An introduction to derived (algebraic) geometry

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Abstract

These are notes from an introductory lecture course on derived geometry, given by the second author, mostly aimed at an audience with backgrounds in geometry and homological algebra. The focus is on derived algebraic geometry, mainly in characteristic 0, but we also see the tweaks which extend most of the content to analytic and differential settings.

Preface

These are notes taken by both authors from a course given by the second author in Edinburgh in spring 2021, with some material from courses given in Cambridge in 2013 and 2011.

The course consisted of twelve 90 minute lectures, and the material here mainly follows their pattern, with details, references and in some cases additional related content (notably some more down-to-earth characterisations of derived stacks), added in several sections.

The main background topics assumed are homological algebra, sheaves, basic category theory and algebraic topology, together with some familiarity with typical notation and terminology in algebraic geometry. A lot of the motivation will be clearer for those familiar with moduli spaces, but they are not essential background.

The perspective of the course was to try to present the subject as a natural, concrete development of more classical geometry, instead of merely as an opportunity to showcase ∞-topos theory (a topic we only encounter indirectly in these notes). The main moral of the later sections is that if you are willing to think of geometric objects in terms of Čech nerves of atlases rather than as ringed topoi, the business of developing higher and derived generalisations becomes much simpler.

These notes are only intended as an introduction to the subject, and are far from being a comprehensive survey. We have tried to include more detailed references throughout, with the original references where we know them. Readers may be surprised at how old most of the references are, but the basics have not changed in a decade, though as terminology becomes more specialised, researchers can tend to overestimate the originality of their ideas[1]. We have probably overlooked precursors for many phenomena in the supersymmetry literature, for which we apologise in advance.

We would like to thank the audience members, and particularly Sebastian Schlegel-Mejia, for very helpful comments, without which many explanations would be missing from the text.

- Conventions that are set in the text and hold from there onwards are currently in a "Notation"-environment.

[1] potentially compounded by Maslow's hammer and Disraeli's maxim on reading books

- Footnotes tend to contain details and comments which are tangential to the main thread of the notes; they are excessive in number.
- We adhere strictly to the standard convention that the indices in chain complexes and simplicial objects, and related operations and constructions, are denoted with subscripts, while those in cochain complexes and cosimplicial objects are denoted with superscripts; to do otherwise would invite chaos.
- We intermittently write chain complexes V as $V_\bullet$ to emphasise the structure, and similarly for cochain, simplicial and cosimplicial structures. The presence or absence of bullets in a given expression should not be regarded as significant.
- We denote shifts of chain and cochain complexes by $[n]$, and always follow the convention originally developed for cochains, so we have $M[n]^i := M^{n+i}$ for cochain complexes, but $M[n]_i := M_{n-i}$ for chain complexes.

Contents

1 Introduction and dg algebras

The idea behind derived geometries, and in particular derived algebraic geometry (DAG for short), is to endow rings of functions with extra structure, making families of geometric objects behave better. For example, singular points start behaving more like smooth ones as observed in [Kon94b, Kon94a], a philosophy known as *hidden smoothness*.

The most fundamental formulation of the theory would probably be in terms of simplicial rings, but in characteristic 0 these give the same theory as commutative differential graded algebras (dg-algebras), which we will focus on most in these notes, as they are simpler to work with.

Remark 1.1. Spectral algebraic geometry[2] (SAG) is another powerful closely related framework and is based on commutative ring spectra; it is studied amongst other homotopical topics in [Lur18]. In characteristic 0 this gives the same theory as DAG, but different geometric behaviour appears in characteristics $p > 0$. While DAG is mostly used to apply methods of algebraic topology to algebraic geometry, SAG is mainly used the other way around, an example being elliptic cohomology as in [Lur07].

The motivation for SAG is that cohomology theories come from symmetric spectra, and you try to cook up more exotic cohomology theories by replacing rings in the theory of schemes/stacks with E_∞-ring spectra. There's a functor H embedding discrete rings in E_∞-ring spectra [TV04, p. 185], but it doesn't preserve smoothness: even the morphism $H\mathbb{F}_p \to H(\mathbb{F}_p[t])$ is not formally smooth.

This is just a side note and we won't use spectra in these lecture notes, although most of the results of §6 also hold in SAG[3].

Notation 1.2. Henceforth (until we start using simplicial rings), we fix a commutative ring k containing $\mathbb{Q}$, i.e. we work in equal characteristics $(0, 0)$[4]

1.1 dg-algebras

In this section we define dg-algebras and affine dg-schemes, as well as analogues in differential and analytic geometry.

Definition 1.3. A *differential graded k-algebra* (dga or dg-algebra for short) A consists of a chain complex with an associative multiplication. Concretely, that is a family of k-modules $\{A_i\}_{i\in\mathbb{Z}}$, an associative k-linear multiplication $(- \cdot -) : A_i \times A_j \to A_{i+j}$ (for all i, j) and a differential $\delta : A_i \to A_{i-1}$ (for all i) which is k-linear, satisfies $\delta^2 = 0$ and is derivation with respect to the multiplication $\delta(a \cdot b) = \delta(a) \cdot b + (-1)^{\deg(a)} a \cdot \delta(b)$. Without a differential δ, we simply call A a *graded algebra*.

A k-algebra A is *graded-commutative* if $a \cdot b = (-1)^{\deg(a)\cdot\deg(b)} b \cdot a$. We write *cdga* for differential graded-commutative algebras.

Definition 1.4. A dg-algebra $A_\bullet$ is *discrete* if $A_n = 0$ for all $n \neq 0$.

Notation 1.5. We usually denote a graded algebra with $A_* := \{A_i\}_i$ while we use the notation $A_\bullet := (\{A_i\}_i, \delta)$ to denote a differential graded algebra, where usually the δ is implicit/suppressed. Moreover, if a (differential) graded algebra is discrete with ring A in

[2]often confusingly referred to as derived algebraic geometry following [Lur09a], and originally dubbed Brave New Algebraic Geometry in [TV03, TV04], "brave new algebra" then being well-established Huxleian terminology, dating at least to Waldhausen's plenary talk "Brave new rings" at the conference [Mah88])

[3]Indeed, [Pri09] was explicitly couched in sufficient generality to apply to ring spectra.

[4]cdga don't behave nicely in other characteristics because the symmetric powers of example 1.8 don't preserve quasi-isomorphisms of chain complexes.

degree zero, we just denote the it by A, i.e. like the ring in degree zero and without any subscript.

Remark 1.6. Usually we are restricting ourselves to the case where these cdga are *concentrated in non-negative chain degree*, i.e. $A_i = 0$ for all $i < 0$.

Note that often in algebraic geometry one rather works with co-chains instead of chains. The two main reasons for using chain notation here are to assist the comparison with simplicial objects and to help distinguish the indices from those arising from sheaf cohomology, which we will encounter in later sections.

Notation 1.7. In concrete examples we will often denote cdga concentrated in non-negative chain degree like $(A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \ldots)$, assuming that the first written entry is degree zero. For example if $f : A \twoheadrightarrow B$ is a surjective map of rings then $(A \hookleftarrow \ker(f) \leftarrow 0 \leftarrow \ldots)$ would be a chain complex with A in degree zero, $\ker(f)$ in degree 1 and 0 everywhere else. This chain complex is quasi-isomorphic to the induced chain complex B (see later lectures).

Example 1.8. Let M be a graded k-module. The *free graded k-algebra* generated by M is $k[M] := (\bigoplus \mathrm{Symm}^n M_{\mathrm{even}}) \otimes (\bigoplus \bigwedge^n M_{odd})$.

Example 1.9. Take a free graded-commutative k-algebra A on three generators X, Y, Z where $\deg(X) = 0, \deg(Y) = \deg(Z) = 1$. Then we get

- $A_0 = k[X]$
- $A_1 = k[X]Y \oplus k[X]Z$
- $A_2 = k[X]YZ$
- $A_i = 0$ for $i < 0$ and $i \geq 3$.

which we can see by computing that $ZY = -YZ$ and $Y^2 = Z^2 = 0$.

A differential of A is then completely determined by its values $f := \delta(Y), g := \delta(Z) \in A_0 = k[X]$. So for example for $a, b, c \in k[X]$ we have $\delta(aY + bZ) = af + bg$ and $\delta(cYZ) = c(Zf - Yg)$.[5]

In fact, we get $H_0(A) = k[X]/(f, g)$, so A is the ring of functions on the *derived vanishing locus* of the map $(f, g) : \mathbb{A}^1 \to \mathbb{A}^2, x \mapsto (f(x), g(x))$.

Remark 1.10. Example 1.9 shows what happens in algebraic geometry. However, it is straightforward to adjust the example to differential or analytic geometry. All that's needed is to put extra structure on A_0. For differential geometry, A_0 ought to be a *$\mathcal{C}^\infty$-ring* [Dub81], which means that for any $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ there is an n-ary operation $A_0 \times \ldots \times A_0 \to A_0$, and these operations need to satisfy some natural consistency conditions.[6] This approach allows for singular spaces, and is known as *synthetic* differential geometry.

For analytic geometry, A_0 should be a ring with entire functional calculus (EFC-ring), meaning for any holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ there is again an operation $A_0 \times \ldots \times$

[5]Some readers might recognise this as a variant of a Koszul complex.

[6]As an example, finitely generated C^∞-rings just take the form $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})/I$ where I is an ideal; these include $\mathcal{C}^\infty(X, \mathbb{R})$ for manifolds X. Hadamard's lemma ensures that the operations descend to the quotient.

A $\mathcal{C}^\infty$-ring homomorphism $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})/I \to \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})/J$ is then just given by elements $f_1, \ldots, f_m \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})/J$ satisfying $g(f_1, \ldots, f_m) = 0$ for all $g \in I$; think of this as a smooth morphism from the vanishing locus of J to the vanishing locus of I.

Arbitrary $\mathcal{C}^\infty$ rings arise as quotient rings of nested unions $\bigcup_{T \subset S, \text{ finite}} \mathcal{C}^\infty(\mathbb{R}^T, \mathbb{R})$ for infinite sets S.

$A_0 \to A_0$, with these satisfying some natural consistency conditions — there are similar definitions for non-Archimedean analytic geometry.

For more details and further references on this approach, see [CR12, Nui18] in the differential setting, and [Pri18b] in the analytic setting.[7]

Example 1.11. We recall that, as with any chain complex, we can define the homology $H_*(A_\bullet)$ of a cdg-algebra $A_\bullet$ by $H_i(A_\bullet) = \ker(\delta : A_i \to A_{i-1})/\mathrm{Im}\,(\delta : A_{i+1} \to A_i)$, which is a graded-commutative algebra.

Definition 1.12. A *morphism of dg-algebras* is a map $f : A_\bullet \to B_\bullet$ that respects the differentials (i.e. $f\delta_A = \delta_B f : A_i \to B_{i-1}$ for all $i \in \mathbb{Z}$), and the multiplication (i.e. $f(a \cdot_A b) = f(a) \cdot_B f(b) \in B_{i+j}$ for all $a \in A_i, b \in A_j$ for all i, j).

Definition 1.13. We denote by $dg_+\mathrm{Alg}_k$ the *category of graded-commutative differential graded k-algebras* which are concentrated in non-negative degree.

The opposite category $(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})^{\mathrm{op}}$ is the *category of affine dg-schemes*, also denoted with $\mathrm{DG}^+\mathrm{Aff}$. We denote elements in this opposite category formally by $\mathrm{Spec}\,(A_\bullet)$.

The notation $\mathrm{Spec}\,(A_\bullet)$ is used to stress the similarity to rings and affine schemes. However, at this stage the construction of an affine dg-scheme is purely in a categorical sense, meaning we do not use any of the explicit constructions such as the prime spectrum of a ring or locally ringed spaces.

Remark 1.14. In geometric terms, one should think of the ”points” of a dg-scheme just as the points in $\mathrm{Spec}\,(H_0(A_\bullet))$ (which is a classic affine spectrum). The rest of the structure of a dg-scheme is in some sense infinitesimal.

In analytic and $\mathcal{C}^\infty$ settings, we can make similar definitions for dg analytic spaces or dg $\mathcal{C}^\infty$ spaces, but it is usual to impose some restrictions on the EFC-rings and $\mathcal{C}^\infty$-rings being considered, since not all are of geometric origin; we should restrict to those coming from *closed* ideals in affine space, with some similar restriction on the A_0-modules A_i.

Definition 1.15. Let $A_\bullet, B_\bullet \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$. A morphism $f : A_\bullet \to B_\bullet$ of cdga is a *quasi-isomorphism* (or *weak-equivalence*) if it induces an isomorphism on homology $H_*(A_\bullet) \xrightarrow{\simeq} H_*(B_\bullet)$. $A_\bullet$ and $B_\bullet$ are *equivalent* if there exists a quasi-isomorphism $A_\bullet \to B_\bullet$.

1.2 Global structures

As a next step, one would like to globalise the concept of an affine dg-scheme to get a dg-scheme (or a dg analytic space or dg $\mathcal{C}^\infty$-space in other contexts). There's a straightforward approach to achieve this: instead of a ring in degree 0 and more structure above it, we can take a scheme (or analogous geometric object) in degree 0 and a sheaf of dg-algebras above it. This definition is due to [CFK99] after Kontsevich [Kon94a, Lecture 27].[8]

Definition 1.16. A *dg-scheme* consists of a scheme X^0 and quasi-coherent sheaves $\mathscr{O}_X := \{\mathscr{O}_{X,i}\}_{i\geq 0}$ on X^0 such that $\mathscr{O}_{X,0} = \mathscr{O}_{X^0}$ (i.e. the structure sheaf of X^0), equipped with a cdga structure, i.e. $\delta : \mathscr{O}_{X,i} \to \mathscr{O}_{X,i-1}$ and $\cdot : \mathscr{O}_{X,i} \otimes \mathscr{O}_{X,j} \to \mathscr{O}_{X,i+j}$ satisfying the usual conditions.

Although we have given this definition in the algebraic setting, obvious analogues exist replacing schemes with other types of geometric object in $\mathcal{C}^\infty$ and analytic settings.

[7] In particular, this is shown in [Pri18b] to be equivalent to the approach via pregeometries in [Lur11a], classical theorems in analysis rendering most of the pregeometric data redundant.

[8] These dg schemes should not be confused with the DG schemes of [Gai11], which are an alternative characterisation of the derived schemes of Definition 1.21.

Definition 1.17. A *morphism of dg-schemes* $f : (X^0, \mathscr{O}_X) \to (Y^0, \mathscr{O}_Y)$ consists of a morphism of schemes $f^0 : X^0 \to Y^0$ and a morphism of sheaves of cdga $f^\sharp : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

Definition 1.18. Define the underived truncation $\pi^0 X \subseteq X^0$ to be $\underline{\mathrm{Spec}}_{X^0}(H_0(\mathscr{O}_X))$, the closed subscheme of X^0 on which δ vanishes, or equivalently defined by the ideal $\delta\mathscr{O}_{X,1} \subset \mathscr{O}_{X,0}$. [9]

Definition 1.19. A morphism of dg-schemes is a *quasi-isomorphism* if $\pi^0 f : \pi^0 X \to \pi^0 Y$ is an isomorphism of schemes and $\mathrm{H}_*\mathscr{O}_Y \to \mathrm{H}_*\mathscr{O}_X$ is an isomorphism of quasi-coherent sheaves on $\pi^0 X = \pi^0 Y$.

Remark 1.20. A problem with definition 1.16 is that X^0 has no geometrical meaning, in the sense that we can replace it with any open subscheme containing $\pi^0 X$ and get a quasi-isomorphic dg-scheme. Moreover, the ambient scheme X^0 gets in the way when we want to glue multiple dg-schemes together.

Gluing tends not to be an issue for analogous constructions in differential geometry, because a generalised form of Whitney's embedding theorem holds: a derived $\mathcal{C}^\infty$ space has a quasi-isomorphic dg $\mathcal{C}^\infty$ space with $X^0 = \mathbb{R}^N$ whenever its underived truncation $\pi^0 X$ admits a closed embedding in $\mathbb{R}^N$.

However, in algebraic and analytic settings this definition turns out to be too restrictive in general, which can be resolved by working with derived schemes.

The following definition incorporates the flexibility needed to allow gluing constructions, and gives a taste of the sort of objects we will be encountering towards the end of the notes.

Definition 1.21. A *derived scheme* X consists of a scheme $\pi^0 X$ and a presheaf $\mathscr{O}_X$ on the site of affine open subschemes of $\pi^0 X$, with values in cdga $dg_+\mathrm{Alg}_k$, such that $H_0\mathscr{O}_X = \mathscr{O}_{\pi^0 X}$ in degree zero and all $H_i(\mathscr{O}_X)$ are quasi-coherent $\mathscr{O}_{\pi^0 X}$-modules for all $i \geq 0$.[10]

Remark 1.22. To get from a dg-scheme to a derived scheme one looks at the canonical embedding $i : \pi^0 X \hookrightarrow X^0$ and takes $(\pi^0 X, i^{-1}\mathscr{O}_X)$, which is a derived scheme.

In the other direction, observe that on each open affine subscheme $U \subset \pi^0 X$, we have an affine dg scheme $\mathrm{Spec}\,\mathscr{O}_X(U)$, but that the schemes $\mathrm{Spec}\,\mathscr{O}_{X,0}(U) \supset U$ will not in general glue together to give an ambient affine scheme $X^0 \supset \pi^0 X$.

Remark 1.23. By [Pri09, Thm 6.42], these objects are equivalent to objects usually described in a much fancier way: those derived Artin or Deligne–Mumford ∞-stacks in the sense of [TV04, Lur04a] whose underlying underived stacks are schemes.[11]

To generalise the definition to derived algebraic spaces (or even derived Deligne–Mumford stacks), let $\pi^0 X$ be an algebraic space and let U run over affine schemes étale over $\pi^0 X$.

Definition 1.24. A dg-scheme is a *dg-manifold* if X^0 is smooth and as a graded-commutative algebra $\mathscr{O}_X$ is freely generated over $\mathscr{O}_{X^0}$ by a finite rank projective module (i.e. a graded vector bundle).

[9] In [CFK99], the notation π_0 is used for this construction, but subscripts are more appropriate for quotients than kernels, and using π_0 would cause confusion in simplicial constructions.

[10] Here $\mathscr{O}_{\pi^0 X}$ is the structure sheaf of the scheme $\pi^0 X$ and $H_i\mathscr{O}_X$ is a presheaf of homology groups.

[11] Beware that this is not the same as the notion of a derived scheme in [Lur04a, Definition 4.5.1], which gives a notion more general than a derived algebraic space (see [Lur04a, Proposition 5.1.2]), out of step with the rest of the literature.

Note that the second condition says that the morphism $\mathscr{O}_{X,0} \to \mathscr{O}_{X,\bullet}$ is given by finitely generated cofibrations of cdgas.

Remarks 1.25. Every affine dg-scheme with perfect cotangent complex is quasi-isomorphic to an affine dg-manifold. (We can drop perfect condition if we drop finiteness in the definition of a dg-manifold.)

The "manifold" terminology alludes to the locally free generation of $\mathscr{O}_{X,\bullet}$ by co-ordinate variables.

There is a more extensive literature on dg manifolds in the setting of differential geometry, often in order to study supersymmetry and supergeometry in mathematical physics; these tend to be $\mathbb{Z}/2$- or $\mathbb{Z}$-graded and are often known as Q-manifolds (their Q corresponding to our differential δ), following [AKSZ95, Kon97]; also see [DM99, Vor07]. The Q-manifold literature tends to place less emphasis on homotopy-theoretical phenomena (and especially quasi-isomorphism invariance) than the derived geometry literature.

When the sheaf $\mathscr{O}_{X^0}$ of functions is enriched in the opposite direction to Definition 1.24, i.e. $\delta \colon \mathscr{O}_{X,0} \to \mathscr{O}_{X,-1} \to \ldots$, the resulting object behaves very differently from the dg-manifolds we will be using, and corresponds to a stacky (rather than derived) enrichment, giving a form of derived Lie algebroid or s.h. Lie–Rinehart algebra. In differential settings, these tend to be known as NQ-manifolds or (confusingly) dg-manifolds. For more on their relation to derived geometry, see [Nui18, Pri18a, Pri19] and references therein.

1.3 Quasi-coherent complexes

Definition 1.26. Let $(A_\bullet, \delta_A) \in dg_+\mathrm{Alg}_{\mathrm{k}}$. An *A-module in complexes* consists of a chain complex $(M_\bullet, \delta)$ of k-modules and a scalar multiplication $(A \otimes_k M)_\bullet \to M_\bullet$ which is compatible with the multiplication on A.

Explicitly, for all i, j we have a k-bilinear map $A_i \times M_j \to M_{i+j}$ satisfying $(ab)m = a(bm)$, $1m = m$, and the chain map condition $\delta(am) = \delta_A(a)m + (-1)^{\deg(a)} a\delta(m)$.

Definition 1.27. A morphism of A-modules $M_\bullet \to N_\bullet$ is a *quasi-isomorphism* if it induces an isomorphism on homology $H_*(M_\bullet) \xrightarrow{\simeq} H_*(N_\bullet)$.

Definition 1.28 (Global version)**.** Let $(\pi^0 X, \mathscr{O}_X)$ be a derived scheme. We can look at $\mathscr{O}_X$-modules $\mathscr{F}$ in complexes of presheaves. We say they are *homotopy-Cartesian* modules (following [TV04]), or *quasi-coherent complexes* (following [Lur04a]), if for every inclusion $U \hookrightarrow V$ of open affine subschemes in $\pi_0 X$, the maps

$$\mathscr{O}_X(U) \otimes^{\mathbb{L}}_{\mathscr{O}_X(V)} \mathscr{F}(V) \to \mathcal{U}$$

are quasi-isomorphisms; equivalently, this says that the homology *presheaves* $\mathrm{H}_i\mathscr{F}$ are all quasi-coherent $\mathscr{O}_{\pi^0 X}$-modules.

1.4 What about morphisms and gluing?

We want to think of derived schemes X, Y as equivalent if they can be connected by a zigzag of quasi-isomorphisms.

$$\begin{tikzcd}
X & & W_1 & & & \ldots & & W_n & & Y \\
& Z_0 \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & & Z_1 \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \ldots & & {} \arrow[ur, "\sim"'] & & Z_n \arrow[ul, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

How should we define morphisms compatibly with this notion of equivalence?[12] What about gluing data?

We could forcibly invert all quasi-isomorphisms, giving the "homotopy category" $\mathrm{Ho}(dg_+\mathrm{Alg}_k)$ in the affine case. That doesn't have limits and colimits, or behave well with gluing.

For any small category I, we might also want to look at the category $dg_+\mathrm{Alg}_k^I$ of I-shaped diagrams of cdgas (e.g. taking I to be a poset of open subschemes as in the definition of a derived scheme). There is then a homotopy category $\mathrm{Ho}(dg_+\mathrm{Alg}_k^I)$ of diagrams, given by inverting objectwise quasi-isomorphisms.

But: The natural functor

$$\mathrm{Ho}(dg_+\mathrm{Alg_k}^I) \to \mathrm{Ho}(dg_+\mathrm{Alg_k})^I$$

(from the homotopy category of diagrams to diagrams in the homotopy category) is seldom an equivalence; it goes wrong for everything but for discrete diagrams, i.e. when I is a set. This means that constructions such as sheafification are doomed to fail if we try to formulate everything in terms of the homotopy category $\mathrm{Ho}(dg_+\mathrm{Alg_k})$.

To fix this, we will need some flavour of infinity (i.e. $(\infty, 1)$) category, this description in terms of diagrams being closest to Grothendieck's derivators.[13]

[12] The first global constructions [CFK99, CFK00] of derived moduli spaces did not come with functors of points partly because morphisms are so hard to define; it was not until [Pri11b] that those early constructions were confirmed to parametrise the "correct" moduli functors.

[13] An early attempt to address the problems of morphisms and gluing for dg schemes was [Beh02], which used 2-categories to avoid the worst pathologies.

2 Infinity categories and model categories (a bluffer's guide)

2.1 Infinity categories

There are many equivalent notions of ∞-categories. We start by looking at a few different ones as it can be quite useful to have different ways to think about ∞-categories at hand.

This entire section is meant to merely give an overview of the more accessible notions of ∞-categories and is in no way meant to be a complete or rigorous introduction.

For equivalences between these and some other models of ∞-categories, see for instance [JT07, Joy07]. For the general theory of ∞-categories, with slightly different emphasis, see [Hin17, Cis19].

2.2 Different notions of ∞-categories

We continue with some constructions of ∞-categories.

1. Arguably *topological categories* are conceptually among the easiest notions. A *topological category* is a category enriched in topological spaces (i.e. for any two objects $X, Y \in \mathcal{C}$ the morphism between them $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ form a topological space and composition is a continuous operation).

   Given a topological category, $\mathcal{C}$, the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ *of* $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and the morphisms are given by path components of morphisms in $\mathcal{C}$, i.e. $\pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y))$.

   A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ (assumed to respect the extra structure, so everything is continuous) is a *quasi-equivalence* if

   (a) for all $X, Y \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ is a weak homotopy equivalence of topological spaces (i.e. an isomorphism on homotopy groups).

   (b) $\mathcal{F}$ induces an equivalence on the homotopy categories $\mathrm{Ho}(\mathcal{F})\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$.

2. Topological spaces contain a lot of data, so a more combinatorially efficient model with much of the same intuition is given by *simplicial categories*, which have a simplicial set of morphisms between each pair of objects. We will be defining simplicial sets later. The behaviour is much the same as for topological categories but simplicial categories have much less data to handle.

3. By far the easiest to construct are *relative categories* (see Dwyer–Kan [DK80a, DK87], Barwick–Kan [BK10]).

   These consist of pairs $(\mathcal{C}, \mathcal{W})$ where $\mathcal{C}$ is a category and $\mathcal{W}$ is a subcategory. That's it![14] The idea is that the morphisms in $\mathcal{W}$ should be encode some notion of equivalence weaker than isomorphism. The homotopy category $\mathrm{Ho}(\mathcal{C})$ is a localisation of $\mathcal{C}$ given by forcing all the morphisms in $\mathcal{W}$ to be isomorphisms, and the associated simplicial category $L_{\mathcal{W}}\mathcal{C}$ arises as a fancier form of localisation. (i.e. it is a simplicial category whose path components of morphisms recover the homotopy category)

   Examples of subcategories $\mathcal{W}$ are homotopy equivalences or weak homotopy equivalences for topological spaces, quasi-isomorphisms of chain complexes, and equivalences of categories. The drawback is that quasi-equivalences of relative categories are hard to describe.

[14] ignoring cardinality issues/Russell's paradox

4. *Grothendieck's derivators* provide another useful perspective: Given a small category I, we can look at the ∞-category of I-shaped diagrams $\mathcal{C}^I$ in an ∞-category $\mathcal{C}$, and then there is a natural functor $\mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C})^I$ from the homotopy category of diagrams to diagrams in the homotopy category, which is usually **not an equivalence**; instead, these data essentially determine the whole ∞-category.

   Concretely, a *derivator* is an assignment $I \mapsto \mathrm{Ho}(\mathcal{C}^I)$ for all small categories I. There are several accounts of the theory written by Maltsiniotis and others. It turns out that a derivator determines the ∞-category structure on $\mathcal{C}$, up to essentially unique quasi-equivalence, by [Ren06]. This can be a useful way to think about ∞-functors $\mathcal{C} \to \mathcal{D}$, since they amount to giving compatible functors $\mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{D}^I)$ for all I.

Remark 2.1. Especially (3) illustrates how much data one needs to specify an $\infty-category$. While topological categories suggest that there are entire topological spaces to choose, relative categories show that in practice once a notion of weak equivalence has been picked, everything else is determined.

Remark 2.2. Model categories don't belong in this list. They are relative categories equipped with some extra structure (two more subcategories in addition to $\mathcal{W}$) which makes many calculations feasible — a bit like a presentation for a group — and avoids Russell's paradox. See [Qui67, Hov99, Hir03] and §2.4 below.

If anyone gives you an infinity category, you can assume it's a topological or simplicial category. If someone asks you for an infinity category, it's enough to give them a relative category.

2.3 Derived functors

Although derived functors are often just defined in the setting of model categories, they only depend on relative category structures, as in the approach of [DHKS04]:

Definition 2.3. If $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{D}, \mathcal{V})$ are relative categories and $F\colon \mathcal{C} \to \mathcal{D}$ is a functor of the underlying categories, we say that $F'\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is a *right-derived functor* of F if:

1. there is a natural transformation $\eta\colon \lambda_{\mathcal{D}} \circ F \to F' \circ \lambda_{\mathcal{C}}$, for $\lambda_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ and $\lambda_{\mathcal{D}}\colon \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$.

2. Any natural transformation $\lambda_{\mathcal{D}} \circ F \to G \circ \lambda_{\mathcal{C}}$ factors through η, and this factorisation is unique up to natural isomorphism in $\mathrm{Ho}(\mathcal{D})$ — this condition ensures that F' is unique up to weak equivalence.

In this case we denote F' by $\mathbf{R}F$. The dual notion is called *left-derived functor* and denoted by $\mathbf{L}F$.

Warning 2.4. The notation $\mathbf{R}F$ is also used to denote derived ∞-functors: $L_{\mathcal{W}}\mathcal{C} \to L_{\mathcal{V}}\mathcal{D}$.[15]

Under the derivator philosophy, for nice enough $\mathcal{C}$ and $\mathcal{D}$ this corresponds to having compatible derived functors $\mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{D}^I)$ for all small diagrams I.

Then most homology/cohomology theories arise as left/right derived functors.

Examples 2.5.

[15] See [Rie19, §4.1] for more on this view on derived functors. The results there are stated for homotopical categories, which are relative categories with extra restrictions (always satisfied in practice).

1. Consider the global sections functor Γ from sheaves of non-negative cochain complexes on a topological space X to cochain complexes of abelian groups. If we take weak equivalences being quasi-isomorphisms on both sides, then Γ has a right-derived functor $\mathbf{R}\Gamma$, whose cohomology groups are just sheaf cohomology.

2. For any category $\mathcal{C}$, consider the functor $\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set} \subset \mathrm{Top}$ (or simplicial sets, if you prefer). For a subcategory $\mathcal{W} \subset \mathcal{C}$ and for π_*-equivalences in Top, we get a right-derived functor, the derived mapping space $\mathbf{R}\mathrm{Map} : \mathrm{Ho}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathrm{Top})$. That's essentially how simplicial and topological categories are associated to relative categories — the spaces of morphisms in the topological category associated to the relative category $(\mathcal{C}, \mathcal{W})$ are then just $\mathbf{R}\mathrm{Map}(X, Y)$ (up to weak homotopy equivalence).

3. For a category $\mathcal{C}$ of chain complexes, we have an enriched Hom functor $\underline{\mathrm{Hom}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{CochainCpx}$ (with $\mathrm{Hom} = \mathrm{Z}^0\underline{\mathrm{Hom}}$). If we take weak equivalences to be quasi-isomorphisms on both sides, this then leads to a right-derived functor $\mathbf{R}\underline{\mathrm{Hom}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{CochainCpx}$, which has cohomology groups $H^i\mathbf{R}\underline{\mathrm{Hom}}(X, Y) \cong \mathbb{E}\mathrm{xt}^i(X, Y)$. The space $\mathbf{R}\mathrm{Map}$ is then just the topological space associated to the good truncation of this complex, which satisfies $\pi_j\mathbf{R}\mathrm{Map}(X, Y) \cong \mathrm{Ext}_{\mathcal{C}}^{-j}(X, Y)$ for $j \geq 0$.[16]

4. As a more exotic example, if F is the functor sending a topological space X to the free topological abelian group generated by X, then taking weak equivalences to be π_*-isomorphisms on both sides, we have a left-derived functor $\mathbf{L}F$ with homotopy groups $\pi_i\mathbf{L}F(X) \cong \mathrm{H}_i(X)$ given by singular homology.

2.4 Model categories

A standard reference for this section is [Hov99].

The idea is to endow a model category with extra structure aiding computations. This is similar in flavour to presentations of a group; once the weak equivalences are chosen all the homotopy theory is determined, but the extra structure (fibrations, cofibrations, etc.) makes it much more accessible.

Definition 2.6. A *model category* is a relative category $(\mathcal{C}, \mathcal{W})$ together with two choices of classes of morphisms, called *fibrations* and *cofibrations*. These classes of morphisms are required to satisfy several further axioms.

A *trivial (co)fibration* is a (co)fibration that is also in $\mathcal{W}$, i.e. also a weak equivalence.

Definition 2.7. Let $f : X \to Y$ be a morphism in a category and S a class of morphisms in this category. We say that f has the *left lifting property* with respect to S (LLP for short) if for any map $g : A \to B$ in S and any commutative diagram like below, there is a lift as indicated.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "g \in S"] \\ Y \arrow[r] \arrow[ur, dotted, "\exists"'] & B \end{tikzcd}$$

Dually, a map f has the *right lifting property* with respect to S (RLP for short) if the dual statement holds, i.e. where f and g are swapped in the above diagram.

[16]This last statement follows by combining the Dold–Kan equivalence with composition of right-derived functors, using that the right-derived functor of Z^0 is the good truncation $\tau^{\leq 0}$.

Example 2.8. Any category with limits and colimits has a trivial model structure, in which all morphisms are both fibrations and cofibrations, while the weak equivalences are just the isomorphisms.

Example 2.9 (Model structure on $\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$). There is a model structure on $\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$, due to Quillen [Qui69].[17]

On $\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$ weak equivalences are quasi-isomorphisms. Fibrations are maps which are surjective in strictly positive chain degree, i.e. $f : A_i \to B_i$ is surjective for all $i > 0$.

Cofibrations are maps $f : P_\bullet \to Q_\bullet$ which have the left lifting property with respect to trivial fibrations. Explicitly, if $Q_\bullet$ is *quasi-free* over $P_\bullet$ in the sense that it is freely generated as a graded-commutative algebra, then f is a cofibration. An arbitrary cofibration is a retract of a quasi-free map.

