\geq m}$  denotes brutal truncation in the cochain direction. In fact, we can write

$$\varprojlim_m \varinjlim_n \text{Tot}((\sigma^{\geq -n} V)/(\sigma^{\geq m} V)) = \widehat{\text{Tot}} V = \varinjlim_n \varprojlim_m \text{Tot}((\sigma^{\geq -n} V)/(\sigma^{\geq m} V)).$$

The latter description also shows that there is a canonical map  $(\widehat{\text{Tot}} U) \otimes (\widehat{\text{Tot}} V) \rightarrow \widehat{\text{Tot}}(U \otimes V)$  — the same is not true of the product total complex  $\text{Tot}^\Pi$  in general.

**Definition 2.18.** Given a super dg NQ-manifold  $X$ , define the super cochain complex of  $n$ -shifted polyvector fields (resp.  $n$ -shifted polyvector fields of reversed parity) on  $X$

respectively by

$$\begin{aligned}\widehat{\text{Pol}}(X, n) &:= \prod_{j \geq 0} \widehat{\text{Tot}} \text{Symm}_{\mathcal{C}^\infty(X)}^j(T_{\mathcal{C}^\infty(X)}^{[-n-1]}), \\ \widehat{\text{Pol}}(X, \Pi n) &:= \prod_{j \geq 0} \widehat{\text{Tot}} \text{Symm}_{\mathcal{C}^\infty(X)}^j(\Pi T_{\mathcal{C}^\infty(X)}^{[-n-1]}).\end{aligned}$$

These have filtrations by super cochain complexes

$$\begin{aligned}\text{Fil}^p \widehat{\text{Pol}}(X, n) &:= \prod_{j \geq p} \widehat{\text{Tot}} \text{Symm}_{\mathcal{C}^\infty(X)}^j(T_{\mathcal{C}^\infty(X)}^{[-n-1]}), \\ \text{Fil}^p \widehat{\text{Pol}}(X, \Pi n) &:= \prod_{j \geq p} \widehat{\text{Tot}} \text{Symm}_{\mathcal{C}^\infty(X)}^j(\Pi T_{\mathcal{C}^\infty(X)}^{[-n-1]})\end{aligned}$$

respectively, with  $[\text{Fil}^i, \text{Fil}^j] \subset \text{Fil}^{i+j-1}$  and  $\text{Fil}^i \text{Fil}^j \subset \text{Fil}^{i+j}$ , where the commutative product and Schouten–Nijenhuis bracket are defined as before.

We now define the spaces  $\mathcal{P}(X, n), \mathcal{P}(X, \Pi n)$  of Poisson structures by the formulae of Definition 2.13.

**Definition 2.19.** We say that a Poisson structure  $\pi \in \mathcal{P}(X, n)$  is non-degenerate if the map

$$\pi_2^\sharp: \text{Tot}^\Pi(\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^0)^{[n]} \rightarrow T_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^0$$

defined by contraction is a quasi-isomorphism.

The definitions of shifted symplectic structures from §2.1 now carry over:

**Definition 2.20.** Define the space  $\text{PreSp}(X, n)$  of  $n$ -shifted pre-symplectic structures on  $X$  by regarding the de Rham complex of Definition 1.30 as an abelian filtered DGLA, and writing

$$\begin{aligned}\text{PreSp}(X, n) &:= \varprojlim_i \underline{\text{MC}}(\text{Fil}^2 \text{DR}(X)^{[n+1],=} / \text{Fil}^{i+2}) \\ \text{PreSp}(X, \Pi n) &:= \varprojlim_i \underline{\text{MC}}(\text{Fil}^2 \text{DR}(X)^{[n+1],\neq} / \text{Fil}^{i+2}).\end{aligned}$$

Let  $\text{Sp}(X, n) \subset \text{PreSp}(X, n)$  (resp.  $\text{Sp}(X, \Pi n) \subset \text{PreSp}(X, \Pi n)$ ) consist of the symplectic structures in the sense of Definition 1.32 — this is a union of path-components.

**Theorem 2.21.** *For a super dg NQ-manifold  $X$ , there are canonical weak equivalences*

$$\text{Sp}(X, n) \simeq \mathcal{P}(X, n)^{\text{nondeg}} \quad \text{Sp}(X, \Pi n) \simeq \mathcal{P}(X, \Pi n)^{\text{nondeg}}$$

*of simplicial sets.*

*In particular, the sets of equivalence classes of (parity-reversed)  $n$ -shifted symplectic structures and of (parity-reversed) non-degenerate  $n$ -shifted Poisson structures on  $X$  are isomorphic.*

*Proof.* This proof follows along the lines of Theorem 2.16, with constructions adapted from [Pri6, §3]. The space of Poisson structures still has a canonical tangent vector  $\sigma$ , and good properties of  $\widehat{\text{Tot}}$  with respect to tensor products ensure that each Poisson structure  $\pi$  gives a compatibility map  $\mu(-, \pi)$  from de Rham cohomology to Poisson cohomology. The non-degeneracy condition is formulated to ensure that this is a quasi-isomorphism, and the proof of Theorem 2.16 then adapts verbatim.  $\square$

### 3. DEFORMATION QUANTISATION

We now consider quantisation for  $n$ -shifted symplectic and Poisson structures. In this section, we will not consider parity-reversed Poisson structures, since they have no natural notion of quantisation. For  $k \geq 1$ , the natural notion of quantisation for  $P_k$ -algebras is given by  $E_k$ -algebras. An  $E_k$ -algebra can be thought of as a cochain complex with  $k$  associative multiplications which commute with each other up to homotopy, and the commutators then give rise to a Lie bracket of cochain degree  $1 - k$ . For instance,  $k = 2$  can be modelled by brace algebras.

But for  $k \geq 2$ , Kontsevich formality gives an equivalence (for any choice of Drinfeld associator) between  $E_k$ -algebras and  $P_k$ -algebras, so quantisations of  $n$ -shifted Poisson structures automatically exist for all  $n \geq 2$ . We will focus here on the non-trivial cases of 0-shifted and  $(-1)$ -shifted symplectic structures, with quantisations of  $P_1$ -algebras being given by  $E_1$  (i.e.  $A_\infty$ )-algebras, and quantisations of  $P_0$ -algebras being given by  $BV_\infty$ -algebras. The situation for  $(-2)$ -shifted structures is even more subtle, with [BJ] developing the beginnings of a theory.

#### 3.1. Quantisation of 0-shifted Poisson structures.

##### 3.1.1. Polydifferential operators.

**Definition 3.1.** Given a super dg NQ-manifold  $X$ , we write  $\mathcal{D}_{\mathcal{C}^\infty(X)} \subset \underline{\text{Hom}}_{\mathbb{R}}(\mathcal{C}^\infty(X), \mathcal{C}^\infty(X))$  for the super chain cochain complex of  $\mathcal{C}^\infty$  differential operators. This consists of homomorphisms which can be written locally as the  $\mathcal{O}_X$ -linear span of

$$\partial_{i_1, \dots, i_m} := \partial_{x_{i_1}} \dots \partial_{x_{i_m}}$$

for homogeneous co-ordinates  $x_i \in \mathcal{O}_X$ . We denote by  $F_k \mathcal{D}_{\mathcal{C}^\infty(X)} \subset \mathcal{D}_{\mathcal{C}^\infty(X)}$  the space of differential operators of order  $\leq k$ , i.e. the span of  $\{\partial_{i_1, \dots, i_m} : \sum i_r \leq k\}$ .

Given  $\mathcal{C}^\infty$ -modules  $M, N$  in super chain cochain complexes, we write  $F_k \text{Diff}_{\mathcal{C}^\infty(X)}(M, N)$  for the space of  $\mathcal{C}^\infty$  differential operators from  $M$  to  $N$  of order  $\leq k$ , i.e.

$$F_k \text{Diff}_{\mathcal{C}^\infty(X)}(M, N) := \underline{\text{Hom}}_{\mathcal{C}^\infty(X)}(M, N \otimes_{\mathcal{C}^\infty(X)} \mathcal{D}_{\mathcal{C}^\infty(X)}),$$

where  $\underline{\text{Hom}}$  is taken with respect to the right  $\mathcal{C}^\infty(X)$ -module structure on the target. We then write  $\text{Diff}_{\mathcal{C}^\infty(X)}(M, N) := \varinjlim_k F_k \text{Diff}_{\mathcal{C}^\infty(X)}(M, N)$ .

*Remark 3.2.* For co-ordinate-free descriptions of  $\mathcal{D}_{\mathcal{C}^\infty(X)}$  and  $\text{Diff}_{\mathcal{C}^\infty(X)}(M, N)$ , we may adapt the standard algebraic descriptions. The super chain cochain algebra  $\mathcal{C}^\infty(X \times X)$  of Definition 1.34 has a left  $\mathcal{C}^\infty(X)$ -module structure coming from the projection map  $X^n \times X \rightarrow X$  to the first factor, and a morphism  $\Delta^\sharp: \mathcal{C}^\infty(X \times X) \rightarrow \mathcal{C}^\infty(X)$  coming from the diagonal embedding. If we write  $I := \ker(\Delta^\sharp)$ , then we just have

$$F_k \mathcal{D}_{\mathcal{C}^\infty(X)} \cong \underline{\text{Hom}}_{\mathcal{C}^\infty(X)}(\mathcal{C}^\infty(X \times X) / \overbrace{(I \cdots I)}^{k+1}, \mathcal{C}^\infty(X)),$$

$$F_k \text{Diff}_{\mathcal{C}^\infty(X)}(M, N) \cong \underline{\text{Hom}}_{\mathcal{C}^\infty(X)}(M \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X \times X) / \overbrace{(I \cdots I)}^{k+1}, N).$$

Alternatively, we can describe  $\mathcal{C}^\infty$ -differential operators as algebraic differential operators with additional conditions. If for  $a \in \mathcal{C}^\infty(X)$  and  $\theta \in \underline{\text{Hom}}_{\mathbb{R}}(M, N)$ , we write

$\text{ad}_a(\theta)$  for the commutator  $a \circ \theta \mp \theta \circ a$ , then algebraic differential operators from  $M$  to  $N$  of order  $\leq k$  are elements  $\theta \in \underline{\text{Hom}}_{\mathbb{R}}(M, N)$  satisfying

$$\text{ad}_{a_0}(\text{ad}_{a_1}(\dots(\text{ad}_{a_k}(\theta))\dots)) = 0$$

for all  $(k+1)$ -tuples  $(a_0, a_1, \dots, a_k) \in \mathcal{C}^\infty(X)^{k+1}$ . To be a  $\mathcal{C}^\infty$ -differential operator,  $\theta$  must also satisfy a generalisation of the condition of [Joy1, §5.3] for  $\mathcal{C}^\infty$  1-forms, namely that for all  $v \in M$ ,  $f \in \mathcal{C}^\infty(\mathbb{R}^m)$  and  $a_1, \dots, a_m \in \mathcal{C}^\infty(X)_0^{0,=}$ , we have a generalised Taylor series expansion

$$\theta(f(a_1, \dots, a_n)v) = \sum_{I=(i_1, \dots, i_n)} \frac{1}{i_1! \dots i_n!} \frac{\partial^I f}{\partial x^I}(a_1, \dots, a_n) \text{ad}_{a_1}^{i_1}(\dots(\text{ad}_{a_n}^{i_n}(\theta))\dots)(v)$$

(the sum is necessarily finite, since only terms with  $|I| \leq k$  contribute when  $\theta$  has order  $\leq k$ ).

We now generalise the complex of polydifferential operators from [Kon2, 4.6.1].