Example 2.10 (Model structure on $\mathrm{DG}^+\mathrm{Aff}$). When considering the opposite category $\mathrm{DG}^+\mathrm{Aff} = (\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})^{\mathrm{op}}$ one takes the opposite model structure, so cofibrations in $\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$ correspond to fibrations in $DG^+\mathrm{Aff}$ and vice versa.

Example 2.11. Another model structure is the projective model structure on non-negatively graded chain complexes of R-modules: Weak equivalences are quasi-isomorphisms, fibrations are surjective in strictly positive chain degrees, cofibrations are maps $f : M \hookrightarrow N$ such that N/M is a complex of projective R-modules.

The resulting homotopy category is the full subcategory of the derived category $\mathcal{D}(A)$ on non-negatively graded chain complexes.[18]

Example 2.12. Dually there is an injective model structure for non-negatively graded cochain complexes of R-modules: weak equivalences are quasi-isomorphisms, cofibrations have trivial kernel in strictly positive degrees, and fibrations are surjective maps with levelwise injective kernel.

The resulting homotopy category is the full subcategory of the derived category $\mathcal{D}(A)$ on non-negatively graded cochain complexes.

Remark 2.13. There are also $\mathbb{Z}$-graded versions of the two examples above, but cofibrations (resp. fibrations) have extra restrictions[19].

In both cases, the resulting homotopy category is the derived category $\mathcal{D}(A)$.

Remark 2.14. Here we list some of the key properties of model structures, though this is not an exhaustive list of required axioms:

- (Lifting A) Cofibrations have the LLP with respect to all trivial fibrations.
- (Lifting B) Trivial cofibrations have the LLP with respect to all fibrations.
- (Lifting B') Dually, fibrations have the RLP with respect to trivial cofibrations.
- (Lifting A') Dually, trivial fibrations have the RLP with respect to all cofibrations.

[17] Quillen's proof is for dg-Lie algebras, but he observed that the same proof works for other types of algebras. For associative algebras and other algebras over non-symmetric operads, our characteristic 0 hypothesis becomes unnecessary.

[18] Here, we are using "homotopy category" in the homotopy theory sense of inverting weak equivalences (i.e. quasi-isomorphisms); beware that this clashes with the usage in homological algebra which refers to the category $\mathbf{K}(A)$ of [Wei94, §10.1] in which only strong homotopy equivalences are inverted.

[19] Specifically in the projective case, there should exist an ordering on the generators x by some ordinal such that each δx lies in the span of generators of lower order. For complexes bounded below, we can just order by degree; in general, the total complex of a Cartan–Eilenberg resolution as in [Wei94, §5.7] is cofibrant.

- (Factorisation A) Every morphism $f : A \to B$ can be factorised as $A \to \tilde{A} \to B$ where the first map is a trivial cofibration and the second one a fibration. (In some respects, this can be regarded as a generalisation of injective resolutions.)
- (Factorisation B) Every morphism $f : A \to B$ can be factorised as $A \to \hat{B} \to B$ where the first map is a cofibration and the second one a trivial fibration. (In some respects, this can be regarded as a generalisation of projective resolutions.)

Examples 2.15. Let R be a commutative k-algebra, $a \in R$ not a zero-divisor, and consider the map $R \to R/(a) =: S$. There is a way of resolving this as $R \to \hat{S} \to S$ such that $R \to \hat{S}$ is a cofibration and $\hat{S} \to S$ a trivial fibration.

The explicit construction for this is by choosing $\hat{S} := (R[t], \delta t = a)$ so this is a chain complex of the form $0 \to Rt \xrightarrow{\delta} R$. The cofibration $R \to \hat{S}$ is just the canonical inclusion and the trivial fibration sends t to 0.

Remark 2.16. We will follow the modern convention for model categories in assuming that the factorisations A and B above can be chosen functorially. However, beware that the functorial factorisations tend to be huge.

Example 2.17. On topological spaces, there is a model structure in which weak equivalences are π_*-equivalences; note that these really are weak, not distinguishing between totally disconnected (e.g. p-adic) and discrete topologies. Fibrations are Serre fibrations, which have RLP with respect to the inclusions $S^n_+ \to B^{n+1}$ of the closed upper n-hemisphere in an n-ball, for $n \geq 0$ (see Figure 4). Cofibrations are then defined via LLP, or generated by $S^{n-1} \to B^n$ for $n \geq 0$ — these include all relative CW complexes.

Example 2.18. We've already seen commutative dg k-algebras in non-negative chain degree. There are variants for dg EFC and $\mathcal{C}^\infty$-algebras. Weak equivalences are quasi-isomorphisms. Fibrations are maps which are surjective in strictly positive chain degree, i.e. $f : A_i \to B_i$ is surjective for all $i > 0$.

Cofibrations are again defined by LLP, the property being satisfied whenever the morphism is freely generated as a graded EFC or $\mathcal{C}^\infty$-algebra. Free as a graded $\mathcal{C}^\infty$-algebra means $\mathcal{C}^\infty(\mathbb{R}^n)[x_1, x_2, \ldots]$, with $\deg x_i > 0$ (taking exterior powers for odd variables); nested unions of these are also free.

2.5 Computing the homotopy category using model structures

Definition 2.19. We say that an object in a model category is *fibrant* if the map to the final object is a fibration, and *cofibrant* if the map from the initial object is a cofibration.

Given an object A and a weak equivalence $A \to \hat{A}$ with $\hat{A}$ fibrant, we refer to $\hat{A}$ as a *fibrant replacement* of A. Dually, if we have a weak equivalence $\tilde{A} \to A$ with $\tilde{A}$ cofibrant, we refer to $\tilde{A}$ as a *cofibrant replacement* of A.

Example 2.20. With the model structure on dg_+Alg_k from example 2.9 every object is fibrant.

Definition 2.21. Given a fibrant object X, a *path object* PX for X is an object PX together with a diagram

$$\begin{tikzcd} X \arrow[r, "w"] \arrow[dr, "\text{diag.}"'] & PX \arrow[d, "f"] \\ & X \times X \end{tikzcd}$$

where w a weak equivalence and f a fibration.

Remark 2.22. Note that path objects always exist, by applying the factorisation axiom in 2.14 to the diagonal $X \to X \times X$.

Theorem 2.23 (Quillen)**.** *Let $A \in \mathcal{C}$ be a cofibrant object and $X \in C$ a fibrant object. Morphisms in the homotopy category* $\mathrm{Ho}(\mathcal{C})$ *are given by* $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, X)$ *being the co-equaliser (i.e. quotient) of the diagram*

$$\mathrm{Hom}_{\mathcal{C}}(A, PX) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(A, X)$$

induced by the two possible projections $PX \to X \times X \rightrightarrows X$.

Example 2.24. In the model category $dg\mathrm{Mod}(A)$ of unbounded chain complexes of A-modules, a path object PM for M is given by $(PM)_n := M_n \oplus M_n \oplus M_{n+1}$, with $\delta(a, b, c) = (\delta a, \delta b, \delta c + (-1)^n(a-b))$. The map $M \to PM$ is $a \mapsto (a, a, 0)$, and the map $PM \to M \times M$ is $(a, b, c) \mapsto (ab)$.

Thus for Q levelwise projective, two morphisms $f, g \colon Q \to M$ are homotopic if and only if there exists a graded morphism $h \colon Q \to M[-1]$ such that $f - g = \delta \circ h + h \circ \delta$.

To modify this example for chain complexes concentrated in non-negative degrees, apply the good truncation $\tau_{\geq 0}$ in non-negative chain degrees[20], so that PM is still in the same category; the description of homotopic morphisms is unaffected.

Example 2.25. In topological spaces, we can just take PX to be the space of paths in X, i.e. the space of continuous maps $[0, 1] \to X$, with

$$X \xrightarrow{\text{constant}} PX \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} X \times X.$$

Thus morphisms in the homotopy category are just homotopy classes of morphisms.

Example 2.26. In $dg_+\mathrm{Alg}_k$, a choice of path object is given by taking $PA = \tau_{\geq 0}(A[t, \delta t])$, for t of degree 0. The map $A \to PA$ is the inclusion of constants, and the map $PA \to A \times A$ given by $a(t) \mapsto (a(0), a(1))$ and $b(t)\delta t \mapsto 0$.

Explicitly,

$$(PA)_n = \begin{cases} A_n[t] \oplus A_{n+1}[t]\delta t & n > 0 \\ \ker(\delta \colon A_0[t] \oplus A_{n+1}[t]\delta t \to A_0[t]\delta t & n = 0, \end{cases}$$

where $\delta(\sum_i a_i t^i) = \sum_i (\delta a_i) t^i + \sum_i (-1)^{\deg a_i} i a_i t^{i-1} \delta t$.

Thus for C cofibrant, $\mathrm{Hom}_{\mathrm{Ho}(dg_+\mathrm{Alg}_k)}(C, A)$ is the quotient

$$\mathrm{Hom}_{dg_+\mathrm{Alg}_k}(C, A)/\mathrm{Hom}_{dg_+\mathrm{Alg}_k}(C, PA).$$

Taking a cofibrant replacement can be nuisance, but there are Quillen equivalent model structures with more cofibrant objects but fewer fibrant objects, with the fibrant replacement functor for a CDGA A being its completion, Henselisation or localisation over $\mathrm{H}_0 A$;[21]existence of all these follows from [Pri09, Lemma 6.37], with details for the complete case in [Pri10b, Proposition 2.7] and the others (and $\mathcal{C}^\infty$ and analytic versions) in [Pri18b, Proposition 3.12]. For the complete and Henselian model structures, all smooth k-algebras are cofibrant.

[20]Explicitly, this means $(\tau_{\geq 0} V)_i = \begin{cases} V_i & i > 0 \\ \mathrm{Z}_0 V & i = 0 \\ 0 & i < 0 \end{cases}$, where $\mathrm{Z}_0 V = \ker(\delta \colon V_0 \to V_1)$.

[21]Specifically, cofibrations in the local (resp. Henselian) model structure are generated by cofibrations in the standard model structure together with localisations (resp. étale morphisms). Fibrations are those fibrations $A \to B$ in the standard model structure for which $A_0 \to B_0 \times_{H_0 B} \mathrm{H}_0 A$ is conservative (resp. Henselian) in the terminology of [Ane09, §4]. The identity functor from the standard model structure to the local or Henselian model structure is then a left Quillen equivalence.

Example 2.27. For dg $\mathcal{C}^\infty$-algebras, a similar description applies, with $PA = \tau_{\geq 0}(A \odot \mathcal{C}^\infty(\mathbb{R})[\delta t])$, with $t \in \mathcal{C}^\infty(\mathbb{R})$ the co-ordinate, and $\odot$ the $\mathcal{C}^\infty$ tensor product, given by

$$(\mathcal{C}^\infty(\mathbb{R}^m)/(f_1, f_2, \ldots)) \odot (\mathcal{C}^\infty(\mathbb{R}^n)/(g_1, g_2, \ldots)) \cong (\mathcal{C}^\infty(\mathbb{R}^{m+n})/(f_1, g_1, f_2, g_2, \ldots)),$$

so in particular $\mathcal{C}^\infty(X) \odot \mathcal{C}^\infty(Y) \cong \mathcal{C}^\infty(X \times Y)$.

Again, the map $A \to PA$ is given by inclusion of constants, and the map $PA \to A \times A$ by evaluation at $t = 0$ and $t = 1$.

There is an entirely similar description for EFC algebras using analytic functions.

2.6 Derived functors

A way to compute derived functors:

Definition 2.28. A functor $G \colon \mathcal{C} \to \mathcal{D}$ of model categories is *right-Quillen* if it has a left-adjoint F and preserves fibrations and trivial fibrations.

Dually, F is *left-Quillen* if it has a right-adjoint and F preserves cofibrations and trivial cofibrations.

$F \dashv G$ is in that case called a *Quillen adjunction*.

Indeed a Quillen adjunction is well-defined:

Lemma 2.29. *Let $F \dashv G$ be an adjunction of functors of model categories. F is left-Quillen if and only if G is right-Quillen*

Theorem 2.30 (Quillen)**.** *If G is right Quillen, then the right-derived functor $\mathbf{R}G$ exists and is given on objects by $A \mapsto G\hat{A}$, for $A \to \hat{A}$ a fibrant replacement.*

Dually, left Quillen functors give left-derived functors by cofibrant replacement. Left Quillen dually.

Remark 2.31. To get a functor, we can take fibrant replacements functorially, but on objects the choice of fibrant replacement doesn't matter (and in particular need not be functorial), because it turns out that right Quillen functors preserve weak equivalences between fibrant objects. The proof is an exercise with path objects.

Example 2.32. We can thus interpret sheaf cohomology in terms of derived functors, because fibrant replacement in the model category of non-negatively graded cochain complexes of sheaves corresponds to taking an injective resolution.

Definition 2.33. A Quillen adjunction $F \dashv G$ is said to be a Quillen equivalence if $\mathbf{R}G \colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is an equivalence of categories, with quasi-inverse $\mathbf{R}F$.

Explicitly, this says that for all fibrant objects $A \in \mathcal{C}$ and cofibrant objects $B \in \mathcal{D}$, the unit and co-unit give rise to weak equivalences $F(\widetilde{GA}) \to A$ and $B \to G(\widehat{FB})$, where $\widehat{()}, \widetilde{()}$ are fibrant and cofibrant replacement.

Note that this implies that $\mathrm{Ho}(\mathcal{C}) \simeq \mathrm{Ho}(\mathcal{D})$, and gives weak equivalences on $\mathbf{R}\mathrm{Map}$.

2.7 Homotopy limits and fibre products

Definition 2.34. Homotopy limits $\mathrm{ho}\varprojlim_I$ or $\mathbf{R}\varprojlim_I$ are right-derived functors of the limit functors $\varprojlim_I \colon \mathcal{C}^I \to \mathcal{C}$ (weak equivalences in $\mathcal{C}^I$ defined objectwise).

For diagrams of the form $X \to Y \leftarrow Z$, we denote the homotopy fibre product by $X \times^h_Y Z$.

Lemma 2.35. *If Y is fibrant, the homotopy fibre product $X \times_Y^h Z$ is given by $\hat{X} \times_Y \hat{Z}$, where $\hat{X} \to Y$ and $\hat{Z} \to Y$ are fibrant replacements over Y. In "right proper" model categories (almost everything we work with), it suffices to take $\hat{X} \times_Y Z$.*

Example 2.36. An explicit construction of $X \times_Y^h Z$ is given by $X \times_{Y,\mathrm{ev}_0} PY \times_{\mathrm{ev}_1,Y} Z$. In particular, for topological spaces we get $\{x\} \times_Y^h \{z\} = P(Y; x, z)$, the space of paths from x to z. Hence $\{y\} \times_Y^h \{y\} = \Omega(Y; y)$, the space of loops based at Y.

This leads to a long exact sequence

$$\pi_i(X \times_Y^h Z) \to \pi_i X \times \pi_i Y \to \pi_i Z \to \pi_{i-1}(X \times_Y^h Z) \to \ldots \to \pi_0 Z$$

of homotopy groups and sets.

Example 2.37. Similarly, in cdgas, we get a long exact sequence

$$\mathrm{H}_i(A \times_B^h C) \to \mathrm{H}_i A \times \mathrm{H}_i B \to \mathrm{H}_i C \to \mathrm{H}_{i-1}(A \times_B^h C) \to \ldots.$$

We can evaluate this as $A \times_B PB \times_B C$, though $\hat{A} \times_B C$ for any fibrant replacement $\hat{A} \to B$ will do.

3 Consequences for dg algebras

We have an embedding of algebras in cdgas

$$\begin{aligned}\mathrm{Alg}_k &\subset \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}\\ A &\mapsto (A \leftarrow 0 \leftarrow 0 \leftarrow \dots)\end{aligned}$$

which induces a map $\mathrm{Alg}_k \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})$ by composition with the map $\mathrm{dg}_+\mathrm{Alg}_\mathrm{k} \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})$.

Lemma 3.1. *The induced functor* $\mathrm{Alg}_k \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})$ *is full and faithful.*

Proof. First, we observe the following. For any $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$ and $B \in \mathrm{Alg}_k$ we have

$$\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}(A_\bullet, B) = \mathrm{Hom}_{\mathrm{Alg}_k}(\mathrm{H}_0(A_\bullet), B)$$

because for any $f \in \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}(A_\bullet, B)$ anything positive $a \in A_{>0}$ has to map to zero $f(a) = 0 \in B_i$ and thus $f(\delta a') = \delta f(a') = 0$ for all $a' \in A_1$. In particular we can also replace $A_\bullet$ with a cofibrant replacement $\tilde{A}_\bullet$ to obtain

$$\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}(\tilde{A}_\bullet, B) = \mathrm{Hom}_{\mathrm{Alg}_k}(\mathrm{H}_0(\tilde{A}_\bullet), B) = \mathrm{Hom}_{\mathrm{Alg}_k}(\mathrm{H}_0(A_\bullet), B)$$

Next, we observe that for any $B \in \mathrm{Alg}_k$ the map $B \to B \times B$ is a fibration (as there is nothing in positive degrees), so any such B is a path object for itself.

With these two observations we can show that the functor is full; let $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$ (for the proof it would be enough to take $A \in \mathrm{Alg}_k$). and $B \in \mathrm{Alg}_k$. We calculate

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})}(A_\bullet, B) &= \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})}(\tilde{A}_\bullet, B)\\ &= \mathrm{coeq}(\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}(\tilde{A}_\bullet, B) \rightrightarrows \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}(\tilde{A}_\bullet, B))\\ &= \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}(\tilde{A}_\bullet, B)\\ &= \mathrm{Hom}_{\mathrm{Alg}_k}(H_0(A_\bullet), B)\end{aligned}$$

The second step is theorem 2.23 together with the observation that B is a path object for itself. Step three is then the observation that both maps in this coequaliser are simply the identity.

Faithfulness follows because for $A \in \mathrm{Alg}_k$, we have $\mathrm{H}_0(A) = A$. □

Remark 3.2. The same result and proof hold for $\mathcal{C}^\infty$-rings and EFC-rings.

Remark 3.3. Geometrically, we can rephrase that statement as saying that given an affine scheme X and a derived affine scheme Y, we have

$$\mathrm{Hom}_{\mathrm{Ho}(DG^+\mathrm{Aff})}(X, Y) \cong \mathrm{Hom}_{\mathrm{Aff}}(X, \pi^0 Y);$$

a similar statement holds for non-affine X and Y.

3.1 Derived tensor products (derived pullbacks and intersections)

Definition 3.4. Let $A_\bullet, B_\bullet \in dg_+\mathrm{Alg}_k$. The *graded tensor product* $A_\bullet \otimes_k B_\bullet$ is defined by

$$(A_\bullet \otimes_k B_\bullet)_n = \bigoplus_{i+j=n} A_i \otimes_k B_j$$

with differential $\delta(aa \otimes b) := \delta a + (-1)^{\deg(a)} \delta b$, and multiplication $(a \otimes b) \cdot (a' \otimes b') := (-1)^{\deg(a')\deg(b)}(aa' \otimes bb')$.

Lemma 3.5. *The functor $\otimes_k : \mathrm{dg_+Alg_k} \times \mathrm{dg_+Alg_k} \to \mathrm{dg_+Alg_k}$ is left-Quillen, with right adjoint $A \mapsto (A, A)$.*

Proof. It is immediate to see that this is the correct right adjoint functor:

$$\mathrm{Hom}_{\mathrm{dg_+Alg_k}}(A_\bullet \otimes_k B_\bullet, C_\bullet) \cong \mathrm{Hom}_{\mathrm{dg_+Alg_k} \times \mathrm{dg_+Alg_k}}((A_\bullet, B_\bullet), (C_\bullet, C_\bullet))$$

This right adjoint is right-Quillen as it clearly preserves fibrations and trivial fibrations. Thus, by lemma 2.29 the left-adjoint is left-Quillen. □

Section 2.6 told us that a functor F being left-Quillen means that the left-derived functor LF exists.

Definition 3.6. Define $\otimes_k^{\mathbf{L}} : \mathrm{Ho}(\mathrm{dg_+Alg_k}) \times \mathrm{Ho}(\mathrm{dg_+Alg_k}) \to \mathrm{Ho}(\mathrm{dg_+Alg_k})$ to be the left-derived functor of $\otimes_k$: $dg_+\mathrm{Alg}_k \times dg_+\mathrm{Alg}_k \to dg_+\mathrm{Alg}_k$.

Remark 3.7. Recall that the base k can be any $\mathbb{Q}$-algebra, not only a field. Therefore this construction is less trivial that it might seem at first glance.

From this one could expect that one would need to take cofibrant replacements on both sides to calculate $\otimes_k^{\mathbf{L}}$, which could be really complicated. The following simplifying lemma shows that one gets away with much less.

Definition 3.8. Given a cdga A and an A-module M in chain complexes, say that M is quasi-flat if the underlying graded module is flat over the graded algebra underlying A.[22]

Lemma 3.9. *To calculate $A_\bullet \otimes_k^{\mathbf{L}} B_\bullet$ it is enough to take a quasi-flat replacement of one of the two factors. In particular, if $A_\bullet$ is a complex of flat k-modules, then $A_\bullet \otimes_k B_\bullet$ is a model*[23] *for $A_\bullet \otimes_k^{\mathbf{L}} B_\bullet$.*

Proof. The assumptions imply that the i^{th} homology groups of the tensor product are simply

$$H_i(A_\bullet \otimes_k B_\bullet) = \mathrm{Tor}_i^k(A_\bullet, B_\bullet)$$

Now if $\tilde{A}_\bullet, \tilde{B}_\bullet$ are cofibrant replacements, they also satisfy the flatness condition, so we get

$$H_i(A_\bullet \otimes_k^{\mathbf{L}} B_\bullet) = H_i(\tilde{A}_\bullet \otimes_k \tilde{B}_\bullet) = \mathrm{Tor}_i^k(A_\bullet, B_\bullet)$$

Therefore $A_\bullet \otimes_k^{\mathbf{L}} B_\bullet \to A_\bullet \otimes_k B_\bullet$ is a quasi-isomorphism. □

One can generalise this result by choosing an arbitrary base $C_\bullet \in \mathrm{dg_+Alg_k}$ instead of k. This just induces another grading but the proof goes through the same way:

Lemma 3.10. *If $A_\bullet$ is quasi-flat over $C_\bullet$, then $A_\bullet \otimes_{C_\bullet} B_\bullet \simeq A_\bullet \otimes_{C_\bullet}^{\mathbf{L}} B_\bullet$.*

In the opposite category we denote these as *homotopy-pullbacks*, i.e. we write $X \times_Z^h Y := \mathrm{Spec}\,(A_\bullet \otimes_{C_\bullet}^{\mathbf{L}} B_\bullet)$ where $X = \mathrm{Spec}\,(A_\bullet), Y = \mathrm{Spec}\,(B_\bullet), Z = \mathrm{Spec}\,(C_\bullet)$.

Example 3.11. Consider the self-intersection

$$\{0\} \times_{\mathbb{A}^1}^h \{0\}$$

of the origin in the affine line, or equivalently look at $k \otimes_{k[t]}^{\mathbf{L}} k$. There is a quasi-flat (in fact cofibrant) resolution of k over $k[t]$ given by $(k[t] \cdot s \to k[t])$ with $\delta s = 1$. In other words,

[22] we say "quasi-flat" rather than just "flat" to avoid a clash with Definition 3.44

[23] in other words, $A_\bullet \otimes_k B_\bullet$ (which is defined up to isomorphism) is quasi-isomorphic to $A_\bullet \otimes_k^{\mathbf{L}} B_\bullet$ (which is defined up to quasi-isomorphism)

this is the graded algebra $k[t,s]$ with $\deg(t) = 0$, $\deg(s) = 1$ and $\delta s = 1$. (and since we are in a commutative setting we automatically have $s^2 = 0$). We calculate

$$k[t,s] \otimes_{k[t]} k = k[s]$$

where $\deg(s) = 1$ and $\delta s = 0$.

The underived intersection corresponds to an underived tensor product, taking H_0 of this to just give k, corresponding to $\operatorname{Spec} k \cong \{0\}$. On the other hand, the virtual number of points of this derived scheme $\operatorname{Spec} k[s]$ is given by taking the Euler characteristic, giving $1 - 1 = 0$, so we can think of this as a negatively thickened point.

It also makes sense to talk of the virtual dimension of an example such as this, informally given by taking the Euler characteristic of the generators. Since s is in odd degree, the virtual dimension of $\operatorname{Spec} k[s]$ is -1, which is consistent with the usual rules for intersections.

Example 3.12. More generally, we can look at the derived intersection $\{a\} \times^h_{\mathbb{A}^1} \{0\} = \operatorname{Spec}(k \otimes^{\mathbb{L}}_{a,k[t],0} k)$.

By Lemma 3.9, to compute $k \otimes^{\mathbb{L}}_{k[t]} k$ we need to replace one of the copies of k with a quasi-flat $k[t]$-algebra that is quasi-isomorphic to k. For this, consider the cdga A generated by variables t, s with $\deg(t) = 0$ and $\deg(s) = 1$ and differential defined by $\delta s = t - a$. We have $A_0 = k[t]$ and $A_1 = k[t]s$ and $A_i = 0$ for $i > 1$. Thus the morphism $f\colon k[t] \to A$ is quasi-flat, and is in fact cofibrant: A is free as a graded algebra over $k[t]$. Now we can compute the derived intersection

$$\{a\} \times_{\mathbb{A}^1} \{0\} = \operatorname{Spec}(k[t]/(t-a) \otimes^{\mathbb{L}}_{k[t],0} k) = \operatorname{Spec}(A \otimes_{k[t],0} k) = \operatorname{Spec}(k[s], \delta s = -a).$$

When a is a unit, this means the derived intersection is quasi-isomorphic to $\operatorname{Spec} 0 = \emptyset$, but when $a = 0$ we have $k[s] = k \oplus k.s$ with $\delta s = 0$.

The Euler characteristic of $k[s]$ is equal to zero, regardless of δ. If we think of the Euler characteristic of a finite dimensional cdga as the (virtual) number of points, then this corresponds to our intuition for intersecting two randomly chosen points in $\mathbb{A}^1$.

Contrast this with the classical intersection, which is not constant under small changes, since $\{a\} \times_{\mathbb{A}^1} \{0\}$ is $\emptyset$ if $a \neq 0$ and $\{0\}$ if $a = 0$. Our derived self-intersection is categorifying Serre's intersection numbers [Ser65].

Definition 3.13. Denote the *derived loop space* of $X \in \mathrm{DG}^+\mathrm{Aff}$ as $\mathcal{L}X := X \times^h_{X \times X} X$, i.e. the pullback via the diagonal.[24]

Example 3.14. Look at $\mathcal{L}\mathbb{A}^1 = \mathbb{A}^1 \times^h_{\mathbb{A}^1 \times \mathbb{A}^1} \mathbb{A}^1$, i.e. a self-intersection of a line in a plane. Equivalently, we are looking at

$$(k[x,y]/(x-y)) \otimes^L_{k[x,y]} (k[x,y]/(x-y)).$$

A cofibrant replacement for $k[x,y]/(x-y)$ over $k[x,y]$ is given by $k[x,y,s]$ with $\deg(x) = \deg(y) = 0$, $\deg(s) = 1$ and $\delta s = x - y$. Then $\mathbb{A}^1 \times^h_{\mathbb{A}^1 \times \mathbb{A}^1} \mathbb{A}^1$ is $\operatorname{Spec}(k[x,s])$ with $\deg(s) = 1$ and $\delta s = x - x = 0$.

More generally, what happens if we take the loop space $X \times^h_{X \times X} X$ in $DG^+\mathrm{Aff}$?

[24]These loop spaces don't look like loop spaces in topology; the reason is that here the notion of equivalence is a completely different one.

Example 3.15. For any smooth affine scheme X of dimension d, we can calculate $\mathcal{L}X$ as

$$\mathcal{L}X = \mathrm{Spec}\,(\mathscr{O}_X \xleftarrow{0} \Omega^1_X \xleftarrow{0} \Omega^2_X \ldots \xleftarrow{0} \Omega^d_X).$$

This is a strengthening of the HKR isomorphism; for more details and generalisations, see e.g. [BZN07, TV09][25], which were inspired by precursors in the supergeometry literature, where the right-hand side corresponds to $\Pi TX = \mathrm{Map}(\mathbb{R}^{0|1}, X)$, as in [Kon94a, Lectures 4 & 5] or [Kon97, §7].

3.1.1 Analogues in differential and analytic contexts

There are analogues of derived tensor products for the C^∞-case and EFC-case; one needs to tweak things slightly but not very much.

The basic problem is that the abstract tensor product of two rings of smooth or analytic functions won't be a ring of smooth or analytic functions. So there are $\mathcal{C}^\infty$ and EFC tensor products $\odot$ as in Example 2.27, satisfying

$$\mathcal{C}^\infty(X) \,\odot\, \mathcal{C}^\infty(Y) = \mathcal{C}^\infty(X \times Y)$$

and similarly for EFC-rings. To extend these to dg-rings, we set

$$A_\bullet \odot B_\bullet := A_\bullet \otimes_{A_0} (A_0 \odot B_0) \otimes_{B_0} B_\bullet;$$

for example

$$\mathcal{C}^\infty(X)[s_1, s_2, \ldots] \,\odot\, \mathcal{C}^\infty(Y)[t_1, t_2, \ldots] = \mathcal{C}^\infty(X \times Y)[s_1, t_1, s_2, t_2, \ldots]$$

There are similar expressions for EFC rings (and any indeed any Fermat theory in the sense of [CR12, DK84]).

3.2 Tangent and obstruction spaces

An area where derived techniques are particularly useful is obstruction theory. To begin with, we recall the dual numbers and how they give rise to tangent spaces.

Definition 3.16. We define the *dual numbers* by $k[\epsilon]$ with $\deg(\epsilon) = 0$ and $\epsilon^2 = 0$, so $k[\epsilon] = k \oplus k\epsilon$.

Remark 3.17. Note that this is naturally a $\mathcal{C}^\infty$-ring when $k = \mathbb{R}$ and an a EFC-ring when $k = \mathbb{C}$, since

$$\mathcal{C}^\infty(\mathbb{R})/(t^2) \cong \mathbb{R}[\epsilon], \qquad \mathscr{O}^{\mathrm{hol}}(\mathbb{C})/(z^2) \cong \mathbb{C}[\epsilon].$$

for co-ordinates t on $\mathbb{R}$ and z on $\mathbb{C}$.

Construction 3.18. If X is a smooth scheme, a $\mathcal{C}^\infty$-space (e.g. a manifold) or a complex analytic space (e.g. a complex manifold), then maps $\mathrm{Spec}\,(k[\epsilon]) \to X$ correspond to *tangent vectors*. That means

$$X(k[\epsilon]) \cong \{(x, v) \;:\; x \in X(k),\; v \text{ a tangent vector at } x\}.$$

i.e. the set of $k[\epsilon]$-valued points forms a *tangent space*. More generally, for any ring A and any A-module I, we have that $X(A \oplus I)$ consists of I-valued *tangent vectors* at A-valued points of X.

[25]The cotangent complex $\mathbb{L}_A$ of §3.4 gives a generalisation to all cdgas A, with $A \otimes^{\mathbf{L}}_{A \otimes^{\mathbf{L}} A} A \simeq \bigoplus_p \Lambda^p \mathbb{L}_A[p]$. The easiest way to prove this is to observe that the functors have derived right adjoints sending B to $B \times^h_{B \times B} B$ and $B \oplus B[1]$ respectively; for PB as in Example 2.26, inclusion of constants then gives a quasi-isomorphism $B \oplus B[1] \to PB \times_{B \times B} B \simeq B \times^h_{B \times B} B$.

In this construction the ring $A \oplus I$ has multiplication determined by setting $I \cdot I = 0$.

Definition 3.19. A *square-zero extension* of commutative rings is a surjective map $f\colon A \twoheadrightarrow B$ such that $xy = 0$ for all $x, y \in \ker(f)$.

Notation 3.20. For the rest of this section we define $I := \ker(f)$ where $f : A \twoheadrightarrow B$ is always the square-zero extension under consideration.

Note that any nilpotent surjection of rings can be written as a composite of finitely many square-zero extensions, which is why deformation theory focuses on the latter.

There is a way of thinking about square-zero extensions in terms of torsors. Note that

$$\begin{aligned} A \times_B A &\cong A \times_B (B \oplus I) \\ (a, a') &\mapsto (a, (f(a), a - a')) \end{aligned}$$

which is a ring homomorphism.

For a smooth scheme X this means that

$$\begin{aligned} X(A) \times_{X(B)} X(A) &\cong X(A \times_B A) \\ &\cong X(A \times_B (B \oplus I)) \\ &\cong X(A) \times_{X(B)} X(B \oplus I) \end{aligned}$$

so we get I-valued tangent vectors (from the tangent space $X(B \oplus I)$) acting transitively on the fibres of $X(A) \to X(B)$.

Note that $X(A) \to X(B)$ is only surjective for X smooth (assuming finite type). Singularities in X give obstructions to lifting B-valued points to A-valued points. It had long been observed that obstruction spaces tend to exist, measuring this failure to lift. Specifically, the image of $X(A) \to X(B)$ tends to be the vanishing locus of a section of some bundle over $X(B)$, known as the obstruction space.

Here is an analogy with homological algebra. If $f : A^\bullet \twoheadrightarrow B^\bullet$ is a surjective map of cochain complexes with kernel $I^\bullet$, then in the derived category we have a map $B^\bullet \to I^\bullet[1]$ with homotopy kernel $A^\bullet$. For instance, the image of $H^0(B^\bullet) \to H^0(I^\bullet[1]) = H^1(I^\bullet)$ gives the obstruction to lifting elements from $H^0(B^\bullet)$ to $H^0(A^\bullet)$.

Now we want to construct a non-abelian version of this, leading to the miracle of derived deformation theory: that tangent spaces are obstruction spaces. This accounts for the well-known phenomenon that when a tangent space is given by a cohomology group, the natural obstruction space tends to be the next group up.

Almost everything we have seen in the lectures so far is essentially due to Quillen. However, the first instance of our next argument is apparently [Man99, proof of Theorem 3.1, step 3], although its consequences already featured in [Ill71, III 1.1.7], with a more indirect proof.