**Definition 3.3.** Given a super dg NQ-manifold  $X$ , define the super chain cochain complex  $\mathcal{D}^{\text{poly}}(X)$  of polydifferential operators in terms of the products  $X^n := \overbrace{X \times \dots \times X}^n$  by

$$\mathcal{D}^{\text{poly}}(X)_{\#} := \prod_{n \geq 0} \text{Diff}_{\mathcal{C}^\infty(X^n)}(\mathcal{C}^\infty(X^n), \mathcal{C}^\infty(X))_{[n]},$$

with cochain differential  $Q$  and chain differential  $\delta \pm b$ , for the Hochschild differential  $b$  determined by

$$\begin{aligned} (bf)(a_1, \dots, a_n) &= a_1 f(a_2, \dots, a_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i f(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_n) \\ &\quad + (-1)^n f(a_1, \dots, a_{n-1}) a_n. \end{aligned}$$

We define an increasing filtration  $\tau^{HH}$  on  $\mathcal{D}^{\text{poly}}(X)$  by good truncation in the Hochschild direction, so  $\tau_p^{HH} \mathcal{D}^{\text{poly}}(X) \subset \mathcal{D}^{\text{poly}}(X)$  is the subspace

$$\begin{aligned} &\prod_{n=0}^{p-1} \text{Diff}_{\mathcal{C}^\infty(X^n)}(\mathcal{C}^\infty(X^n), \mathcal{C}^\infty(X))_{[n]} \\ &\times \ker(b: \text{Diff}_{\mathcal{C}^\infty(X^p)}(\mathcal{C}^\infty(X^p), \mathcal{C}^\infty(X)) \rightarrow \text{Diff}_{\mathcal{C}^\infty(X^{p+1})}(\mathcal{C}^\infty(X^{p+1}), \mathcal{C}^\infty(X)))_{[p]}. \end{aligned}$$

For the brace operad  $\text{Br}$  of [Vor1], regarded as an operad in chain complexes,  $\mathcal{D}^{\text{poly}}(X)$  is then naturally a Br-algebra. In other words, it has a cup product in the form of a map

$$\mathcal{D}^{\text{poly}}(X) \otimes \mathcal{D}^{\text{poly}}(X) \xrightarrow{\smile} \mathcal{D}^{\text{poly}}(X),$$

of super cochain chain complexes, and braces in the form of maps

$$\{-\}\{-, \dots, -\}_r: \mathcal{D}^{\text{poly}}(X) \otimes \mathcal{D}^{\text{poly}}(X)^{\otimes r} \rightarrow \mathcal{D}^{\text{poly}}(X)_{[r]}$$

of super cochain complexes, satisfying the conditions of [Vor1, §3.2] with respect to the chain differential  $b$ . The commutator of the brace  $\{-\}\{-\}_1$  is a Lie bracket, so  $\mathcal{D}^{\text{poly}}(X)_{[-1]}$  is naturally a Lie algebra in super cochain chain complexes.

The following definitions are adapted from [Pri8], replacing Hochschild complexes with complexes of polydifferential operators:

**Definition 3.4.** Define the complex of quantised 0-shifted polyvector fields on  $X$  by

$$Q\widehat{\text{Pol}}(X, 0) := \prod_{p \geq 0} \widehat{\text{Tot}} \tau_p^{HH} \mathcal{D}^{\text{poly}}(X) \hbar^{p-1}.$$

Properties of the filtration  $\tau^{HH}$  as in [Pri8, Lemma 1.14] ensure that  $Q\widehat{\text{Pol}}(X, 0)^{[1]}$  is a super DGLA.

**Definition 3.5.** Define a decreasing filtration  $\tilde{\tau}_{HH}$  on  $Q\widehat{\text{Pol}}(X, 0)$  by the subcomplexes

$$\tilde{\tau}_{HH}^i Q\widehat{\text{Pol}}(X, 0) := \prod_{j \geq i} \tau_j^{HH} \text{CC}_R^\bullet(A) \hbar^{j-1}.$$

This filtration is complete and Hausdorff, with  $[\tilde{\tau}_{HH}^i, \tilde{\tau}_{HH}^j] \subset \tilde{\tau}_{HH}^{i+j-1}$ . In particular, this makes  $\tilde{\tau}_{HH}^2 Q\widehat{\text{Pol}}(X, 0)^{[1]}$  into a pro-nilpotent filtered super DGLA.

**Definition 3.6.** Define an  $E_1$  quantisation of  $X$  to be a Maurer–Cartan element

$$\Delta \in \text{MC}(\tilde{\tau}_{HH}^2 Q\widehat{\text{Pol}}(X, 0)^\equiv),$$

and define the space of  $E_1$  quantisations of  $X$  by

$$Q\mathcal{P}(X, 0) := \underline{\text{MC}}(\tilde{\tau}_{HH}^2 Q\widehat{\text{Pol}}(X, 0)^\equiv).$$

When  $X$  is just a dg supermanifold or a super NQ-manifold, this gives a curved  $A_\infty$ -algebra structure  $\mathcal{O}'_X$  on  $\mathcal{O}_X[[\hbar]]$  with  $\mathcal{O}'_X/\hbar = \mathcal{O}_X$ , because  $\hbar \mid \Delta$ ; for dg supermanifolds the curvature is necessarily 0 for degree reasons. For more general super dg NQ-manifolds, the stacky and derived structures interact in a non-trivial way for quantisations, and indeed for Poisson structures; a quantisation gives rise to a curved  $A_\infty$ -algebra structure on  $\widehat{\text{Tot}} \mathcal{O}_X[[\hbar]]$ , but each component of the  $A_\infty$  structure must be bounded below in the cochain direction.

**3.1.2. Quantisation of 0-shifted Poisson structures on dg manifolds.** We now explain how [Pri5] adapts to give quantisations of 0-shifted Poisson structures on dg manifolds. As in the Kontsevich–Tamarkin approach [Tam1, Kon1, Yek1, Yek2, VdB] to quantisation, we begin by making use of the  $E_2$ -algebra structure on polydifferential operators and formality of the  $E_2$ -operad. Where their quantisation for manifolds hinges on invariance of the Hochschild complex under affine transformations, an argument which will not adapt to dg manifolds, we instead exploit the observation that the Hochschild complex carries an anti-involution, and that such anti-involutive deformations of the complex of polyvectors are essentially unique.

**Theorem 3.7.** *Given a dg supermanifold  $X$ , the space  $Q\mathcal{P}(X, 0)$  of  $E_1$  quantisations of  $X$  is equivalent to the Maurer–Cartan space*

$$\underline{\text{MC}}((\hbar \mathcal{C}^\infty(X)_{[-1]})^\equiv \times \hbar T_{\mathcal{C}^\infty(X)}^\equiv \times \prod_{p \geq 2} \Lambda_{\mathcal{C}^\infty(X)}^p (T_{\mathcal{C}^\infty(X)}^\equiv)_{[p-1]}^\equiv \hbar^{p-1})[[\hbar]].$$

*In particular, there exists a quantisation for every Poisson structure*

$$\pi \in \mathcal{P}(X, 0) = \underline{\text{MC}}\left(\prod_{p \geq 2} \Lambda_{\mathcal{C}^\infty(X)}^p (T_{\mathcal{C}^\infty(X)}^\equiv)_{[p-1]}^\equiv \hbar^{p-1}\right),$$

*in the form of a curved  $A_\infty$ -deformation of  $\mathcal{O}_X$ .*

*Proof.* We adapt the proof of [Pri5, Theorem 2.10 and Corollary 2.12]. As in [Pri4, Remark 1.17], we first note that we may replace  $\tau^{HH}$  with a quasi-isomorphic filtration  $\gamma$  analogous to that in [Pri4, Definition 1.16]. Explicitly,  $\gamma_r \mathcal{D}iff_{\mathcal{C}^\infty(X^n)}(\mathcal{C}^\infty(X^n), \mathcal{C}^\infty(X)) \subset \mathcal{D}iff_{\mathcal{C}^\infty(X^n)}(\mathcal{C}^\infty(X^n), \mathcal{C}^\infty(X))$  consists of differential operators  $\theta$  which for all  $(r+1)$ -tuples  $(n_0, \dots, n_r)$  with  $n_i > 0$  and  $\sum n_i = n$  satisfy  $\sum_\sigma \pm \sigma(f) = 0$ , when  $\sigma$  runs over the set of  $(n_0, \dots, n_r)$ -shuffle permutations.

Adapting [Pri8, Lemma 1.15], following [Bra, §2.1],  $(\mathcal{D}^{\text{poly}}, \gamma)$  is then an almost commutative anti-involutive brace algebra in super chain complexes (in the sense of [Pri5, Definition 2.4]). As in [Pri5, Definition 2.9], for Levi decompositions  $w$  of the Grothendieck–Teichmüller group corresponding to even associators, we have an  $\infty$ -functor  $p_w$  giving an equivalence between almost commutative anti-involutive brace algebras and almost commutative anti-involutive  $P_2$ -algebras. The associated graded object  $\text{gr}^\gamma \mathcal{D}^{\text{poly}}$  is a graded super chain complex resembling symmetric powers of Harrison cohomology, which is thus quasi-isomorphic as a graded super chain  $P_2$ -algebra to

$$\text{Pol}(X, 0) := \bigoplus_i \text{Symm}_{\mathcal{C}^\infty(X)}^i((T_{\mathcal{C}^\infty(X)})_{[1]}).$$

(cf. [Pri8, Lemma 1.14] or the proof of [Pri5, Theorem 2.10]).

Now, the anti-involutive graded super chain  $P_2$ -algebra  $\text{Pol}(X, 0)$  satisfies the conditions of [Pri5, Theorem 1.18] (the same is not true for NQ-manifolds in general), giving a filtered  $L_\infty$ -quasi-isomorphism

$$(\mathcal{D}^{\text{poly}}(X), \tau^{HH}) \simeq (\widehat{\text{Pol}}(X, 0), \text{Fil}),$$

and the desired expressions follow by substitution.  $\square$

**3.1.3. Quantisation of 0-shifted symplectic structures on dg NQ-manifolds.** Although Theorem 3.7 does not give quantisations of 0-shifted Poisson structures on NQ-manifolds, the approach of [Pri8] adapts to show that non-degenerate 0-shifted Poisson structures do quantise. Whereas deformation quantisation for manifolds [DWL, Fed, Del, Kon2, Tam2] relies on the reduction locally to  $\mathbb{R}^n$ , this is not an option for NQ-manifolds, so [Pri8] develops a new approach to show that all non-degenerate Poisson structures can be quantised even if the Hochschild complex is not formal.

**Definition 3.8.** Define an involution  $Q\widehat{\text{Pol}}(X, 0) \xrightarrow{*} Q\widehat{\text{Pol}}(X, 0)$  by  $\Delta^*(\hbar) := i(\Delta)(-\hbar)$ , for the involution

$$i(f)(a_1, \dots, a_m) = -(-1)^{\sum_{i < j} \bar{a}_i \bar{a}_j} (-1)^{m(m+1)/2} f(a_m, \dots, a_1).$$

of  $\mathcal{D}^{\text{poly}}(X)$ .

By [Bra, §2.1], it follows that  $*$  is a morphism of super DGLAs, and we define the space  $Q\mathcal{P}(X, 0)^{sd} \subset Q\mathcal{P}(X, 0)$  of self-dual quantisations to be the fixed points of the involution  $*$ .

In particular, this means that when  $X$  is a supermanifold, elements of  $Q\mathcal{P}(X, 0)^{sd}$  can be represented by associative algebra deformations  $(\mathcal{O}_X[[\hbar]], \star_\hbar)$  of  $\mathcal{O}_X$ , with

$$a \star_{-\hbar} b = b \star_\hbar a.$$

**Theorem 3.9.** *For  $X$  a super dg NQ-manifold and any Levi decomposition  $w$  of the Grothendieck–Teichmüller group, there is a canonical weak equivalence*

$$Q\mathcal{P}(X, 0)^{\text{nondeg}, sd} \simeq \mathcal{P}(X, 0)^{\text{nondeg}} \times \underline{\text{MC}}(\hbar^2 \text{DR}(X)^{[1], =}[[\hbar^2]]).$$

In particular,  $w$  gives a canonical choice of self-dual quantisation for any non-degenerate 0-shifted Poisson structure on  $X$ , and the set of equivalence classes of all such quantisations is isomorphic to  $\hbar^2\mathrm{H}^2\mathrm{DR}(X)^\hbar$ .

*Proof.* The approach of [Pri8] adapts to this context, with this result being the analogue of [Pri8, Theorem 2.20].

The idea of the correspondence is similar to that of Theorem 2.16. Given a quantisation  $\Delta$ , we form a complex  $T_\Delta\widehat{Q\mathrm{Pol}}(X, 0)$  given by first taking  $\widehat{\hbar Q\mathrm{Pol}}(X, 0)$  then adding  $[\Delta, -]$  to the differential. Reducing modulo  $\hbar$  recovers the Poisson cohomology complex of the associated Poisson structure  $\pi$ , so we regard  $T_\Delta\widehat{Q\mathrm{Pol}}(X, 0)$  as the *quantised* Poisson cohomology complex.