We start with an analogue of the homological construction above. Given a square-zero extension[26] $A \twoheadrightarrow B$ with kernel I, let $\tilde{B}_\bullet := \mathrm{cone}(I \to A)$, i.e. $\tilde{B}_\bullet = (A \hookleftarrow I \leftarrow 0 \leftarrow \ldots) \in dg_+\mathrm{Alg}_k$; the multiplication on $\tilde{B}_\bullet$ is the obvious one. There is a natural quasi-isomorphism $\tilde{B}_\bullet \to B$.

Now we have a cdga map $u : \tilde{B}_\bullet \to (B \xleftarrow{0} I \leftarrow 0 \leftarrow \ldots) =: B \oplus I[1]$ where we just kill the image of I.[27] Observe that u is surjective and that

$$\tilde{B}_\bullet \times_{u,(B \otimes I[1]),0} B = A$$

[26]For simplicity, you can assume that A and B are commutative rings, but exactly the same argument holds for cdgas and for (dg) $\mathcal{C}^\infty$ or EFC rings.

[27]This is where we need I to be square-zero; otherwise, the map would not be multiplicative.

which gives us

$$A = \tilde{B}_\bullet \times^h_{B\otimes I[1]} B \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}\,. \qquad (\dagger)$$

For a sufficiently nice functor on $\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$, we can use this to generate obstructions to lifting elements. The first functors we can look at are representable functors on the homotopy category $\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})$, i.e. $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})}(S,-)$ for cdgas S, the functors associated to derived affine schemes.

Limits in the homotopy category tend not to exist, but we do have homotopy fibre products, which have a weak limit property and permit the following definition (c.f [Hel81]).

Definition 3.21. A functor $F : \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}) \to \mathsf{Set}$ is *half-exact*[28] if for any $A_\bullet, B_\bullet, C_\bullet \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$ we have

1. $F(0) \cong *$,
2. $F(A_\bullet \times B_\bullet) \cong F(A_\bullet) \times F(B_\bullet)$,
3. $F(A_\bullet \times^h_{B_\bullet} C_\bullet) \twoheadrightarrow F(A_\bullet) \times_{F(B_\bullet)} F(C_\bullet)$.

Lemma 3.22. *Any representable functor F on $\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})$ is half-exact.*

Proof (sketch). The reason for this is that $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_\mathrm{k})}(S,-)$ is given by path components π_0 of a topological space-valued functor $\mathbf{R}\mathrm{Map}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}(S,-)$, with the latter preserving homotopy limits. The first two properties then follow quickly, with the final property following by noting that if we take a homotopy fibre product of spaces, then its path components map surjectively onto the fibre product of the path components:

$$\pi_0(X \times^h_Y Z) \twoheadrightarrow \pi_0(X) \times_{\pi_0(Y)} \pi_0(Z). \qquad \square$$

Remark 3.23. It will turn out that non-affine geometric objects such as derived schemes and stacks still satisfy a weakened half-exactness property, with the final condition only holding when $A \to B$ is a nilpotent surjection, which is all we will need for the consequences in this section to hold.

Returning to the obstruction question, if we apply a half-exact F to our square-zero extension $A \twoheadrightarrow B$, then the expression $(\dagger)$ gives

$$\begin{aligned} F(A) \twoheadrightarrow & F(\tilde{B}_\bullet) \times_{u,F(B\oplus I[1]),0} F(B) \\ & \cong F(B) \times_{u,F(B\oplus I[1]),0} F(B), \end{aligned}$$

so the theory has given us a map $u : F(B) \to F(B \oplus I[1])$ such that

$$u(x) = (x,0) \quad \text{if and only if} \quad x \in \mathrm{Im}\,(F(A) \to F(B)).$$

In other words, by working over $\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$, we have acquired an obstruction theory

$$(F(B \oplus I[1]), u)$$

for free. In contrast to classical deformation theory, this means obstruction spaces exist automatically in derived deformation theory.

Remark 3.24. Whereas $F(B \oplus I)$ is a tangent space over $F(B)$, we think of $F(B \oplus I[1])$ as a higher degree tangent space. In due course, we'll work with tangent complexes instead of tangent spaces, and this then becomes the first cohomology group H^1.

[28]If restricting to Artinian objects, readers may notice the similarity of the resulting half-exactness property to Schlessinger's conditions [Sch68] in the underived setting (also see [Gro60, Art74]), and to Manetti's characterisation of extended deformation functors in [Man99].

3.3 Postnikov towers

Pick for this entire section a cdga $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$. Postnikov towers will give us the justification for thinking of derived structure as being infinitesimal.

Notation 3.25. We recall the notations $\mathrm{B}_n A := \mathrm{Im}\,(\delta : A_{n+1} \to A_n)$ for the image of the differential and $\mathrm{Z}_n A := \ker(\delta : A_n \to A_{n-1})$ for the kernel. In particular we have that $\mathrm{B}_n A \cong A_{n+1}/\mathrm{Z}_{n+1}A$ and $H_n(A_\bullet) = \mathrm{Z}_n/\mathrm{B}_n$.

Definition 3.26. The *n^{th} coskeleton* $(\mathrm{cosk}_n A)_\bullet \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$ of $A_\bullet$ is given by

$$(\mathrm{cosk}_n A)_i = \begin{cases} A_i & i < n+1 \\ \mathrm{Z}_n A & i = n+1 \\ 0 & i > n+1 \end{cases}$$

with the differential in degrees $i < n$ being the differential of $A_\bullet$ (i.e. $\delta_{(\mathrm{cosk}_n A)} = \delta_A : A_{i+1} \to A_i$) and the differential in degree n being $\delta_{(\mathrm{cosk}_n A)} : (\mathrm{cosk}_n A)_{n+1} \to (\mathrm{cosk}_n A)_n$ given by the inclusion $\mathrm{Z}_n A \to A_n$. The multiplication on $(\mathrm{cosk}_n A)_\bullet$ is given by

$$a \cdot b = \begin{cases} ab & \deg(a) + \deg(b) < n+1 \\ \delta_A(ab) & \deg(a) + \deg(b) = n+1 \\ 0 & \deg(a) + \deg(b) > n+1 \end{cases}$$

The canonical map $A_\bullet \to (\mathrm{cosk}_n A)_\bullet$ is given in degree $n+1$ by $\delta_A : A_{n+1} \to \mathrm{Z}_n A$ and by the identity in degrees $\leq n$.

Remark 3.27. The idea of coskeleta is to give quotients truncating $A_\bullet$ without changing its lower homology groups, i.e. $H_i((\mathrm{cosk}_n A)_\bullet) = H_i(A_\bullet)$ for $i < n$ and $H_i((\mathrm{cosk}_n A)_\bullet) = 0$ for $i \geq n$.

The following gives an adjoint characterisation of the coskeleton:

Lemma 3.28. $\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}(A_\bullet, (\mathrm{cosk}_n B)_\bullet) \cong \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_\mathrm{k}}((A_{\leq n})_\bullet, B_\bullet)$, *where $(A_{\leq n})_\bullet$ is the* brutal truncation *in degrees $\leq n$ (also known as the* n-skeleton*).*

Definition 3.29. Let $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$. The *Moore-Postnikov tower* is the family of cdga $\{(P_n A)_\bullet\}_{n\in\mathbb{N}}$ given by $(P_n A)_\bullet = \mathrm{Im}\,(A_\bullet \to (\mathrm{cosk}_n A)_\bullet) = \mathrm{Im}\,((\mathrm{cosk}_{n+1} A)_\bullet \to (\mathrm{cosk}_n A)_\bullet) \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$, so

$$(P_n A)_i = \begin{cases} A_i & i \leq n \\ \mathrm{B}_n A & i = n+1 \\ 0 & i > n+1. \end{cases}$$

Remark 3.30. We then have maps

$$A_\bullet \to \ldots \to (P_n A)_\bullet \to (P_{n-1} A)_\bullet \to \ldots \to (P_0 A)_\bullet.$$

Lemma 3.31. *The morphism $(P_n A)_\bullet \to (P_{n-1} A)_\bullet$ is the composition of a trivial fibration and a square-zero extension.*

Proof. Define $C_\bullet \in \mathrm{dg}_+\mathrm{Alg}_\mathrm{k}$ by

$$C_i := \begin{cases} A_i & i < n \\ A_n/\mathrm{B}_n A & i = n \\ 0 & i > n, \end{cases}$$

and note that the map $(P_nA)_\bullet \to (P_{n-1}A)_\bullet$ factors as $(P_nA)_\bullet \to C_\bullet \to (P_{n-1}A)_\bullet$, with $(P_nA)_\bullet \to C_\bullet$ a trivial fibration, and $C_\bullet \to (P_{n-1}A)_\bullet$ a square-zero extension (with kernel $(\mathrm{H}_n(A_\bullet))[-n]$). ☐

Remark 3.32. Thus Spec $(A_\bullet)$ is like a formal infinitesimal neighbourhood of Spec $(H_0(A_\bullet))$, since we have characterised it as a direct limit of a sequence of square-zero thickenings.

Assuming some finiteness conditions, we now strengthen these results, relating $A_\bullet$ to a genuine completion over $\mathrm{H}_0(A_\bullet)$.

Definition 3.33. Let $A_\bullet \in dg_+\mathrm{Alg}_\mathbb{k}$. The *completion* of $A_\bullet$ is given by

$$\hat{A}_\bullet := \varprojlim_n A_\bullet/I^n A_\bullet$$

where $I := \ker(A_0 \to \mathrm{H}_0(A_\bullet))$.

Lemma 3.34. *If A_0 is Noetherian and each A_n is a finite A_0-module, then $A_\bullet \to \hat{A}_\bullet$ is also a quasi-isomorphism.*

Proof. This is [Pri09, Lemma 6.37], proved using fairly standard commutative algebra. If A_0 is Noetherian, then [Mat89, Thm. 8.8] implies that $A_0 \to \hat{A}_0$ is flat. If A_n is a finite A_0-module, then [Mat89, Thm 8.7] implies that $\hat{A}_n = \hat{A}_0 \otimes_{A_0} A_n$. Thus

$$\mathrm{H}_*(\hat{A}_\bullet) \cong \mathrm{H}_*(A_\bullet) \otimes_{A_0} \hat{A}_0,$$

and applying [Mat89, Thm 8.7] to the A_0-module $\mathrm{H}_0(A_\bullet)$ gives that $\mathrm{H}_*(\hat{A}_\bullet) \cong \mathrm{H}_*(A_\bullet)$, as required. ☐

3.4 The cotangent complex

The cotangent complex is one of the earliest applications of abstract homotopy theory, due to Quillen [Qui70][29], using [Qui67]. Until then, tangent and obstruction spaces for relative extensions only fitted in the nine-term long exact sequence of [LS67]. For more history, see [Bar04].

Definition 3.35. Given a morphism $R \to A$ in $dg_+\mathrm{Alg}_\mathbb{k}$, the complex $\Omega^1_{A/R} \in dg_+\mathrm{Mod}_A$ of *Kähler differentials* is given by I/I^2, where $I = \ker(A \otimes_R A \to A)$.

Example 3.36. If $A = (R[x_1, \dots, x_n], \delta)$ for variables x_i in various degrees, then $\Omega^1_{A/R} = (\bigoplus_{i=1}^n A dx_i, \delta)$.

In general, we always have a derivation $d\colon A \to \Omega^1_{A/R}$ given by $a \mapsto a \otimes 1 - 1 \otimes a + I$.

The idea behind the cotangent complex is that we want to take left-derived functor, but this isn't a functor as such, since the codomain depends on A. Instead, we take the slice category $dg_+\mathrm{Alg}_R{\downarrow}A$ of A-augmented R-algebras, and look at the functor $B \mapsto \Omega^{B/R} \otimes_B A$ from $dg_+\mathrm{Alg}_R{\downarrow}A$ to $dg_+\mathrm{Mod}_A$; this is left adjoint to the functor $M \mapsto A \oplus M\epsilon$, for $\epsilon^2 = 0$. These form a Quillen pair, and taking the left-derived functor gives the cotangent complex $\mathbb{L}^{A/R} := \mathbb{L}(\Omega^1_{-/R} \otimes_- A)(A)$.

In other words, take a cofibrant replacement $\tilde{A} \to A$ in $dg_+\mathrm{Alg}_R$, and then set the *cotangent complex* to be $\mathbb{L}^{A/R} := \Omega^1_{\tilde{A}/R} \otimes_{\tilde{A}} A$. Note that $\mathrm{H}_0\mathbb{L}^{A/R} = \Omega_{\mathrm{H}_0 A/\mathrm{H}_0 R}$.

Explicitly,

[29]The two manuscripts with the greatest influence on derived geometry are probably [Qui70] and [Kon94a], though practitioners tend to encounter their contents indirectly.

Lemma 3.37. *The cotangent complex $\mathbb{L}^{A/R} \simeq \Omega^1_{\tilde{A}/R} \otimes_{\tilde{A}} A$ can be calculated by letting $J = \ker(\tilde{A} \otimes_R A \to A)$, and setting $\mathbb{L}^{A/R} := J/J^2$.*

Remark 3.38. It follows from results below that we can take $\tilde{A}$ with just $R_0 \to \tilde{A}_0$ ind-smooth and $\tilde{A}$ cofibrant over $R \otimes_{R_0} \tilde{A}_0$ (i.e. underlying graded freely generated by a graded projective module).

Also note that the functor $- \otimes_{\tilde{A}} A\colon dg_+\mathrm{Mod}_{\tilde{A}} \to dg_+\mathrm{Mod}_A$ is a left Quillen equivalence, and in particular that $\Omega^1_{\tilde{A}} \to \Omega^{\tilde{A}/R} \otimes_{\tilde{A}} A$ is a quasi-isomorphism of $\tilde{A}$-modules, but beware that the domain is not an A-module.

Definition 3.39. André–Quillen (or Harrison — they agree in characteristic 0) cohomology is defined to be $D^i_R(A, M) := \mathbb{E}\mathrm{xt}^i_A(\mathbb{L}^{A/R}, M)$.

In interpreting this, note that $\mathrm{Hom}_A(\Omega^1_{A/R}, M)$ consists of R-linear derivations from A to M.

The homotopy fibre of $\mathbf{R}\mathrm{Map}(A, B \oplus M) \to \mathbf{R}\mathrm{Map}(A, B)$ has homotopy groups $\pi_i = D^i_R(A, M)$. In particular, the obstruction space in §3.2 is $D^0_R(S, I[1]) = D^1_R(S, I)$.

Theorem 3.40 (Quillen)**.** *If S is a smooth R-algebra (concentrated in degree 0), then $\mathbb{L}^{S/R} \simeq \Omega^1_{S/R}$.*

Moreover, if $T = S/I$, for I an ideal generated by a regular sequence $a_1, a_2, \ldots$, then

$$\mathbb{L}^{T/R} \simeq \mathrm{cone}(I/I^2 \to \Omega^1_{S/R} \otimes_S T).$$

We begin with a key lemma from [Qui70], which follows from universal properties of derived functors.

Lemma 3.41. *Given morphisms $A \to B \to C$ of cdgas, we have an exact triangle*

$$\mathbb{L}^{C/B}[-1] \to \mathbb{L}^{B/A} \otimes_B C \to \mathbb{L}^{C/A} \to \mathbb{L}^{C/B}.$$

Sketch proof of theorem.

1. If $S = R[x_1, \ldots, x_n]$, then it is cofibrant over R, so the conclusion holds.

2. Next, reduce to the étale case (smooth of relative dimension 0). A smooth morphism is étale locally affine space:

$$\begin{tikzcd} U \arrow[r, "f", "\text{étale}"'] \arrow[d, "g"', "\text{étale}"] & \mathrm{Spec}\, S \arrow[d] \\ \mathbb{A}^n_R \arrow[r] & \mathrm{Spec}\, R. \end{tikzcd}$$

   If the statement holds for étale morphisms, then the lemma gives $f^*\mathbb{L}_{S/R} \cong \mathbb{L}_{U/R}$ and $\mathbb{L}^{U/R} \simeq g^*\mathbb{L}^{\mathbb{A}^n_R/R} \simeq \Omega^1_{U/R}$.

   Thus the map $\mathbb{L}^{S/R} \to \Omega^1_{S/R}$ is a quasi-isomorphism étale locally, so must be a quasi-isomorphism globally.

3. Now, reduce to open immersions. If $U \to Y$ is an étale map of affine schemes, then the relative diagonal

$$\Delta\colon U \to U \times_Y U$$

   is an open immersion. If the statement holds for open immersions, this gives

$$\Delta^*\mathbb{L}^{(U \times_Y U)/Y} \cong \mathbb{L}^{U/Y}.$$

But $\mathbb{L}^{(U\times_Y U)/Y} \cong \mathrm{pr}_1^*\mathbb{L}^{U/Y} \oplus \mathrm{pr}_2^*\mathbb{L}^{U/Y}$, so we would then have

$$\mathbb{L}^{U/Y} \oplus \mathbb{L}^{U/Y} \cong \mathbb{L}^{U/Y},$$

and thus $\mathbb{L}^{U/Y} \simeq 0 = \Omega^1_{U/Y}$.

4. Every open immersion is given by repeated composition and pullback of the open immersion $\mathrm{Spec}\, R[x,x^{-1}] \to \mathrm{Spec}\, R[x]$, so it suffices to prove the theorem for this morphism.

5. The abstract nonsense has taken us this far, but now we have to dirty our hands. A cofibrant replacement $\tilde{A}$ for $A := R[x,x^{-1}]$ over $B := R[x]$ is given by $R[x,y,t]$ with $\delta t = xy - 1$, for t of degree 1 and y of degree 0, i.e.

$$R[x,y] \xleftarrow{\delta} R[x,y]t.$$

Then $\Omega^1_{\tilde{A}/B} = \tilde{A}dy \oplus \tilde{A}dt$, with $\delta(dt) = xdy$. Thus $\mathbb{L}^{A/B} \simeq (Ady \oplus Adt, \delta(dt) = xdy)$. Since $x \in A$ is a unit, this gives $\mathbb{L}^{A/B} \simeq 0 = \Omega^1_{A/B}$, completing the proof for smooth algebras.

6. For the regular sequence, we observe that a cofibrant replacement $\tilde{T}$ for $T = S/(a_1, a_2, \ldots)$ over S is given by $(S[t_1, t_2, \ldots], \delta)$ with $\delta t_i = a_i$, for T_i of degree 1; this is effectively a Koszul complex calculation. Then

$$\Omega^1_{\tilde{T}/S} \cong (\bigoplus_i \tilde{T}dt_i, \delta),$$

and $\mathbb{L}^{T/S} \simeq \Omega^1_{\tilde{T}/S} \otimes_{\tilde{T}} T \cong \bigoplus_i Tdt_i \cong (I/I^2)[1]$.

□

As a consequence, in general, we don't need cofibrant replacement to calculate $\mathbb{L}^{A/R}$, it suffices for $R \to A$ to be the composite of a cofibration and a smooth morphism.[30]

Remark 3.42. Analogues in differential and analytic settings work in much the same way for all the results in this section, giving $\Omega^1_{A/R}$ as the module of smooth or analytic differentials. The definition just uses the analytic or $\mathcal{C}^\infty$ tensor product $\odot$ instead of $\otimes$.[31] For instance, $\Omega^1_{\mathcal{C}^\infty(\mathbb{R}^n)}$ has to be $\bigoplus_i \mathcal{C}^\infty(\mathbb{R}^n)dx_i$.

The proof of the last theorem also works much the same: simpler in the differential setting, but harder in the analytic setting.

Remark 3.43. A map $f\colon A \to B$ in $dg_+\mathrm{Alg}_R$ is a weak equivalence if and only if $\mathrm{H}_0 f$ is an isomorphism and $\mathbb{L}^{B/A} \otimes^{\mathbf{L}}_B \mathrm{H}_0 B \simeq 0$. The "only if" direction follows by definition; to prove the "if" direction, look at maps from both to arbitrary $C \in \mathrm{Ho}(dg_+\mathrm{Alg}_R)$, and use the Postnikov tower to break C down into square-zero extensions over $\mathrm{H}_0 C$. For details, see Lemma 6.24.

The same is true for dg $\mathcal{C}^\infty$-rings and dg EFC-rings, with exactly the same reasoning.

[30] The natural name for this concept, as used for instance in [Kon94a, Man99, Pri07a] is *quasi-smoothness*, and it was simply called smoothness in [CFK99, CFK00]. However, quasi-smooth is more commonly used in the later DAG literature (apparently originating with [Toë06]) to mean virtually LCI in the sense that the cotangent complex is generated in degrees 0, 1, so the term is now best avoided altogether. Both usages have their roots in the *hidden smoothness* philosophy of [Kon94b] and [Kon94a, Lecture 27], with the motivating examples from the former (but *not* the latter) being virtually LCI as well as quasi-smooth in the original sense.

[31] In fact, cotangent modules were formulated in [Qui70] for arbitrary algebraic theories, taking values in Beck modules [Bec67].

Definition 3.44. ([TV04]) A morphism $f\colon A \to B$ in $dg_+\mathrm{Alg}_R$ is *strong* if $\mathrm{H}_i B \cong \mathrm{H}_i A \otimes_{\mathrm{H}_0 A} \mathrm{H}_0 B$. We then say a morphism is homotopy-(flat, resp. open immersion, resp. étale, resp. smooth) if it is strong and $\mathrm{H}_0 A \to \mathrm{H}_0 B$ is (flat, resp. open immersion, resp. étale, resp. smooth).[32]

[TV04, Def 1.2.7.1 and Theorem 2.2.2.6] then characterises homotopy-étale and homotopy-smooth as follows:

$$A \to B \text{ is homotopy-étale} \iff \mathbb{L}^{B/A} \simeq 0$$
$$A \to B \text{ is homotopy-smooth} \iff \mathbb{L}^{B/A} \otimes^{\mathbf{L}}_B H_0 B \simeq \text{ projective } \mathrm{H}_0 B\text{-module in degree } 0.$$

Remark 3.45. The cotangent complex functor $\mathbb{L}$ can be constructed using functorial cofibrant replacements, so it sheafifies (Illusie [Ill71, Ill72]).

Lemma 3.46. *For any morphism $f\colon X \to Y$ of derived schemes, the presheaf $\mathbb{L}^{\mathscr{O}_{X,\bullet}/f^{-1}\mathscr{O}_{Y,\bullet}}$ is a homotopy-Cartesian dg $\mathscr{O}_{X,\bullet}$-module.*

Proof. For any inclusion $U \to V$ of open affines in $\pi^0 X$, the map $\mathscr{O}_{X,\bullet}(V) \to \mathscr{O}_{X,\bullet}(U)$ is homotopy-open immersion, so $\mathbb{L}^{\mathscr{O}_{X,\bullet}(U)/\mathscr{O}_{X,\bullet}(V)} \simeq 0$, and $\mathbb{L}^{f^{-1}\mathscr{O}_{Y,\bullet}(U)/f^{-1}\mathscr{O}_{Y,\bullet}(V)} \simeq 0$ similarly. The exact triangle for $\mathbb{L}$ thus gives

$$\mathbb{L}^{\mathscr{O}_{X,\bullet}(U)/f^{-1}\mathscr{O}_{Y,\bullet}(U)} \simeq \mathscr{O}_{X,\bullet}(U) \otimes^{\mathbf{L}}_{\mathscr{O}_{X,\bullet}(V)} \mathbb{L}^{\mathscr{O}_{X,\bullet}(V)/f^{-1}\mathscr{O}_{Y,\bullet}(V)},$$

as required. □

Although defined in terms of deformations of morphisms, the cotangent complex also governs deformations of objects:

Lemma 3.47. *Given a cdga S over B and a surjection $A \twoheadrightarrow B$ with kernel I, the potential obstruction to lifting S to a cdga S' over A with $S' \otimes^{\mathbf{L}}_A B \simeq S$ lies in $\mathrm{Ext}^2_S(\mathbb{L}^{S/B}, S \otimes^{\mathbf{L}}_B I)$. If the obstruction vanishes, then the set of equivalence classes of lifts is a torsor for $\mathrm{Ext}^1_S(\mathbb{L}^{S/B}, S \otimes^{\mathbf{L}}_B I)$.*

Proof. This is essentially contained in [Ill71, III 1.2.5], but there is a more direct proof based on [Kon94a, Lectures 13–14]. Without loss of generality, we may assume that S is cofibrant. Since free algebras don't deform, there is a free graded-commutative A-algebra S' with $S' \otimes_A B \cong S$.

The obstruction in Ext^2 then comes from lifting δ to a derivation δ' on S' and looking at $(\delta')^2$, while the parametrisation in terms of Ext^1 comes from different choices of lift δ'. Most of the work is then in checking quasi-isomorphism-invariance. For details of this argument and global generalisations, see [Pri09, §8.2]. □

[32] In [TV04], these properties are simply called flat, smooth, étale, etc., but we prefer to emphasise their homotopy-invariant nature and avoid potential confusion with notions such as Definition 3.8, or the smoothness of [CFK99, CFK00] (i.e. quasi-smoothness in the original sense).

3.5 Derived de Rham cohomology

This originates in [Ill72, §VIII.2]. We have a functor from cdgas to double complexes (a.k.a. bicomplexes), sending A to

$$\begin{aligned}\Omega_A^\bullet := & \quad (A \xrightarrow{d} \Omega_A^1 \xrightarrow{d} \Omega_A^2 \xrightarrow{d} \ldots), \\ = & \left(\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \iddots \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\ A_2 & \xrightarrow{d} & \Omega^1_{A,2} & \xrightarrow{d} & \Omega^2_{A,2} & \xrightarrow{d} & \ldots \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\ A_1 & \xrightarrow{d} & \Omega^1_{A,1} & \xrightarrow{d} & \Omega^2_{A,1} & \xrightarrow{d} & \ldots \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\ A_0 & \xrightarrow{d} & \Omega^1_{A,0} & \xrightarrow{d} & \Omega^2_{A,0} & \xrightarrow{d} & \ldots \end{array}\right)\end{aligned}$$

where $\Omega_A^p := \Lambda_A^p \Omega_A^1$ is the alternating power, taken in the graded sense. Beware that when A has terms of odd degree, alternating powers go on forever.

Our notion of weak equivalence for double complexes will be quasi-isomorphism on the columns, so

$$(U^0 \xrightarrow{d} U^1 \xrightarrow{d} \ldots) \to (V^0 \xrightarrow{d} V^1 \xrightarrow{d} \ldots)$$

is an equivalence if $\mathrm{H}_*(U^i) \cong \mathrm{H}_*(V^i)$ for all i.

The idea behind derived de Rham cohomology is to then take the left derived functor, giving the double complex $\mathbf{L}\Omega_A^\bullet := \Omega_{\tilde{A}}^\bullet$, for a cofibrant replacement $\tilde{A}$ of A (cofibrant over smooth suffices — we just need $\Omega^1 \simeq \mathbb{L}$).[33]

Then we take the *derived de Rham complex* to be the *product* total complex $\mathrm{Tot}^{\Pi}\mathbf{L}\Omega_A^\bullet$ (i.e. $\mathrm{Tot}^{\Pi}(V)_i := (\prod_p V^p_{p+i}, \delta \pm d)$). In fact, for our notion of weak equivalences, Tot^{Π} is just the right-derived functor of the functor $\mathrm{Z}^0 \colon V \mapsto \ker(d \colon V^0 \to V^1)$ on double complexes in non-negative cochain degrees; it preserves weak equivalences by [Wei94, §5.6].

Theorem 3.48 ([Ill72] (with restrictions), [FT85] (omitting details), [Bha12])**.** *The cohomology groups $\mathrm{H}^*(\pi^0 X, \mathrm{Tot}^{\Pi}\mathbf{L}\Omega_X^\bullet)$ are Hartshorne's algebraic de Rham cohomology groups [Har72][34]. In particular, these are the singular cohomology groups $\mathrm{H}^*(X(\mathbb{C})_{\mathrm{an}}, \mathbb{C})$ of $X(\mathbb{C})$ with the analytic topology when working over $\mathbb{C}$.*

One proof proceeds by taking a cofibrant resolution and killing variables x of non-zero degree, thus identifying dx with $\pm\delta x$; this generates power series in δx when $\deg x = 1$, giving the comparison with [Har72]. The same arguments work in differential and $\mathbb{C}$-analytic settings.

[33] Note that $\Omega^p_{\tilde{A}} \simeq \Lambda^p_A \mathbb{L}^{A/R}$ (just apply $\otimes_{\tilde{A}} A$), but also that the de Rham differential d does not descend to the latter objects.

[34] The algebraic de Rham cohomology of Z is defined by taking a closed embedding of Z in a smooth scheme Y, then looking at the completion $\hat{\Omega}_Y^\bullet$ of the de Rham complex of Y with respect to the ideal $\mathscr{I}_Z$.

3.5.1 Shifted symplectic structures

Any complex or double complex admits a filtration by brutal truncation, i.e.

$$F^p(V^0 \xrightarrow{d} V^1 \xrightarrow{d} \ldots) = (0 \to \ldots \to 0 \to V^p \xrightarrow{d} V^{p+1} \xrightarrow{d} \ldots);$$

on the de Rham (double) complex, this is called the *Hodge filtration*. Then $(\mathrm{Tot}^{\Pi} F^p)[p]$ is a right-derived functor $\mathbf{R}\mathrm{Z}^p$ of $\mathrm{Z}^p\colon V \mapsto \ker(V^p \to V^{p+1})$,[35] so the homologically correct analogue of closed p-forms is given by the complex $(\mathrm{Tot}^{\Pi} F^p \mathbf{L}\Omega^{\bullet}_A)[p]$.

Example 3.49. Classically, when X is a smooth scheme (in the algebraic setting) or a manifold (in the $\mathcal{C}^\infty$ and analytic settings), then we just have $\mathbf{L}\Omega^p_X \simeq \Omega^p_X$, and hence $(\mathrm{Tot}^{\Pi} F^p \mathbf{L}\Omega^{\bullet}_X)[p] \simeq F^p\Omega^{\bullet}_X[p]$.

In $\mathcal{C}^\infty$ and analytic settings, we can say more, because the Poincaré lemma implies that $F^p\Omega^{\bullet}_X[p]$ is quasi-isomorphic to the sheaf $\mathrm{Z}^p\Omega^{\bullet}_X = \ker(d\colon \Omega^p_X \to \Omega^{p+1}_X)$ of closed p-forms on X, so the derived constructions reduce to the naïve underived object.

In algebraic settings, the sheaf $\mathrm{Z}^p\Omega^{\bullet}_X$ of closed algebraic p-forms on the Zariski site is poorly behaved, but the GAGA principle [Ser56] applied to the graded pieces shows that for smooth proper complex varieties X, analytification gives a quasi-isomorphism

$$\mathbf{R}\Gamma(X, F^p\Omega^{\bullet}_X)[p] \simeq \mathbf{R}\Gamma(X(\mathbb{C})_{\mathrm{an}}, F^p\Omega^{\bullet}_{X_{\mathrm{an}}})[p] \simeq \mathbf{R}\Gamma(X(\mathbb{C})_{\mathrm{an}}, \mathrm{Z}^p\Omega^{\bullet}_{X_{\mathrm{an}}}),$$

and hence an isomorphism between hypercohomology of the algebraic Hodge filtration and cohomology of closed analytic p-forms.

Thus even in the absence of derived structure, one immediately looks to the Hodge filtration in algebraic geometry when seeking to mimic closed forms in analytic geometry.

Definition 3.50 ([KV08, Bru10, PTVV11][36]). The complex of *n-shifted pre-symplectic structures* is $\tau^{\le 0}((\mathrm{Tot}^{\Pi} F^2\mathbf{L}\Omega^{\bullet}_A)[n+2])$.[37]

Hence the set of homotopy classes of such structures is $\mathrm{H}^{n+2}(\mathrm{Tot}^{\Pi} F^2\mathbf{L}\Omega^{\bullet}_A)$, each element consisting of an infinite sequence $(\omega_i \in (\Omega^i_{\tilde{A}})_{n-2+i})$ with $d\omega_i = \pm\delta\omega_{i+1}$, where $\tilde{A}$ is a cofibrant (or cofibrant over smooth) replacement for A.

We say ω is *shifted symplectic* if it is non-degenerate in the sense that the maps $\mathbb{E}\mathrm{xt}^i_{\tilde{A}}(\Omega^1_{\tilde{A}}, \tilde{A}) \to \mathrm{H}_{-i-n}\Omega^1_{\tilde{A}}$ from the tangent complex to the cotangent complex induced by contraction with $\omega_2 \in \mathrm{H}_{-n}(\Omega^2)$ is a quasi-isomorphism.[38]

In the global case (for a derived scheme, or even derived algebraic space or DM stack), the complex of n-shifted pre-symplectic structures is

$$\tau^{\le 0}\mathbf{R}\Gamma(X, (\mathrm{Tot}^{\Pi} F^2\mathbf{L}\Omega^{\bullet}_X)[n+2]),$$

so homotopy classes are elements of $\mathrm{H}^{n+2}(X, \mathrm{Tot}^{\Pi} F^2\mathbf{L}\Omega^{\bullet}_X)$, and are regarded as symplectic if they are locally non-degenerate. (Derived Artin stacks are treated similarly, but non-degeneracy becomes a more global condition — see Definition 6.27.)

[35] Explicitly, we have a quasi-isomorphisms $V^q \to \mathit{cone}(\mathrm{Tot}^{\Pi} F^{q+1} \to \mathrm{Tot}^{\Pi} F^q)[q]$ given by $v \mapsto (\pm dv, v)$ combining to give a levelwise quasi-isomorphism of double complexes, with Z^p of the codomain being $(\mathrm{Tot}^{\Pi} F^p V)[p]$.

[36] In [KV08], working in the $\mathcal{C}^\infty$ setting, chain complexes were $\mathbb{Z}/2$-graded rather than $\mathbb{Z}$-graded, only even shifts were considered, and δ was zero. Their definition and results were extended to odd shifts in [Bru10]. The definition in [PTVV11] is only formulated inexplicitly as a homotopy limit, obscuring the similarity with earlier work; the Hodge filtration is not mentioned.

[37] The terminology here follows [Pri15], differing slightly from both sources. In [PTVV11], pre-symplectic structures are called closed 2-forms, terminology we avoid because it refers more naturally to Z^2 than to $\mathbf{R}\mathrm{Z}^2$. Also beware that ibid. refers to double complexes as "graded mixed complexes".

[38] In particular this implies that n-shifted symplectic structures on derived schemes only exist for $n \le 0$; positively shifted structures can however exist on derived Artin stacks.

Example 3.51. For Y a smooth scheme, the *shifted cotangent bundle*

$$T^*Y[-n] := \mathbf{Spec}_Y(\mathrm{Symm}_{\mathscr{O}_Y}(\mathscr{T}_Y[n]), \delta = 0)$$

is $(-n)$-shifted symplectic, with ω given in local co-ordinates by $\sum_i dy_i \wedge d\eta_i$, for $\eta_i = \partial_{y_i} \in \mathscr{T}_Y$, the tangent sheaf. Thus $\omega \in \Omega^2$ with $\delta\omega = 0$ and $d\omega = 0$.

There are also twisted versions, e.g. twist $T^*Y[-1]$ by taking the differential δ to be given by contraction with df, and we still have a (-1)-shifted symplectic structure. That derived scheme is the *derived critical locus* of f, i.e. the derived vanishing locus of df,

$$T \times^h_{df, T^*Y, 0} Y.$$

Remark 3.52. There is a related notion of shifted Poisson structures [KV08, Pri15, CPT+15][39]. In this setting, such a structure just amounts to a shifted L_∞-algebra structure on $\mathscr{O}_X$, with the brackets all being multiderivations, assuming we have chosen a cofibrant (or cofibrant over smooth) model for $\mathscr{O}_X$. The equivalence between shifted symplectic and non-degenerate shifted Poisson structures is interpreted in [KV08] as a form of Legendre transformation, and the comparison in [Pri15] can be interpreted as a homotopical generalisation, but the comparison in [CPT+15] takes a much less direct approach.[40]

[39] introductory slides available at https://www.maths.ed.ac.uk/∼jpridham/edbpoisson.pdf

[40] There are also notions of deformation quantisation for n-shifted Poisson structures, mostly summarised in [Pri18a]. For $n > 0$ (generally existing on derived stacks rather than schemes), quantisation is an immediate consequence of formality of the little $(n+1)$-discs operad [Kon99, Theorem 2], as observed in [CPT+15, Theorem 3.5.4]. The problem becomes increasingly difficult as n decreases, unless one is willing to break the link with BV quantisation and redefine quantisation for $n < 0$ as in [CPT+15, Definition 3.5.8] so that it also becomes a formality.

4 Simplicial structures

References for this section include [Wei94, §8] and [GJ99], among others.

4.1 Simplicial sets

Motivation:

- half-exact functors don't behave well enough to allow gluing, so we'll need to work with some flavour of ∞-categories instead of homotopy categories
- the category $s\mathrm{Set}$ of simplicial sets is much more manageable to work with than the category Top of topological spaces

In algebraic geometry, the idea of looking at simplicial set-valued functors to model derived phenomena goes back at least as far as [Hin98].

Definition 4.1. Let $|\Delta^n| \subset \mathbb{R}_+^{n+1}$ be the subspace $\{(x_0, \ldots, x_n) : \sum x_i = 1\}$; this is the *geometric n-simplex* . See Figure 1.

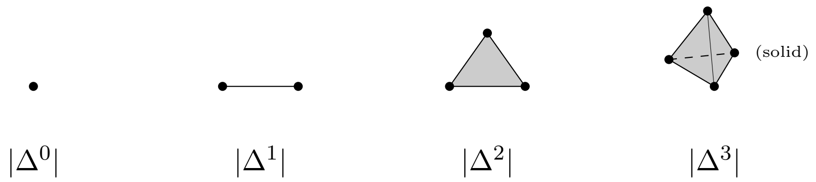


Figure 1: Geometric n-simplices.

Definition 4.2. Given a topological space X, we then have a system

$$\mathrm{Sing}(X)_n := \mathrm{Hom}(|\Delta^n|, X)$$

of sets, known as the *singular functor*, fitting into a diagram

$$\begin{tikzcd}
\mathrm{Sing}(X)_0 \arrow[r, dotted, "\sigma_0" description] & \mathrm{Sing}(X)_1 \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] \arrow[r, dotted, shift left=1] \arrow[r, dotted, shift right=1] & \mathrm{Sing}(X)_2 \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] & \vdots & \mathrm{Sing}(X)_3 \arrow[ll, bend right] \arrow[ll, bend left] & \cdots & \cdots,
\end{tikzcd}$$

where the maps $\partial_i : X_n \to X_{n-1}$ come from inclusion of the ith face $\partial^i \colon |\Delta^{n-1}| \to |\Delta^n|$, and the maps $\sigma_i : X_n \to X_{n+1}$ come from the degeneracy map $\sigma^i \colon |\Delta^{n+1}| \to |\Delta^n|$ given by collapsing the edge $(i, i+1)$.

These operations satisfy the following identities:

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad \text{for} \quad i < j,$$

$$\sigma_i \sigma_j = \sigma_{j+1} \sigma_i \quad \text{for} \quad i \leq j,$$

and

$$\partial_i \sigma_j = \begin{cases} \mathrm{id} & i = j, j-1 \\ \sigma_{j-1} \partial_i & i < j \\ \sigma_j \partial_{i-1} & i > j+1. \end{cases}$$

Sing(X) has given us a contravariant functor from a category Δ to sets, where the ordinal number category Δ has objects $\mathbf{n} := \{0, 1, \ldots, n\}$ for $n \geq 0$, and morphisms f given by non-decreasing maps between them (i.e. $f(i+1) \geq f(i)$ for every $i \in [0, n]$). The correspondence comes by labelling the vertices of $|\Delta^n|$ from 0 to n according to the non-zero co-ordinate, with [Wei94, Lemma 8.1.2] expressing every morphism in Δ as a composition of degeneracy and face maps.

Definition 4.3. The category sSet of *simplicial sets* consists of functors $Y \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. Write Y_n for $Y(n)$. Thus objects are just diagrams

$$\begin{tikzcd} Y_0 \arrow[r, dotted, "\sigma_0" description] & Y_1 \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & Y_2 \arrow[l] \arrow[l, bend right] \arrow[l, bend left] & Y_3 \arrow[l, bend right] \arrow[l, bend left, "\vdots" description] & \cdots & \cdots, \end{tikzcd}$$

satisfying the relations of Definition 4.2.

Definition 4.4. Define the *combinatorial n-simplex $\Delta^n \in s\mathrm{Set}$* by the property that $\Delta^n := \mathrm{Hom}_\Delta(-, n)$.

For example, Δ^0 is the constant diagram[41]

$$\begin{tikzcd} \bullet \arrow[r, dotted, "\sigma_0" description] & \bullet \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & \bullet \arrow[l] \arrow[l, bend right] \arrow[l, bend left] & \bullet \arrow[l, bend right] \arrow[l, bend left, "\vdots" description] & \cdots & \cdots \end{tikzcd}$$

on the one-point set, while $(\Delta^1)_i$ has $i+2$ elements, of which only the two elements in $(\Delta^1)_0$ and one of those in $(\Delta^1)_1$ are non-degenerate (i.e. not in the image of any degeneracy map σ_i).

Lemma 4.5. *The functor* Sing : Top $\to s$Set *has a left adjoint $Y \mapsto |Y|$, determined by $\Delta^n \mapsto |\Delta^n|$, and the need to preserve coproducts and pushouts.*

Explicitly, $|Y|$ is the quotient of $\coprod_n (Y_n \times |\Delta^n|)$ by the relations $(\partial_i y, a) \sim (y, \partial^i a)$ and $(\sigma_i y, a) \sim (y, \sigma^i a)$.

4.1.1 The Kan–Quillen model structure

Definition 4.6. We say that a morphism $X \to Y$ in sSet is a *weak equivalence* if $|X| \to |Y|$ is a weak equivalence (i.e. π_*-equivalence) of topological spaces.

Theorem 4.7 (Quillen)**.** *There is a model structure on sSet with the weak equivalences above, with cofibrations just being maps $f \colon X \to Y$ which are injective in each level. Fibrations are then those maps with RLP with respect to all trivial cofibrations (i.e. cofibrations which are weak equivalences):*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "triv.\ cof."'] & X \arrow[d, "fib."] \\ B \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

Definition 4.8. For $n \geq 0$, define $\partial\Delta^n \subset \Delta^n$ to be $\bigcup_i \partial^i(\Delta^{n-1})$ $(n \geq 0)$. See Figure 2.

[41] Note that this constant diagram is the smallest possible simplicial set with a point in degree 0, since the degeneracy maps are necessarily injective.

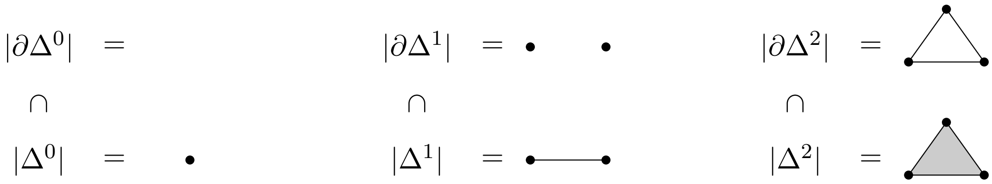


Figure 2: Realisations of $\partial\Delta^n$

Definition 4.9. For $n \geq 1$, define the kth horn $\Lambda^{n,k} \subset \Delta^n$ to be $\bigcup_{i \neq k} \partial^i(\Delta^{n-1}) \subset \Delta^n$ $(n \geq 1)$. See Figure 3.

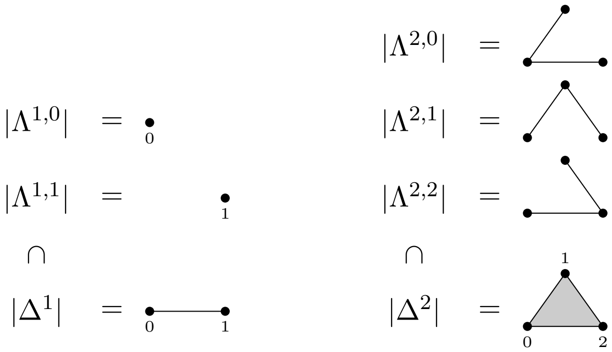


Figure 3: Realisations of $\Lambda^{n,k}$

Theorem 4.10 (Kan)**.** *Fibrations, resp. trivial fibrations, correspond to maps with RLP with respect to $\Lambda^{n,k} \to \Delta^n$ (generating trivial cofibrations), resp. $\partial\Delta^n \to \Delta^n$ (generating cofibrations).*

See Figure 4.

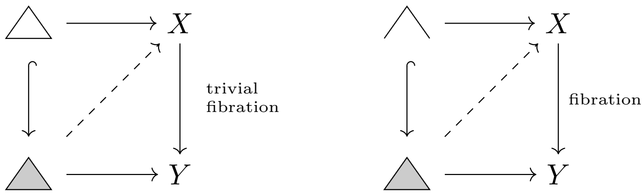


Figure 4: Existence of boundary-fillers and horn-fillers

Definition 4.11. Say that a simplicial set is a *Kan complex* if it is fibrant.

$$\begin{tikzcd}
\Lambda^{n,k} \arrow[r] \arrow[d] & X \\
\Delta^n \arrow[ur, dotted] &
\end{tikzcd}$$

Theorem 4.12. *The adjunction*

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, shift left, "\mathrm{Sing}"] & s\mathrm{Set} \arrow[l, shift left, "|-|"]
\end{tikzcd}$$

is a Quillen equivalence. In particular, $\mathrm{Ho}(\mathrm{Top}) \simeq \mathrm{Ho}(s\mathrm{Set})$.

This also gives rise to an equivalence between the category of topological categories and the category of simplicial categories, up to weak equivalence in both cases. Here, a *simplicial category* is a category enriched in simplicial sets, meaning that for any two objects $X, Y \in \mathcal{C}$ there is a simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them, and a composition operation defined levelwise.

The *homotopy category* $\pi_0\mathcal{C}$ of a simplicial or topological category has the same objects, but morphisms given by path components $\pi_0\mathcal{C}(x, y)$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is then a *weak equivalence* of simplicial or topological categories if the functor $\pi_0 F\colon \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories and the maps $\mathcal{C}(X, Y) \to \mathcal{C}(FX, FY)$ are all weak equivalences of simplicial sets or topological spaces.

4.1.2 Matching objects

Definition 4.13. Given $X \in s\mathrm{Set}$, we define the *nth matching space* by $M_{\partial\Delta^n}(X) := \mathrm{Hom}_{s\mathrm{Set}}(\partial\Delta^n, X)$; this is often simply denoted by $M_n(X)$. Explicitly, this means

$$M_{\partial\Delta^n}(X) = \{x \in \prod_{i=0}^{n} X_{n-1} \,:\, \partial_i x_j = \partial_{j-1} x_i \text{ if } i < j\}$$

for $n > 0$, with $M_{\partial\Delta^0} X = *$.

Define the *(n, k)th partial matching space* by $M_{\Lambda^{n,k}}(X) := \mathrm{Hom}_{s\mathrm{Set}}(\Lambda^{n,k}, X)$. Explicitly, this means

$$M_{\Lambda^{n,k}}(X) = \{x \in \prod_{i=0, i\neq k}^{n} X_{n-1} \,:\, \partial_i x_j = \partial_{j-1} x_i \text{ if } i < j\}.$$

The inclusions $\partial\Delta^n \to \Delta^n$ and $\Lambda^{n,k} \to \Delta^n$ induce matching maps and partial matching maps

$$X_n \to M_{\partial\Delta^n}(X) \quad \text{and} \quad X_n \to M_{\Lambda^{n,k}}(X),$$

sending x to

$$(\partial_0 x, \partial_1 x, \ldots, \partial_n x) \quad \text{and} \quad (\partial_0 x, \partial_1 x, \ldots, \cancel{\partial_k x}, \ldots \partial_n x), \quad \text{respectively.}$$

Thus $X \to Y$ a fibration says the relative partial matching maps

$$X_n \to Y_n \times_{M_{\Lambda^{n,k}}(Y)} M_{\Lambda^{n,k}}(X)$$

are all surjective, while $X \to Y$ a trivial fibration says the relative matching maps

$$X_n \to Y_n \times_{M_{\partial\Delta^n}(Y)} M_{\partial\Delta^n}(X)$$

are all surjective.

4.1.3 Diagonals

A *bisimplicial set* is just a simplicial simplicial set, i.e. a functor $X\colon (\Delta \times \Delta)^{\mathrm{op}} \to \mathrm{Set}$. There is then a *diagonal functor*

$$\mathrm{diag}: ss\mathrm{Set} \to s\mathrm{Set}$$

from bisimplicial sets to simplicial sets given by $\mathrm{diag}\,(X)_n := X_{n,n}$, with the maps $\partial_i\colon \mathrm{diag}\,(X)_{n+1} \to \mathrm{diag}\,(X)_n$ and $\sigma_i\colon \mathrm{diag}\,(X)_{n-1} \to \mathrm{diag}\,(X)_n$ given by composing the corresponding horizontal ($\partial_i^h\colon X_{m,n} \to X_{m-1,n}$, $\sigma_i^h\colon X_{m,n} \to X_{m+1,n}$) and vertical ($\partial_i^v\colon X_{m,n} \to X_{m,n-1}$, $\sigma_i^v\colon X_{m,n} \to X_{m,n+1}$) maps in X.

It turns out that $\operatorname{diag} X$ is a model for the homotopy colimit $\operatorname{ho}\varinjlim_{n\in\Delta^{\mathrm{op}}} X_{n,\bullet}$. As a consequence, its homotopical behaviour is just like the total complex of a double complex, even though the diagonal seems much larger.[42]

4.2 The Dold–Kan equivalence

If A is a simplicial abelian group, then $\delta := \sum(-1)^i\partial_i$ satisfies $\delta^2 = 0$, so (A, δ) becomes a chain complex.

Definition 4.14. The *normalisation* NA of a simplicial abelian group is the chain complex given by $N_mA := \{a \in A_m \ :\ \partial_i a = 0\ \forall i > 0\}$, with differential given by $\partial_0\colon N_{m+1}A \to N_mA$ (squares to zero because $\partial_0(\partial_0 a) = \partial_0(\partial_1 a) = \partial_0(0)$).

In fact, the inclusion $NA \to (A, \delta)$ is a quasi-isomorphism of chain complexes. Also, the homology groups H_*NA are just the homotopy groups $\pi_*(A, 0) := \pi_*(|A|, 0)$ of the simplicial set underlying A.[43]

Theorem 4.15 (Dold–Kan)**.** *The functor N gives an equivalence of categories between simplicial abelian groups and chain complexes in non-negative degrees.*

The inverse functor N^{-1} is just given by throwing in degenerate elements $\sigma_{i_1}\cdots\sigma_{i_n}a$. (For non-positively graded cochain complexes $V^\bullet$, we will occasionally write $N^{-1}V$ for the simplicial abelian group given by applying N^{-1} to the chain complex $i \mapsto V^{-i}$.)

$N : s\mathrm{Ab} \to dg_+\mathrm{Ab}$ is an equivalence of categories, $d = \partial_0$. $(NA, \partial_0) \to (A, d)$ a quasi-isomorphism.

4.3 The Eilenberg–Zilber correspondence

Given a bisimplicial abelian group A, we can normalise in both directions to get a double complex $\underline{N}A$, and we can also take the diagonal to give a simplicial abelian group $\operatorname{diag}(A)$.

There is a quasi-isomorphism, known as the *Eilenberg–Zilber shuffle map*,

$$\nabla\colon\ \operatorname{Tot}\underline{N}A \to N\operatorname{diag}A,$$

given by summing signed shuffle permutations of the horizontal and vertical degeneracy maps σ_i in A.

This map is symmetric with respect to swapping the horizontal and vertical bisimplicial indices. The homotopy inverse of ∇ is given by the Alexander–Whitney cup product, which sums the maps

$$(\partial_{i+1}^h)^j(\partial_0^v)^i\colon\ A_{i+j,i+j} \to A_{ij},$$

and is not symmetric.

Remark 4.16. One consequence of the shuffle map is to give a functor from simplicial commutative rings A (i.e. each A_i a commutative ring) to cdgas. If we write $\underline{\otimes}$ for the

[42] The analogous statements for semi-simplicial sets are not true: although the degeneracy maps σ_i might feel superfluous much of the time, they are vital for results such as these to hold.

[43] These should not be confused with the homology groups $\mathrm{H}_*(X, \mathbb{Z})$ of a simplicial set X, which correspond to homotopy groups of the free simplicial abelian group $\mathbb{Z}.X$ on generators X, with $N(\mathbb{Z}.X)$ then being the complex of normalised chains on X.

external tensor product $(U\underline{\otimes}V)_{i,j} := U_i \otimes V_j$, then we can characterise the multiplication on A as a map $\mu\colon \operatorname{diag}(A\underline{\otimes}A) \to A$, so we have a composite

$$NA \otimes NA \cong \mathrm{Tot}\,(NA\underline{\otimes}NA) = \mathrm{Tot}\,(\underline{N}(A\underline{\otimes}A)) \xrightarrow{\nabla} N\mathrm{diag}\,(A\underline{\otimes}A) \xrightarrow{\mu} NA,$$

giving our graded-commutative multiplication on the chain complex NA.

Another consequence of the Alexander–Whitney is to give us a simplicial ring $N^{-1}A$ associated to any dg algebra A in non-negative chain degrees, but this does not preserve commutativity. A generalisation of this construction allows us to associate simplicial categories to dg categories (i.e. categories enriched in chain complexes as in [Kel06]), after truncation if necessary, as in Lemma 4.23.

4.4 Simplicial mapping spaces

Given a category $\mathcal{C}$ with weak equivalences, we write $\mathbf{R}\mathrm{Map}_{\mathcal{C}}$ for the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to s\mathrm{Set}$ given by right-deriving Hom (if it exists). In model categories, $\mathbf{R}\mathrm{Map}_{\mathcal{C}}$ always exists, and we now show how to calculate it using function complexes as in [DK80b] or [Hov99, §5.4].

Definition 4.17. Given a model category $\mathcal{C}$ and a object $Y \in \mathcal{C}$, we can define a *simplicial fibrant resolution* of Y to be a simplicial diagram $\hat{Y}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ and a map from the constant diagram Y to $\hat{Y}$ (equivalently, a map $Y \to \hat{Y}_0$ in $\mathcal{C}$) such that

1. the maps $Y \to \hat{Y}_n$ are all weak equivalences,
2. the matching maps $\hat{Y}_n \to M_{\partial\Delta^n}(\hat{Y})$ (defined by the same formulae as §4.1.2) are fibrations in $\mathcal{C}$ for all $n \geq 0$ (in particular, $\hat{Y}_0$ is fibrant).

Exercise 4.18. $\hat{Y}_1$ is a path object for $\hat{Y}_0$, via $\sigma_0\colon \hat{Y}_0 \to \hat{Y}_1$ and $\hat{Y}_1 \xrightarrow{(\partial_0,\partial_1)} \hat{Y}_0 \times \hat{Y}_0$.

Examples 4.19.

1. In Top, we can take $\hat{Y}_n$ to be the space $Y^{|\Delta^n|}$ of maps from $|\Delta^n|$ to Y.
2. In sSet, if Y is fibrant, we can take $\hat{Y}_n := Y^{\Delta^n}$, where $(Y^K)_i := \mathrm{Hom}_{s\mathrm{Set}}(\Delta^i \times K, Y)$.
3. In cochain complexes, we can take $\hat{V}_n := V \otimes \mathrm{C}^\bullet(\Delta^n, \mathbb{Z})$ (simplicial cochains on the n-simplex).
4. In cdgas $dg_+\mathrm{Alg}_{\mathrm{k}}$, we can take $\hat{A}_n := \tau_{\geq 0}(A \otimes \Omega^\bullet(\Delta^n))$, where

$$\Omega^\bullet(\Delta^n) = \mathbb{Q}[x_0, \ldots, x_n, \delta x_0, \ldots, \delta x_n]/(\sum x_i - 1, \sum \delta x_i),$$

for x_i of degree 0 (the polynomial de Rham complex of the n-simplex).

The reason this works is that the matching object $M_{\partial\Delta^n}(\hat{A})$ is isomorphic to the cdga $\tau_{\geq 0}(A \otimes \Omega^\bullet(\partial\Delta^n))$, where

$$\Omega^\bullet(\partial\Delta^n) = \Omega^\bullet(\Delta^n)/(\prod_i x_i, \delta(\prod_i x_i));$$

since $\Omega^\bullet(\Delta^n) \to \Omega^\bullet(\partial\Delta^n)$ is surjective, the matching map $\hat{A}_n \to M_{\partial\Delta^n}(\hat{A})$ is surjective in strictly positive degrees, so a fibration.

Theorem 4.20. *If X is cofibrant and $\hat{Y}$ is a fibrant simplicial resolution of Y, then the right function complex $\mathbf{R}\mathrm{Map}_r(X,Y)$, given by*

$$n \mapsto \mathrm{Hom}_{\mathcal{C}}(X, \hat{Y}_n)$$

gives a model for the right-derived functor $\mathbf{R}\mathrm{Map}_{\mathcal{C}}$ of $\mathrm{Hom}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to s\mathrm{Set}$.

Proof (sketch). By [DK80b] or [Hov99, §5.4], function complexes preserve weak equivalences, and are independent of the choice of resolution (so in particular we may assume $\hat{Y}$ is chosen functorially). There is an obvious natural transformation $\mathrm{Hom}_{\mathcal{C}} \to \mathbf{R}\mathrm{Map}_{\mathcal{C}}$, so it suffices to prove universality.

If we have a natural transformation $\mathrm{Hom}_{\mathcal{C}} \to F$ with F preserving weak equivalences, then the maps $F(X,Y) \to F(X, \hat{Y}_i)$ are weak equivalences for all i, so the map $F(X,Y) \to \mathrm{diag}\,(i \mapsto F(X, \hat{Y}_i))$ is a weak equivalence. But we have a map $\mathrm{Hom}_{\mathcal{C}}(X, \hat{Y}_i) \to F(X, \hat{Y}_i)$, so taking diagonals gives

$$\mathbf{R}\mathrm{Map}_r(X,Y) \to \mathrm{diag}\,(i \mapsto F(X, \hat{Y}_i)) \xleftarrow{\sim} F(X,Y),$$

hence the required morphism in the homotopy category, as required. □

Note that derived functors send $\mathbf{R}\mathrm{Map}_{\mathcal{C}}$ to $\mathbf{R}\mathrm{Map}_{\mathcal{D}}$, and that Quillen equivalences induce weak equivalences on $\mathbf{R}\mathrm{Map}$.

Here are some explicit examples of mapping spaces of cdgas:

Examples 4.21.

1. Consider the affine line $\mathbb{A}^1 = \mathrm{Spec}\, R[x]$. A model for $\mathbf{R}\mathrm{Map}_{dg_+\mathrm{Alg}_R}(R[x], B)$ is given by $n \mapsto \mathrm{Z}_0(\Omega^\bullet(\Delta^n) \otimes B)$, since A is cofibrant. However, a smaller model is given by Dold–Kan denormalisation: $\mathbf{R}\mathrm{Map}_{dg_+\mathrm{Alg}_R}(R[x], B) \simeq N^{-1}B$.

2. Consider the affine group $\mathrm{GL}_n = \mathrm{Spec}\, A$, where

$$A = R[x_{ij} \ :\ 1 \le i, j, \le n][(\det(x_{ij}))^{-1}].$$

A cofibrant replacement for A is given by $\tilde{A} := R[x_{ij}, y, t]$ with t in degree 1 satisfying $\delta t = y \det(x_{ij}) - 1$, so

$$\mathrm{Hom}(\tilde{A}, B) = \{(M, c, h) \in \mathrm{Mat}_n(B_0) \times B_0 \times B_1 \ :\ \delta h = c \det M - 1\},$$

and then simplicial level r of $\mathbf{R}\mathrm{Map}(A,B)$ is given by applying this to $\tau_{\ge 0}(\Omega^\bullet(\Delta^r) \otimes B)$.

However, when B Noetherian, we may just take $\mathrm{GL}_n(\widehat{\mathrm{Z}_0(\Omega^\bullet(\Delta^r) \otimes B)})$ in level r, where $\widehat{(-)}$ is completion along $\mathrm{Z}_0(\Omega^\bullet(\Delta^r) \otimes B) \to \mathrm{H}_0 B$, using the (Quillen equivalent) complete model structure of [Pri10b, Proposition 2.7].

In fact, since GL_n is Zariski locally affine space, instead of completing we can just localise away from $\mathrm{H}_0 B$, and drop the Noetherian hypothesis; this follows by using the local model structure, a special case of [Pri18b, Proposition 3.12].

Remark 4.22. The expressions for cdgas adapt to dg $\mathcal{C}^\infty$ and EFC algebras, using $\odot$ instead of $\otimes$ and $\mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathscr{O}^{\mathrm{hol}}(\mathbb{C}^n)$ instead of $\Omega^0(\Delta^n) \cong R[x_1, \ldots, x_n]$.

Lemma 4.23. *In categories like $dg\mathrm{Mod}_A$ or $dg_+\mathrm{Mod}_A$, the simplicial abelian groups $\mathbf{R}\mathrm{Map}(M,P)$ normalise to give $N\mathbf{R}\mathrm{Map}(M,P) \simeq \tau^{\le 0}\mathbf{R}\underline{\mathrm{Hom}}_A(M,P)$ for $\underline{\mathrm{Hom}}$ the dg* Hom *functor.*

Proof. One approach is just to take the function complex $\hat{P}(n) \cong P \otimes \bar{C}^\bullet(\Delta^n)$ (normalised *chains* on the n-simplex).

Alternatively, note that $\tau_{\leq 0}\mathbf{R}\underline{\mathrm{Hom}}$ is the right-derived bifunctor of the composition of Hom with the inclusion of abelian groups in non-negatively graded chain complexes.

Normalisation preserves weak equivalences, as does the forgetful functor from simplicial abelian groups to simplicial sets, so Dold–Kan denormalisation N^{-1} gives the simplicial set-valued functor $N^{-1}\tau_{\geq 0}\mathbf{R}\underline{\mathrm{Hom}}_A(M,P)$ as the right-derived functor of Hom, and thus

$$\mathbf{R}\mathrm{Map}(M,P) \simeq N^{-1}\tau_{\geq 0}\mathbf{R}\underline{\mathrm{Hom}}_A(M,P).$$

□

The following is a consequence of Theorem 4.20:

Corollary 4.24. *If $F\colon \mathcal{C} \to \mathcal{D}$ is left Quillen, with right adjoint G, then*

$$\mathbf{R}\mathrm{Map}_{\mathcal{C}}(A, \mathbf{R}GB) \simeq \mathbf{R}\mathrm{Map}_{\mathcal{D}}(\mathbf{L}FA, B);$$

in particular, the derived functors $\mathbf{L}F, \mathbf{R}G$ give an adjunction of the associated infinity categories.

Example 4.25. The homotopy fibre of $\mathbf{R}\mathrm{Map}_{dg_+\mathrm{Alg}_k}(A, B \oplus I) \to \mathbf{R}\mathrm{Map}_{dg_+\mathrm{Alg}_k}(A,B)$ over f is $\mathbf{R}\mathrm{Map}_{dg_+\mathrm{Mod}_A}(\mathbb{L}^{A/k}, f_*M) \simeq N^{-1}\tau_{\geq 0}\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbb{L}^{A/k}, f_*M)$, so

$$\pi_i\mathbf{R}\mathrm{Map}_{dg_+\mathrm{Mod}_A}(\mathbb{L}^{A/k}, f_*M) \cong \mathrm{Ext}^{-i}_A(\mathbb{L}^{A/k}, f_*M).$$

This accounts for all of the obstruction maps seen in §3.2.

Another feature of $\mathbf{R}\mathrm{Map}$ is that it interacts with homotopy limits in the obvious way, so

$$\mathbf{R}\mathrm{Map}(A, \operatorname{ho}\varprojlim_{i\in I} B(i)) \simeq \operatorname{ho}\varprojlim_{i\in I} \mathbf{R}\mathrm{Map}(A, B(i))$$
$$\mathbf{R}\mathrm{Map}(\operatorname{ho}\varinjlim_{j\in J} A(j), B) \simeq \operatorname{ho}\varprojlim_{j\in J} \mathbf{R}\mathrm{Map}(A(j), B).$$

4.5 Simplicial algebras

If we don't want our base R to contain $\mathbb{Q}$, then we have to use simplicial rings instead of dg algebras, giving the primary viewpoint of [Qui70].

4.5.1 Definitions

Definition 4.26. Define the category $s\mathrm{Alg}_R$ to consist of simplicial commutative R-algebras, i.e. functors $A \to \Delta^{\mathrm{op}} \to \mathrm{Alg}_R$.

Thus each A_n is a commutative R-algebra and the operations ∂_i, σ_i are R-algebra homomorphisms.

Quillen [Qui69, Qui67] gives $s\mathrm{Alg}_R$ a model structure in which fibrations and weak equivalences are inherited from the corresponding properties for the underlying simplicial sets.

Theorem 4.27 (Quillen)**.** *For $\mathbb{Q} \subset R$, Dold–Kan denormalisation gives a right Quillen* **equivalence** *$N\colon s\mathrm{Alg}_R \to dg_+\mathrm{Alg}_R$, where the multiplication on NA is defined using shuffles.*

Remarks 4.28. The theorem tells us that cdgas and simplicial algebras have equivalent homotopy theory in characteristic 0, but simplicial algebras still work in finite and mixed characteristic. Our focus is on cdgas, though, because they give much more manageable objects — the degeneracies in a simplicial diagram generate a lot of elements.

For an explicit homotopy inverse to $N\colon s\mathrm{Alg}_R \to dg_+\mathrm{Alg}_R$, instead of taking the derived left Quillen functor, we can just take the model for $\mathbf{R}\mathrm{Map}(R[x], -)$ from Examples 4.21.

Remark 4.29. We can also consider simplicial EFC-algebras and $\mathcal{C}^\infty$-algebras (i.e. simplicial diagrams in the respective categories of algebras, so all structures are defined levelwise). Dold–Kan normalisation again gives a right Quillen functor to dg EFC or dg $\mathcal{C}^\infty$-algebras, and this is a right Quillen equivalence by [Nui18], hence our focus on the latter.

Cotangent complexes are formulated for any algebraic theory in [Qui70], so the results there can be applied directly to these more exotic settings, but again they reduce to the differential graded constructions by [Nui18].

4.5.2 Simplicial modules

Definition 4.30. Given a simplicial ring A, we define the category $s\mathrm{Mod}_A$ of simplicial A-modules to consist of A-modules M in simplicial sets.

Thus each M_n is an A_n-module, with the obvious compatibilities between the face and degeneracy maps ∂_i, σ_i on A and on M.

Theorem 4.31 (Quillen)**.** *For $A \in s\mathrm{Alg}_R$ Dold–Kan denormalisation gives a right Quillen* **equivalence** $N\colon s\mathrm{Mod}_A \to dg_+\mathrm{Alg}_A$, *where the multiplication of NA on NM is defined using shuffles.*

Note that this statement does *not* need any restriction on the characteristic, essentially because modules do not care whether an algebra is commutative.

4.5.3 Consequences

The various constructions we have seen for dg algebras carry over to simplicial algebras, extending results beyond characteristic 0. Such constructions include the cotangent complex $\mathbb{L}^{S/R} \in s\mathrm{Mod}_S$ (equivalently, $dg_+\mathrm{Mod}_{NS}$), which has the same properties for smooth morphisms, étale morphisms and regular embeddings as before, though the calculation in the proof of Theorem 3.40 becomes a little dirtier. The cotangent complex is then used to define André–Quillen cohomology D^*. In characteristic 0, these are all (quasi-)isomorphic to our earlier cdga constructions. For details, see [Qui70].