We then find that  $\hbar^2\frac{\partial\Delta}{\partial\hbar} \in T_\Delta\widehat{Q\mathrm{Pol}}(X, 0)^{2,=}$  is closed, so defines a quantised Poisson cohomology class. The choice  $w$  of Levi decomposition turns the brace algebra  $T_\Delta\widehat{Q\mathrm{Pol}}(X, 0)$  into a  $P_2$ -algebra  $p_w T_\Delta\widehat{Q\mathrm{Pol}}(X, 0)$ , which in particular has a commutative product, and we then define a compatibility map

$$\begin{aligned} \mu_w(-, \Delta): \mathrm{DR}(X)^\hbar &\rightarrow p_w T_\Delta\widehat{Q\mathrm{Pol}}(X, 0) \\ \mathrm{ad}f_1 \wedge \dots \wedge \mathrm{d}f_p &\mapsto a[\Delta, f_1] \dots [\Delta, f_p]. \end{aligned}$$

When  $\pi$  is non-degenerate,  $\mu_w(-, \Delta)$  is a quasi-isomorphism, so to  $\Delta$  we may associate a power series

$$\mu_w(\Delta, -)^{-1}(\hbar^2\frac{\partial\Delta}{\partial\hbar}) \in \mathrm{H}^2\mathrm{Fil}^2\mathrm{DR}(X)^\hbar \times \hbar\mathrm{H}^2\mathrm{DR}(X)^\hbar.$$

We then attempt to solve the equation  $\mu_w(\omega, \Delta) \simeq \hbar^2\frac{\partial\Delta}{\partial\hbar}$  up to coherent homotopy for a given  $\omega \in \mathrm{H}^2\mathrm{Fil}^2\mathrm{DR}(X)^\hbar \times \hbar\mathrm{H}^2\mathrm{DR}(X)^\hbar$ , expressing  $\Delta$  as a power series in  $\hbar$  and solving for the coefficients inductively.

The leading term is given by the correspondence between non-degenerate Poisson structures and symplectic structures in Theorem 2.21. For higher terms, we filter by powers of  $\hbar$ , and use the obstruction theory associated to filtered DGLAs. Calculation as in [Pri8, Proposition 2.17] shows that the only potential obstruction or ambiguity is in the first-order deformation of the Poisson structure, but as in [Pri8, Lemma 1.35], this vanishes when we restrict to self-dual quantisations, showing that they are parametrised by

$$\mathrm{H}^2\mathrm{Fil}^2\mathrm{DR}(X)^\hbar \times \hbar^2\mathrm{H}^2\mathrm{DR}(X)^\hbar. \quad \square$$

*Remark 3.10.* In [Pri5, Corollary 3.26], existence of deformation quantisations is established for all 0-shifted Poisson structures on derived Artin stacks. As in [Pri5, Remark 3.29], the proof should adapt to the  $\mathcal{C}^\infty$  setting to extend Theorem 3.7 to give quantisations for 0-shifted Poisson structures on super dg NQ-manifolds. The proof would proceed by constructing Hochschild complexes in an appropriate operad of stacky  $\mathcal{C}^\infty$ -differential operators analogous to that in [Pri5, §3].

**3.2. Quantisation of  $(-1)$ -shifted symplectic structures on dg NQ-manifolds.** Fix a super dg NQ-manifold  $X$ .

**3.2.1. Formulation of  $(-1)$ -shifted quantisations.** The following definitions are adapted from [Pri9, Definitions 3.9, 1.10, 1.11 and 1.18]:



**Definition 3.11.** Define a strict line bundle over  $X$  to be a  $\mathcal{C}^\infty(X)$ -module  $M$  in super chain cochain complexes such that  $M_\#^\#$  is a projective module of rank 1 over the super bigraded-commutative algebra  $\mathcal{C}^\infty(X)_\#^\#$  underlying  $\mathcal{C}^\infty(X)$ .

What we are calling a line bundle should really be thought of as the module of global sections of a line bundle. For each such  $M$ , there is an associated sheaf  $M \otimes_{\mathcal{C}^\infty(X)} \mathcal{O}_X$  of sections.

**Definition 3.12.** Given a strict line bundle  $M$  over  $X$ , define the complex of quantised  $(-1)$ -shifted polyvector fields on  $M$  by

$$Q\widehat{\text{Pol}}(M, -1) := \prod_{p \geq 0} \widehat{\text{Tot}} F_p \mathcal{D}\text{iff}_{\mathcal{C}^\infty(X)}(M, M) \hbar^{p-1},$$

for differential operators and the order filtration from Definition 3.1.

Multiplication of differential operators gives us a product

$$Q\widehat{\text{Pol}}(M, -1) \times Q\widehat{\text{Pol}}(M, -1) \rightarrow \hbar^{-1} Q\widehat{\text{Pol}}(M, -1),$$

but because  $M$  is a line bundle, the associated commutator  $[-, -]$  takes values in  $Q\widehat{\text{Pol}}(M, -1)$ , so  $Q\widehat{\text{Pol}}(M, -1)$  is a super DGLA.

Define a decreasing filtration  $\tilde{F}$  on  $Q\widehat{\text{Pol}}(M, -1)$  by

$$\tilde{F}^i Q\widehat{\text{Pol}}(M, -1) := \prod_{j \geq i} F_j \mathcal{D}\text{iff}_{\mathcal{C}^\infty(X)}(M, M) \hbar^{j-1};$$

this has the properties that  $Q\widehat{\text{Pol}}(M, -1) = \varprojlim_i \widehat{\text{Pol}}(M, -1) / \tilde{F}^i$ , with  $[\tilde{F}^i, \tilde{F}^j] \subset \tilde{F}^{i+j-1}$ ,  $\delta \tilde{F}^i \subset \tilde{F}^i$ , and  $\tilde{F}^i \tilde{F}^j \subset \hbar^{-1} \tilde{F}^{i+j}$ .

**Definition 3.13.** Define the space  $QP(M, -1)$  of  $E_0$  quantisations of a strict line bundle  $M$  on  $X$  to be given by the simplicial set

$$QP(M, -1) := \varprojlim_i \underline{\text{MC}}(\tilde{F}^2 Q\widehat{\text{Pol}}(M, -1) / \tilde{F}^{i+2}).$$

Thus an  $E_0$  quantisation is a deformation of  $M$  given by differential operators, with some constraints on their orders. As in [Pri9, Remark 1.14],  $E_0$  quantisations  $\Delta$  of  $\mathcal{C}^\infty(X)$  with  $\Delta(1) = 0$ , give rise to commutative  $BV_\infty$ -algebras in the sense of [Kra, Definition 9].

**3.2.2. Quantisation for spin structures.** The module  $\mathcal{C}^\infty(X)$  naturally has the structure of a left  $\mathcal{D}_{\mathcal{C}^\infty(X)}$ -module (via the embedding of  $\mathcal{D}_{\mathcal{C}^\infty(X)}$  in  $\underline{\text{Hom}}_{\mathbb{R}}(\mathcal{C}^\infty(X), \mathcal{C}^\infty(X))$ ); the same is true for any vector bundle equipped with a flat connection. Right  $\mathcal{D}$ -modules are more subtle to construct, but on a super NQ-manifold  $X$ , the orientation bundle (i.e. the determinant, or Berezinian, of  $\Omega_{\mathcal{C}^\infty(X)}^1$ ) is naturally a right  $\mathcal{D}_{\mathcal{C}^\infty(X)}$ -module, via the identification

$$\det \Omega_{\mathcal{C}^\infty(X)}^1 \simeq (\text{Ext}_{\mathcal{D}_{\mathcal{C}^\infty(X)}^\#}^p(\mathcal{C}^\infty(X)^\#, \mathcal{D}_{\mathcal{C}^\infty(X)}^\#), Q)$$

(i.e. turn off the differential  $Q$ , calculate Ext, then restore  $Q$ ), where  $p$  is the number of local generators of  $\mathcal{O}_X$  of even parity; this identification follows along the same lines as the construction of the Berezinian in [DM].

Similarly,  $(\text{Ext}_{\mathcal{D}_{C^\infty(X),\#}^\#}^p(C^\infty(X)^\#, \mathcal{D}_{C^\infty(X),\#}^\#), Q, \delta)$  gives a right  $\mathcal{D}$ -module when  $X$  is a super dg NQ-manifold, but the expression is not usually invariant under the equivalences of Remarks 1.13. It does, however, behave when the only even parity generators are in chain degree 0, in which case it broadly corresponds to the dualising line bundle of [Gai, §5.6].

Now, if  $\omega$  is a strict line bundle with a right  $\mathcal{D}$ -module structure, there is a standard isomorphism

$$\mathcal{D}_{C^\infty(X)}^{\text{opp}} \cong \text{Diff}_{C^\infty(X)}(\omega, \omega)$$

of super chain cochain associative algebras, following from the observation that the elements of  $\mathcal{D}_{C^\infty(X)}^{\text{opp}}$  acting on  $\omega$  on the right must act as differential operators. Moreover, a right  $\mathcal{D}$ -module structure on any vector bundle  $M$  then corresponds to a flat connection on  $M \otimes_{C^\infty(X)} \omega^*$ .

We now proceed as in [Pri9, §4]:

If  $L$  is a strict line bundle with a right  $\mathcal{D}$ -module structure on  $L^{\otimes 2}$  (so  $L$  broadly corresponds to a spin structure), we then have

$$\begin{aligned} \text{Diff}_{C^\infty(X)}(L, L)^{\text{opp}} &\cong (L \otimes \mathcal{D}_{C^\infty} \otimes L^*)^{\text{opp}} \\ &\cong L^* \otimes \mathcal{D}_{C^\infty}^{\text{opp}} \otimes L \\ &\cong L^* \otimes L^{\otimes 2} \otimes \mathcal{D}_{C^\infty} \otimes (L^*)^{\otimes 2} \otimes L \\ &\cong \text{Diff}_{C^\infty(X)}(L, L). \end{aligned}$$

**Definition 3.14.** For a line bundle  $L$  with a right  $\mathcal{D}$ -module structure on  $L^{\otimes 2}$ , writing

$$(-)^t: \text{Diff}_{C^\infty(X)}(L, L)^{\text{opp}} \rightarrow \text{Diff}_{C^\infty(X)}(L, L)$$

for the natural anti-involution above, define

$$(-)^*: Q\widehat{\text{Pol}}(L, -1) \rightarrow Q\widehat{\text{Pol}}(L, -1)$$

by

$$\Delta^*(\hbar) := -\Delta^t(-\hbar).$$

We then define  $Q\widehat{\text{Pol}}(L, -1)^{sd}$  to be the fixed points for the involution  $*$ , and set

$$Q\mathcal{P}(L, -1)^{sd} := \varprojlim_i \underline{\text{MC}}(\tilde{F}^2 Q\widehat{\text{Pol}}(L, -1)^{sd,=} / \tilde{F}^{i+2})$$

The reason for the choice of sign  $-\hbar$  in the definition of  $\Delta^*$  is that on the associated graded  $\text{gr}_p^F \mathcal{D}_X(\mathcal{E}) \cong \text{Sym}^p \mathcal{T}_X$ , the operation  $(-)^t$  is given by  $(-1)^p$ . Thus the underlying Poisson structures satisfy  $\pi_{\Delta^*} = \pi_\Delta$ .

**Theorem 3.15.** *For a super dg NQ-manifold  $X$  and a line bundle  $L$  on  $X$  with  $L^{\otimes 2}$  a right  $\mathcal{D}$ -module (such as any square root of the orientation bundle), there is a canonical weak equivalence*

$$Q\mathcal{P}(L, -1)^{\text{nondeg}, sd} \simeq \mathcal{P}(L, -1)^{\text{nondeg}} \times \underline{\text{MC}}(\hbar^2 \text{DR}(X/R)[[\hbar^2]]).$$

*In particular, every non-degenerate  $(-1)$ -shifted Poisson structure gives a canonical choice of self-dual quantisation of  $L$ .*

*Proof.* The main results of [Pri9] combine and adapt to give this statement. The key is to modify the argument from Theorems 2.16 and 3.9 via a compatibility map defined on a variant of the de Rham complex. As in [Pri9, Definition 1.26], the first step is to

let  $\mathcal{C}^\infty(X^n)^\wedge$  be the completion of  $\mathcal{C}^\infty(X^n)$  with respect to the ideal of the diagonal map  $\mathcal{C}^\infty(X^n) \rightarrow \mathcal{C}^\infty(X)$ .

We then let  $\mathcal{C}^\infty(X^{\bullet+1})^\wedge$  be the total super cochain complex of the super double cochain complex

$$\mathcal{C}^\infty(X)^\wedge \xrightarrow{d} \mathcal{C}^\infty(X^2)^\wedge \xrightarrow{d} \mathcal{C}^\infty(X^3)^\wedge \xrightarrow{d} \dots,$$

with boundary map  $d: \mathcal{C}^\infty(X^{m+1})^\wedge \rightarrow \mathcal{C}^\infty(X^{m+2})^\wedge$  given by

$$df(x^0, \dots, x^{m+1}) = \sum_{i=0}^{m+1} (-1)^i f(x^0, \dots, x^{i-1}, x^{i+1}, \dots, x^{m+1}).$$

Then  $\mathcal{C}^\infty(X^{\bullet+1})^\wedge$  has an associative product given by the Alexander–Whitney cup product

$$(f \smile g)(x^0, \dots, x^{m+n+1}) = f(x^0, \dots, x^m)g(x^m, \dots, x^{m+n}).$$

The next step is to set  $\mathrm{DR}'(X) \subset \mathcal{C}^\infty(X^{\bullet+1})^\wedge$  be the super cochain subcomplex given by cosimplicial conormalisation, so we only consider functions in  $\mathcal{C}^\infty(X^{m+1})^\wedge$  which vanish on all big diagonals  $X^m \subset X^{m+1}$ . As in [Pri9, Lemma 1.27], there is a natural quasi-isomorphism  $\mathrm{DR}'(X) \rightarrow \mathrm{DR}(X)$ .