Mapping spaces for simplicial algebras are in fact simpler to describe than those for dg algebras, since a fibrant simplicial resolution of A is given by $n \mapsto A^{\Delta^n}$, defined in the same way as for simplicial sets in Examples 4.19.

4.6 n-Hypergroupoids

References for this section include [Dus75, Gle82, Get04], or [Pri09] for the relative and trivial statements; we follow the treatment in [Pri11a].

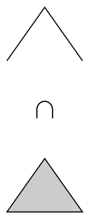
Definition 4.32. Given $Y \in s\mathrm{Set}$, define a *relative n-hypergroupoid* over Y to be a morphism $f : X \to Y$ in $s\mathrm{Set}$, such that the relative partial matching maps

$$X_m \to M_{\Lambda^{m,k}}(X) \times_{M_{\Lambda^{m,k}}(Y)} Y_m$$

are surjective for all k, m (i.e. f is a Kan fibration), and isomorphisms for all $m > n$. In the terminology of [Gle82], this says that f is a Kan fibration which is an exact fibration in all dimensions $> n$.

When $Y = *$ (the constant diagram on a point), we simply say that X is an *n-hypergroupoid*.

In other words, the definition says, "Relative horn fillers exist for all m, and are unique for $m > n$": the dashed arrows in figure 5 making the triangles commute always exist, and are unique for $m > n$.



$$\begin{tikzcd} \Lambda^{m,k} \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^m \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Figure 5: Horn-filling conditions

Examples 4.33.

1. A 0-hypergroupoid X is just a set $X = X_0$ regarded as a constant simplicial object, in the sense that we set $X_m = X_0$ for all n.

2. [Gle82, §2.1] (see also [GJ99, Lemma I.3.5]): 1-hypergroupoids are precisely nerves $B\Gamma$ of groupoids Γ, given by

$$(B\Gamma)_n = \coprod_{x_0,\dots,x_n} \Gamma(x_0, x_1) \times \Gamma(x_1, x_2) \times \ldots \times \Gamma(x_{n-1}, x_n),$$

with the face maps ∂_i given by multiplications or discarding the ends, and the degeneracy maps σ_i by inserting identity maps.

3. A relative 0-hypergroupoid $f\colon X \to Y$ is a Cartesian morphism, in the sense that the maps

$$X_n \xrightarrow{(\partial_i, f)} X_{n-1} \times_{Y_{n-1}, \partial_i} Y_n$$

are all isomorphisms.

$$\begin{tikzcd} X_n \arrow[r, "f"] \arrow[d, "\partial_i"'] & Y_n \arrow[d, "\partial_i"] \\ X_{n-1} \arrow[r, "f"] & Y_{n-1}. \end{tikzcd}$$

Given $y \in Y_0$, we can write $F(y) := f_0^{-1}\{y\}$, and observe that f is equivalent to a local system on Y with fibres F.

Properties 4.34.

1. For an n-hypergroupoid X, we have $\pi_m X = 0$ for all $m > n$.

2. Conversely, if $Y \in s\mathrm{Set}$ with $\pi_m Y = 0$ for all $m > n$, then there exists a weak equivalence $Y \to X$ for some n-hypergroupoid X (given by taking applying the fundamental n-groupoid construction of [Gle82] to a fibrant replacement).

3. [Pri09, Lemma 2.12]: An n-hypergroupoid X is completely determined by its truncation $X_{\le n+1}$. Explicitly, $X = \mathrm{cosk}_{n+1}X$, where the m-coskeleton $\mathrm{cosk}_m X$ has the universal property that $\mathrm{Hom}_{s\mathrm{Set}}(Y, \mathrm{cosk}_m X) \cong \mathrm{Hom}(Y_{\le m}, X_{\le m})$ for all $Y \in s\mathrm{Set}$, so in particular has $(\mathrm{cosk}_m X)_i = \mathrm{Hom}((\Delta^i)_{\le m}, X_{\le m})$.

   Moreover, a simplicial set of the form $\mathrm{cosk}_{n+1}X$ is an n-hypergroupoid if and only if it satisfies the conditions of Definition 4.32 up to level $n+2$.

   When $n = 1$, these statements amount to saying that a groupoid is uniquely determined by its objects (level 0), morphisms and identities (level 1) and multiplication (level 2). However, we do not know we have a groupoid until we check associativity (level 3).

4. Under the Dold–Kan correspondence between non-negatively graded chain complexes and simplicial abelian groups, n-hypergroupoids in abelian groups correspond to chain complexes concentrated in degrees $[0, n]$. One implication is easy to see because all simplicial groups are fibrant and $N_m A = \ker(A_m \to M_{\Lambda^{m,0}}(A))$; the reverse implication uses the characterisation $N_m A \cong A_m / \sum \sigma_i A_{m-1}$.

Remark 4.35. There are also versions for categories instead of groupoids, with just inner horns — drop the conditions for $\Lambda^{m,0}$ and $\Lambda^{m,m}$. These give a model for n-categories (i.e. $(n,1)$-categories) instead of n-groupoids (i.e. $(n,0)$-categories). Taking $n = \infty$ then gives Boardman and Vogt's weak Kan complexes [BV73], called quasi-categories by Joyal [Joy02].[44]

4.6.1 Trivial hypergroupoids

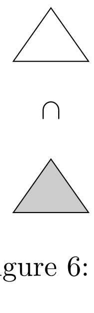
When is a groupoid contractible? When does a relative hypergroupoid correspond to an equivalence?

Definition 4.36. Given $Y \in s\mathrm{Set}$, define a *trivial relative n-hypergroupoid* over Y to be a morphism $f : X \to Y$ in $s\mathrm{Set}$, such that the relative matching maps

$$X_m \to M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

are surjective for all m (i.e. f is a trivial Kan fibration), and isomorphisms for all $m \ge n$.

In other words, the dashed arrows in figure 6 making the triangles commute always exist, and are unique for $m \ge n$.



$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^m \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Figure 6: Simplex-filling conditions

Note that if X is a trivial n-hypergroupoid over a point, then $X = \mathrm{cosk}_{n-1}X$, so X is determined by $X_{<n}$. The converse needs conditions to hold for $X_{<n}$.

[44] Nowadays, these are often known simply as ∞-categories following the usage in [Lur09b, Lur18], whereas [Lur03, Lur04a] use that term exclusively for simplicial categories, which give an equivalent theory by [Joy07]. While quasi-categories lead to efficient proofs in the general theory of ∞-categories, they tend to be less convenient when working in a specific ∞-category.

Examples 4.37.

1. A trivial relative 0-hypergroupoid is an isomorphism.

2. A trivial 1-hypergroupoid over a point is the nerve of a contractible groupoid.

5 Geometric n-stacks

References for this section are [Pri11a, Pri09]. Apparently, the approach we will be taking was first postulated by Grothendieck in [Gro83]. Familiarity with the theory of algebraic stacks [DM69, Art74, LMB00] is not essential to follow this section, as we will construct everything from scratch in a more elementary way.

So far, we've mostly looked at derived affine schemes; they arise as homotopy limits of affine schemes.

Now, we want to glue or take quotients, so we want homotopy colimits, which means we look to enrich objects in the opposite direction.

Warning 5.1. Whereas the simplicial algebras of §4.5 correspond to functors from Δ to affine schemes, i.e. *cosimplicial* affine schemes

$$\begin{tikzcd} X^0 \arrow[r, shift left=2, "\partial^0"] \arrow[r, shift right=2, "\partial^1"'] & X^1 \arrow[l, dotted, "\sigma^0" description] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & X^2 \arrow[l, dotted, shift left=1.5] \arrow[l, dotted, shift right=1.5] \arrow[r, bend left] \arrow[r, phantom, "\vdots"] \arrow[r, bend right] & X^3 \arrow[r, bend left] \arrow[r, phantom, "\vdots"] \arrow[r, bend right] & \cdots, \end{tikzcd}$$

(one model for derived affine schemes), we now look at simplicial affine schemes

$$\begin{tikzcd} Y_0 \arrow[r, dotted, "\sigma_0" description] & Y_1 \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] \arrow[r, dotted, shift left=1.5] \arrow[r, dotted, shift right=1.5] & Y_2 \arrow[l, shift left=3] \arrow[l] \arrow[l, shift right=3] & Y_3 \arrow[l, bend right] \arrow[l, phantom, "\vdots"] \arrow[l, bend left] & \cdots & \cdots \end{tikzcd}$$

as models for higher stacks. These constructions behave *very* differently from each other.[45]

Simplicial resolutions of schemes will be familiar to anyone who has computed Čech cohomology. Given a quasi-compact scheme Y which is semi-separated (i.e. the diagonal map $Y \to Y \times Y$ is affine), we may take a finite affine cover $U = \coprod_i U_i$ of Y, and define the simplicial affine scheme $\check{Y}$ to be the Čech nerve $\check{Y} := \operatorname{cosk}_0(U/Y)$. Explicitly,

$$\check{Y}_n = \overbrace{U \times_Y U \times_Y \ldots \times_Y U}^{n+1} = \coprod_{i_0,\ldots,i_n} U_{i_0} \cap \ldots \cap U_{i_n},$$

so $\check{Y}_n$ is an affine scheme, and $\check{Y}$ is the unnormalised Čech resolution of Y.[47]

Given a quasi-coherent sheaf $\mathscr{F}$ on Y, we can then form a cosimplicial abelian group $\check{C}^n(Y, \mathscr{F}) := \Gamma(\check{Y}_n, \mathscr{F})$, and of course Zariski cohomology is given by

$$\mathrm{H}^i(Y, \mathscr{F}) \cong \mathrm{H}^i(\check{C}^\bullet(Y, \mathscr{F}), \sum_i (-1)^i \partial^i).$$ [48]

[45] At this point, someone usually asks whether we can replace these simplicial schemes with cdgas, and the answer is no. Although denormalisation gives a right Quillen equivalence from cdgas in non-negative *cochain* degrees to cosimplicial algebras (an analogue of Theorem 4.27), this would only be applicable if we were willing to declare morphisms $X \to Y$ to be equivalences whenever they induce isomorphisms $\mathrm{H}^*(Y, \mathscr{O}_Y) \to \mathrm{H}^*(X, \mathscr{O}_X)$.[46] Infinitesimally, there is a correspondence as in [Hin98] and [Pri07a, §§4.5–4.6], but even this requires a more subtle notion of equivalence than quasi-isomorphism.

[46] Anyone thinking this sounds as harmless as rational homotopy theory [Qui69, Sul77] should reflect that it would force the projective spaces $\mathbb{P}^n$ and the stacks $B\mathrm{GL}_n$ to all be equivalent to points. If you have to do that, please don't try to call it algebraic geometry.

[47] The nerve of a groupoid that we saw in §4.6 is also a form of Čech nerve, as $B\Gamma \cong \operatorname{cosk}_0(\operatorname{Ob}\Gamma/\Gamma)$, where $B\Gamma$ is the groupoid of objects of Γ with only identity morphisms, provided all fibre products in the Čech nerve are taken as 2-fibre products of groupoids.

[48] The Dold–Kan normalisation gives a quasi-isomorphic subcomplex, restricting to terms for which the indices $i_0, \ldots, i_n$ are all distinct. The standard Čech complex (with $i_0 < \ldots < i_n$) is a quasi-isomorphic quotient of that.

Likewise, if $\mathfrak{Y}$ is a quasi-compact semi-separated Artin stack, we can choose a presentation $U \to \mathfrak{Y}$ with U an affine scheme, and take the Čech nerve $\check{Y} := \mathrm{cosk}_0(U/\mathfrak{Y})$, so

$$\check{Y}_n = \overbrace{U \times_{\mathfrak{Y}} U \times_{\mathfrak{Y}} \ldots \times_{\mathfrak{Y}} U}^{n+1}.$$

For example, if G is an affine group scheme acting on an affine scheme U, we can take the quotient stack $\mathfrak{Y} = [U/G]$, and then we get $\check{Y}_n \cong U \times G^n$:

$$U \Leftarrow U \times G \Lleftarrow U \times G \times G \ldots.$$

Resolutions of this sort were used by Olsson in [Ols07] to study quasi-coherent sheaves on Artin stacks, fixing an error in [LMB00]. They also appear extensively in the theory of cohomological descent [SGA4b, Exposé vbis]. The analogous notion in differential geometry is a differentiable stack, with a specific presentation of the form $U \times_{\mathfrak{Y}} U \rightrightarrows U$ corresponding to a Lie groupoid; Deligne–Mumford stacks roughly correspond to orbifolds.

- Which simplicial affine schemes correspond to schemes, Artin stacks or Deligne–Mumford stacks in this way?
- What about higher stacks?
- What about derived schemes and stacks?
- How do we then define morphisms?
- How can we characterise quasi-coherent sheaves in terms of these resolutions?

5.1 Definitions

Given a simplicial set K and a simplicial affine scheme X (i.e. a functor $\Delta^{\mathrm{op}} \to \mathrm{Aff}$), there is an affine scheme $M_K(X)$ (the K-matching object) with the property that for all rings A, we have $M_K(X)(A) = M_K(X(A))$, i.e. $\mathrm{Hom}_{s\mathrm{Set}}(K, X(A))$. Explicitly, when $K = \Lambda^{m,k}$ this is given by the equaliser of a diagram

$$\prod_{\substack{0 \le i \le m \\ i \ne k}} X_{m-1} \Longrightarrow \prod_{\substack{0 \le i < j \le m \\ i,j \ne k}} X_{m-2},$$

and when $K = \partial\Delta^m$, it is given by the equaliser of a diagram

$$\prod_{0 \le i \le m} X_{m-1} \Longrightarrow \prod_{0 \le i < j \le m} X_{m-2}$$

for $m > 0$; the idea being that we have to specify a value for each face of $\Lambda^{n,k}$ or $\partial\Delta^n$, in such a way that they agree on the overlaps. We also have $M_{\partial\Delta^0}X = M_{\emptyset}X \cong *$.

The following definition gives objects which can be used to model higher stacks, and idea apparently originally due to Grothendieck, buried somewhere in [Gro83]:

Definition 5.2. Define an Artin (resp. Deligne–Mumford) n-hypergroupoid to be a simplicial affine scheme $X_\bullet$ for which the partial matching maps

$$X_m \to M_{\Lambda^{m,k}}(X)$$

are smooth (resp. étale) surjections for all m, k (i.e. $m \ge 1$ and $0 \le k \le m$), and isomorphisms for all $m > n$ and all k.

Remark 5.3. Note that hypergroupoids can be defined in any category containing pullbacks along covering morphisms.

In [Zhu08], Zhu uses this to define Lie n-groupoids (taking the category of manifolds, with coverings given by surjective submersions), and hence differentiable n-stacks. A similar approach could be used to define higher topological stacks (generalising [Noo05]), taking surjective local fibrations as the coverings in the category of topological spaces.[49]

Similar constructions can be made in non-commutative geometry (where the main difficulty is in deciding what a surjection should be) [Pri20] and in synthetic differential geometry and analytic geometry.[50] In the last two, descent can become more complicated than for algebraic geometry because affine objects are no longer compact.[52]

We could also extend our category to allow formal affine schemes as building blocks, allowing us to model functors such as the de Rham stack X_{dR} of [Sim96b, §7].

The reasons we take affine schemes as our building blocks, rather than schemes or algebraic spaces, are twofold: firstly, we know what a derived affine scheme is, but the other two are tricky, so this will generalise readily; secondly, quasi-coherent sheaves and quasi-coherent cohomology work much better if we can reduce to affine objects. From a conceptual point of view, it also feels more satisfying to reduce to an algebraic theory in an elementary way.

Remark 5.4. Other generalisations of higher stacks exist by taking more structured objects than simplicial sets as the foundation; for details see [Bal17].

Examples 5.5.

1. The Čech nerve as above of a quasi-compact semi-separated scheme gives a DM (in fact Zariski) 1-hypergroupoid. The same construction for a quasi-compact semi-separated algebraic space gives a DM 1-hypergroupoid. (Imposing the extra condition that $X_1 \to M_{\partial\Delta^1}(X)$ be an immersion characterises such nerves.)

2. The Čech nerve of a quasi-compact semi-separated DM stack is a DM 1-hypergroupoid.

3. The Čech nerve of a quasi-compact semi-separated Artin stack is an Artin 1-hypergroupoid. This applies to BG or $[U/G]$ for smooth affine group schemes G (e.g. GL_n).

4. Given a smooth affine commutative group scheme A (e.g. $\mathbb{G}_m$, $\mathbb{G}_a$), we can form a simplicial affine scheme $K(A,n)$ as follows. First take $A[n]$, regarded as a chain complex of commutative group schemes, then apply the Dold–Kan denormalisation functor to give a simplicial commutative group scheme

$$K(A,n) := N^{-1}A[n],$$

[49] this is in marked contrast to the derived story, there being no non-trivial notion of a derived topological space: see https://mathoverflow.net/questions/291093/derived-topological-stacks

[50] The main reason for this difference from algebraic geometry is that the Zariski topology has more points than the analytic and smooth topologies. In analytic (resp. differential) geometry, the EFC- (resp $\mathcal{C}^\infty$-) ring $\mathbb{C}^{\mathbb{N}}$ (resp. $\mathbb{R}^{\mathbb{N}}$)[51] usually corresponds to the discrete space $\mathbb{N}$. By contrast, $\mathrm{Spec}\,\mathbb{C}^{\mathbb{N}}$ is the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$, with the corona $\beta\mathbb{N}\setminus\mathbb{N} \cong \mathrm{Spec}\,(\mathbb{C}^{\mathbb{N}}/\mathbb{C}^{\infty})$, for the ideal $\mathbb{C}^\infty$ of finite sequences.

[51] Note that these are even finitely presented in these settings, being isomorphic to $\mathscr{O}^{\mathrm{hol}}(\mathbb{C})/(\exp z)$ and $\mathcal{C}^\infty(\mathbb{R})/(\sin x)$, respectively.

[52] A solution in the analytic setting is to take the building blocks to be compact Stein spaces [Tay02, Proposition 11.9.2] endowed with overconvergent functions. This seems a lot of effort to exclude points Grothendieck taught us to embrace, so an alternative solution might allow a compact building block for every EFC-ring, with the space associated to a Stein algebra $\mathscr{O}^{\mathrm{hol}}(X)$ perhaps being $\mathrm{Im}\,(\beta X \to \mathrm{Spec}\,\mathscr{O}^{\mathrm{hol}}(X))$ with the quotient topology; Stein spaces could still be built as countable unions of compact Stein spaces.

which is given in level m by $K(A,n)_m \cong A^{\binom{m}{n}}$. This is an example of an Artin n-hypergroupoid, and will give rise to an Artin n-stack.

We also have a relative notion:

Definition 5.6. Given $Y \in s\mathrm{Aff}$, define a (relative) Artin (resp. DM) n-hypergroupoid over Y to be a morphism $X_\bullet \to Y_\bullet$ in $s\mathrm{Aff}$, for which the partial matching maps

$$X_m \to M_{\Lambda^{m,k}}(X) \times_{M_{\Lambda^{m,k}}(Y)} Y_m$$

are smooth (resp. étale) surjections for all k, m, and are isomorphisms for all $m > n$ and all k.

The following gives rise a notion of equivalence for hypergroupoids:

Definition 5.7. Given $Y \in s\mathrm{Aff}$, define a trivial Artin (resp. DM) n-hypergroupoid over Y to be a morphism $X \to Y$ in $s\mathrm{Aff}$ for which the matching maps

$$X_m \to M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

are smooth (resp. étale) surjections for all $m \geq 0$, and are isomorphisms for all $m \geq n$.

When $n = \infty$, this is called a smooth (resp. étale) simplicial hypercover.

Note in particular that the $m = 0$ term above implies that $X_0 \to Y_0$ is a smooth (resp. étale) surjection.

Example 5.8. If $\{V_j\}_j$ and $\{U_i\}_i$ are finite open affine covers of a semi-separated quasi-compact scheme Y, then for $V := \coprod_j V_j$ and $U := \coprod_i U_i$, with $W := \coprod_{i,j} U_i \cap V_j$, the morphisms

$$\mathrm{cosk}_0(W/Y) \to \mathrm{cosk}_0(U/Y)$$
$$(W \Leftarrow W \times_Y W \Lleftarrow W \times_Y W \times_Y W \ldots) \to (U \Leftarrow U \times_Y U \Lleftarrow U \times_Y U \times_Y U \ldots)$$

and $\mathrm{cosk}_0(W/Y) \to \mathrm{cosk}_0(V/Y)$ are trivial relative DM (in fact Zariski) 1-hypergroupoids.

Example 5.9. If we think about how we calculate morphisms between schemes or algebraic spaces, we have:

$$\mathrm{Hom}(X,Y) = \varinjlim_{X' \to \check{X}} (X', \check{Y}),$$

for Čech nerves $\check{X} := \mathrm{cosk}_0(U/X)$ and $\check{Y} := \mathrm{cosk}_0(V/Y)$ for some affine covers U, V, with $X' \to \check{X}$ then ranging over all trivial Zariski or DM 1-hypergroupoids.

5.2 Main results

For our purposes, we can use the following as the definition of an $(n-1)$-geometric stack. It is a special case of [Pri09, Theorem 4.15].[53]

Theorem 5.10. *The homotopy category of strongly quasi-compact $(n-1)$-geometric Artin stacks is given by taking the full subcategory of $s\mathrm{Aff}$ consisting of Artin n-hypergroupoids $X_\bullet$, and formally inverting the trivial relative Artin n-hypergroupoids $X_\bullet \to Y_\bullet$.*

[53] Beware that [Pri09, Pri11a] use the terminology from an earlier version of [TV04] in which the indices were 1 higher for strongly quasi-compact objects, so that n-geometric in [Pri09] corresponds to $(n-1)$-geometric in the final version of [TV04], whereas our terminology in these notes conforms with the latter.

In fact, a model for the ∞-category of strongly quasi-compact $(n-1)$-geometric Artin stacks is given by the relative category $(\mathcal{C}, \mathcal{W})$ with $\mathcal{C}$ the full subcategory of $s\mathrm{Aff}$ on Artin n-hypergroupoids $X_\bullet$ and $\mathcal{W}$ the subcategory of trivial relative Artin n-hypergroupoids $X_\bullet \to Y_\bullet$.

The same results hold true if we substitute "Deligne–Mumford" for "Artin" throughout.

In particular, this means we obtain the simplicial category of such stacks by simplicial localisation of Artin/DM n-hypergroupoids at the class of trivial relative Artin/DM n-hypergroupoids.

Examples 5.11. We have already seen several fairly standard examples of hypergroupoid resolutions in Examples 5.5, and we now describe some more involved constructions, for those who are interested.

1. The method of split resolutions from [Del74, §6.2] can be adapted to give resolutions for schemes, algebraic spaces, and even Deligne–Mumford n-stacks, but not for Artin stacks because the diagonal of a smooth morphism is not smooth.

2. Take a smooth group scheme G which is quasi-compact and semi-separated, but not affine, so an elliptic curve, for instance. The simplicial scheme BG (given by $(BG)_m = G^m$) is an Artin 1-hypergroupoid in non-affine schemes which resolves the classifying stack $BG^\sharp$ of G.

   Next, we have to take a finite affine cover $\{U_i\}_{i\in I}$ for G and set $U = \coprod_i U_i$, writing $p\colon U \to G$. To proceed further, we introduce the simplicial schemes $U^{\Delta^\bullet_r}$ (*not* to be confused with the simplicial schemes U^{Δ^r} we meet in §6.4), which are given by $(U^{\Delta^\bullet_r})_m := U^{\Delta^m_r} \cong U^{\binom{m+r+1}{m}}$, and have the property that maps $X_\bullet \to U^{\Delta^\bullet_r}$ correspond to maps $X_r \to U$.

   We can then form an affine Artin 2-hypergroupoid resolving $BG^\sharp$ by taking

   $$BG \times_{G^{\Delta^\bullet_1}} U^{\Delta^\bullet_1}$$

   Explicitly, this looks like

   $$p^{-1}(e) \Leftarrow p^{-1}(e)^2 \times U \Lleftarrow p^{-1}(e)^3 \times \{(x,y,z) \in U^3 \ : \ p(x)p(y) = p(z)\} \quad \cdots,$$

   with the affine scheme in level m being

   $$\{\underline{x} \in \prod_{0\le i\le j\le m} U \ : \ p(x_{ij})p(x_{jk}) = p(x_{ik}) \ \forall i \le j \le k\}$$

   (in particular, $p(x_{ii}) = e$ for all i).

3. As a higher generalisation of the previous example, if G is moreover commutative, then we can form the simplicial scheme $K(G,n) = N^{-1}G[n]$, which is an Artin n-hypergroupoid in non-affine schemes, and then form the resolution

   $$K(G,n) \times_{G^{\Delta^\bullet_n}} U^{\Delta^\bullet_n}$$

   to give an affine Artin $(n+1)$-hypergroupoid.

Most examples are however not so simple, and the algorithm from [Pri09] takes $2^n - 1$ steps to construct an n-hypergroupoid resolution in general.

An ∞-stack over R is a functor $\mathrm{Alg}_R \to s\mathrm{Set}$ satisfying various conditions, so we need to associate such functors to Artin/DM hypergroupoids. The solution (not explicit) is to take

$$X^\sharp(A) = \mathbf{R}\mathrm{Map}_{\mathcal{W}}(\operatorname{Spec} A, X),$$

where $\mathbf{R}\mathrm{Map}_{\mathcal{W}}$ is the right-derived functor of $\mathrm{Hom}_{s\mathrm{Aff}}$, with respect to trivial Artin/DM n-hypergroupoids.

For an explicit solution:

5.2.1 Morphisms

Definition 5.12. Define the simplicial Hom functor on simplicial affine schemes by letting $\underline{\mathrm{Hom}}_{s\mathrm{Aff}}(X, Y)$ be the simplicial set given by

$$\underline{\mathrm{Hom}}_{s\mathrm{Aff}}(X, Y)_n := \mathrm{Hom}_{s\mathrm{Aff}}(\Delta^n \times X, Y),$$

where $(X \times \Delta^n)_i$ is given by the coproduct of Δ^n_i copies of X_i.

Definition 5.13. Given $X \in s\mathrm{Aff}$, say that an inverse system $\tilde{X} = (\tilde{X}(0) \leftarrow \tilde{X}(0) \leftarrow \ldots)$ (possibly transfinite) over X is an n-Artin (resp. n-DM) universal cover[54] if:

1. the morphisms $\tilde{X}_0 \to X$ and $\tilde{X}_{a+1} \to \tilde{X}_a$ are trivial Artin (resp. DM) n-hypergroupoids;
2. for any limit ordinal a, we have $\tilde{X}(a) = \varprojlim_{b<a} \tilde{X}(b)$;
3. for every a and every trivial Artin (resp. DM) n-hypergroupoid $Y \to \tilde{X}(a)$, there exists $b \geq a$ and a factorisation

$$\begin{tikzcd} \tilde{X}(b) \arrow[rr] \arrow[dr, dashed] & & \tilde{X}(a) \\ & Y. \arrow[ur] & \end{tikzcd}$$

These always exist, by [Pri09, Proposition 3.24]. Moreover, [Pri09, Corollary 3.32] shows that every n-DM universal cover is in fact an n-Artin universal cover.

Definition 5.14. Given an Artin n-hypergroupoid Y and $X \in s\mathrm{Aff}$, define

$$\underline{\mathrm{Hom}}^\sharp_{s\mathrm{Aff}}(X, Y) := \varinjlim \underline{\mathrm{Hom}}_{s\mathrm{Aff}}(\tilde{X}(i), Y),$$

where the colimit runs over the objects $\tilde{X}(i)$ of any n-Artin universal cover $\tilde{X} \to X$.

The following is [Pri09, Corollary 4.10]:

Theorem 5.15. *If $X \in s\mathrm{Aff}$ and Y is an Artin n-hypergroupoid, then the derived* Hom *functor on the associated hypersheaves (a.k.a. n-stacks) $X^\sharp, Y^\sharp$ is given (up to weak equivalence) by*

$$\mathbf{R}\mathrm{Map}(X^\sharp, Y^\sharp) \simeq \underline{\mathrm{Hom}}^\sharp_{s\mathrm{Aff}}(X, Y).$$

[54] Think of this as being somewhat similar to a universal cover of a topological space.

Remarks 5.16. Given a ring A, set $X = \operatorname{Spec} A$, and note that $\underline{\mathrm{Hom}}^{\sharp}_{s\mathrm{Aff}}(X,Y) = Y^{\sharp}(A)$, the hypersheafification of the functor $Y : \mathrm{Alg}_R \to s\mathrm{Set}$, so we can take Definition 5.14 as a definition of hypersheafification for Artin hypergroupoids, giving an explicit description of $Y^{\sharp}$. The $n = 1$ case should be familiar as the standard definition of sheafification.

Between them, Theorems 5.10 and 5.15 recover the simplicial category of strongly quasi-compact $(n-1)$-geometric Artin stacks, with Theorem 5.10 giving the objects and Theorem 5.15 the morphisms. We could thus take those theorems to be a definition of that simplicial category.

5.2.2 Truncation considerations

Remark 5.17. Recall from Properties 4.34 that an n-hypergroupoid Y is determined by $Y_{\leq n+1}$, and in fact that $Y \cong \mathrm{cosk}_{n+1} Y$. This implies that

$$\mathrm{Hom}(X,Y) \cong \mathrm{Hom}(X_{\leq n+1}, Y_{\leq n+1})$$

for any X, and hence

$$\mathrm{Hom}(X \times \Delta^m, Y) \cong \mathrm{Hom}((X \times \Delta^m)_{\leq n+1}, Y_{\leq n+1}),$$

which greatly simplifies the calculation of $\underline{\mathrm{Hom}}(X,Y)$.

Example 5.18. A trivial 0-hypergroupoid is just an isomorphism, so the 0-DM universal cover of any X is just X. For Y an affine scheme, and any $X \in s\mathrm{Aff}$, this means that

$$\begin{aligned}\underline{\mathrm{Hom}}^{\sharp}_{s\mathrm{Aff}}(X,Y) &= \underline{\mathrm{Hom}}_{s\mathrm{Aff}}(X,Y),\\ &= \underline{\mathrm{Hom}}(X_{\leq 1}, Y_{\leq 1}).\end{aligned}$$

$$\begin{tikzcd} X_0 \arrow[r, dotted, shift left=0] \arrow[d] & X_1 \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] \arrow[d] \\ Y \arrow[r, dotted] & Y \arrow[l, shift right=2, "\mathrm{id}"'] \arrow[l, shift left=2, "\mathrm{id}"] \end{tikzcd}$$

In level 0, this is just the equaliser of $\mathrm{Hom}(X_0, Y) \rightrightarrows \mathrm{Hom}(X_1, Y)$, so is $\mathrm{Hom}(\pi_0^{\mathrm{Aff}} X, Y)$, where $\pi_0^{\mathrm{Aff}} X$ is the equaliser of $X_1 \rightrightarrows X_0$ in the category of affine schemes. For example, when X is the Čech nerve of a scheme, algebraic space, or even algebraic stack, Z, we have $\pi_0^{\mathrm{Aff}} X \cong \operatorname{Spec} \Gamma(Z, \mathscr{O}_Z)$.

For higher n, we get $\underline{\mathrm{Hom}}(X,Y)_n \cong \mathrm{Hom}(\pi_0^{\mathrm{Aff}}(X \times \Delta^n), Y) \cong \mathrm{Hom}(\pi_0^{\mathrm{Aff}} X, Y)$, so

$$\underline{\mathrm{Hom}}^{\sharp}_{s\mathrm{Aff}}(X,Y) \cong \mathrm{Hom}(\pi_0^{\mathrm{Aff}} X, Y)$$

(constant simplicial structure).

Example 5.19. Take a 1-hypergroupoid Y, and an affine scheme U, then look at $\underline{\mathrm{Hom}}^{\sharp}_{s\mathrm{Aff}}(U,Y)$. Relative 1-hypergroupoids over U are just Čech nerves $\mathrm{cosk}_0(U'/U) \to U$ for étale surjections $U' \to U$:

$$(U' \Leftarrow U' \times_U U' \Lleftarrow U' \times_U U' \times_U U' \ldots) \to U.$$

Then

$$\begin{aligned}\underline{\mathrm{Hom}}^{\sharp}_{s\mathrm{Aff}}(U,Y) &= \varinjlim_{U'} \underline{\mathrm{Hom}}_{s\mathrm{Aff}}(\mathrm{cosk}_0(U'/U), Y),\\ &= \varinjlim_{U'} \underline{\mathrm{Hom}}(\mathrm{cosk}_0(U'/U)_{\leq 2}, Y_{\leq 2}),\end{aligned}$$

so an element f of the mapping space consists of maps f_i making the diagram

$$\begin{tikzcd}
U' \arrow[r, dotted, shift left=0] & U' \times_U U' \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, dotted, shift left=1] \arrow[r, dotted, shift right=1] & U' \times_U U' \times_U U' \arrow[l, shift left=3] \arrow[l, shift right=3] \arrow[l] \\
Y_0 \arrow[from=1-1, "f_0"] \arrow[r, dotted] & Y_1 \arrow[from=1-2, "f_1"] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, dotted, shift left=1] \arrow[r, dotted, shift right=1] & Y_2 \arrow[from=1-3, "f_2"] \arrow[l, shift left=3] \arrow[l, shift right=3] \arrow[l]
\end{tikzcd}$$

commute.

When Y is a Čech nerve $\mathrm{cosk}_0(V/Z)$ (for V a cover of a scheme, algebraic space or algebraic stack Z), the target becomes

$$\begin{tikzcd}
V \arrow[r, dotted] & V \times_Z V \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, dotted, shift left=1] \arrow[r, dotted, shift right=1] & V \times_Z V \times_Z V, \arrow[l, shift left=3] \arrow[l, shift right=3] \arrow[l]
\end{tikzcd}$$

giving the expected data for a morphism f, as in the expression of [LMB00, Lemma 3.2] for U-valued points of the stack $Z \simeq [V \times_Z V \Rightarrow V]$. Note that when Y is an algebraic space, f_0 determines f_1 because then the map $Y_1 \to M_{\partial\Delta^1}(Y)$ (i.e. $V \times_Z V \to V \times V$) is an immersion, so even f_2 then becomes redundant.