For any  $\Delta \in Q\mathcal{P}(L, -1)$ , we write  $T_\Delta Q\widehat{\mathrm{Pol}}(L, -1)$  for the complex given by first taking  $\hbar Q\widehat{\mathrm{Pol}}(L, -1)$  then adding  $[\Delta, -]$  to the differential. We regard its cohomology as quantised  $(-1)$ -shifted Poisson cohomology, and it contains a canonical 1-cocycle  $\hbar^2 \frac{\partial \Delta}{\partial \hbar}$ . As in [Pri9, Lemmas 1.31 and 1.32], any  $\Delta \in Q\mathcal{P}(L, -1)$  gives rise to a compatibility map

$$\mu(-, \Delta): \mathrm{DR}'(X)[[\hbar]] \rightarrow T_\Delta Q\widehat{\mathrm{Pol}}(L, -1),$$

induced by the continuous multiplicative map on  $\mathcal{C}^\infty(X^{\bullet+1})^\wedge$  determined by the properties that  $\mu(1 \otimes 1, \Delta) = \Delta$  and  $\mu(a, \Delta) = a$  for  $a \in \mathcal{C}^\infty(X)$ . When  $\Delta$  is non-degenerate, this map is a quasi-isomorphism, so to  $\Delta$  we may associate the power series

$$[\mu(\Delta, -)^{-1} \hbar^2 \frac{\partial \Delta}{\partial \hbar}] \in \mathrm{H}^1 \mathrm{Fil}^2 \mathrm{DR}(X) \times \hbar \mathrm{H}^1 \mathrm{DR}(X)[[\hbar]].$$

Proposition 4.6 gives vanishing for self-dual.

If we start from a power series in  $\mathrm{H}^1 \mathrm{DR}(X)$  and attempt to solve for  $\Delta$ , then the leading term is given by the correspondence between non-degenerate Poisson structures and symplectic structures in Theorem 2.21. For higher terms, we filter powers of  $\hbar$ , and use the obstruction theory associated to filtered DGLAs. Calculation as in [Pri9, Proposition 1.41] shows that the only potential obstruction or ambiguity is in the first-order deformation of the Poisson structure, but as in [Pri9, Lemma 4.6], this vanishes when we restrict to self-dual quantisations, showing that the latter are parametrised by

$$\mathrm{H}^1 \mathrm{Fil}^2 \mathrm{DR}(X)^\simeq \times \hbar^2 \mathrm{H}^1 \mathrm{DR}(X)^\simeq [[\hbar^2]].$$

□

*Example 3.16.* For  $M$  a manifold of dimension  $p$ , Examples 1.14 give a canonical  $(-1)$ -shifted symplectic structure  $\omega$  on the derived critical locus  $X = \mathrm{DCrit}(M, f)$ , and pulling back the determinant bundle  $\Omega_M^p$  to  $X$  gives a line bundle  $L$  satisfying the conditions

of Theorem 3.15. A natural self-dual quantisation of

$$\begin{aligned} L &= \Omega_{\mathcal{C}^\infty(M)}^p \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(\mathrm{DCrit}(M, f)) \\ &= \Omega_{\mathcal{C}^\infty(M)}^p \otimes_{\mathcal{C}^\infty(M)} \left( \bigoplus_i (\Lambda_{\mathcal{C}^\infty(M)}^i T_{\mathcal{C}^\infty(M)}[-i], \lrcorner df) \right) \\ &\cong \left( \bigoplus_j \Omega_{\mathcal{C}^\infty(M), [j]}^j, df \wedge - \right)_{[-p]} \end{aligned}$$

over this symplectic structure is then given by the twisted de Rham complex

$$\left( \bigoplus_j \Omega_{\mathcal{C}^\infty(M), [j]}^j \llbracket \hbar \rrbracket, \hbar d + df \wedge - \right)_{[-p]},$$

and as in [Pri9, Lemma 4.8], this is the quantisation associated by Theorem 3.15 to the constant power series  $\omega$ .

A volume form  $\mu$  on  $M$  is the same as a choice of isomorphism  $\Omega_M^p \cong \mathcal{O}_M$ . This leads to an isomorphism  $L \cong \mathcal{O}_X$ , and the quantisation above then becomes a quantisation of the trivial line bundle on  $X$ . This quantisation is precisely the quantum BV complex as described on [the nlab](#).

**3.3. Quantisation of shifted Lagrangians.** In the smooth setting, there is a natural analogue of the shifted Lagrangians of [PTVV]:

**Definition 3.17.** Given an  $n$ -shifted pre-symplectic structure  $\omega$

$$\omega \in Z^{n+2} \mathrm{Fil}^2 \mathrm{DR}(X)^\sphericalangle,$$

on a super dg NQ-manifold  $X$ , and a morphism  $Z \rightarrow X$  of super dg NQ-manifolds, an isotropic structure on  $Z$  relative to  $\omega$  is an element  $(\omega, \lambda)$  of

$$Z^{n+2} \mathrm{cocone}(\mathrm{Fil}^2 \mathrm{DR}(X) \rightarrow \mathrm{Fil}^2 \mathrm{DR}(Z))^\sphericalangle$$

lifting  $\omega$ . This structure is called Lagrangian if  $\omega$  is symplectic and if contraction of  $T_{\mathcal{C}^\infty(Z)}$  with the image  $\bar{\lambda}_2$  of  $\lambda$  in  $Z^{n-1}(\mathrm{Tot}^\Pi \Omega_{\mathcal{C}^\infty(Z)}^2)^\sphericalangle$  induces a quasi-isomorphism

$$\begin{aligned} \lambda_2^\sharp: \hat{\mathrm{Tot}} \mathrm{cone}(T_{\mathcal{C}^\infty(Z)} \otimes_{\mathcal{C}^\infty(Z)} \mathcal{C}^\infty(Z)^0 \rightarrow T_{\mathcal{C}^\infty(X)} \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(Z)^0) \\ \rightarrow \mathrm{Tot}^\Pi(\Omega_{\mathcal{C}^\infty(Z)}^1 \otimes_{\mathcal{C}^\infty(Z)} \mathcal{C}^\infty(Z)^0)^{[n]}. \end{aligned}$$

We define  $n$ -shifted structures of reversed parity similarly, replacing  $=$  with  $\neq$  and applying the parity reversion operator  $\Pi$  to the target of  $\lambda_2^\sharp$ .

In particular, this means that Lagrangian submanifolds of symplectic manifolds are Lagrangians with respect to 0-shifted symplectic structures, but note that the morphism  $Z \rightarrow X$  in Definition 3.17 need not be in any sense injective. For many examples of  $n$ -shifted Lagrangians on NQ-manifolds, see [PS]; the prototypical example is given by the embedding of a supermanifold  $M$  in its  $n$ -shifted cotangent bundle  $T^*M[n]$ .

**3.3.1. Deformation quantisations.** Coisotropic structures are harder to describe than Poisson structures, because they rely on the notion of a  $P_{k+1}$ -algebra acting on a  $P_k$ -algebra. In the algebraic setting they are formulated in [MS1], and those results should translate to the smooth setting.

On the other hand, quantisations of an  $n$ -shifted coisotropic structures on  $Z \rightarrow X$  can be understood for  $n > 0$  as an  $E_{n+1}$ -algebra deformation of  $\mathcal{O}_X$  acting on an  $E_n$ -algebra deformation of  $\mathcal{O}_Z$ . This action takes the form of an  $E_{n+1}$ -algebra morphism to the  $E_n$ -Hochschild complex, which is naturally an  $E_{n+1}$ -algebra. For instance, a quantisation

of a 1-shifted coisotropic structure on  $Z \rightarrow X$  could be formulated as a suitable brace algebra deformation of  $\mathcal{O}_X$  equipped with a brace algebra map to the brace algebra of polydifferential operators on an associative algebra deformation of  $\mathcal{O}_Z$ . For  $n > 1$ , [MS2] show that quantisation of coisotropic structures follows from formality of the  $E_n$  operad and an unpublished result of Rozenblyum. Again, those results should translate to the smooth setting.

We now focus on the case  $n = 0$ , in which case [Pri4] (or in the classical setting [BGKP]) defines a quantisation of  $Z \rightarrow X$  to be given by an  $E_1$ -quantisation of  $(X, \mathcal{O}_X)$  in an analogous sense to Definition 3.6 (so for us an associative deformation  $\tilde{\mathcal{O}}_X$  of  $\mathcal{O}_X$  given by polydifferential operators), an  $E_0$ -quantisation of  $(Z, \mathcal{O}_Z)$  in the sense of Definition 3.13 (so a deformation of the sheaf  $\tilde{\mathcal{O}}_Z$  given by differential operators) and a suitable  $\tilde{\mathcal{O}}_X$ -module structure on  $\tilde{\mathcal{O}}_Z$ .

Since the precise definitions of quantised polyvectors and quantisations [Pri4, Definitions 2.10 and 2.13] are quite involved, we omit them here. There is again a notion of self-dual quantisation, combining those of §§3.1.3, 3.2, and [Pri4, Theorem 4.20] will adapt to the smooth setting, replacing Hochschild complexes with polydifferential operators, to give:

**Theorem 3.18.** *Take a morphism  $Z \rightarrow X$  of super dg NQ-manifolds, and a strict line bundle  $M$  on  $Z$  with a right  $\mathcal{D}$ -module structure on  $M^{\otimes 2}$ . Then for any Lagrangian structure  $(Z, \lambda)$  over a 0-shifted symplectic structure  $(X, \omega)$ , a Levi decomposition  $w$  for the Grothendieck–Teichmüller group corresponding to an even associator gives a parametrisation of self-dual quantisations of  $(Z, M, \lambda) \rightarrow (X, \omega)$  by the group*

$$\hbar^2 H^1 \text{cone}(\text{DR}(X) \rightarrow \text{DR}(Z))[[\hbar^2]].$$

*In particular,  $w$  associates a canonical choice of self-dual quantisation of  $(Z, M)$  to every Lagrangian structure.*

*Remark 3.19.* The characterisation of quantised Lagrangians on  $X$  as modules over an associative deformation  $\tilde{\mathcal{O}}_X$  of a 0-shifted symplectic structure gives rise, for each choice of  $\tilde{\mathcal{O}}_X$  to a dg category whose objects are quantised Lagrangians with spin structures, cf. [Pri4, §5]. As in [Pri4, Lemma 5.13], when a Lagrangian  $(Z, \lambda)$  is compact, the complex of morphisms from  $(Z, \lambda, L)$  to another Lagrangian  $(Z', \lambda', L')$  with spin structure is given by a quantisation of a line bundle on the derived Lagrangian intersection  $Z \times_X^h Z'$ , which necessarily carries a  $(-1)$ -shifted symplectic structure  $\lambda' - \lambda$ . As in Examples 3.16, this tends to mean that the complex of morphisms is a twisted de Rham complex, so as discussed in [BBD<sup>+</sup>, Remark 6.15] and [Sch, §3.3], the resulting category resembles a Fukaya category, but is defined algebraically and includes objects corresponding to exotic Lagrangians.

**3.4. Quantisation of  $(-2)$ -shifted symplectic structures on dg NQ-manifolds.** Fix a super dg NQ-manifold  $X$ .

**3.4.1. Formulation of  $(-2)$ -shifted quantisations.** The following definitions are adapted from [Pri7, Definitions 1.6, 1.8, 1.11 and 1.18]:

**Definition 3.20.** We define a homotopy right  $\mathcal{D}$ -module structure (or flat right connection) on  $\mathcal{C}^\infty(X)$  to be a sequence of maps  $\nabla_{p+1} \in \text{Tot } \underline{\text{Hom}}_{\mathbb{R}}(\Lambda^p T_{\mathcal{C}^\infty(X)}, \mathcal{C}^\infty(X))^{1-p,=}$  for  $p \geq 1$ , satisfying the following conditions:

- (1) For  $a \in \mathcal{C}^\infty(X)$  and  $\xi \in T_{\mathcal{C}^\infty(X)}$ , we have  $\nabla_2(a\xi) = a\nabla_2(\xi) - \xi(da)$ ;

- (2) For  $p \geq 2$ , the maps  $\nabla_{p+1}$  are  $\mathcal{C}^\infty(X)$ -linear;  
(3) The operations  $(\nabla_2 - \text{id}, \nabla_3, \nabla_4, \dots)$  define an  $L_\infty$ -morphism from the DGLA  $\widehat{\text{Tot}} T_{\mathcal{C}^\infty(X)}$  to the DGLA  $\widehat{\text{Tot}}(\mathcal{C}^\infty(X) \oplus T_{\mathcal{C}^\infty(X)}^{\text{opp}}) = \widehat{\text{Tot}} F_1 \mathcal{D}_{\mathcal{C}^\infty(X)}^{\text{opp}}$  of first-order differential operators with bracket given by negating the commutator.

**Definition 3.21.** Given a flat right connection  $\nabla$  on  $\mathcal{C}^\infty(X)$ , we define the right de Rham complex  $\text{DR}^r(X, \nabla)$  associated to  $\nabla$ , and its increasing filtration  $F$ , by

$$F_i \text{DR}^r(X, \nabla) := \bigoplus_{p \leq i} \widehat{\text{Tot}} \Lambda^p T_{\mathcal{C}^\infty(X)}[p],$$

equipped with differential  $D^\nabla = \sum_{k \geq 1} D_k^\nabla$  as follows. Define  $D_k^\nabla: \Lambda^p T_{\mathcal{C}^\infty(X)} \rightarrow \Lambda^{p+1-k} T_{\mathcal{C}^\infty(X)}$  by setting (for  $\omega \in \Omega_A^{p+1-k}$ )

$$D_k^\nabla(\pi \lrcorner \omega) := \begin{cases} \nabla_k(\pi \lrcorner \omega) & k > 2, \\ \nabla_2(\pi \lrcorner \omega) + (-1)^{\deg \pi} \pi \lrcorner d\omega & k = 2, \\ \delta(\pi \lrcorner \omega) \pm \partial(\pi \lrcorner \omega) & k = 1, \end{cases}$$

where  $d$  is the de Rham differential and  $\delta, \partial$  are induced by the differentials  $\delta, \partial$  on  $\mathcal{C}^\infty(X)$ .

**Definition 3.22.** Given a flat right connection  $\nabla$  on  $\mathcal{C}^\infty(X)$ , define the complex of quantised  $(-2)$ -shifted polyvector fields on  $X$  by

$$Q\widehat{\text{Pol}}(X, \nabla, -2) := \prod_j \hbar^{j-1} F_j \text{DR}^r(X, \nabla).$$

Define a decreasing filtration  $\tilde{F}$  on  $Q\widehat{\text{Pol}}(X, \nabla, -2)$  by

$$\tilde{F}^i Q\widehat{\text{Pol}}(X, \nabla, -2) := \prod_{j \geq i} \hbar^{j-1} F_j \text{DR}^r(X, \nabla).$$

It follows as in [Pri7, Lemma 1.11] (following [Kra, Vit]) that  $\text{DR}^r(X, \nabla)$  is a form of filtered  $BV_\infty$ -algebra. This induces an  $L_\infty$ -algebra structure on  $\text{DR}^r(X, \nabla)[-1]$  with brackets

$$[a_1, \dots, a_k]_{\nabla, k} := [\dots [\nabla, a_1], \dots, a_k](1),$$

which extends naturally to an  $\mathbb{R}[[\hbar]]$ -linear  $L_\infty$ -algebra structure on  $Q\widehat{\text{Pol}}(X, \nabla, -2)[-1]$ .

**Definition 3.23.** Define the space  $Q\mathcal{P}(X, \nabla, -2)$  of  $E_{-1}$  quantisations of  $(X, \nabla)$  to be given by the simplicial set

$$Q\mathcal{P}(X, \nabla, -2) := \varprojlim_i \underline{\text{MC}}(\tilde{F}^2 Q\widehat{\text{Pol}}(X, \nabla, -2)^{[-1]} / \tilde{F}^{i+2}).$$

**3.4.2. Quantisation for flat right connections.** Note that  $Q\widehat{\text{Pol}}(X, \nabla, -2)/\hbar \cong \widehat{\text{Pol}}(X, -2)$ , giving a map  $Q\mathcal{P}(X, \nabla, -2) \rightarrow \mathcal{P}(X, -2)$ . We regard the fibres of this map over a shifted Poisson structure  $\pi$  as quantisations of  $\pi$ . When  $\pi$  is non-degenerate, we also refer to the fibre as the space of deformation quantisations of the corresponding shifted symplectic structure relative to  $\nabla$ .

Note that the de Rham complex featuring is a left de Rham complex, as in Definition 1.30:

**Theorem 3.24.** *Given a  $(-2)$ -shifted symplectic structure  $\omega$  on  $X$ , the space of pairs  $(\nabla, S)$ , where  $\nabla$  is a flat right connection on  $C^\infty(X)$  and  $S$  is a deformation quantisation of  $\omega$  relative to  $\nabla$ , is either empty or equivalent to*

$$\hbar^2 H^0(\pi^0 X_0^-, \mathbb{R})[[\hbar]],$$

*depending on whether any flat right connections on  $C^\infty(X)$  exist; the potential obstruction lies in  $H^2(F^1 \mathrm{DR}(X))$ .*

*In particular, this shows that when  $\pi^0 X_0^-$  is connected and flat right connections exist, pairs  $(\nabla, S)$  are essentially unique up to addition by  $(0, \hbar^2 \mathbb{R}[[\hbar]])$ .*

*Proof.* Combining [Pri7, Propositions 1.37, 1.45, 2.1], adapted to our setting along similar lines to §3.2, gives the obstruction and shows that the space is equivalent to  $\pi_j \prod_{i \geq 2} \underline{\mathrm{MC}}(\hbar^i \mathrm{DR}(X)^{[-1]})$ .

Homotopy groups of this space are given by  $\pi_j \prod_{i \geq 2} \underline{\mathrm{MC}}(\hbar^i \mathrm{DR}(X)^{[-1]}) \cong \hbar^2 H^{-j} \mathrm{DR}(X)[[\hbar]]$ . Since we are willing to forget the filtration, we may compute  $\mathrm{DR}(X)$  by completing  $\mathrm{DR}(X^0)$  along  $\pi^0 X$  (cf. [FT, Bha]). Thus  $\mathrm{DR}(X)$  has no negative cohomology groups, and our space of pairs  $(\nabla, S)$  is discrete. Moreover, the proof of [Har, Theorem IV.1.1] generalises from the holomorphic setting to the smooth setting to show that for a super dg manifold  $Y$ , we have  $\mathrm{DR}(Y) \simeq \mathbf{R}\Gamma(\pi^0 Y^-, \mathbb{R})$ , so  $H^0 \mathrm{DR}(X) \cong H^0(\pi^0 X_0^-, \mathbb{R})$ .  $\square$

**3.4.3. Virtual fundamental classes.** By [BL, Theorem 3.7], there is an  $L_\infty$ -isomorphism from  $\mathrm{DR}^r(X, \nabla)[-1]$  with the  $L_\infty$ -structure  $[-]_\nabla$  to the complex  $\mathrm{DR}^r(X, \nabla)[-1]$  with abelian  $L_\infty$  structure. In particular, for  $S \in \mathrm{DR}^r(A, \nabla)^0$ , [BL, Remark 3.6] shows that the expression  $\sum_n [S, \dots, S]_{n, \nabla} / n!$  can be rewritten as  $e^{-S} D_S^\nabla(e^S)$ , so the Maurer–Cartan equation  $\sum_n [S, \dots, S]_{n, \nabla} / n! = 0$  is equivalent to the quantum master equation  $D^\nabla(e^S) = 0$ .

Our complex  $Q\widehat{\mathrm{Pol}}(X, \nabla, -2)$  is not itself a  $BV_\infty$ -algebra, but it is an  $L_\infty$ -subalgebra of  $(\hbar(\mathrm{DR}^r(X, \nabla)[\hbar]/\hbar^r)[-1]; [-]_\nabla)$ . Therefore sending  $S$  to  $e^S - 1$  gives natural maps

$$\pi_i Q\mathcal{P}(X, \nabla, -2) \rightarrow \hbar H^{-i}(\mathrm{DR}^r(X, \nabla))[[\hbar]]$$

from quantisations to power series in right de Rham cohomology. Note that these are not isomorphisms, since quantisations have additional restrictions in terms of the filtration  $F$ , which in particular allow them to recover Poisson structures.

In the algebraic setting, [Pri7, §3] then interprets these as classes in Borel–Moore homology. However, that approach does not adapt to the smooth setting. We can still reduce to a right de Rham complex of a dg manifold generated in chain degrees  $[0, 1]$  by the methods of [Pri7, §3.1.2], but this no longer has an interpretation as Borel–Moore homology, essentially because for co-ordinates  $y_j$  in degree 1, the right de Rham complex involves polynomials, rather than smooth functions, in the vectors  $\partial_{y_j}$ .

## 4. FUNCTORIALITY, DERIVED AND HIGHER LIE GROUPOIDS

**4.1. Étale functoriality for dg NQ-manifolds.** Any morphism  $X \rightarrow Y$  of super dg NQ-manifolds gives rise to a filtered morphism  $\mathrm{DR}(Y) \rightarrow \mathrm{DR}(X)$  of de Rham complexes, so functoriality for shifted symplectic structures and Lagrangians is straightforward. Shifted Poisson structures and quantisations are functorial with respect to a generalisation of local diffeomorphisms, but as in [Pri6, §§2.1.2, 3.4], this is subtle to formulate.

Given a morphism  $f: X \rightarrow Y$ , we can classify  $n$ -shifted Poisson structures on  $X$  and  $Y$  which are strictly compatible with  $f$  by replacing the space  $\mathrm{Symm}_{\mathcal{C}^\infty(X)}^j(T_{\mathcal{C}^\infty(X)}^{[-n-1]})$  of polyvectors in Definition 2.13 with the fibre product given by the limit of the diagram

$$\begin{array}{ccc} \mathrm{Symm}_{\mathcal{C}^\infty(X)}^j(T_{\mathcal{C}^\infty(X)}^{[-n-1]}) & \longrightarrow & \mathcal{C}^\infty(X) \otimes_{\mathcal{C}^\infty(Y)} \mathrm{Symm}_{\mathcal{C}^\infty(Y)}^j(T_{\mathcal{C}^\infty(Y)}^{[-n-1]}). \\ \mathrm{Symm}_{\mathcal{C}^\infty(Y)}^j(T_{\mathcal{C}^\infty(Y)}^{[-n-1]}) & \longrightarrow & \end{array}$$

This fibre product only behaves well when it is a homotopy fibre product, or equivalently when one of the maps in the diagram is surjective. There are two main ways this can occur: either because the map  $\mathcal{C}^\infty(Y) \rightarrow \mathcal{C}^\infty(X)$  is surjective, or because the map  $T_{\mathcal{C}^\infty(X)} \rightarrow \mathcal{C}^\infty(X) \otimes_{\mathcal{C}^\infty(Y)} T_{\mathcal{C}^\infty(Y)}$  is surjective. The former is the approach taken in [Pri6], but we take the latter to avoid having to use rings which are  $\mathbb{Z}$ -graded in the chain direction.

**Definition 4.1.** Say that a morphism  $f: X \rightarrow Y$  of super dg NQ-manifolds is a quasi-submersion if  $X_0^{0,=} \rightarrow Y_0^{0,=}$  is a submersion and  $\mathcal{C}^\infty(X)_{\#}^{\#}$  is locally freely generated over  $\mathcal{C}^\infty(Y)_{\#}^{\#} \otimes_{\mathcal{C}^\infty(Y)_0^{0,=}} \mathcal{C}^\infty(X)_0^{0,=}$ .

Note that this condition is not invariant under chain quasi-isomorphisms; quasi-submersions are finitely presented cofibrations in an injective model structure on super chain cochain  $\mathcal{C}^\infty$ -rings (cf. [Pri6, Lemma 3.4] and [CR]), so should be thought of as a computational convenience. Super dg NQ-manifolds themselves correspond to finitely presented cofibrant objects in such a model structure. (See for instance [Hov] for background on model categories. The difference between our approach and that of [Pri6] is that we are taking an injective model structure, defining cofibrations levelwise in the chain direction, rather than a projective model structure defining fibrations levelwise.)