Example 5.20. Now let X be a scheme, with $\check{X} := \mathrm{cosk}_0(U/X)$, for $U \to X$ an étale cover. Let $Y = BG$, for G a smooth affine group scheme (e.g. GL_n, $\mathbb{G}_m$ or $\mathbb{G}_a$ but not μ_p in characteristic p). Set

$$\check{\mathrm{C}}^n(X, G) := \Gamma(\check{X}_n, G(\mathscr{O}_X)) = \mathrm{Hom}(\overbrace{U \times_X \ldots \times_X U}^{n+1}, G);$$

for $\mathbb{G}_a$ and $\mathbb{G}_m$, this gives $\Gamma(\check{X}_n, \mathscr{O}_X)$ and $\Gamma(\check{X}_n, \mathscr{O}_X^*)$ respectively.

Then $\mathrm{Hom}(\check{X}, BG)$ is isomorphic to

$$\check{Z}^1(U, G) = \{\omega \in \check{C}^1(U, G) \,:\, \partial^2\omega \cdot \partial^0\omega = \partial^1\omega \in \check{C}^2(U, G)\};$$

in other words, ω satisfies the cocycle condition, so determines a G-torsor P on X with $P \times_X U \cong G \times U$. Meanwhile,

$$\mathrm{Hom}(\Delta^1 \times \check{X}, BG) \cong \{(\omega_0, g, \omega_1) \in \check{Z}^1(U, G) \times \check{C}^0(U, G) \times \check{Z}^1(U, G) \,:\, \partial^1(g) \cdot \omega_0 = \omega_1 \cdot \partial^0(g)\},$$

so g is a gauge transformation between ω_0 and ω_1; on the corresponding G-torsors, this amounts to giving an isomorphism $\phi \colon P_1 \cong P_2$ (which need not respect the trivialisation on U).

Thus $\underline{\mathrm{Hom}}(\check{X}, BG)$ is the nerve of the groupoid $[\check{Z}^1(U, G)/\check{C}^0(U, G)]$ of G-torsors on X which become trivial on pullback to U, and passing to the colimit over all étale covers U' of X, we get that $\underline{\mathrm{Hom}}^\sharp(\check{X}, BG)$ is equivalent to the nerve of the groupoid of all étale G-torsors on X, as expected.

Example 5.21. For E an elliptic curve, the Čech complex of BE is a (1-truncated) Artin 2-hypergroupoid, but $\mathrm{Map}(X, (BE)^\sharp)$ still classifies E-torsors on X.

Examples 5.22. Example 5.20 tells us that

$$\pi_i(BG)^\sharp(X) = \varinjlim \mathrm{H}^{1-i}(X'_\bullet, G) = \mathrm{H}^{1-i}_{\text{ét}}(X, G)$$

for $i = 0, 1$, where $X'_\bullet \to X$ runs over étale hypercovers.

If A is a smooth commutative affine group scheme (such as $\mathbb{G}_m, \mathbb{G}_a$ or μ_n for $n^{-1} \in R$), we can generalise this to the higher $K(A, n)$. We have

$$\mathrm{Hom}(\check{X}, K(A, n)) \cong \mathrm{Z}^n N\check{C}(X, A),$$

and using a path object for $K(A,n)$, we then get

$$\pi_i \underline{\mathrm{Hom}}(X, K(A,n)^\sharp) \cong \pi_i K(A,n)^\sharp(X) \cong \mathrm{H}^{n-i}_{\text{ét}}(X,A),$$

which is reminiscent of Brown representability in topology. Note that for $A = \mathbb{G}_a$, this is just $\mathrm{H}^{n-i}(X, \mathscr{O}_X)$.

Definition 5.23. We say that a functor $F\colon \mathrm{Alg}_k \to s\mathrm{Set}$ is n-truncated if $\pi_i(F(A)) = 0$ for all $i > n$ and all $A \in \mathrm{Alg}_k$.

For a hypergroupoid X, n-truncatedness of $X^\sharp$ amounts to saying that the the maps

$$X_i \to M_{\partial\Delta^i}(X)$$

are monomorphisms (i.e. immersions) of affine schemes for all $i > n$.

Warning 5.24. Beware that there are slight differences in terminology between [TV04] and [Lur04a]. In the former, affine schemes are (-1)-representable, so arbitrary schemes might only be 1-geometric, while Artin stacks are 0-geometric stacks if and only if they have affine diagonal. In the latter, algebraic spaces are 0-stacks.[55]

An n-stack $\mathfrak{X}$ in the sense of [Lur04a] is an n-truncated functor which is m-geometric for some m.

It follows easily from Property 4.34.3 that every n-geometric stack in [TV04] is $(n+1)$-truncated; conversely, any n-truncated stack $\mathfrak{X}$ is $(n+1)$-geometric[56]. Any Artin stack with affine diagonal (in particular any separated algebraic space) is 0-geometric.

The conditions can be understood in terms of higher diagonals. If for simplicial sets or topological spaces K, we write $\mathfrak{X}^{hK}$ for the functor $A \mapsto \mathbf{R}\mathrm{Map}_{s\mathrm{Set}}(K, \mathfrak{X}(A))$, then we get $\mathfrak{X}^{hS^n} \simeq \mathfrak{X} \times^h_{\mathfrak{X}^{hS^{n-1}}} \mathfrak{X}$, and we think of the diagonal map $\mathfrak{X} \to \mathfrak{X}^{hS^n}$ as the nth higher diagonal.

Being n-geometric then amounts to saying that the higher diagonal morphism $\mathfrak{X} \to \mathfrak{X}^{hS^n}$ to the iterated loop space is affine (where $S^{-1} = \emptyset$ and $S^0 = \{-1, 1\}$), while being n-truncated amounts to saying that the morphism $\mathfrak{X} \to \mathfrak{X}^{S^{n+1}}$ is an equivalence.

If we took quasi-compact, quasi-separated algebraic spaces instead of affine schemes in Definition 5.2, then Theorem 5.10 would adapt to give a characterisation of n-truncated Artin stacks. Our main motivation for using affine schemes as the basic objects is that they will be easier to translate to a derived setting.

Remark 5.25. The strong quasi-compactness condition in Theorem 5.10 is terminology from [TV05] which amounts to saying that the objects are quasi-compact, quasi-separated, and so on (all the higher diagonals $\mathfrak{X} \to \mathfrak{X}^{hS^n}$ we saw in Warning 5.24 are quasi-compact). We can drop this assumption if we expand our category of building blocks by allowing arbitrary disjoint unions of affine schemes.

[55] The situation is further complicated by earlier versions of [TV04] using higher indices, and the occasional use as in [Toë08] of n-algebraic stacks, intermediate between $(n-1)$-geometric and n-truncated stacks. In [Toë06], n-geometric Artin stacks are simply called n-Artin stacks, and distinguished from Lurie's Artin n-stacks. Finally, beware that [Lur04b] and its derivatives refer to geometric stacks, by which they mean 0-geometric stacks (apparently in the belief this is standard algebro-geometric terminology for semi-separated).

[56] This is *not* a typo for $(n-1)$; a non-semi-separated scheme such as $\mathbb{A}^2 \cup_{\mathbb{A}^2 \setminus \{0\}} \mathbb{A}^2$ is 0-truncated but 1-geometric, while an affine scheme is 0-truncated and (-1)-geometric

5.3 Quasi-coherent sheaves and complexes

The following is part of [Pri09, Corollary 5.12]:

Proposition 5.26. *For an Artin n-hypergroupoid X, giving a quasi-coherent sheaf (a.k.a Cartesian sheaf) on the associated n-geometric stack $X^\sharp$ is equivalent to giving a Cartesian sheaf on X, i.e.:*

1. *a quasi-coherent sheaf $\mathscr{F}^n$ on X_n for each n, and*
2. *isomorphisms $\partial^i \colon \partial_i^* \mathscr{F}^{n-1} \to \mathscr{F}^n$ for all i and n, satisfying the usual cosimplicial identities*
$$\partial^j \partial^i = \partial^{i+1} \partial^j \qquad j \leq i.$$

It is not too hard to see that $\mathscr{F}$ is determined by the sheaf $\mathscr{F}^0$ on X_0 and the isomorphism $\partial_0^* \mathscr{F}^0 \cong \partial_1^* \mathscr{F}^0$ on X_1, which satisfies a cocycle condition on X_2.

This has the following generalisation (again, [Pri09, Corollary 5.12]):

Proposition 5.27. *For an Artin n-hypergroupoid X, giving a quasi-coherent complex on the associated n-geometric stack $X^\sharp$ is equivalent to giving a homotopy-Cartesian module, i.e.:*

1. *a complex $\mathscr{F}^n_\bullet$ of quasi-coherent sheaves on X_n for each n,*
2. *quasi-isomorphisms $\partial^i \colon \partial_i^* \mathscr{F}^{n-1}_\bullet \to \mathscr{F}^n_\bullet$ for all i and n, and*
3. *morphisms $\sigma^i \colon \sigma_i^* \mathscr{F}^{n+1}_\bullet \to \mathscr{F}^n_\bullet$ for all i and n,*

where the operations ∂^i, σ^i satisfy the usual cosimplicial identities,[57] *and a morphism $\{\mathscr{E}^n_\bullet\}_n \to \{\mathscr{F}^n_\bullet\}_n$ is a weak equivalence if the maps $\mathscr{E}^n_\bullet \to \mathscr{F}^n_\bullet$ are all quasi-isomorphisms.*

Note that because the maps ∂_i are all smooth, they are flat, so we do not need to left-derive the pullback functors ∂_i^*. Also note that because the maps ∂^i are only quasi-isomorphisms, they do not have inverses, which is why we have to include the morphisms σ^i as additional data. The induced morphisms $\sigma^i \colon \mathbf{L}\sigma_i^* \mathscr{F}^{n+1}_\bullet \to \mathscr{F}^n_\bullet$ are then automatically quasi-isomorphisms. We can also rephrase the quasi-isomorphism condition as saying that $\{\mathrm{H}_i(\mathscr{F}^n_\bullet)\}_n$ is a Cartesian sheaf on X for all i.

Remark 5.28. Inclusion gives a canonical functor $\mathcal{D}(\mathcal{QC}oh(\mathscr{O}_X)) \to \mathrm{Ho}(\mathrm{Cart}(\mathscr{O}_X))$ from the derived category of complexes of quasi-coherent sheaves on X to the homotopy category of homotopy-Cartesian complexes, and this is an equivalence when X is a quasi-compact semi-separated scheme, by [Hüt08, Theorem 4.5.1].

Under the same hypotheses, $\mathcal{D}(\mathcal{QC}oh(\mathscr{O}_X))$ is in turn equivalent to the derived category $\mathcal{D}_{\mathcal{QC}oh}(\mathscr{O}_X)$ of complexes of sheaves of $\mathscr{O}_X$-modules with quasi-coherent homology *sheaves*, by [BN93, Corollary 5.5] (or just [SGA6, Exp. II, Proposition 3.7b] if X is Noetherian).

To compare $\mathrm{Ho}(\mathrm{Cart}(\mathscr{O}_X))$ and $\mathcal{D}_{\mathcal{QC}oh}(\mathscr{O}_X)$ directly, observe that since sheafification is exact, it gives us a functor $\mathrm{Ho}(\mathrm{Cart}(\mathscr{O}_X)) \to \mathcal{D}_{\mathcal{QC}oh}(\mathscr{O}_X)$. This is always an equivalence, with quasi-inverse given by the derived right adjoint, sending each complex of sheaves to an injective resolution (or more precisely, to a fibrant replacement in the injective model structure).

[57] These identities are all required to hold on the nose, so that for instance $\sigma^i \circ \partial^i$ must equal the identity, not just be homotopic to it.

5.3.1 Inverse images

Given a morphism $f\colon X \to Y$ of Artin n-hypergroupoids, inverse images of are easy to compute: if $\mathscr{F}$ is a Cartesian quasi-coherent sheaf on Y, then the sheaf $f^*\mathscr{F}$ on X given levelwise by $(f^*\mathscr{F})^n := f_n^*\mathscr{F}^n$ is also Cartesian. Similarly, if $\mathscr{F}_\bullet$ is a homotopy-Cartesian quasi-coherent complex on Y, then the there is a complex $\mathbf{L}f^*\mathscr{F}_\bullet$ on X, given levelwise by $(\mathbf{L}f^*\mathscr{F}_\bullet)^n \simeq \mathbf{L}f_n^*\mathscr{F}_\bullet^n$, which is also homotopy-Cartesian.

5.3.2 Derived global sections

Direct images are characterised as right adjoints to inverse images, but are much harder to construct, because taking f_* levelwise destroys the Cartesian property. There is an explicit description in [Pri09, §5.4.3] of the derived direct image functor $\mathbf{R}f_*^{\mathrm{cart}}$.

The special case of derived global sections is much easier to describe. If $\mathscr{F}_\bullet$ is homotopy-Cartesian on X, then $\mathbf{R}\Gamma(X^\sharp, \mathscr{F}_\bullet)$ is just the *product* total complex of the double complex

$$\Gamma(X_0, \mathscr{F}_\bullet^0) \xrightarrow{\partial^0-\partial^1} \Gamma(X_1, \mathscr{F}_\bullet^1) \xrightarrow{\partial^0-\partial^1+\partial^2} \Gamma(X_2, \mathscr{F}_\bullet^2) \to \ldots$$

(or equivalently of its Dold–Kan normalisation, restricting to $\bigcap_i \ker \sigma^i$ in each level).

5.4 Hypersheaves

Definition 5.29. A functor

$$\mathscr{F}\colon \mathrm{Aff}^{\mathrm{op}} \to \mathcal{C}$$

to a model category (or more generally an ∞-category) $\mathcal{C}$ is said to be an étale hypersheaf if for any trivial DM ∞-hypergroupoid (a.k.a. étale hypercover) $U_\bullet \to X$, the map

$$\mathscr{F}(X) \to \operatorname{ho}\varprojlim_{n\in\Delta} \mathscr{F}(U_n)$$

is a weak equivalence, and for any X, Y the map

$$\mathscr{F}(X \sqcup Y) \to \mathscr{F}(X) \times \mathscr{F}(Y)$$

is a weak equivalence, with $\mathscr{F}(\emptyset)$ contractible.

Note that since $U_\bullet$ is a simplicial scheme, contravariance means that the functor $n \mapsto \mathscr{F}(U_n)$ is a *cosimplicial* diagram in $\mathcal{C}$ (i.e. a functor $\Delta \to \mathcal{C}$). On the category $\mathcal{C}^\Delta$ of cosimplicial objects, $\operatorname{holim}_{n\in\Delta}$ is a right-derived functor $\mathbf{R}\pi^0$ of the functor π^0, which sends a cosimplicial object $A^\bullet$ to the equaliser of $\partial^0, \partial^1\colon A^0 \to A^1$.

Examples 5.30.

1. If $\mathcal{C}$ is a category with trivial model structure (all morphisms are fibrations and cofibrations, isomorphisms are the only weak equivalences), then $\operatorname{ho}\varprojlim_{n\in\Delta} A^n$ is the equaliser in $\mathcal{C}$ of the maps $\partial^0, \partial^1 : A^0 \to A^1$, so hypersheaves in $\mathcal{C}$ are precisely $\mathcal{C}$-sheaves.

2. For the category Ch of unbounded chain complexes and $V \in \mathrm{Ch}^\Delta$,

$$\operatorname{ho}\varprojlim_{n\in\Delta} V^n \simeq \operatorname{Tot}^\Pi(V^0 \xrightarrow{\partial^0-\partial^1} V^1 \to \ldots),$$

the product total complex of the double complex, with reasoning as in §3.5.1.

3. On the category $(s\mathrm{Set})^{\Delta}$ of cosimplicial simplicial sets, $\mathrm{ho}\varprojlim_{n\in\Delta}$ is the functor $\mathbf{R}\mathrm{Tot}_{s\mathrm{Set}}$, where $\mathbf{R}$ means "take (Reedy) fibrant replacement first", and $\mathrm{Tot}_{s\mathrm{Set}}$ is the total complex Tot of a cosimplicial space defined in [GJ99, §VII.5]; see Figure 7.

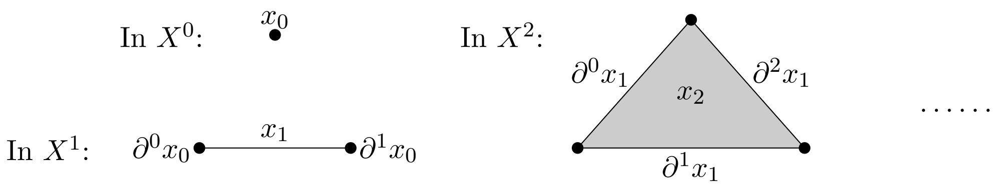


Figure 7: an element x of $\mathrm{Tot}_{s\mathrm{Set}} X^{\bullet}$.[58]

Explicitly,

$$\mathrm{Tot}_{s\mathrm{Set}} X^{\bullet} = \{x \in \prod_n (X^n)^{\Delta^n} \,:\, \partial^i_X x_n = (\partial^i_{\Delta})^* x_{n+1},\, \sigma^i_X x_n = (\sigma^i_{\Delta})^* x_{n-1}\}.$$

Homotopy groups $\pi_i \mathbf{R}\mathrm{Tot}_{s\mathrm{Set}} X^{\bullet}$ of the total space are related to the homotopy groups $\pi_{i+n} X^n$ by a spectral sequence given in [GJ99, §VII.6].

4. For the category $dg_+\mathrm{Alg}$ of non-negatively graded cdgas, a model for $\mathrm{ho}\varprojlim_{n\in\Delta}$ is given by taking good truncation of the functor of Thom–Sullivan (a.k.a Thom–Whitney) cochains [HS87, §4], defined using de Rham polynomial forms.

Remarks 5.31. A sheaf $\mathscr{F}$ of modules is a hypersheaf when regarded as a presheaf of non-negatively graded chain complexes, but not a hypersheaf when regarded as a presheaf of unbounded or non-positively graded chain complexes unless $\mathrm{H}^i(U, \mathscr{F}) = 0$ for all $i > 0$ and all U. Beware that the sheafification (as opposed to hypersheafification) of a hypersheaf will not, in general, be a hypersheaf.

The construction $X \rightsquigarrow X^{\sharp}$ introduced after Theorem 5.10 is an example of hypersheafification:

Definition 5.32. Given a functor $\mathscr{F} \colon \mathrm{Aff}^{\mathrm{op}} \to s\mathrm{Set}$, the étale hypersheafification $\mathscr{F}^{\sharp}$ of $\mathscr{F}$ is the universal étale hypersheaf $\mathscr{F}^{\sharp}$ equipped with a map $\mathscr{F} \to \mathscr{F}^{\sharp}$ in the homotopy category of simplicial presheaves.

In other words, hypersheafification is the derived left adjoint to the forgetful functor from hypersheaves to presheaves.

Remark 5.33. Terminology is disastrously inconsistent between references. Hypersheaves are often known as ∞-stacks or ∞-sheaves, but are referred to as stacks in [TV02], and as sheaves in [Lur09b, Lur04a] (where *stacks* refer only to algebraic stacks). They are also sometimes known as homotopy sheaves, but we avoid this terminology for fear of confusion with homotopy groups of a simplicial sheaf.

5.5 The conventional approach to higher stacks

Instead of defining n-stacks using hypergroupoids, [TV04, Lur04a] use an inductive definition, following the approach set out by Simpson in [Sim96a], which he attributed to Walter.

[58] Note that the vertices of the 2-simplex match up because $\partial^0\partial^0 x_0 = \partial^1\partial^0 x_0$, $\partial^0\partial^1 x_0 = \partial^2\partial^0 x_0$ and $\partial^1\partial^1 x_0 = \partial^2\partial^0 x_0$.

In that approach, n-geometric stacks are defined inductively by saying that an étale hypersheaf F is n-geometric if

1. there exists a smooth covering $\coprod_i U_i \to F$ from a family $\{U_i\}_i$ of affine schemes, and
2. the diagonal $F \to F \times F$ is relatively representable by $(n-1)$-geometric stacks.

If we take the families $\{U_i\}_i$ to be finite at each stage in the definition above, then we obtain the definition of a strongly quasi-compact n-geometric stack. In the final version of [TV04], the induction starts by setting affine schemes to be (-1)-geometric.

Beware that the n-stacks of [Lur04a] are indexed slightly differently, taking 0-stacks to be algebraic spaces, leading to the differences explained in Warning 5.24.

By [Pri09, Theorem 4.15] (see Theorem 5.10 above), n-geometric stacks correspond to Artin $(n+1)$-hypergroupoids.

In practice, this characterisation feels like a halfway house, and it iterates the least satisfactory aspects of the definition of an algebraic stack. To prove a general statement about geometric n-stacks, it is usually easier to work with hypergroupoids, while to prove that a hypersheaf is a geometric n-stack, it is usually easier to use a representability theorem (see §6.5).

6 Derived geometric n-stacks

There is nothing special about affine schemes as building blocks, so now we will use derived affine schemes. The only real change is that we have to replace all limits with homotopy limits, but since homotopy limits are practically useless without a means to compute them, we will start out with a more elementary characterisation. The constructions and results prior of §§6.1–6.4,6.9 all have natural analogues in differential and analytic contexts, by using dg $\mathcal{C}^\infty$ or EFC rings instead of derived commutative rings, replacing smooth morphisms with submersions and étale morphisms with local diffeomorphisms or local biholomorphisms.

6.1 Definitions

Notation 6.1. Since we want statements applying in all characteristics, we will let the category $d\mathrm{Aff}_R$ of derived affine schemes be the opposite category to either cdgas $dg_+\mathrm{Alg}_R$ (if $\mathbb{Q} \subset \mathbb{R}$) or to simplicial algebras $s\mathrm{Alg}_R$; denote that opposite category by $d\mathrm{Alg}_R$.

We give this the opposite model structure, so a morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ in $d\mathrm{Aff}_R$ is a fibration/cofibration/weak equivalence if and only if $A \to B$ is a cofibration/fibration/weak equivalence of cdgas or simplicial algebras; in particular, fibration means the corresponding map of cdgas or simplicial algebras is a retract of a quasi-free map[59].

Notation 6.2. From now on, for consistency with the dg setting, we denote the homotopy groups of a simplicial algebra A by $\pi_i A = \mathrm{H}_i(A, \sum(-1)^i \partial_i)$ by $\mathrm{H}_i A$. Beware that this notation is highly abusive, since these are the homotopy groups, *not* the homology groups, of the underlying simplicial set.

Definition 6.3. We define the category $sd\mathrm{Aff}_R$ of simplicial derived affine schemes to be $(d\mathrm{Aff}_R)^{\Delta^{\mathrm{op}}}$, so it consists of diagrams

$$\begin{tikzcd} X_0 \arrow[r, dotted, "\sigma_0"] & X_1 \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] \arrow[r, dotted, shift left=1] \arrow[r, dotted, shift right=1] & X_2 \arrow[l, bend right] \arrow[l] \arrow[l, bend left] & Y_3 \arrow[l, bend right] \arrow[l, phantom, "\vdots"] \arrow[l, bend left] & \cdots & \cdots, \end{tikzcd}$$

for derived affine schemes X_i.

Equivalently, this is the opposite category to the category of *cosimplicial* cdgas or of cosimplicial simplicial algebras.

Recall that we write $\pi^0\mathrm{Spec}\, A = \mathrm{Spec}\, \mathrm{H}_0 A$ for $A \in dg_+\mathrm{Alg}_R$, and similarly write $\pi^0\mathrm{Spec}\, A := \mathrm{Spec}\, \pi_0 A$ for $A \in s\mathrm{Alg}_R$.

Definition 6.4. We say that a simplicial derived affine scheme $X_\bullet$ is a *homotopy derived Artin (resp. DM) n-hypergroupoid* if:

1. the simplicial affine scheme $\pi^0 X_\bullet$ is an Artin (resp. DM) n-hypergroupoid (Definition 5.2);
2. the sheaves $\mathrm{H}_j(\mathscr{O}_{X_\bullet})$ on $\pi^0 X_\bullet$ are all Cartesian; explicitly, for the morphisms $\partial_i\colon X_{m+1} \to X_m$, we have isomorphisms
$$\partial^i\colon \partial_i^{-1}\mathrm{H}_j(\mathscr{O}_{X_m}) \otimes_{\partial_i^{-1}\mathrm{H}_0(\mathscr{O}_{X_m})} \mathrm{H}_0(\mathscr{O}_{X_{m+1}}) \to \mathrm{H}_j(\mathscr{O}_{X_{m+1}})$$
for all i, m, j.

[59]or at least ind-quasi-smooth in the original sense, if you prefer to use the Henselian model structure of [Pri18b, Proposition 3.12], as discussed in Example 2.26

Equivalently, the second condition says that the morphisms $\partial_i\colon X_{m+1} \to X_m$ are strong for all i, m, hence homotopy-smooth (resp. homotopy-étale).

As in the underived setting, we have a relative notion:

Definition 6.5. Given $Y_\bullet \in sd\mathrm{Aff}$, a morphism $f\colon X_\bullet \to Y_\bullet$ of simplicial derived affine schemes is said to be a *homotopy derived Artin (resp. DM) n-hypergroupoid* over $Y_\bullet$ if:

1. the morphism $\pi^0 X_\bullet \to \pi^0 Y_\bullet$ of simplicial affine schemes is an Artin (resp. DM) n-hypergroupoid (Definition 5.6);
2. the morphisms $(f, \partial_i)\colon X_{m+1} \to Y_{m+1} \times^h_{\partial_i, Y_m} X_m$ are strong (Definition 3.44) for all i, m.

The morphism $X_\bullet \to Y_\bullet$ is then said to be homotopy-smooth (resp. homotopy-étale, resp. surjective) if in addition $X_0 \to Y_0$ is homotopy-smooth (resp. homotopy-étale, resp. surjective).

Remarks 6.6.

1. If X_0 is underived in the sense that $X_0 \simeq \pi^0 X_0$, and if $X_\bullet$ is a homotopy derived Artin n-hypergroupoid, then the morphism $\pi^0 X_\bullet \to X_\bullet$ is a weak equivalence by homotopy-smoothness, because anything homotopy-smooth over an underived base is itself underived.
2. Similarly, if Y_0 is underived then for $X \to Y$ to be homotopy-smooth (resp. homotopy-étale) just means that X_0 is quasi-isomorphic to a smooth (resp. étale) underived affine scheme over Y_0.

Examples 6.7.

1. Every Artin/DM n-hypergroupoid is a homotopy derived Artin/DM n-hypergroupoid.
2. Saying that $X_\bullet$ is a homotopy 0-hypergroupoid is equivalent to saying that $X_0 \to X_\bullet$ is a weak equivalence, i.e. that $X_\bullet$ is equivalent to a derived affine scheme (with constant simplicial structure).
3. If a smooth affine group scheme G acts on a derived affine scheme U, then the simplicial derived affine scheme
$$B[U/G] := (U \Leftarrow U \times G \Lleftarrow U \times G \times G \ldots)$$
is a homotopy derived Artin 1-hypergroupoid.
4. If $(\pi^0 X, \mathscr{O}_X)$ is a derived scheme (Definition 1.21), with $\pi^0 X$ quasi-compact and semi-separated, take a finite cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $\pi^0 X$ by open affine subschemes, and consider the simplicial derived affine scheme $\check{X}_{\mathcal{U}}$ given by the Čech nerve
$$(\check{X}_{\mathcal{U}})_m := \mathrm{Spec}\,(\prod_{i_0,\ldots,i_m \in I} \Gamma(U_{i_0} \cap \ldots \cap U_{i_m}, \mathscr{O}_X))$$
with the obvious face and degeneracy maps.

   Since $\pi^0 \check{X}_{\mathcal{U}}$ is the Čech nerve of $\coprod U_i$ over $\pi^0 X$ and $\mathscr{O}_X$ is homotopy-Cartesian by definition, it follows that $\check{X}_{\mathcal{U}}$ is a homotopy derived DM (in fact Zariski) 1-hypergroupoid. Similar statements hold for semi-separated derived algebraic spaces and derived DM stacks with affine diagonal.

As in the underived setting, we have the following notion giving rise to equivalences for hypergroupoids:

Definition 6.8. Given $Y_\bullet \in s d\mathrm{Aff}$, a morphism $f\colon X_\bullet \to Y_\bullet$ in $s d\mathrm{Aff}$ is a *homotopy trivial derived Artin (resp. DM) n-hypergroupoid* over $Y_\bullet$ if and only if:

1. the morphism $\pi^0 f\colon \pi^0 X_\bullet \to \pi^0 Y_\bullet$ of simplicial affine schemes is a trivial Artin (resp. DM) n-hypergroupoid;
2. for all j, m, the maps $\mathrm{H}_0(\mathscr{O}_{X_m}) \otimes_{f^{-1}\mathrm{H}_0(\mathscr{O}_{Y_m})} f^{-1}\mathrm{H}_j(\mathscr{O}_{Y_m}) \to \mathrm{H}_j(\mathscr{O}_{X_m})$ are isomorphisms.

Because the morphisms $\pi^0 f_m$ are all smooth (resp. étale), note that the second condition is equivalent to saying that the maps f_m are all strong, hence homotopy-smooth (resp. homotopy-étale).

Example 6.9. If $(\pi^0 X, \mathscr{O}_X)$ is a derived scheme, with $\pi^0 X$ quasi-compact and semi-separated, take finite covers $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ of $\pi^0 X$ by open affine subschemes, and let $\mathcal{W} := \{U_i \cap V_j\}_{(i,j) \in I \times J}$. Then in the notation of Example 6.7.(4), the resulting morphisms

$$\check{X}_{\mathcal{U}} \leftarrow \check{X}_{\mathcal{W}} \to \check{X}_{\mathcal{V}}$$

are both trivial DM (in fact Zariski) 1-hypergroupoids.

Warning 6.10. A homotopy *derived* Artin/DM n-hypergroupoid X isn't determined by $X_{\le n+1}$ (whereas we have $X \cong \mathrm{cosk}_{n+1} X$ for underived Artin n-hypergroupoids).[60]

However, a homotopy *trivial* derived Artin/DM n-hypergroupoid X over Y does satisfy $X \simeq \mathrm{cosk}^h_{n-1}(X) \times^h_{\mathrm{cosk}^h_{n-1}(Y)} Y$ for the homotopy $(n-1)$-coskeleton cosk^h_{n-1} (the right-derived functor of cosk_{n-1}), so is determined by $X_{<n}$ over Y, up to homotopy.

6.2 Main results

6.2.1 Derived stacks

For our purposes, we can use the following as the definition of a derived $(n-1)$-geometric stack. It is a special case of [Pri09, Theorem 4.15], as strengthened in [Pri11a, Theorem 5.11].[61]

Theorem 6.11. *The homotopy category of strongly quasi-compact $(n-1)$-geometric derived Artin stacks is given by taking the full subcategory of $s d\mathrm{Aff}$ consisting of homotopy derived Artin n-hypergroupoids $X_\bullet$, and formally inverting the homotopy trivial relative Artin n-hypergroupoids $X_\bullet \to Y_\bullet$.*

In fact, a model for the ∞-category of strongly quasi-compact $(n-1)$-geometric derived Artin stacks is given by the relative category $(\mathcal{C}, \mathcal{W})$ with $\mathcal{C}$ the full subcategory of $s d\mathrm{Aff}$ consisting of homotopy derived Artin n-hypergroupoids $X_\bullet$ and $\mathcal{W}$ the subcategory of homotopy trivial relative derived Artin n-hypergroupoids $X_\bullet \to Y_\bullet$.

The same results hold true if we substitute "Deligne–Mumford" for "Artin" throughout.

[60] The reason the previous, underived argument fails is that a section need not be a weak equivalence if its left inverse is the homotopy pullback of a section.

[61] As in Theorem 5.10, we are using the terminology from later versions of [TV04], so indices are 1 higher than in [Pri09, Pri11a].

In particular, this means we obtain the simplicial category of such derived stacks by simplicial localisation of homotopy derived n-hypergroupoids at the class of homotopy trivial relative derived n-hypergroupoids.

Remark 6.12. We can extend Theorem 6.11 to non-quasi-compact objects if we expand our ∞-category of building blocks by allowing arbitrary disjoint unions of derived affine schemes (which form a full subcategory of the ind-category $\mathrm{ind}(d\mathrm{Aff})$).

An derived ∞-stack over R is a functor $d\mathrm{Alg}_R \to s\mathrm{Set}$ satisfying various conditions, so we need to associate such functors to homotopy derived Artin/DM n-hypergroupoids. Similarly to the underived setting, the solution (not explicit) is to take

$$X^\sharp(A) = \mathbf{R}\mathrm{Map}_{\mathcal{W}}(\mathrm{Spec}\, A, X),$$

where $\mathbf{R}\mathrm{Map}_{\mathcal{W}}$ is the right-derived functor of $\mathrm{Hom}_{sd\mathrm{Aff}}$ with respect to homotopy trivial derived Artin/DM n-hypergroupoids. When X is a 0-hypergroupoid, we simply write $\mathbf{R}\mathrm{Spec}\, A := (\mathrm{Spec}\, A)^\sharp$; this is just given by the functor $\mathbf{R}\mathrm{Map}_{d\mathrm{Alg}}(A, -)$.

We will give explicit formulae for the mapping spaces $\mathbf{R}\mathrm{Map}(X^\sharp, Y^\sharp)$ in §6.9.3, and describe some of their structure in §6.4.1.

Definition 6.13. A derived stack $F\colon d\mathrm{Alg}_R \to s\mathrm{Set}$ is said to be *n-truncated* if the restriction $\pi^0 F\colon \mathrm{Alg}_R \to s\mathrm{Set}$ is so.

Warning 6.14. Beware that this does *not* mean that $\pi_i F(A) = 0$ for $A \in d\mathrm{Alg}_R$ and $i > n$; that is only true if A is underived, i.e. $A \in \mathrm{Alg}_R$. We have already seen that the first statement fails even for the affine line, with $\pi_i(\mathbb{A}^1)^\sharp(A) \cong \mathrm{H}_i A$.