**Definition 4.2.** Given a quasi-submersion  $f: X \rightarrow Y$  of super dg NQ-manifolds, we define  $\widehat{\mathrm{Pol}}(X \xrightarrow{f} Y, n)$  to be the product  $\prod_{j \geq 0}$  of semi-infinite total complexes  $\widehat{\mathrm{Tot}}$  of the fibre products given by limits of the diagrams

$$\begin{array}{ccc} \mathrm{Symm}_{\mathcal{C}^\infty(X)}^j((T_{\mathcal{C}^\infty(X)})^{[-n-1]}) & \longrightarrow & \mathcal{C}^\infty(X) \otimes_{\mathcal{C}^\infty(Y)} \mathrm{Symm}_{\mathcal{C}^\infty(Y)}^j((T_{\mathcal{C}^\infty(Y)})^{[-n-1]}). \\ \mathrm{Symm}_{\mathcal{C}^\infty(Y)}^j((T_{\mathcal{C}^\infty(Y)})^{[-n-1]}) & \longrightarrow & \end{array}$$

We then follow Definition 2.13 in defining the space of Poisson structures on the diagram  $f: X \rightarrow Y$  as

$$\mathcal{P}(X, n) := \varprojlim_i \underline{\mathrm{MC}}(\mathrm{Fil}^2 \widehat{\mathrm{Pol}}(X, n)^{[n+1]} / \mathrm{Fil}^{i+2}).$$

We define  $\mathcal{P}(X \xrightarrow{f} Y, \Pi n)$  and  $Q\mathcal{P}(X \xrightarrow{f} Y, n)$  ( $n = 0, 1$ ) similarly, adapting Definitions 2.18, 3.13, 3.6.

Observe that restriction to either of the factors gives morphisms

$$\widehat{\mathrm{Pol}}(X, n) \leftarrow \widehat{\mathrm{Pol}}(X \xrightarrow{f} Y, n) \rightarrow \widehat{\mathrm{Pol}}(Y, n).$$

The following is the key to functoriality statements:



**Proposition 4.3.** *If a quasi-submersion  $f: X \rightarrow Y$  of super dg NQ-manifolds is a homotopy local diffeomorphism in the sense that the map*

$$\mathrm{Tot}(\Omega_{\mathcal{C}^\infty(Y)}^1 \otimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(X)^0) \rightarrow \mathrm{Tot}(\Omega_{\mathcal{C}^\infty(X)}^1 \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X)^0)$$

*is a quasi-isomorphism, then the natural maps*

$$\begin{aligned} \mathcal{P}(f: X \rightarrow Y, n) &\rightarrow \mathcal{P}(Y, n) \\ \mathcal{P}(f: X \rightarrow Y, \Pi n) &\rightarrow \mathcal{P}(Y, \Pi n) \\ Q\mathcal{P}(f: X \rightarrow Y, n) &\rightarrow \mathcal{P}(Y, n) \end{aligned}$$

*are weak equivalences.*

*If  $f$  is a levelwise quasi-isomorphism in the sense that the morphisms  $\mathcal{C}^\infty(Y)^i \rightarrow \mathcal{C}^\infty(X)^i$  are all chain quasi-isomorphisms, then the maps*

$$\begin{aligned} \mathcal{P}(f: X \rightarrow Y, n) &\rightarrow \mathcal{P}(X, n) \\ \mathcal{P}(f: X \rightarrow Y, \Pi n) &\rightarrow \mathcal{P}(X, \Pi n) \\ Q\mathcal{P}(f: X \rightarrow Y, n) &\rightarrow \mathcal{P}(X, n) \end{aligned}$$

*are also weak equivalences.*

*Proof.* As in [Pri6, Lemma 3.26] or the proof of [Pri6, Proposition 3.19], the homotopy local diffeomorphism hypothesis leads to quasi-isomorphisms

$$\widehat{\mathrm{Tot}} \mathrm{Symm}_{\mathcal{C}^\infty(X)}^j(T_{\mathcal{C}^\infty(X)}^{[-n-1]}) \rightarrow \widehat{\mathrm{Tot}}(\mathcal{C}^\infty(X) \otimes_{\mathcal{C}^\infty(Y)} \mathrm{Symm}_{\mathcal{C}^\infty(Y)}^j(T_{\mathcal{C}^\infty(Y)}^{[-n-1]})),$$

and hence the map  $\widehat{\mathrm{Pol}}(X \xrightarrow{f} Y, n) \rightarrow \widehat{\mathrm{Pol}}(Y, n)$  is a pullback along a surjective quasi-isomorphism, so is a quasi-isomorphism. The first statement for  $\mathcal{P}$  now follows by applying MC, and the others follow similarly.

The hypotheses for the second statement guarantee that those for the first hold, and also ensure that the maps

$$\widehat{\mathrm{Tot}} \mathrm{Symm}_{\mathcal{C}^\infty(Y)}^j(T_{\mathcal{C}^\infty(Y)}^{[-n-1]}) \rightarrow \widehat{\mathrm{Tot}}(\mathcal{C}^\infty(X) \otimes_{\mathcal{C}^\infty(Y)} \mathrm{Symm}_{\mathcal{C}^\infty(Y)}^j(T_{\mathcal{C}^\infty(Y)}^{[-n-1]}))$$

are quasi-isomorphisms, yielding the second set of equivalences.  $\square$

On equivalence classes of Poisson structures or of quantisations, functoriality for quasi-submersions which are homotopy local diffeomorphisms is now clear, with a Poisson structure on  $Y$  giving rise to a Poisson structure on  $X$  via the maps

$$\pi_0 \mathcal{P}(Y, n) \cong \pi_0 \mathcal{P}(f: X \rightarrow Y, n) \rightarrow \pi_0 \mathcal{P}(X, n).$$

Abstract homotopy theory then permits us to extend this functor to all homotopy local diffeomorphisms of super dg NQ-manifolds, because the homotopy category of a model category is the same as that of the subcategory of cofibrations. Explicitly, for any  $Y$ , there exists a ‘‘path object’’  $PY$  (cf. [Hov, Definition 1.2.4]), which comes with a quasi-submersion  $PY \rightarrow Y \times Y$  and a levelwise quasi-isomorphism  $Y \rightarrow PY$  which is a section of both the projection maps  $PY \rightarrow Y$ . Then for any map  $f: X \rightarrow Y$ , the first projection gives a quasi-submersive levelwise quasi-isomorphism  $X \times_Y PY \rightarrow X$  admitting a section, which combines with the second projection  $X \times_Y PY \rightarrow Y$  to factorise  $f$ . When  $X$  is a homotopy local diffeomorphism, repeated application of

Lemma 4.3 gives

$$\begin{aligned} \pi_0 \mathcal{P}(Y, n) &\rightarrow \pi_0 \mathcal{P}(X \times_Y PY \rightarrow Y, n) \\ &\cong \pi_0 \mathcal{P}(X \times_Y PY, n) \\ &\cong \pi_0 \mathcal{P}(X \times_Y PY \rightarrow X, n) \\ &\cong \pi_0 \mathcal{P}(X, n). \end{aligned}$$

This suffices in cases such as 2-shifted structures in Example 2.8 where  $\mathcal{P}$  has no homotopy groups, but in general descent and gluing arguments for these structures are required, involving  $\infty$ -categories, or  $m$ -categories if the higher homotopy groups  $\pi_{>m} \mathcal{P}(X, n)$  vanish. As in [Pri6, §2.1.2], the data here are best suited to construct complete Segal spaces in the sense of [Rez, §6].

**4.2. Lie groupoids.** Say we have a Lie groupoid  $\mathfrak{X} := (X_1 \rightrightarrows X_0)$ , so  $X_0$  and  $X_1$  are manifolds (regarded as the spaces of objects and of morphisms), we have an identity section  $\sigma_0: X_0 \rightarrow X_1$ , source and target maps  $\partial_0, \partial_1: X_1 \rightarrow X_0$ , and an associative multiplication  $X_1 \times_{\partial_0, X_0, \partial_1} X_1 \rightarrow X_1$ . We can then form the nerve

$$B\mathfrak{X} := (X_0 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X_1 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X_2 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \vdots \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X_3 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \vdots \dots)$$

by setting

$$X_m := \overbrace{X_1 \times_{\partial_0, X_0, \partial_1} X_1 \times_{\partial_0, X_0, \partial_1} \dots \times_{\partial_1, X_0, \partial_0} X_1}^m,$$

the space of  $m$ -strings of morphisms.

If we wish to define Poisson structures and quantisations for our Lie groupoid  $\mathfrak{X}$ , then we encounter the difficulty that Proposition 4.3 only applies to local diffeomorphisms rather than submersions, so we could only apply descent arguments directly if the source and target maps were local diffeomorphisms, meaning  $\mathfrak{X}$  would be an orbifold.

As in [Pri6, §3.2] or [CPT<sup>+</sup>], the solution is to resolve the Lie groupoid by Lie algebroids, in the form of NQ-manifolds.

**Definition 4.4.** Given a Lie groupoid  $\mathfrak{X} := (X_1 \rightrightarrows X_0)$ , we define the NQ-manifold  $(\hat{X}_1 \rightrightarrows X_0) = (X_0, \mathcal{O}_{(\hat{X}_1 \rightrightarrows X_0)})$  by first forming the Lie algebroid  $A\mathfrak{X}$  associated to the Lie groupoid  $\mathfrak{X}$  as in [Mac], then taking the associated NQ-manifold as in [LWX].

*Example 4.5.* If  $G$  is a Lie group acting on a manifold  $M$ , there is a Lie groupoid  $[M/G] := (G \times M \rightrightarrows M)$ , and then we have

$$(\widehat{G \times M} \rightrightarrows M) \cong [M/\mathfrak{g}],$$

for the NQ-manifold  $[M/\mathfrak{g}] := (M, \mathcal{O}_{[M/\mathfrak{g}]})$ , where  $\mathcal{O}_{[M/\mathfrak{g}]}$  is given by the Chevalley–Eilenberg complex

$$\mathcal{O}_{[M/\mathfrak{g}]} := \text{CE}(\mathfrak{g}, \mathcal{O}_M) = (\mathcal{O}_M \xrightarrow{Q} \mathcal{O}_M \otimes \mathfrak{g}^* \xrightarrow{Q} \mathcal{O}_M \otimes \Lambda^2 \mathfrak{g}^* \xrightarrow{Q} \dots).$$

**Definition 4.6.** Given a Lie groupoid  $\mathfrak{X} := (X_1 \rightrightarrows X_0)$  and a submersion  $Y_0 \rightarrow X_0$ , we define the NQ-manifold  $\hat{\mathfrak{X}}_{Y_0}$  (the completion of  $\mathfrak{X}$  along  $Y_0$ ) by first setting  $Y_1 := Y_0 \times_{X_0, \partial_1} X_1 \times_{\partial_0, X_0} Y_0$  and then

$$\hat{\mathfrak{X}}_{Y_0} := (\hat{Y}_1 \rightrightarrows Y_0),$$

for the Lie groupoid  $(Y_1 \rightrightarrows Y_0)$  homotopy equivalent to  $\mathfrak{X}$  with objects  $Y_0$ .

In particular, note that  $\hat{\mathfrak{X}}_{X_0} = (\hat{X}_1 \rightrightarrows X_0)$  and that the construction of  $\hat{\mathfrak{X}}_{Y_0}$  is invariant if we replace  $\mathfrak{X}$  with a homotopy equivalent Lie groupoid.

The examples we are interested in are  $\hat{\mathfrak{X}}_{X_j}$ , which can alternatively be written as  $\widehat{((X^{\Delta^j})_1 \rightrightarrows (X^{\Delta^j})_0)}$ , the completion of the groupoid

$$((X^{\Delta^j})_1 \rightrightarrows (X^{\Delta^j})_0) = (X_j \times_{(X_0)^{j+1}, \partial_0} (X_1)^{j+1} \rightrightarrows X_j)$$

of  $j$ -strings of morphisms and commutative diagrams between them.