Since truncation is a condition on the underlying algebras, Warning 5.24 applies in the derived setting for comparing truncation with geometricity.

6.2.2 Quasi-coherent complexes

For derived n-stacks, the behaviour of quasi-coherent complexes is entirely similar to that for n-stacks in §5.3.

We take a homotopy derived Artin n-hypergroupoid $X_\bullet$:

$$\begin{tikzcd} X_0 \arrow[r, "\sigma_0", dotted] & X_1 \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & X_2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \vdots & X_3 \arrow[lll, bend right] \arrow[lll, bend left] & \ldots & \ldots, \end{tikzcd}$$

for derived affine schemes X_i.

Equivalently, writing $O(X)^i_\bullet$ for the cdga $O(X_i)_\bullet$ associated to X_i,[62] we have a cosimplicial cdga

$$\begin{tikzcd} O(X)^0_\bullet \arrow[r, shift left=2, "\partial^0"] \arrow[r, shift right=2, "\partial^1"'] & O(X)^1_\bullet \arrow[l, dotted, "\sigma^0"] \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & O(X)^2_\bullet \arrow[l, dotted, shift left] \arrow[l, dotted, shift right] \arrow[rr, bend left] \arrow[rr, bend right] & \vdots & O(X)^3_\bullet & \ldots & \ldots, \end{tikzcd}$$

so we can look at modules

$$\begin{tikzcd} M^0_\bullet \arrow[r, shift left=2, "\partial^0"] \arrow[r, shift right=2, "\partial^1"'] & M^1_\bullet \arrow[l, dotted, "\sigma^0"] \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & M^2_\bullet \arrow[l, dotted, shift left] \arrow[l, dotted, shift right] \arrow[rr, bend left] \arrow[rr, bend right] & \vdots & M^3_\bullet & \ldots & \ldots, \end{tikzcd}$$

[62]note that contravariance produces cosimplicial objects from simplicial objects, so turns subscripts into superscripts

over it, with each $M^r_\bullet$ being an $O(X)^r_\bullet$-module in chain complexes.

As in the underived setting of Proposition 5.27, [Pri09, Corollary 5.12] says that giving a quasi-coherent complex on the associated derived n-geometric stack $X^\sharp$ is equivalent to giving a module on X which is *homotopy-Cartesian*:

1. an $O(X)^m_\bullet$-module $\mathscr{F}^m_\bullet$ in chain complexes for each m, and
2. morphisms $\partial^i \colon \partial_i^* \mathscr{F}^{m-1}_\bullet \to \mathscr{F}^m_\bullet$ and $\sigma^i \colon \sigma_i^* \mathscr{F}^{m+1}_\bullet \to \mathscr{F}^m_\bullet$, for all i and m, satisfying the usual cosimplicial identities[57], such that
3. the quasi-coherent sheaves[63] $(m \mapsto \mathrm{H}_j(\mathscr{F}^m_\bullet))$ on the simplicial scheme $m \mapsto \pi^0 X_m$ are Cartesian for all j, i.e. the maps[64]
$$\partial^i \colon (\pi^0 \partial_i)^* \mathrm{H}_j(\mathscr{F}^{m-1}_\bullet) \to \mathrm{H}_j(\mathscr{F}^m_\bullet),$$
for $\pi^0\partial_i \colon \pi^0 X_m \to \pi^0 X_{m-1}$, are isomorphisms of quasi-coherent sheaves on $\pi^0 X_m$ for all i, j, m.

A morphism $\{\mathscr{E}^m_\bullet\}_m \to \{\mathscr{F}^m_\bullet\}_m$ is a weak equivalence if the maps $\mathscr{E}^m_\bullet \to \mathscr{F}^m_\bullet$ are all quasi-isomorphisms.

Note that because the maps ∂_i are homotopy-smooth, the Cartesian condition is equivalent to saying that the composite maps
$$\mathbf{L}\partial_i^* \mathscr{F}^m_\bullet \to \partial_i^* \mathscr{F}^m_\bullet \xrightarrow{\partial^i} \mathscr{F}^{m+1}_\bullet$$
are quasi-isomorphisms, which implies that the morphisms $\sigma^i \colon \mathbf{L}\sigma_i^* \mathscr{F}^{m+1}_\bullet \to \mathscr{F}^m_\bullet$ are also automatically quasi-isomorphisms. We have to left-derive the pullback functors ∂_i^* in this version of the statement because homotopy-smoothness does not imply quasi-flatness.

When X is an Artin n-hypergroupoid with no derived structure, observe that the statement above just recovers Proposition 5.27. We now consider simple examples with derived structure.

Example 6.15. Take a derived scheme $(\pi^0 X, \mathscr{O}_{X,\bullet})$ with $\pi^0 X$ quasi-compact and semi-separated, and let $\mathscr{F}_\bullet$ be a homotopy-Cartesian presheaf of $\mathscr{O}_{X,\bullet}$-modules in chain complexes, in the sense of Definition 1.28. Then for any finite affine cover $\mathcal{U} := \{U_i\}_{i \in I}$ of $\pi^0 X$, we can form chain complexes
$$\check{\mathrm{C}}^m(\mathcal{U}, \mathscr{F}_\bullet) := \prod_{i_0, \ldots, i_m \in I} \Gamma(U_{i_0} \cap \ldots \cap U_{i_m}, \mathscr{F}_\bullet),$$
and these fit together to give a cosimplicial chain complex
$$\check{\mathrm{C}}^0(\mathcal{U}, \mathscr{F}_\bullet) \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} \check{\mathrm{C}}^1(\mathcal{U}, \mathscr{F}_\bullet) \to \check{\mathrm{C}}^2(\mathcal{U}, \mathscr{F}_\bullet) \quad \vdots \quad \check{\mathrm{C}}^3(\mathcal{U}, \mathscr{F}_\bullet) \qquad \ldots \qquad \ldots,$$
which is a module over the cosimplicial cdga
$$\check{\mathrm{C}}^0(\mathcal{U}, \mathscr{O}_{X,\bullet}) \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} \check{\mathrm{C}}^1(\mathcal{U}, \mathscr{O}_{X,\bullet}) \to \check{\mathrm{C}}^2(\mathcal{U}, \mathscr{O}_{X,\bullet}) \quad \vdots \quad \check{\mathrm{C}}^3(\mathcal{U}, \mathscr{O}_{X,\bullet}) \qquad \ldots \qquad \ldots.$$

[63] since $\pi^0 X_m = \mathrm{Spec}\, O(X^m_\bullet)$, we associate quasi-coherent sheaves on $(\pi^0 X_0 \Leftarrow \pi^0 X_1 \Lleftarrow \pi^0 X_2 \cdots)$ to the module $(\mathrm{H}_j(\mathscr{F}^0_\bullet) \Rightarrow \mathrm{H}_j(\mathscr{F}^1_\bullet) \Rrightarrow \mathrm{H}_j(\mathscr{F}^2_\bullet) \cdots)$ over $(\mathrm{H}_0(O(X)^0_\bullet) \Rightarrow \mathrm{H}_0(O(X)^1_\bullet) \Rrightarrow \mathrm{H}_0(O(X)^2_\bullet) \cdots)$

[64] i.e. $\partial^i \colon \partial_i^{-1}\mathrm{H}_j(\mathscr{F}^{m-1}_\bullet) \otimes_{\partial_i^{-1}\mathrm{H}_0(\mathscr{O}_{X_{m-1},\bullet})} \mathrm{H}_0(\mathscr{O}_{X_m,\bullet}) \to \mathrm{H}_j(\mathscr{F}^m_\bullet)$,

The latter is just $O(\check{X}_{\mathcal{U}})$ for the homotopy derived DM 1-hypergroupoid $\check{X}_{\mathcal{U}}$ from Example 6.7.(4), and the homotopy-Cartesian hypothesis on $\mathscr{F}_\bullet$ ensures that $\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathscr{F}_\bullet)$ is a homotopy-Cartesian module on $\check{X}_{\mathcal{U}}$.

Example 6.16. Let's look at what happens when X is a homotopy derived 0-hypergroupoid, so the morphisms $\partial_i \colon X_m \to X_{m-1}$, $\sigma_i \colon X_{m-1} \to X_m$ are all quasi-isomorphisms. Then the definition simplifies to say that a homotopy-Cartesian module on X is an $\{O(X)^m_\bullet\}_m$-module $\{\mathscr{F}^m_\bullet\}_m$ for which the morphisms $\partial^i \colon \mathscr{F}^m_\bullet \to \mathscr{F}^{m+1}_\bullet$ (and hence $\sigma^i \colon \mathscr{F}^{m+1}_\bullet \to \mathscr{F}^m_\bullet$) are all quasi-isomorphisms.

This gives us an equivalence of ∞-categories between homotopy-Cartesian modules on X, and $O(X)^0_\bullet$-modules in chain complexes. The correspondence sends a module $\{\mathscr{F}^m_\bullet\}_m$ over X to the $O(X_0)_\bullet$-module $\mathscr{F}^0_\bullet$, with quasi-inverse functor given by the right adjoint, which sends an $O(X_0)_\bullet$-module $\mathscr{E}_\bullet$ to $(\mathscr{E}_\bullet \rightrightarrows \mathscr{E}_\bullet \Rrightarrow \mathscr{E}_\bullet \cdots)$, i.e. to itself, given constant cosimplicial structure, with the $O(X_n)_\bullet$-actions coming via the degeneracy maps in X. The unit $\{\mathscr{F}^m_\bullet\}_m \to \{\mathscr{F}^0_\bullet\}_m$ of the adjunction is then manifestly a levelwise quasi-isomorphism by the reasoning above, because $\mathscr{F}$ is homotopy-Cartesian.

As in the underived setting, for any morphism $f \colon X_\bullet \to Y_\bullet$ of homotopy derived Artin n-hypergroupoids we have a functor $\mathbf{L}f^*$ on quasi-coherent complexes, given levelwise by $(\mathbf{L}f^*\mathscr{F}_\bullet)^m \simeq \mathbf{L}f^*_m\mathscr{F}^m_\bullet$.

6.3 Tangent and obstruction theory

We follow the treatment in [Pri10b, §1.2].

Lemma 6.17. *Given a derived geometric Artin n-stack $F \colon d\mathrm{Alg}_R \to s\mathrm{Set}$ and maps $A \to B \leftarrow C$ in $d\mathrm{Alg}_R$, with $A \twoheadrightarrow B$ a surjection with nilpotent kernel, we have*

$$F(A \times_B C) \xrightarrow{\sim} FA \times^h_{FB} FC.$$

For a proof, see Corollary 6.61. As in [Pri10b], we call this condition *homotopy-homogeneity*, by analogy with the notion of homogeneity [Man99]; it is best thought of as a derived form of Schlessinger's conditions.[65]

Lemma 6.18. *If F is homotopy-homogeneous, then we have a surjection*

$$\pi_0 F(A \times_B C) \twoheadrightarrow \pi_0 FA \times_{\pi_0 FB} \pi_0 FC$$

for all maps $A \to B \leftarrow C$ in $d\mathrm{Alg}_R$ with $A \twoheadrightarrow B$ a nilpotent surjection, and

$$\pi_0 F(A \times C) \simeq \pi_0 FA \times \pi_0 FC.$$

You may recognise these as generalisations of two of Schlessinger's conditions [Sch68].

Example 6.19. For a homotopy derived 0-hypergroupoid given by a derived affine scheme U (with constant simplicial structure), the associated derived stack is given by $U^\sharp \simeq \mathbf{R}\mathrm{Map}_{d\mathrm{Aff}}(-, U)$, so we've seen these results before, with tangent and obstruction theory as in §3.2.

[65] Most people nowadays say "infinitesimally cohesive on one factor" for this notion (or a slight variant), because the notion of infinitesimally cohesive in [Lur04a] imposes the nilpotent surjectivity condition to $C \to B$ as well; our notion of homotopy-homogeneity more closely resembles Artin's generalisation [Art74, 2.2 (S1)] of Schlessinger's conditions, which unsurprisingly leads to a more usable representability theorem.

6.3.1 Tangent spaces

Now, take $A \in dg_+\mathrm{Alg}$, and $M \in dg_+\mathrm{Mod}_A$, with $F\colon dg_+\mathrm{Alg} \to s\mathrm{Set}$.

Definition 6.20. For $x \in F(A)_0$, define the tangent space of F at x with coefficients in M to be the homotopy fibre $T_x(F, M) := F(A \oplus M) \times_{F(A)} \{x\}$, where the product of elements M is set to be 0.

If F is homotopy-homogeneous, we have an additive structure on tangent spaces $T_x(F, M)$, via the composition

$$\begin{aligned} &F(A \oplus M) \times^h_{FA} F(A \oplus M) \\ &\simeq F((A \oplus M) \times_A (A \oplus M)) \\ &= F(A \oplus (M \oplus M)) \xrightarrow{+} F(A \oplus M). \end{aligned}$$

Moreover we have a short exact sequence $0 \to M \to \mathrm{cone}(M \to M) \to M[-1] \to 0$, so $M = \mathrm{cone}(M \to M) \times_{M[-1]} 0$, and thus

$$F(A \oplus M) \simeq F(A \oplus \mathrm{cone}(M \to M)) \times^h_{F(A \oplus M[-1])} F(A),$$

since F is homotopy-homogeneous and $A \oplus \mathrm{cone}(M \to M) \to A \oplus M[-1]$ is a square-zero extension.

If F is also homotopy-preserving in the sense that it preserves weak equivalences, then $F(A \oplus \mathrm{cone}(M \to M)) \simeq F(A)$, so we have

$$F(A \oplus M) \simeq F(A) \times^h_{F(A \oplus M[-1])} F(A).$$

Taking homotopy fibres over $x \in F(A)$, we then get

$$T_x(F, M) = 0 \times^h_{T_x(F, M[-1])} 0,$$

which is a loop space, so $T_x(F, M[-1])$ deloops $T_x(F, M)$ and

$$\pi_i T_x(F, M) = \pi_{i+n} T_x(F, M[-n]).$$

Definition 6.21. We can thus define tangent cohomology groups by

$$D^{n-j}_x(F, M) := \pi_j T_x(F, M[-n]).$$

These generalise the André–Quillen cohomology groups of derived affine schemes.

Definition 6.22. For $x \in F(k)$ for k a field, define $\dim_x(F) := \sum (-1)^i \dim D^i(F, k)$, when finite.

Examples 6.23.

1. If F is the derived stack associated to the Čech complex of a dg scheme X and $x \in X(A)$, then $D^i(F, M) \cong \mathrm{Ext}^i_{\mathscr{O}_X}(\mathbb{L}^X, \mathbf{R}x_* M) \cong \mathrm{Ext}^i_A(\mathbf{L}x^*\mathbb{L}^X, M)$. The dimension of F at $x \in X(k)$ is therefore $\dim_x(F) = \dim x^* T_X = \dim X$ when X is smooth, and by the Euler characteristic $\chi(x^*\mathbb{L}^X)$ in general (when defined).

2. If V is a cochain complex in degrees $\geq -n$, finite-dimensional over k, then

   $$F\colon A \mapsto N^{-1}\tau_{\geq 0}\mathrm{Tot}^{\Pi}(V \otimes_k A)$$

   (Dold–Kan denormalisation of good truncation) is represented by an n-hypergroupoid over k, with $D^i_x(F^\sharp, M) \cong \mathrm{H}^i(\mathrm{Tot}^{\Pi}(V \otimes_k M))$ for all i. At all points $x \in F(k) = N^{-1}\tau_{\geq 0} V$, we thus have $D^i_x(F^\sharp, k) \cong \mathrm{H}^i(V)$, so $\dim_x(F) \cong \chi(V)$, when finite.

6.3.2 The long exact sequence of obstructions

If $A \to B$ is a square-zero extension with kernel I, there is a long exact sequence of groups and sets:

$$\begin{tikzcd}
\cdots \arrow[r, "e_*"] & \pi_n(FA,y) \arrow[r, "f_*"] & \pi_n(FB,x) \arrow[r, "o_e"] & D^{1-n}_y(F,I) \arrow[r, "e_*"] & \pi_{n-1}(FA,y) \arrow[r, "f_*"] & \cdots \\
& \cdots \arrow[r, "f_*"] & \pi_1(FB,x) \arrow[r, "o_e"] & D^0_y(F,I) \arrow[dll, "-*y"'] & & \\
& \pi_0(FA) \arrow[r, "f_*"'] & \pi_0(FB) \arrow[r, "o_e"'] & \Gamma(FB, D^1(F,I)). & &
\end{tikzcd}$$

The first part is the sequence associated to the fibration $F(A) \to F(B)$ as in [GJ99, Lemma I.7.3], but the non-trivial content here is in the final map o_e which gives rise to obstructions.[66]

Here are the details of the construction (following [Pri10b, Proposition 1.17]). Let $C = C(A,I)$ be the mapping cone of $I \to A$. Then $C \to B$ is a square-zero acyclic surjection, so $FC \to FB$ is a weak equivalence, and thus $\pi_i(FC) \to \pi_i(FB)$ is an isomorphism for all i. Now,

$$A = C \times_{B \oplus I[-1]} B,$$

and since $C \to B \oplus I[-1]$ is surjective, this (for $x \in F(B)$) gives a map

$$p' : F_x(C) \to T_x(F, I[-1])$$

in the homotopy category, with homotopy fibre $F_x(A)$ over 0. The sequence above is then just the long exact sequence of a fibre sequence [GJ99, Lemma I.7.3].

Lemma 6.24. *A morphism $F \to G$ of n-geometric derived stacks is a weak equivalence if and only if*

1. *$\pi^0 f \colon \pi^0 F(B) \to \pi^0 G(B)$ is a weak equivalence (of functors $\mathrm{Alg} \to s\mathrm{Set}$), and*

2. *for all discrete A, all A-modules M and all $x \in F(A)$, the maps $f \colon D^i_x(F,M) \to D^i_{fx}(G,M)$ are isomorphisms for all $i > 0$. (Note that for $i \le 0$, we already know that these maps are isomorphisms, from the first condition.)*

Proof. We need to show that $F(B) \to G(B)$ is a weak equivalence for all $B \in d\mathrm{Alg}$. Work up the Postnikov tower $B = \varprojlim_k P_k B$. First note that $F(B) \simeq \mathrm{ho}\varprojlim_k F(P_k B)$, and then that $P_{k+1}B \to P_k B$ is weakly equivalent to a square-zero extension with kernel $\mathrm{H}_{n+1}B[n+1]$. Thus the long exact sequence of obstructions gives inductively (on k) that $\pi_i F(P_k B) \cong \pi_i G(P_k B)$ for all i, k. □

Note that we could relax both conditions in Lemma 6.24 by asking that they only hold for reduced discrete algebras, and then apply a further induction to the quotients of $\mathrm{H}_0 B$ by powers of its nilradical. Also note that the proof applies to any homotopy-preserving homotopy-homogeneous functor F with $F(B) \simeq \mathrm{ho}\varprojlim_k F(P_k B)$.

[66] This phenomenon of central and abelian extensions giving rise to such obstructions arises in many branches of algebra and topology — see [Pri17] for a more general algebraic formulation.

6.3.3 Sample application of derived deformation theory — semiregularity

We now give an application from [Pri12].[67] Take:

- a smooth proper variety X,
- a square-zero extension $A \to B$ of algebras with kernel I,
- a closed LCI subscheme $Z \subset X \otimes B$ of codimension p, flat over B.

Then the obstruction to lifting Z to a subscheme of $X \otimes A$ lies in $\mathrm{H}^1(Z, \mathscr{N}_{Z/X}) \otimes I$. Bloch [Blo72] defined a semiregularity map

$$\tau\colon \mathrm{H}^1(Z, \mathscr{N}_{Z/X}) \to \mathrm{H}^{p+1}(X, \Omega_X^{p-1}),$$

and conjectured that it annihilates all obstructions, giving a reduced obstruction space. There is also a generalisation where X deforms, and then τ measures the obstruction to deforming the Hodge class $[Z]$. These conjectures were extended to perfect complexes in place of $\mathscr{O}_Z$ by [BF03].

In [Pri12], the conjectures were proved by interpreting τ as the tangent map of a morphism of homotopy-preserving homotopy-homogeneous functors, then factoring through something unobstructed. In more detail, the Chern character ch_p gives a map from the moduli functor to

$$\mathscr{J}_X^p(A) := (\mathbf{R}\Gamma(X, \mathrm{Tot}^{\Pi}\mathbf{L}\Omega^\bullet_{X_A/k}) \times^h_{\mathbf{R}\Gamma(X, \Omega^\bullet_{X_A/A})} \mathbf{R}\Gamma(X, F^p\Omega^\bullet_{X_A/A}))[2p],$$

which has derived tangent complex

$$(\{0\} \times^h_{\mathbf{R}\Gamma(X,\Omega^\bullet_X)} \mathbf{R}\Gamma(X, F^p\Omega^\bullet_X))[2p] \simeq \mathbf{R}\Gamma(X, \Omega_X^{<p})[2p-1].$$

The map τ on obstruction spaces then comes from applying H^1 to the derived tangent maps

$$\begin{array}{ccccc} T_{[Z]}\mathbf{R}\mathcal{H}ilb_X & \longrightarrow & T_{[\mathscr{O}_Z]}\mathbf{R}\mathcal{P}erf_X & \xrightarrow{d\mathrm{ch}_p} & T_{\mathrm{ch}_p(\mathscr{O}_Z)}\mathscr{J}_X^p \\ \mathbf{R}\Gamma(Z, \mathscr{N}_{Z/X}) & \longrightarrow & \mathbf{R}\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{O}_Z, \mathscr{O}_Z)[1] & \xrightarrow{d\mathrm{ch}_p} & \mathbf{R}\Gamma(X, \Omega_X^{<p})[2p-1], \end{array}$$

from the derived Hilbert scheme to the derived moduli stack of perfect complexes and then to $\mathscr{J}_X^p$, since $d\mathrm{ch}_p$ factors through $\mathbf{R}\Gamma(X, \Omega_X^{p-1})[p]$ (a summand via the Hodge decomposition). The obstruction in $\mathscr{J}_X^p$ then vanishes, or more generally measures obstructions to deforming $[Z]$ as a Hodge class.[68]

6.4 Cotangent complexes

The cotangent complex (when it exists) of a functor $F\colon d\mathrm{Alg}_R \to s\mathrm{Set}$ represents the tangent functor. Explicitly, it is a quasi-coherent complex[69] $\mathbb{L}_F$ on F such that for all $A \in d\mathrm{Alg}_R$, all points $x \in F(A)$ and all A-modules M, we have

$$\begin{aligned} T_x(F, M) &\simeq \mathbf{R}\mathrm{Map}_{dg\mathrm{Mod}_A}(\mathbf{L}x^*\mathbb{L}_F, M) \\ &\simeq N^{-1}\tau_{\geq 0}\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{L}x^*\mathbb{L}_F, M), \end{aligned}$$

[67] explanatory slides available at http://www.maths.ed.ac.uk/~jpridham/semiregslide.pdf

[68] The key geometric difference between this and earlier approaches is not so much the language of derived deformation theory, but rather the use of derived de Rham cohomology $\mathbf{R}\Gamma(X, \mathrm{Tot}^{\Pi}\mathbf{L}\Omega^\bullet_{X_A/k})$ over the fixed base k to generate horizontal sections, instead of a more classical cohomology theory.

[69] explicitly, this means we have an A-module $\mathbb{L}_{F,x}$ for each $x \in F(A)$, such that the maps $\mathbb{L}_{F,x} \otimes^{\mathbf{L}}_A B \to \mathbb{L}_{F,fx}$ are quasi-isomorphisms for all $f\colon A \to B$

so in particular $D^i_x(F, M) \cong \mathrm{Ext}^i_A(\mathbf{L}x^*\mathbb{L}_F, M)$.

For homotopy derived DM n-hypergroupoids X, the cotangent complex $\mathbb{L}^{X^\sharp}$ of the associated stack $X^\sharp$ corresponds via §6.2.2 to the complex $m \mapsto \mathbb{L}^{X_m}$ on X, which is homotopy-Cartesian because the maps $\partial_i \colon X_{m+1} \to X_m$ are homotopy-étale, so $\mathbb{L}^{X_{m+1}} \simeq \mathbf{L}\partial_i^*\mathbb{L}^{X_m}$.

For homotopy derived Artin n-hypergroupoids X, that doesn't work, but it turns out that the iterated derived loop space X^{hS^n} is a derived 0-hypergroupoid for $n > 0$, and then the complex $m \mapsto (\mathbb{L}^{X^{hS^n}})_m[-n]$ is homotopy-Cartesian on X^{hS^n} and pulls back to give a model for $\mathbb{L}^{X^\sharp}$ on X.

Explicitly, writing $X^K \in sd\mathrm{Aff}$ for the functor $(X^K)_i(A) := \mathrm{Hom}_{s\mathrm{Set}}(K \times \Delta^i, X(A))$, when X is Reedy fibrant as in §6.9.2, a model for the cotangent complex $\mathbb{L}^{X^\sharp}$ is given by $\mathbf{L}i^*\Omega^1_{X^{\Delta^n}/X^{\partial\Delta^n}}[-n]$, for the natural map $i \colon X \to X^{\Delta^n}$.

Example 6.25. If $X = B[U/G]$ (a homotopy derived Artin 1-hypergroupoid), then

$$\begin{aligned} X^{\Delta^1} &= B[(U \times G)/(G \times G)] \\ X^{\partial\Delta^1} &= X \times X = [(U \times U)/(G \times G)] \end{aligned}$$

In level 0 (i.e. on X_0), the complex $\mathbf{L}i^*\mathbf{L}\Omega^\bullet_{X^{\Delta^1}/X^{\partial\Delta^1}}[-1]$ is then $\mathbf{L}e^*\mathbf{L}\Omega^1_{(U\times G)/(U\times U)}[-1]$ for $e \colon U \to U \times G$ given by $u \mapsto (u, e)$, where e is the identity element of the group G. This therefore recovers the formula

$$\mathbb{L}^{[U/G]}|_U \simeq \mathrm{cone}(\mathbb{L}^U \to \mathfrak{g}^* \otimes \mathscr{O}_U)[-1],$$

which readers familiar with cotangent complexes of Artin stacks will recognise.

6.4.1 Morphisms revisited

Given homotopy derived Artin n-hypergroupoids X, Y, what does the space $\mathbf{R}\mathrm{Map}(X^\sharp, Y^\sharp)$ of maps $f \colon X_\bullet \to Y_\bullet$ between the associated derived $(n-1)$-geometric stacks look like?[70]

For a start, we have a map $\mathbf{R}\mathrm{Map}(X^\sharp, Y^\sharp) \to \mathbf{R}\mathrm{Map}(\pi^0 X^\sharp, Y^\sharp)$, and the latter is just the space of maps $(\pi^0 X)^\sharp \to (\pi^0 Y)^\sharp$ of underived $(n-1)$-geometric stacks, as described in §5.2.1. In particular, this is m-truncated whenever $Y^\sharp$ is so.

By the universal property of hypersheafification, we can replace $X^\sharp$ with X. Since $\mathbf{R}\mathrm{Map}(X, Y^\sharp) \simeq \mathrm{ho}\varprojlim_{m\in\Delta} \mathbf{R}\mathrm{Map}(X_m, Y^\sharp)$, any homotopy limit expressions for $Y^\sharp$ as a functor on $d\mathrm{Alg}$ thus apply to the contravariant functor $\mathbf{R}\mathrm{Map}(-, Y^\sharp)$ on $sd\mathrm{Aff}$ as well.

We can now work our way up the Postnikov tower of §3.3, writing $\tau^{\le k}\mathrm{Spec}\, A := \mathrm{Spec}\, P_k A$ and $(\tau^{\le k} X)_m := \tau^{\le k}(X_m)$ (so in particular $\tau^{\le 0} X = \pi^0 X$) to give a tower

$$\ldots \to \mathbf{R}\mathrm{Map}(\tau^{\le k+1} X, Y^\sharp) \to \mathbf{R}\mathrm{Map}(\tau^{\le k} X, Y^\sharp) \to \ldots \to \mathbf{R}\mathrm{Map}(\pi^0 X, Y^\sharp).$$

Lemma 3.31 and §3.2 then give an expression for $P_{k+1}\mathscr{O}_X$ as a homotopy pullback of a diagram $P_k\mathscr{O}_X \xrightarrow{u} \mathrm{H}_0(\mathscr{O}_X) \oplus \mathrm{H}_{k+1}(\mathscr{O}_X)[k+2] \xleftarrow{(\mathrm{id},0)} \mathrm{H}_0(\mathscr{O}_X)$ in the homotopy category,

[70]In the terminology of §6.5, the description we use here in fact adapts to $\mathbf{R}\mathrm{Map}(X^\sharp, F)$ for any $X \in sd\mathrm{Aff}$ and any homotopy-homogeneous nilcomplete functor $F \colon d\mathrm{Alg}_R \to s\mathrm{Set}$.

giving a homotopy pullback square

$$\begin{tikzcd}
\mathbf{R}\mathrm{Map}(\tau^{\le k+1}X, Y^\sharp) \arrow[rr] \arrow[d] & & \mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp) \arrow[d, "u"] \\
\mathbf{R}\mathrm{Map}(\pi^0 X, Y^\sharp) \arrow[r] & \mathbf{R}\mathrm{Map}(\mathbf{Spec}_{\pi^0 X}(\mathscr{O}_{\pi^0 X} \oplus \mathrm{H}_{k+1}(\mathscr{O}_X)[k+2]), Y^\sharp). &
\end{tikzcd}$$

For a fixed element $[g] \in \pi_0\mathbf{R}\mathrm{Map}(\pi^0 X, Y^\sharp)$, with $\pi_0\mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp)_{[g]}$ the fibre over $[g]$, we thus have a long exact sequence

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_1\mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp) \arrow[r] & \mathrm{Ext}^{k+1}_{\mathscr{O}_{\pi^0 X}}(\mathbf{L}g^*\mathbb{L}^Y, \mathrm{H}_{k+1}(\mathscr{O}_X)) \arrow[dll] \\
\pi_0\mathbf{R}\mathrm{Map}(\tau^{\le k+1}X, Y^\sharp)_{[g]} \arrow[r] & \pi_0\mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp)_{[g]} \arrow[r, "u"] & \mathrm{Ext}^{k+2}_{\mathscr{O}_{\pi^0 X}}(\mathbf{L}g^*\mathbb{L}^Y, \mathrm{H}_{k+1}(\mathscr{O}_X))
\end{tikzcd}$$

of homotopy groups and sets . Explicitly, this means that

- a class $[g^{(k)}] \in \pi_0\mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp)_{[g]}$ lifts to a class $[g^{(k+1)}] \in \pi_0\mathbf{R}\mathrm{Map}(\tau^{\le k+1}X, Y^\sharp)$ if and only if $u([g^{(k)}]) = 0$;
- the group $\mathrm{Ext}^{k+1}_{\mathscr{O}_{\pi^0 X}}(\mathbf{L}g^*\mathbb{L}^Y, \mathrm{H}_{k+1}(\mathscr{O}_X))$ acts transitively on the fibre over $[g^{(k)}]$;
- taking homotopy groups at basepoints $g^{(k)}$ and $g^{(k+1)}$, the rest of the sequence is a long exact sequence of groups, ending with the stabiliser of $[g^{(k+1)}]$ in $\mathrm{Ext}^{k+1}_{\mathscr{O}_{\pi^0 X}}(\mathbf{L}g^*\mathbb{L}^Y, \mathrm{H}_{k+1}(\mathscr{O}_X))$.

In particular, since Y is n-truncated, we have $\mathrm{Ext}^{<-n}_{\mathscr{O}_{\pi^0 X}}(\mathbf{L}g^*\mathbb{L}^Y, \mathrm{H}_{k+1}(\mathscr{O}_X)) = 0$, so it follows by induction that $\pi_i\mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp) = 0$ for $i > k+n$.

Finally, we have

$$\mathbf{R}\mathrm{Map}(X, Y^\sharp) \simeq \operatorname{ho}\varprojlim_k \mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp).$$

These homotopy limits behave exactly like derived inverse limits in homological algebra, with the Milnor exact sequence of [GJ99, Proposition VI.2.15] giving us exact sequences

$$* \to \varprojlim_k{}^1 \pi_{i+1}\mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp) \to \pi_i\mathbf{R}\mathrm{Map}(X^\sharp, Y^\sharp) \to \varprojlim_k \pi_i\mathbf{R}\mathrm{Map}(\tau^{\le k}X, Y^\sharp) \to *$$

of groups and pointed sets (basepoints omitted from the notation, but must be compatible).

6.4.2 Derived de Rham complexes

The module $m \mapsto \mathbb{L}^{X_m}$ is not homotopy-Cartesian when X is a derived Artin n-hypergroupoid, so it does not give a quasi-coherent complex on the associated derived stack $\mathfrak{X} := X^\sharp$. However, [Pri09, Lemma 7.8] implies that when X is levelwise fibrant (so $\mathbb{L}^{X_m} \simeq \Omega^1_{X_m}$), the natural map from the homotopy-Cartesian complex $\mathbb{L}^X$ to Ω^1_X does induce a quasi-isomorphism on global sections

$$\mathbf{R}\Gamma(\mathfrak{X}, \mathbb{L}^{\mathfrak{X}}) \simeq \mathbf{R}\Gamma(X, \Omega^1_X) := \mathrm{Tot}^{\Pi}(i \mapsto \Gamma(X_i, \Omega^1_{X_i}))$$

and similarly on tensor powers, including

$$\mathbf{R}\Gamma(\mathfrak{X}, \Lambda^p\mathbb{L}^{\mathfrak{X}}) \simeq \mathbf{R}\Gamma(X, \Omega^p_X).$$

Derived de Rham cohomology can then just be defined as

$$\mathrm{H}^*\mathrm{Tot}^{\Pi}(i \mapsto \Gamma(X_i, \mathrm{Tot}^{\Pi}\Omega^\bullet_{X_i}));$$

over $\mathbb{C}$, this agrees with $\mathrm{H}^*(|\pi^0 X(\mathbb{C})_{\mathrm{an}}|, \mathbb{C})$, for $|\pi^0 X(\mathbb{C})_{\mathrm{an}}|$ the realisation of the simplicial topological space $\pi^0 X(\mathbb{C})_{\mathrm{an}}$.