We are now in a position to define Poisson structures and quantisations for Lie groupoids. We can think of the Lie algebroid  $A\mathfrak{X}$  as being an infinitesimal neighbourhood of  $X_0$  in  $\mathfrak{X}$ , so that in particular we have a local diffeomorphism  $A\mathfrak{X} \rightarrow \mathfrak{X}$ , and then the diagram  $j \mapsto \hat{\mathfrak{X}}_{X_j}$  gives a simplicial resolution of  $\mathfrak{X}$  by NQ-manifolds which are quasi-submersively locally diffeomorphic to it. This allows us to pull back Poisson structures and quantisations, so as in [Pri6, Definition 3.30] we can infer what the spaces of Poisson structures and quantisations on  $\mathfrak{X}$  have to be by looking at the corresponding spaces on the NQ manifolds  $\hat{\mathfrak{X}}_{X_j}$ :

**Definition 4.7.** Given a Lie groupoid  $\mathfrak{X} = (X_1 \rightrightarrows X_0)$ , with nerve  $X_\bullet := B\mathfrak{X}$ , we define the space  $\mathcal{P}(\mathfrak{X}, n)$  of  $n$ -shifted Poisson structures and the space  $Q\mathcal{P}(\mathfrak{X}, n)$  of  $n$ -shifted quantisations (the latter for  $n = 0, -1$ ) by first forming the simplicial NQ-manifold

$$\hat{\mathfrak{X}}_{B\mathfrak{X}} := (\hat{\mathfrak{X}}_{X_0} \begin{array}{c} \xleftarrow{\partial_0} \\ \xrightarrow{\partial_1} \\ \xleftrightarrow{\partial_2} \end{array} \hat{\mathfrak{X}}_{X_1} \begin{array}{c} \xleftarrow{\partial_1} \\ \xrightarrow{\partial_2} \\ \xleftrightarrow{\partial_3} \end{array} \hat{\mathfrak{X}}_{X_2} \begin{array}{c} \xleftarrow{\partial_2} \\ \xrightarrow{\partial_3} \\ \xleftrightarrow{\partial_4} \end{array} \hat{\mathfrak{X}}_{X_3} \begin{array}{c} \xleftarrow{\partial_3} \\ \xrightarrow{\partial_4} \\ \xleftrightarrow{\partial_5} \end{array} \cdots),$$

then observing that the morphisms  $\partial_i$  in the diagram are all homotopy local diffeomorphisms of NQ-manifolds in the sense of Proposition 4.3, giving functoriality and allowing us to take homotopy limits

$$\begin{aligned} \mathcal{P}(\mathfrak{X}, n) &:= \operatorname{holim}_{j \in \Delta} \mathcal{P}(\hat{\mathfrak{X}}_{X_j}, n), \\ Q\mathcal{P}(\mathfrak{X}, n) &:= \operatorname{holim}_{j \in \Delta} Q\mathcal{P}(\hat{\mathfrak{X}}_{X_j}, n). \end{aligned}$$

These homotopy limits of cosimplicial spaces are given by the functor  $\operatorname{Tot}_{\mathfrak{S}}$  of [GJ, §VIII.1]; for generalities on homotopy limits, see [BK, Hov, Hir].

When the spaces  $\mathcal{P}(\hat{\mathfrak{X}}_{X_j}, n)$  have trivial homotopy groups (but non-trivial  $\pi_0$ ), as occurs when  $n = 2$ , our homotopy limit above is just the equaliser of the maps

$$\pi_0 \mathcal{P}(\hat{\mathfrak{X}}_{X_0}, n) \rightrightarrows \pi_0 \mathcal{P}(\hat{\mathfrak{X}}_{X_1}, n),$$

and similarly for  $Q\mathcal{P}(\mathfrak{X}, n)$ .

When the homotopy groups stop at  $\pi_1$ , as occurs for  $n = 1$ , we instead have to include the datum of an isomorphism  $g$  in  $\mathcal{P}(\hat{\mathfrak{X}}_{X_1}, n)$  between the two images  $\partial^0(\pi), \partial^1(\pi)$  of a Poisson structure  $\pi \in \mathcal{P}(\hat{\mathfrak{X}}_{X_0}, n)$ , satisfying the cocycle condition  $\partial^1(g) \simeq \partial^2(g) \circ \partial^0(g)$  in  $\mathcal{P}(\hat{\mathfrak{X}}_{X_2}, n)$ .

In general, since the tangent complex is concentrated in cohomological degrees  $[-1, \infty)$ , the space  $\mathcal{P}(\hat{\mathfrak{X}}_{X_1}, n)$  is empty for  $n \geq 3$  because the governing DGLA  $\widehat{\operatorname{Pol}}(X, n)^{[n+1]}$  is concentrated in cohomological degrees  $[n - 1, \infty)$ . Meanwhile, for  $n < 0$  the DGLA is unbounded below in general, so there is no bound on the homotopy groups, further complicating the description.

*Example 4.8.* When  $\mathfrak{X} = [M/G]$ , then  $X_\bullet := B\mathfrak{X}$  is given by  $X_m = M \times G^m$  and as in [Pri6, Example 3.6],  $\mathfrak{X}_{X_m}$  is the NQ manifold  $[M \times G^m/\mathfrak{g}^{\oplus(m+1)}]$  associated to the Lie groupoid  $[M \times G^m/G^{m+1}]$ , with action

$$(y, h_1, \dots, h_m)(g_0, \dots, g_m) = (yg_0, g_0^{-1}h_1g_1, g_1^{-1}h_2g_2, \dots, g_{m-1}^{-1}h_mg_m).$$

As in [Pri6, Example 3.31], the triviality of homotopy groups then gives the space of 2-shifted Poisson structures as

$$\mathcal{P}([M/G], 2) \cong \{\pi \in (S^2\mathfrak{g} \otimes \mathcal{C}^\infty(M))^G : [\pi, a] = 0 \in \mathfrak{g} \otimes \mathcal{C}^\infty(M) \forall a \in O(M)\}.$$

For more general Lie groupoids, there is a similar description of 2-shifted Poisson structures as invariant 2-shifted Poisson structures on the associated Lie algebroid.

*Example 4.9.* To construct a 1-shifted Poisson structure on  $[M/G]$ , we start with a 1-shifted Poisson structure  $\pi$  on the NQ-manifold  $[M/\mathfrak{g}]$ , which as in Example 2.9 is the same as a quasi-Lie bialgebroid structure. For each of the morphisms  $\partial_i: [M \times G/\mathfrak{g} \oplus \mathfrak{g}] \rightarrow [M/G]$  ( $i = 0, 1$ ) in the nerve, Proposition 4.3 then gives essentially unique quasi-Lie bialgebroid structures  $\partial^i\pi$  on  $[M \times G/\mathfrak{g} \oplus \mathfrak{g}]$ ; in the terminology of [BCLX, §2.2],  $\partial^i\pi$  is projectable to  $[M/\mathfrak{g}]$  along  $\partial_i$ . To complete the data required to define a 1-shifted Poisson structure on  $[M/G]$ , we then need a gauge transformation, i.e. a twist  $\lambda \in \Lambda^2(\mathfrak{g} \oplus \mathfrak{g}) \otimes \mathcal{C}^\infty(M \times G)$  between the two pullbacks  $\partial^0\pi, \partial^1\pi$  of the Poisson structure to  $[M \times G/\mathfrak{g} \oplus \mathfrak{g}]$ , and  $\lambda$  must satisfy the cocycle condition  $\partial^1\lambda = \partial^0\lambda + \partial^2\lambda$  on  $[M \times G^2/\mathfrak{g}^{\oplus(3)}]$ . An isomorphism  $(\pi, \lambda) \rightarrow (\pi', \lambda')$  is then given by a twist  $\tau \in \Lambda^2(\mathfrak{g}) \otimes \mathcal{C}^\infty(M)$  satisfying  $\partial^1\tau + \lambda = \lambda' + \partial^0\tau$ .

When  $M$  is a point,  $\pi$  is trivial and  $\lambda$  is a quasi-Poisson structure on  $G$ , a generalisation of a Poisson-Lie structure; see [Saf1, Theorem 2.9] for the corresponding statement in the setting of [CPT<sup>+</sup>].

By [Saf1, Theorem 3.29], a source-connected smooth algebraic quasi-Poisson groupoid in the sense of [IPLGX] corresponds to a 1-shifted Poisson structure on the Lie groupoid, and a similar argument should apply in the  $\mathcal{C}^\infty$  setting. We can certainly say that every quasi-Poisson structure gives rise to a 1-shifted Poisson structure: by Morita equivalence, a quasi-Poisson structure [BCLX, Theorem 3.11] on a Lie groupoid  $fX := (X_1 \rightrightarrows X_0)$  gives rise to essentially unique quasi-Poisson structures on the Lie groupoids  $((X^{\Delta^j})_1 \rightrightarrows X_j)$ , and hence to compatible quasi-Lie bialgebroid structures on the Lie algebroids  $\hat{\mathfrak{X}}_{X_j}$ , by [IPLGX, Theorem 4.9], which is precisely what it means to give a 1-shifted Poisson structure on  $\mathfrak{X}$ . To complete the equivalence, it would suffice to compare the tangent and obstruction spaces as in [Pri6, §1.4]; in the notation of [BCLX], this would amount to showing that the 2-term complex  $\Sigma^{k+1}(A) \rightarrow \mathcal{T}_{mult}^k \Gamma$  is good truncation of the complex  $C^\bullet(\Gamma, S^{k+1}T_{\mathfrak{X}})$  in degrees  $\leq -k$  for  $k = 1, 2$ , which in the source-connected case should follow from [IPLGX, Proposition 2.35 and Equation (18)] as in [Saf1].

The following is analogous to a special case of [Pri6, Proposition 3.29]. In particular, it ensures that  $n$ -shifted Poisson structures and quantisations are invariant under Morita equivalence, so are invariants of differentiable stacks. For  $\mathcal{P}(\mathfrak{X}, 1)$ , this plays the same role as [BCLX, Theorem 3.11] does for quasi-Poisson structures.

**Proposition 4.10.** *Given a Lie groupoid  $\mathfrak{X} := (X_1 \rightrightarrows X_0)$  and a submersion  $f_0: Y_0 \rightarrow X_0$ , let  $\mathfrak{Y}$  be the Lie groupoid  $(Y_0 \times_{X_0} X_1 \times_{X_0} Y_0 \rightrightarrows Y_0)$  given by pulling  $\mathfrak{X}$  back to  $Y_0$ .*

Then there are natural maps

$$\begin{aligned} f^* &: \mathcal{P}(\mathfrak{X}, n) \rightarrow \mathcal{P}(\mathfrak{Y}, n) \\ f^* &: Q\mathcal{P}(\mathfrak{X}, n) \rightarrow \mathcal{P}(\mathfrak{Y}, n) \end{aligned}$$

in the homotopy category of simplicial sets. If  $f_0$  is surjective, then these maps are weak equivalences.

*Proof.* This follows as in the proof of [Pri6, Proposition 2.17]. By the construction of  $\mathfrak{Y}$ , we have  $\mathfrak{Y}_{Y_j} \cong \hat{\mathfrak{X}}_{Y_j}$ . Since  $f_0$  is a submersion, the maps  $\mathfrak{Y}_{Y_j} \rightarrow \hat{\mathfrak{X}}_{Y_j}$  are then all homotopy local diffeomorphisms, so Proposition 4.3 gives compatible maps  $\mathcal{P}(\hat{\mathfrak{X}}_{X_j}, n) \rightarrow \mathcal{P}(\mathfrak{Y}_{Y_j}, n)$ , and hence  $f^*: \mathcal{P}(\mathfrak{X}, n) \rightarrow \mathcal{P}(\mathfrak{Y}, n)$  on passing to homotopy limits. DGLA obstruction theory gives rise to towers of obstruction spaces for the homotopy limits, and cohomological descent ensures that these are isomorphisms when  $f_0$  is surjective.  $\square$

### 4.3. Higher Lie groupoids.

#### 4.3.1. Super Lie $k$ -groupoids.

**Definition 4.11.** Given a simplicial manifold  $X_\bullet$  and a simplicial set  $K$ , we follow [Zhu] in writing  $\text{hom}(K, X_\bullet)$  for the set of homomorphisms of simplicial manifolds from  $K$  to  $X_\bullet$ , with its natural topology. As in [Zhu, Lemma 2.1], when  $K$  is a finite contractible simplicial set,  $\text{hom}(K, X_\bullet)$  is naturally a manifold.

In particular, note that the combinatorial  $m$ -simplex  $\Delta^m$  is contractible, with  $\text{hom}(\Delta^m, X_\bullet) = X_m$ . Another important class of contractible finite simplicial sets is given by the horns  $\Lambda^{m,i} \subset \Delta^m$  for all  $m \geq 1$ ,  $0 \leq i \leq m$ , which are defined by removing the interior and the  $i$ th face from  $\Delta^m$ .

We now work with Lie  $k$ -groupoids in the sense of [Zhu, Definition 1.2], generalised in the obvious way with supermanifolds rather than just manifolds. In particular, for a simplicial supermanifold  $X_\bullet$  and a finite contractible simplicial set  $K$ , there is a natural supermanifold  $\text{hom}(K, X_\bullet)$ , defined as a limit of the supermanifolds  $X_m$ .

**Definition 4.12.** A super Lie  $k$ -groupoid  $X_\bullet$  is a simplicial diagram

$$X_0 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X_1 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X_2 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X_3 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \cdots$$

of supermanifolds, with the partial matching maps

$$X_m \rightarrow \text{hom}(\Lambda^{m,i}, X_\bullet)$$

being surjective submersions for all  $m \geq 1$ ,  $0 \leq i \leq m$ , and diffeomorphisms for  $m > k$ .

*Examples 4.13.* Giving a super Lie 0-groupoid  $X_\bullet$  is equivalent to giving the supermanifold  $X_0$ , since the matching conditions imply that  $X_m = X_0$  for all  $m$ .

Meanwhile, a super Lie 1-groupoid is just equivalent to a super Lie groupoid: we have supermanifolds  $X_0$  and  $X_1$  (regarded as the objects and the morphisms), an identity  $\sigma_0: X_0 \rightarrow X_1$ , source and target maps  $\partial_0, \partial_1: X_1 \rightarrow X_0$  and diffeomorphisms

$$X_m \cong \overbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}^m,$$

with the face map  $X_1 \times_{X_0} X_1 \cong X_2 \xrightarrow{\partial_1} X_1$  thus giving rise to the multiplication operation, and the higher conditions ensuring associativity.

The partial matching conditions on a Lie  $k$ -groupoid  $X_\bullet$  imply that the boundary maps  $\partial_i: X_m \rightarrow X_{m-1}$  are all submersions. When they are local diffeomorphisms,  $X_\bullet$  is a form of higher orbifold, and we can define Poisson structures and quantisations simply by appealing to Proposition 4.3: we just have to have a compatible system of Poisson structures or quantisations on the diagram  $X_\bullet$ .

We now generalise the construction of Definition 4.7 to apply to super Lie  $k$ -groupoids. For any commutative cochain algebra  $B = B^{\geq 0}$ , the Dold–Kan denormalisation  $DB$  is naturally a cosimplicial commutative algebra via the Eilenberg–Zilber shuffle product, as for instance in [Pri1, Definition 4.20]. This functor has a left adjoint  $D^*$ , which we now describe explicitly. Given a finite set  $I$  of strictly positive integers, write  $\partial^I = \partial^{i_s} \dots \partial^{i_1}$ , for  $I = \{i_1, \dots, i_s\}$ , with  $1 \leq i_1 < \dots < i_s$ .

**Definition 4.14.** Given a commutative cosimplicial super algebra  $A$ , we define the commutative super cochain algebra  $D^*A$  as follows. We first consider the cochain complex  $NA$  given by

$$N^m A := \{a \in A^m : \sigma^j a = 0 \in A^{m-1}, \forall 0 \leq j < m\},$$

with differential  $Qa := \sum (i-1)^i \partial^i a$ . We then define an associative (non-commutative) product  $\smile$  (a variant of the Alexander–Whitney cup product) on  $NA$  by

$$a \smile b := (\partial^{[m+1, m+n]} a) \cdot (\partial^{[1, m]} b)$$

for  $a \in N^m A$ ,  $b \in N^n A$ .

The commutative cochain algebra  $D^*A$  is then the quotient of  $NA$  by the relations

$$(\partial^I a) \cdot (\partial^J b) \sim \begin{cases} (-1)^{(J,I)} (a \smile b) & a \in A^{|J|}, b \in A^{|I|}, \\ 0 & \text{otherwise,} \end{cases}$$

for (possibly empty) sets  $I, J$  with  $I \cap J = \emptyset$ , where for disjoint sets  $S, T$  of integers,  $(-1)^{(S,T)}$  is the sign of the shuffle permutation of  $S \sqcup T$  which sends the first  $|S|$  elements to  $S$  (in order), and the remaining  $|T|$  elements to  $T$  (in order).

*Remarks 4.15.* When a cosimplicial  $\mathbb{R}$ -super algebra  $A^\bullet$  is quasi-freely generated over  $A^0$  in the sense that we have super vector spaces  $V^m \subset A^m$  with  $A^m \cong A^0 \otimes \text{Symm}_{\mathbb{R}} V^m$ , closed under degeneracy maps, so  $\sigma^i(V^m) \subset V^{m-1}$ , then  $D^*A \cong A^0 \otimes \text{Symm}_{\mathbb{R}}(NV)$  if we forget the differential  $Q$ .

As observed in [Pri6, §3.1],  $D^*A$  depends only on the formal completion of  $A$  with respect to  $\ker(A \rightarrow A^0)$ , and then the description above also applies if the completion is a quasi-freely generated cosimplicial power series ring over  $A^0$ .

As in [Pri6, Example 3.6], if  $A$  is the cosimplicial ring of functions on the nerve of the Lie groupoid  $[M/G]$  (so  $A^m = \mathcal{C}^\infty(M \times G^m)$ ), then  $D^*A$  is the Chevalley–Eilenberg complex  $\text{CE}(\mathfrak{g}, \mathcal{C}^\infty(M))$ .

The following generalise the passage from Lie groupoids to Lie algebroids (cf. Definition 4.7):

**Definition 4.16.** Given a simplicial supermanifold  $X_\bullet$ , define the normalisation  $NX$  to be manifold  $X_0^\bullet$  equipped with the super cochain algebra  $D^*((\sigma^0)^{-\bullet} \mathcal{O}_X)$ , where  $(\sigma^0)^{-\bullet} \mathcal{O}_X$  is the cosimplicial sheaf  $((\sigma^0)^{-\bullet} \mathcal{O}_X)^m := ((\sigma^0)^m)^{-1} \mathcal{O}_{X_m}$  on  $X_0^\bullet$ .

**Lemma 4.17.** *If  $X_\bullet$  is a super Lie  $k$ -groupoid, then  $NX$  is a super  $NQ$ -manifold, with  $\mathcal{O}_{NX}$  generated in cochain degrees  $\leq k$ .*

*Proof.* This follows directly from properties of Dold–Kan normalisation, as in [Pri6, Lemma 3.5].  $\square$

We are now in a position to define Poisson structures and quantisations for super Lie  $k$ -groupoids.

**Definition 4.18.** Given a simplicial supermanifold  $X_\bullet$  and a finite contractible simplicial set  $K$ , define the simplicial supermanifold  $X_\bullet^K$  by  $(X_\bullet^K)_m := \text{hom}(K \times \Delta^m, X_\bullet)$ .

For a super Lie  $k$ -groupoid  $X_\bullet$ , we can now form a simplicial super NQ-manifold

$$NX \begin{array}{c} \xleftarrow{\partial_1} \\ \xrightarrow{\sigma_0} \\ \xleftarrow{\partial_0} \end{array} N(X^{\Delta^1}) \begin{array}{c} \xleftrightarrow{\quad} \\ \xleftrightarrow{\quad} \\ \xleftrightarrow{\quad} \end{array} N(X^{\Delta^2}) \begin{array}{c} \xleftrightarrow{\quad} \\ \vdots \\ \xleftrightarrow{\quad} \end{array} N(X^{\Delta^2}) \quad \dots \quad \dots;$$

this resolves  $X_\bullet$  in the sense of [Pri6, Proposition 3.13], and the morphisms  $\partial_i$  are all quasi-submersive homotopy local diffeomorphisms in the sense of Proposition 4.3. Poisson structures and quantisations are functorial with respect to quasi-submersive homotopy local diffeomorphisms, so we proceed as in Definition 4.7, following [Pri6, Definition 3.30]:

**Definition 4.19.** Given a super Lie  $k$ -groupoid  $X_\bullet$ , we define the spaces  $\mathcal{P}(X_\bullet, n)$  of  $n$ -shifted Poisson structures,  $\mathcal{P}(X_\bullet, \Pi n)$  of parity-reversed  $n$ -shifted Poisson structures and the space  $Q\mathcal{P}(X_\bullet, n)$  of  $n$ -shifted quantisations (the latter for  $n = 0, -1$ ) by taking homotopy limits

$$\begin{aligned} \mathcal{P}(X_\bullet, n) &:= \text{holim}_{j \in \Delta} \mathcal{P}(N(X^{\Delta^j}), n), \\ \mathcal{P}(X_\bullet, \Pi n) &:= \text{holim}_{j \in \Delta} \mathcal{P}(N(X^{\Delta^j}), \Pi n), \\ Q\mathcal{P}(X_\bullet, n) &:= \text{holim}_{j \in \Delta} Q\mathcal{P}(N(X^{\Delta^j}), n). \end{aligned}$$

Theorem 2.16 then gives rise to an equivalence between shifted symplectic and non-degenerate Poisson structures on super Lie  $k$ -groupoids, by reasoning as in [Pri6, Theorem 3.33]. Likewise, the results of §3 extend via these constructions from super NQ-manifolds to super Lie  $k$ -groupoids to give quantisation of non-degenerate Poisson structures; they also give quantisations of degenerate 0-shifted Poisson structures on  $k$ -orbifolds (i.e. Lie  $k$ -groupoids for which all the partial matching maps are local diffeomorphisms).

*Remark 4.20.* Since submersions and local diffeomorphisms are the analogues in differential geometry of smooth and étale maps, in the terminology of [Pri3, Pri2] a super Lie  $k$ -groupoid  $X_\bullet$  is an Artin  $k$ -hypergroupoid in supermanifolds, and the construction above has allowed us to replace  $X_\bullet$  with a Deligne–Mumford  $k$ -hypergroupoid  $N(X^{\Delta^\bullet})$  in super NQ-manifolds. A further crucial property of this construction is that it preserves homotopy equivalences and hypercovers, as in Proposition 4.10, so the spaces of Poisson structures and of quantisations depend only on the hypersheafification of the super Lie  $k$ -groupoid on the big site of supermanifolds (with covers generated by surjective submersions); in other words, they depend only on the Morita equivalence class, so are invariants of differentiable superstacks.

4.3.2. *Derived Lie supergroupoids.* We now give a flavour of the global constructions of [Pri6, §3.4] and elsewhere, defining shifted Poisson structures and quantisations on global objects which are both derived and stacky.

**Definition 4.21.** Define a dg super Lie  $k$ -groupoid to be a simplicial dg supermanifold  $X_\bullet$  (so each  $X_m$  is a dg supermanifold in the sense of Definition 1.15) such that

- (1) the simplicial supermanifold  $X_\bullet^0$  is a super Lie  $k$ -groupoid in the sense of Definition 4.12;
- (2) the sheaf  $\mathcal{O}_X$  on  $X_\bullet^{0,=}$  is Cartesian in the sense that for each of the face maps  $\partial_i: X_m \rightarrow X_{m-1}$ , we have

$$\mathcal{O}_{X_m} \cong \mathcal{O}_{X_m^0} \otimes_{(\partial_i^{-1} \mathcal{O}_{X_m^0})} (\partial_i^{-1} \mathcal{O}_{X_{m-1}}).$$

*Remark 4.22.* The conditions ensure that if we take  $\mathcal{S}$  to be the class of surjective homotopy submersions, then every dg super Lie  $k$ -groupoid is a homotopy  $(k, \mathcal{S})$ -hypergroupoid in dg supermanifolds in the sense of [Pri3, Pri2]. However, not all homotopy  $(k, \mathcal{S})$ -hypergroupoids arise in this way, because the second condition in Definition 4.21 is unnecessarily strong. It would suffice to take quasi-isomorphism instead of isomorphism, and to relax the finiteness conditions along the lines of Remark 1.17. Our conditions are chosen to ensure that the formal completion of  $X$  along  $X_0$  is a dg super NQ-manifold in the sense of Definition 1.23, a definition which is itself unnecessarily restrictive. Making changes along the lines of Remark 1.28, our results will all extend to the derived Lie  $n$ -groupoids of [Nui].

**Definition 4.23.** Given a simplicial dg supermanifold  $X_\bullet$ , define the normalisation  $NX$  to be manifold  $X_0^{0,=}$  equipped with the super chain cochain algebra  $D^*((\sigma^0)^{-\bullet} \mathcal{O}_X)$ , where  $(\sigma^0)^{-\bullet} \mathcal{O}_X$  is the cosimplicial sheaf  $((\sigma^0)^{-\bullet} \mathcal{O}_X)^m := ((\sigma^0)^m)^{-1} \mathcal{O}_{X_m}$  on  $X_0^=$ .

Adapting the argument for super Lie  $k$ -groupoids (Lemma 4.17) gives:

**Lemma 4.24.** *If  $X_\bullet$  is a dg super Lie  $k$ -groupoid, then  $NX$  is a super dg NQ-manifold, with  $\mathcal{O}_{NX}$  generated in cochain degrees  $\leq k$ .*

Definitions 4.18 and 4.19 then adapt verbatim to give expressions for shifted Poisson structures and quantisations on dg super Lie  $k$ -groupoids. As in [Pri6, Proposition 3.29], these spaces of depend only on the hypersheafification of the dg super Lie  $k$ -groupoid with respect to surjective submersions (a higher, derived analogue of Morita equivalence). Theorem 2.21 gives rise to an equivalence between shifted symplectic and non-degenerate Poisson structures on dg super Lie  $k$ -groupoids, reasoning as in [Pri6, Theorem 3.33], and the results of §3 all extend via these constructions from dg super NQ-manifolds to give quantisations for shifted Poisson structures on dg super Lie  $k$ -groupoids.

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