Example 6.26. For $X = B\mathbb{G}_m$ over $\mathbb{C}$, this gives derived de Rham cohomology as $\mathrm{H}^*(|B\mathbb{C}^*|, \mathbb{C}) \cong \mathrm{H}^*(|BS^1|, \mathbb{C}) \cong \mathrm{H}^*(K(\mathbb{Z}, 2), \mathbb{C}) \cong \mathrm{H}^*(\mathbb{CP}^\infty, \mathbb{C}) \cong \mathbb{C}[u]$, for u of degree 2.

There is a Hodge filtration $F^p\Omega^\bullet_X$ given by brutal truncation. Since $\mathrm{Tot}^{\Pi} F^p$ is the right derived functor of Z^p, this leads to:

Definition 6.27 ([PTVV11], this characterisation essentially [Pri15])**.** The complex of *n-shifted pre-symplectic structures* on X is $\tau^{\le 0}\mathbf{R}\Gamma(X, (\mathrm{Tot}^{\Pi} F^2\Omega^\bullet_X))[n+2])$. Hence homotopy classes are in $\mathrm{H}^{n+2}(\mathrm{Tot}^{\Pi} F^2\Omega^\bullet_X)$.

We say ω is *symplectic* if it is non-degenerate in the sense that the map $\mathbf{R}\mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathbb{L}^{\mathfrak{X}}, \mathscr{O}_{\mathfrak{X}}) \to \mathbb{L}^{\mathfrak{X}}[n]$ induced by $\omega_2 \in \mathrm{H}^n(X, \Omega^2_X) \cong \mathbf{R}\Gamma(\mathfrak{X}, \Lambda^p\mathbb{L}^{\mathfrak{X}})$ is a quasi-isomorphism of quasi-coherent complexes on $\mathfrak{X}$.

Example 6.28. The trace on GL_n gives rise to a 2-shifted symplectic structure on $B\mathrm{GL}_n$.

There is an equivalent characterisation of shifted symplectic structures in [Pri15, §3] better suited for comparisons with Poisson structures, effectively replacing the derived Artin hypergroupoid X with a derived Deligne–Mumford hypergroupoid $\mathrm{Spec}\, D^*O(X^\Delta)$ which takes values in double complexes with a graded-commutative product, with the extra cochain grading modelling stacky structure as a form of higher Lie algebroid; also see [Pri18a, Pri19].

Shifted Poisson structures are then given by shifted L_∞ structures on these stacky CDGAs, with the brackets all being multiderivations; see [Pri15, Examples 3.31] and [Saf17] for explicit descriptions of the resulting 2-shifted structures on quotient stacks $[Y/G]$ and of 1-shifted structures on BG, respectively.

6.5 Artin–Lurie representability

Anyone familiar with Artin representability for algebraic stacks [Art74] will know that in the underived setting, axiomatising and constructing obstruction theories was one of the hardest steps; also see [BF96]. However, derived algebraic geometry produces obstruction theory for free as in [Man99] or §6.3.2, giving rise to derived representability theorems which can be simpler than their underived counterparts.[71]

The landmark result is the representability theorem of [Lur04a], but it is formulated in a way which can make the conditions onerous to verify, so we will be presenting it in the simplified form established in [Pri10b].

Definition 6.29. A functor $F\colon d\mathrm{Alg}_R \to s\mathrm{Set}$ is said to be *locally of finite presentation* (l.f.p.) if it preserves filtered colimits, or equivalently colimits indexed by directed sets, i.e the natural map

$$\varinjlim_i F(A(i)) \to F(\varinjlim_i A(i))$$

[71]The results from now on have only been developed in the setting of algebraic geometry. There are much weaker derived representability theorems in differential and analytic settings given by adapting [TV04, Appendix C]. Such results only apply when the underlying underived moduli functor is already known to be representable; the main obstacle is in formulating an analogue of algebraisation for formal moduli, since differential and analytic moduli functors are usually only defined on finitely presented objects.

is a weak equivalence.[72]

A functor $F\colon d\mathrm{Alg}_R \to s\mathrm{Set}$ is said to be *almost of finite presentation* (a.f.p.) if it preserves filtered colimits (or equivalently directed colimits) of uniformly bounded objects, in the sense that there exists some n for which the objects are all concentrated in degrees $\le n$.

Example 6.30. If $U = \operatorname{Spec} S$ is a derived affine scheme, then $U^\sharp = \mathbf{R}\mathrm{Map}_{d\mathrm{Alg}_R}(S, -)$ is l.f.p. if and only if S has a finitely generated cofibrant model, whereas $U^\sharp$ is a.f.p. if and only if S has cofibrant model with finitely many generators in each degree.

Beware that a finitely presented algebra is not in general l.f.p as a cdga unless its cotangent complex is perfect, though it will be always be a.f.p. if the base is Noetherian.

More generally, if X is an Artin n-hypergroupoid for which the derived affine scheme X_0 is l.f.p. or a.f.p., then the functor $X^\sharp$ will be l.f.p. or a.f.p., essentially because smooth morphisms are l.f.p.

In order to state the representability theorems, from now on our base ring R will be a derived G-ring admitting a dualising module (in the sense of [Lur04a, Definition 3.6.1]). Examples satisfying this hypothesis are any field, the integers, any Gorenstein local ring, and anything of finite type over any of these.

The following is [Pri10b, Corollary 1.36], substantially simplifying [Lur04a]:

Theorem 6.31. *A functor $F : d\mathrm{Alg}_R \to s\mathrm{Set}$ is an n-truncated geometric derived stack which is almost of finite presentation if and only if the following conditions hold*

1. *F is homotopy-preserving: it maps quasi-isomorphisms to weak equivalences.*
2. *For all discrete H_0R-algebras A, $F(A)$ is n-truncated, i.e. $\pi_i F(A) = 0$ for all $i > n$.*
3. *F is homotopy-homogeneous, i.e. for all square-zero extensions $A \twoheadrightarrow C$ and all maps $B \to C$, the map*
$$F(A \times_C B) \to F(A) \times^h_{F(C)} F(B)$$
is an equivalence.
4. *F is nilcomplete, i.e. for all A, the map*
$$F(A) \to \varprojlim^h F(P_k A)$$
is an equivalence, for $\{P_k A\}$ the Postnikov tower of A.
5. *$\pi^0 F : \mathrm{Alg}_{\mathrm{H}_0 R} \to s\mathrm{Set}$ preserves filtered colimits (equivalently colimits indexed by directed sets), i.e.*
   (a) $\pi_0 \pi^0 F : \mathrm{Alg}_{\mathrm{H}_0 R} \to \mathrm{Set}$ preserves filtered colimits.
   (b) For all $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $x \in F(A)$, the functors $\pi_i(\pi^0 F, x) : \mathrm{Alg}_A \to \mathrm{Set}$ preserve filtered colimits for all $i > 0$.
6. *$\pi^0 F : \mathrm{Alg}_{\mathrm{H}_0 R} \to s\mathrm{Set}$ is a hypersheaf for the étale topology.*

[72] Note that we do not need to write these as homotopy colimits, since filtered colimits are already exact, so are their own left-derived functors.

7. *for all finitely generated integral domains $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$, all $x \in F(A)_0$ and all étale morphisms $f : A \to A'$, the maps*

$$\mathrm{D}^*_x(F, A) \otimes_A A' \to \mathrm{D}^*_{fx}(F, A')$$

on tangent cohomology groups are isomorphisms.

8. *for all finitely generated $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $x \in F(A)_0$, the functors $\mathrm{D}^i_x(F, -) : \mathrm{Mod}_A \to \mathrm{Ab}$ preserve filtered colimits for all $i > 0$.*

9. *for all finitely generated integral domains $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $x \in F(A)_0$, the groups $\mathrm{D}^i_x(F, A)$ are all finitely generated A-modules.*

10. *formal effectiveness: for all complete discrete local Noetherian $\mathrm{H}_0 R$-algebras A, with maximal ideal $\mathfrak{m}$, the map*

$$F(A) \to \varprojlim_n{}^h F(A/\mathfrak{m}^r)$$

is a weak equivalence (see [Pri10b, Remark 1.35] for a reformulation).

F is moreover strongly quasi-compact (so built from $d\mathrm{Aff}$, not $\coprod d\mathrm{Aff}$) if and only if for all sets S of separably closed fields, the map

$$F(\prod_{k \in S} k) \to (\prod_{k \in S} F(k))$$

is a weak equivalence in $s\mathrm{Set}$.

Remarks 6.32. Note that of the conditions in the theorem as stated in this form, only conditions (1), (3) and (4) are fully derived. The conditions (2) (5), (6) and (10) are purely underived in nature, taking only discrete input, and in particular are satisfied if the underived truncation $\pi^0 F$ is representable, while the conditions (7), (8) and (9) relate to tangent cohomology groups. The hardest conditions to check are usually homotopy-homogeneity (3) and formal effectiveness (10).

Because derived algebraic geometry automatically takes care of obstructions, it is often easier to establish representability of the underived moduli functor $\pi^0 F$ by checking the conditions of Theorem 6.31 for F, rather than checking Artin's conditions [Art74] and their higher analogues for $\pi^0 F$. Beware that a natural equivalence of moduli functors does not necessarily give an equivalence of the corresponding derived moduli functors, a classical example being the derived Quot and Hilbert schemes of [CFK99, CFK00].

As we saw back in §3.3, derived structure is infinitesimal in nature, and this now motivates a variant of the representability theorem which just looks at functors on derived rings which are bounded nilpotent extensions of discrete rings.

Definition 6.33. Define $d\mathcal{N}^\flat_R$ to be the full subcategory of $d\mathrm{Alg}_R$ consisting of objects A for which

1. the map $A \to \mathrm{H}_0 A$ has nilpotent kernel.

2. $A_i = 0$ (or $N_i A = 0$ in the simplicial case $A \in s\mathrm{Alg}_R$) for all $i \gg 0$.

Exercise 6.34. Show that any homotopy-preserving a.f.p. nilcomplete functor $F\colon d\mathrm{Alg}_R \to s\mathrm{Set}$ is determined by its restriction to $d\mathcal{N}^\flat_R$, bearing in mind that R is Noetherian.

The following is [Pri10b, Theorem 2.17]; it effectively says that we can restrict to functors on $d\mathcal{N}^\flat_R$ and drop the nilcompleteness condition.

Theorem 6.35. *Let R be a Noetherian G-ring admitting a dualising module.*

Take a functor $F : d\mathcal{N}^\flat_R \to s\mathrm{Set}$. Then F is the restriction of an almost finitely presented derived n-truncated geometric stack $F' : \mathrm{dAlg}_R \to s\mathrm{Set}$ if and only if the following conditions hold

1. *F maps square-zero acyclic extensions to weak equivalences.*
2. *For all discrete rings A, $F(A)$ is n-truncated, i.e. $\pi_i F(A) = 0$ for all $i > n$.*
3. *F is homotopy-homogeneous.*
4. *$\pi^0 F : \mathrm{Alg}_{\mathrm{H}_0 R} \to s\mathrm{Set}$ is a hypersheaf for the étale topology.*
5. *$\pi^0 F : \mathrm{Alg}_{\mathrm{H}_0 R} \to \mathrm{Ho}(s\mathrm{Set})$ preserves filtered colimits.*
6. *for all complete discrete local Noetherian $\mathrm{H}_0 R$-algebras A, with maximal ideal $\mathfrak{m}$, the map $\pi^0 F(A) \to \varprojlim^h \pi^0 F(A/\mathfrak{m}^r)$ is a weak equivalence.*
7. *for all finitely generated integral domains $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$, all $x \in F(A)_0$ and all étale morphisms $f : A \to A'$, the maps $D^*_x(F, A) \otimes_A A' \to D^*_{fx}(F, A')$ are isomorphisms.*
8. *for all finitely generated $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $x \in F(A)_0$, the functors $D^i_x(F, -) : \mathrm{Mod}_A \to \mathrm{Ab}$ preserve filtered colimits for all $i > 0$.*
9. *for all finitely generated integral domains $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $x \in F(A)_0$, the groups $D^i_x(F, A)$ are all finitely generated A-modules.*

Moreover, F' is uniquely determined by F (up to weak equivalence).

Remark 6.36. There is a much simpler representability theorem for functors on dg Artinian rings, essentially requiring only the homogeneity condition. Such functors also tie in with other approaches to derived deformation theory such as DGLAs and L_∞-algebras. The relations between these were proved in [Pri07a] (largely rediscovered as the main result of [Lur10, Lur11b][73]); for a survey see [Mag10].

6.6 Examples

Examples 6.37. The following simplicial-category valued functors $\mathcal{C} \colon d\mathrm{Alg} \to s\mathrm{Cat}$ are homotopy-homogeneous, homotopy-preserving and étale hypersheaves (though too big to be representable). For objects $A \in d\mathrm{Alg}$ and morphisms $A \to B$ in $d\mathrm{Alg}$, we then have the following examples:

1. Take $\mathcal{C}(A)$ to be the simplicial category of strongly quasi-compact n-geometric derived Artin stacks $\mathfrak{X}$ over $\mathrm{Spec}\, A$, with the simplicial functor $\mathcal{C}(A) \to \mathcal{C}(B)$ given by $\mathfrak{X} \mapsto \mathfrak{X} \times^h_{\mathbf{R}\mathrm{Spec}\, A} \mathbf{R}\mathrm{Spec}\, B$ (so $\mathscr{O}_\mathfrak{X} \mapsto \mathscr{O}_\mathfrak{X} \otimes^{\mathbf{L}}_A B$).

[73] The significance of that result has however been somewhat exaggerated in recent years, after [Lur10] conflated moduli problems with derived functors satisfying Schlessinger's conditions (thereby turning a meta-conjecture into a definition). It is hard to imagine an experienced deformation theorist resorting to the theorem to infer the existence of a DGLA governing a given deformation problem; it is almost always easier to write down the governing DGLA than even to formulate the derived version of a deformation problem, let alone verify Schlessinger's conditions, and very general constructions were available off the shelf as early as [KS00, Pri03].

2. For a fixed derived Artin stack $\mathfrak{X}$, take $\mathcal{C}(A)$ to be the simplicial category of quasi-coherent complexes on $\mathfrak{X} \times \operatorname{Spec} A$, with the simplicial functor $\mathcal{C}(A) \to \mathcal{C}(B)$ given by $\mathscr{E} \mapsto \mathscr{E} \otimes^{\mathbf{L}}_A B$.

3. Take $\mathcal{C}(A)$ to be the simplicial category of pairs $(\mathfrak{X}, \mathscr{E})$, for $\mathfrak{X}, \mathscr{E}$ as above.

4. Take $\mathcal{C}(A)$ to be any simplicial category given by taking diagrams in any of the above. For instance, this includes moduli of derived Artin stacks over a base $\mathfrak{Y}$, or of pairs $\mathfrak{X} \to \mathfrak{Y}$, or of maps:

$$A \mapsto \mathbf{R}\mathrm{Map}(\mathfrak{X} \times \operatorname{Spec} A, \mathfrak{Y}).$$

Proof. These all appear in [Pri10a]. The proofs use hypergroupoids intensively. □

In order to associate moduli functors to these simplicial category-valued functors, we first discard any morphisms which are not equivalences, so we restrict to the simplicial subcategory $\mathcal{W}(\mathcal{C}) \subset \mathcal{C}$ of homotopy equivalences, given by

$$\mathcal{W}(C) := \mathcal{C} \times_{\pi_0 \mathcal{C}} \mathrm{core}(\pi_0 \mathcal{C}),$$

for $\mathrm{core}(\pi_0\mathcal{C}) \subset \pi_0\mathcal{C}$ the maximal subgroupoid, which contains all the objects of $\pi_0\mathcal{C}$ with morphisms given by the isomorphisms between them.

Now, the nerve of a category is a simplicial set, and this extends to a construction giving the nerve $B\mathcal{C}$ of a simplicial category $\mathcal{C}$[74]. Taking

$$B\mathcal{W}(\mathcal{C}) \colon d\mathrm{Alg} \to s\mathrm{Set}$$

then gives the moduli stack of objects in $\mathcal{C}$.

For our examples, we now look at tangent cohomology:

1. For moduli of derived Artin n-stacks, at a point $[\mathfrak{X}] \in \mathcal{C}(A)$, the tangent cohomology groups are
$$D^i_{[\mathfrak{X}]}(B\mathcal{W}(\mathcal{C}), M) \cong \mathrm{Ext}^{i+1}_{\mathfrak{X}}(\mathbb{L}^{\mathfrak{X}}, \mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_A M).$$

2. For moduli of quasi-coherent complexes on X, at a point $[\mathscr{E}] \in \mathcal{C}(A)$ the tangent cohomology groups are

$$D^i_{[\mathscr{E}]}(B\mathcal{W}(\mathcal{C}), M) = \mathrm{Ext}^{i+1}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E} \otimes^{\mathbf{L}}_A M).$$

3. For moduli of pairs $(\mathfrak{X}, \mathscr{E})$, we have a long exact sequence

$$\mathrm{Ext}^{i+1}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{E}, \mathscr{E} \otimes^{\mathbf{L}}_A M) \to D^i_{[(\mathfrak{X},\mathscr{E})]}(B\mathcal{W}(\mathcal{C}), M) \to \mathrm{Ext}^{i+1}_{\mathfrak{X}}(\mathbb{L}^{\mathfrak{X}}, \mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_A M) \to \ldots,$$

in which the boundary map is given by the Atiyah class.

4. For moduli of maps $\mathfrak{X} \to \mathfrak{Y}$ (both fixed), we have

$$D^i_{[f \colon \mathfrak{X} \to \mathfrak{Y}]}(B\mathcal{W}(\mathcal{C}), M) = \mathrm{Ext}^i_{\mathscr{O}_{\mathfrak{X}}}(\mathbf{L}f^*\mathbb{L}^{\mathfrak{Y}}, \mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_A M),$$

similarly to §6.4.1.

[74] Explicitly, we first form the bisimplicial set $n \mapsto B\mathcal{C}_n$, then take the diagonal diag, or more efficiently the codiagonal $\bar{W}$ of [CR05], to give a simplicial set

These groups are all far too big to satisfy the finiteness conditions in general, so in each case we have to cut down to some suitably open subfunctor with good finiteness properties:

Example 6.38 (Derived moduli of schemes). Given $A \in d\mathrm{Alg}$, we can look at derived Zariski 1-hypergroupoids X which are homotopy-flat over $\mathrm{Spec}\, A$, with suitable restrictions on $\pi^0 X$ (proper, dimension d, ...). A specific example of this type is given by moduli of smooth proper curves, representable by a 1-truncated derived Artin stack.

Example 6.39 (Moduli of perfect complexes a proper scheme (or stack) X). In this example, given $A \in d\mathrm{Alg}$ we look at quasi-coherent complexes on $X \times \mathrm{Spec}\, A$ which are homotopy-flat over $\mathrm{Spec}\, A$, and perfect on pulling back to $X \times \pi^0\mathrm{Spec}\,(A)$. This is an ∞-geometric derived Artin stack, in the sense that it is a nested union of open n-geometric derived Artin substacks, for varying n. Explicitly, restricting the complexes to live in degrees $[a, b]$ gives an open $(b - a + 1)$-truncated derived moduli stack.

Example 6.40 (Derived moduli of polarised schemes). Given $A \in d\mathrm{Alg}$, look at pairs $(X, \mathscr{L})$, with X a derived Zariski 1-hypergroupoid homotopy-flat over $\mathrm{Spec}\, A$ and $\mathscr{L}$ a quasi-coherent complex on X, such that $(\pi^0 X, \mathscr{L} \otimes^{\mathbf{L}}_A \mathrm{H}_0 A)$ is a polarised projective scheme, so $\mathscr{L}$ is an ample line bundle. We can also fix the Hilbert polynomial to give a smaller open subfunctor.

6.7 Examples in detail

We still follow [Pri10a], in particular §3; the examples are more general than the title of the paper might suggest.[75]

Take a category-valued functor $\mathcal{C} : \mathrm{Alg}_{\mathrm{H}_0 R} \to \mathrm{Cat}$ and an property $\mathbf{P}$ on isomorphism classes of objects of $\mathcal{C}$ which is functorial in the sense that whenever $x \in \mathcal{C}(A)$ satisfies $\mathbf{P}$, its image $\mathcal{C}(f)(x) \in \mathcal{C}(B)$ also satisfies $\mathbf{P}$, for any morphism $f\colon A \to B$ in $\mathrm{Alg}_{\mathrm{H}_0 R}$. Then:

Definition 6.41. Say that $\mathbf{P}$ is an *open property* if it is closed under deformations in the sense that for any square-zero extension $A \to B$, an object of $\mathcal{C}(A)$ satisfies $\mathbf{P}$ whenever its image in $\mathcal{C}(B)$ does.

Definition 6.42. Say that $\mathbf{P}$ is an *étale local property* if for any $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and any étale cover $\{f_i\colon A \to B_i\}_{i \in I}$, an object of $\mathcal{C}(A)$ satisfies $\mathbf{P}$ whenever its images in $\mathcal{C}(B_i)$ all do.

Definition 6.43. Given $\mathcal{C}\colon d\mathcal{N}^\flat_R \to s\mathrm{Cat}$ and a functorial property $\mathbf{P}$ on objects of $\pi^0\mathcal{C}$, extend $\mathbf{P}$ to $\mathcal{C}$ by saying that an object of $\mathcal{C}(A)$ satisfies $\mathbf{P}$ if and only if its image in $\mathcal{C}(\mathrm{H}_0 A)$ does so.[77]

[75] There are some earlier examples of representable derived moduli functors in the literature:

- Stable curves, line bundles and closed subschemes were addressed in [Lur04a], although for stable curves the the derived moduli stack is just the classical underived moduli stack.
- In [TV04], local systems, finite algebras over an operad,[76] and mapping stacks were addressed; the last persist as the most popular way to construct representable functors.
- Representability for perfect complexes over associative dg algebras of finite type was established directly in [TV05], and hence for anything derived Morita equivalent.

[76] The methods of [Pri07b, Pri08, Pri11b] apply to algebras over more general monads.

[77] This convention gives a correspondence between open properties in the sense of Definition 6.41 and the open simplicial subcategories of [Pri10a, Definition 3.8].

Example 6.44. For instance, this means that we would declare a derived Artin stack $\mathfrak{X}$ over $\mathbf{R}\mathrm{Spec}\, A$ to be an algebraic curve if and only if the derived stack $\mathfrak{X} \times^h_{\mathbf{R}\mathrm{Spec}\, A} \mathrm{Spec}\, \mathrm{H}_0 A$ (which has structure sheaf $\mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_A \mathrm{H}_0 A$) is an algebraic curve over $\mathrm{Spec}\, \mathrm{H}_0 A$.

Lemma 6.45. *Take a homotopy-preserving and homotopy-homogeneous étale hypersheaf $\mathcal{C}\colon d\mathcal{N}^\flat_R \to s\mathrm{Cat}$. If $\mathbf{P}$ is a functorial property on $\pi^0\mathcal{C}$ which is open and étale local, then the full subfunctor $\tilde{\mathcal{M}} : d\mathcal{N}^\flat_R \to s\mathrm{Cat}$ of $\mathcal{C}$ on objects satisfying $\mathbf{P}$ is a homotopy-preserving and homotopy-homogeneous étale hypersheaf.*

Proof. [Pri10a, Proposition 2.31, Lemmas 2.23 and 2.26]. □

For criteria to establish homotopy-homogeneity for simplicial category-valued functors $\mathcal{C}$, see [Pri10a, Proposition 2.29].

6.7.1 Moduli of quasi-coherent complexes

The following is [Pri10a, Theorem 4.12]:

Theorem 6.46. *Take a strongly quasi-compact m-geometric derived Artin stack $\mathfrak{X}$ over R.*

Assume that we have an open, étale local condition $\mathbf{P}$ for objects of the functor $A \mapsto \mathcal{D}(\mathfrak{X} \otimes^{\mathbf{L}}_R A)$, the derived category of quasi-coherent complexes on $\mathfrak{X} \otimes^{\mathbf{L}}_R A$, i.e. $\mathrm{Ho}(\mathrm{Cart}(\mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_R A))$ as in §§1.3, 5.3, 6.2.2.

Also assume that this satisfies the following conditions:

1. *for all finitely generated $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $\mathscr{E} \in \mathcal{D}(\mathfrak{X} \otimes^{\mathbf{L}}_R A)$ satisfying $\mathbf{P}$, the functors*
$$\mathbb{E}\mathrm{xt}^i_{\mathfrak{X}\otimes^{\mathbf{L}}_R A}(\mathscr{E}, \mathscr{E} \otimes^{\mathbf{L}}_A -) : \mathrm{Mod}_A \to \mathrm{Ab}$$
preserve filtered colimits (equivalently, colimits indexed by directed sets) for all i.

2. *for all finitely generated integral domains $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $\mathscr{E} \in \mathcal{D}(\mathfrak{X} \otimes^{\mathbf{L}}_R A)$ satisfying $\mathbf{P}$, the groups $\mathbb{E}\mathrm{xt}^i_{\mathfrak{X}\otimes^{\mathbf{L}}_R A}(\mathscr{E}, \mathscr{E})$ are all finitely generated A-modules.*

3. *The functor $|P| : \mathrm{Alg}_{\mathrm{H}_0 R} \to \mathrm{Set}$ of isomorphism classes of objects satisfying $\mathbf{P}$ preserves filtered colimits.*

4. *for all complete discrete local Noetherian $\mathrm{H}_0 R$-algebras A, with maximal ideal $\mathfrak{m}$, the map*
$$|P|(A) \to \varprojlim_r |P|(A/\mathfrak{m}^r)$$
is an isomorphism, as are the maps
$$\begin{aligned}\mathbb{E}\mathrm{xt}^i_{\mathfrak{X}\otimes^{\mathbf{L}}_R A}(\mathscr{E}, \mathscr{E}) &\to \mathbb{E}\mathrm{xt}^i_{\mathfrak{X}\otimes^{\mathbf{L}}_R A}(\mathscr{E}, \mathbf{R}\varprojlim_r \mathscr{E}/\mathfrak{m}^r)\\ &\cong \varprojlim_r \mathbb{E}\mathrm{xt}^i_{\mathfrak{X}\otimes^{\mathbf{L}}_R A}(\mathscr{E}, \mathscr{E}/\mathfrak{m}^r)\end{aligned}$$
for all $\mathscr{E}$ satisfying $\mathbf{P}$ and all $i \leq 0$.

5. *For any $\mathrm{H}_0 R$-algebra A and $\mathscr{E} \in \mathcal{D}(\mathfrak{X} \otimes^{\mathbf{L}}_R A)$ satisfying $\mathbf{P}$, the cohomology groups $\mathbb{E}\mathrm{xt}^i_{\mathfrak{X}\otimes^{\mathbf{L}}_R A}(\mathscr{E}, \mathscr{E})$ vanish for $i \leq -n$.*

Let $\tilde{\mathcal{M}} : d\mathcal{N}^{\flat}_R \to s\mathrm{Cat}$ be given by sending A to the simplicial category of quasi-coherent complexes $\mathscr{E}$ on $\mathfrak{X} \otimes^{\mathbf{L}}_R A$ for which $\mathscr{E} \otimes^{\mathbf{L}}_A \mathrm{H}_0 A \in \mathcal{D}(\mathfrak{X} \otimes^{\mathbf{L}}_R \mathrm{H}_0 A)$ satisfies $\mathbf{P}$. Let $\mathcal{W}\tilde{\mathcal{M}}$ be the full simplicial subcategory of quasi-isomorphisms.

Then the nerve of $\mathcal{W}\tilde{\mathcal{M}}$ is an n-truncated derived Artin stack.

Examples 6.47. One example of an open, étale local condition is to ask that $\mathscr{E}$ be a perfect complex, and we could then impose a further such condition by fixing its Euler characteristic.

Another open, étale local condition would be to impose bounds on $\mathscr{E}$, asking that it only live in degrees $[a, b]$, provided a flatness condition is imposed to ensure functoriality, as we need the derived pullbacks $\mathscr{E} \otimes^{\mathbf{L}}_A B$ to satisfy the same constraint. Similar considerations apply for perverse t-structures, and in particular the moduli stack of objects living in the heart of a t-structure will be 1-truncated because we have no negative Exts.

We can also apply Theorem 6.46 to study derived moduli of Higgs bundles, for instance. One interpretation of a Higgs bundle on a smooth proper scheme X is as a quasi-coherent sheaf $\mathscr{E}$ on the cotangent scheme $T^*X = \mathbf{Spec}_X \mathrm{Symm}_{\mathscr{O}_X} \mathscr{T}_X$ which is a vector bundle when regarded as a sheaf on X. This defines an open, étale local condition on the functor $A \mapsto \mathcal{D}(T^*X \otimes A)$, so gives rise to a derived moduli stack, with tangent spaces given by Higgs cohomology.[78]

6.7.2 Moduli of derived Artin stacks

[Pri10a, Theorem 3.32] gives a similar statement for moduli of derived Artin n-stacks (and thus any subcategories such as derived DM n-stacks, derived schemes, ...), taking open, étale local conditions $\mathbf{P}$ on the homotopy category of n-truncated derived Artin stacks X over a fixed base Y. The relevant cohomology groups are now

$$\mathbb{E}\mathrm{xt}^i_X(\mathbb{L}^{X/Y_A}, \mathscr{O}_X \otimes^{\mathbf{L}}_A -) : \mathrm{Mod}_A \to \mathrm{Ab},$$

and the resulting moduli stack $\tilde{\mathcal{M}}$ is $(n+1)$-truncated.

In detail, the conditions become:

1. for all finitely generated $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all X over A satisfying $\mathbf{P}$, the functors
$$\mathbb{E}\mathrm{xt}^i_X(\mathbb{L}^{X/Y_A}, \mathscr{O}_X \otimes^{\mathbf{L}}_A -) : \mathrm{Mod}_A \to \mathrm{Ab}$$
preserve filtered colimits for all $i > 1$.

2. for all finitely generated integral domains $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all X over A satisfying $\mathbf{P}$, the groups $\mathbb{E}\mathrm{xt}^i_X(\mathbb{L}^{X/Y_A}, \mathscr{O}_X)$ are all finitely generated A-modules.

3. for all complete discrete local Noetherian $\mathrm{H}_0 R$-algebras A, with maximal ideal $\mathfrak{m}$, the map
$$\tilde{\mathcal{M}}(A) \to \varprojlim_r{}^h \tilde{\mathcal{M}}(A/\mathfrak{m}^r)$$
is a weak equivalence.

[78] There is a variant of this example for the derived de Rham moduli space of vector bundles with flat connection, replacing $\mathrm{Symm}_{\mathscr{O}_X} \mathscr{T}_X$ with the ring of differential operators. Since the latter is non-commutative, we cannot appeal directly to Theorem 6.46, but the same proof adapts verbatim.

4. $\tilde{\mathcal{M}} : \mathrm{Alg}_{\mathrm{H}_0 R} \to s\mathrm{Cat}$ preserves filtered colimits (i.e. the simplicial functor $\varinjlim_i \tilde{\mathcal{M}}(A(i)) \to \tilde{\mathcal{M}}(A)$ is a weak equivalence for all systems $A(i)$ indexed by directed sets.)

Example 6.48. If we let Y be the stack $B\mathbb{G}_m$, then one open, étale local condition on derived Artin stacks X over $B\mathbb{G}_m$ is to ask that X be a projective scheme flat over the base R, with $X \to B\mathbb{G}_m$ the morphism associated to an ample line bundle on X. We could also fix the Hilbert polynomial associated to this line bundle. The theorem then gives us representability of derived moduli stacks of polarised projective schemes.

6.8 Pre-representability

Details for this section appear in [Pri10b, §3].

The idea behind pre-representability is to generalise the way we associate derived functors to smooth schemes, which can be useful when constructing things like derived quotient stacks, or morphisms between derived stacks.

Definition 6.49. Given a functor $F : d\mathcal{N}^\flat \to s\mathrm{Set}$, we define a functor $\underline{F} : d\mathcal{N}^\flat \to ss\mathrm{Set}$ to the category of bisimplicial sets by

$$\underline{F}(A)_n := F(A^{\Delta^n}),$$

where for $A \in dg_+\mathrm{Alg}_R$, we set $A^{\Delta^n} := \tau_{\geq 0}(A \otimes \Omega^\bullet(\Delta^n))$, while for $A \in s\mathrm{Alg}$, the simplicial algebra A^{Δ^n} is given by $(A^{\Delta^n})_i := \mathrm{Hom}_{s\mathrm{Set}}(\Delta^i \times \Delta^n, A)$.

The results of [Pri10b, §3] and in particular [Pri10b, Theorem 3.16] then show that if F satisfies the conditions of Theorem 6.35, but mapping acyclic square-zero extensions to surjections rather than weak equivalences, then the diagonal $\mathrm{diag}\,\underline{F}$ is an n-truncated derived Artin stack. We then think of F as being pre-representable, by close analogy with the predeformation functors of [Man99].

One way to interpret the construction is that $\mathrm{diag}\,\underline{F}$ is the right-derived functor of F with respect to quasi-isomorphisms in $d\mathcal{N}^\flat_R$. Note that if F was already representable, then the natural map $F \to \mathrm{diag}\,\underline{F}$ is a weak equivalence.

Constructing morphisms $f\colon \mathfrak{X} \to \mathfrak{Y}$ between derived stacks can be cumbersome to attempt directly because derived stacks encode so much data, but pre-representable functors can provide a simplification. Instead of constructing the morphism f directly, if we can characterise $\mathfrak{X}$ as equivalent to $\mathrm{diag}\,\underline{F}$ for some much smaller functor F, then it suffices to construct a morphism $F \to \mathfrak{Y}$, since

$$\mathfrak{X} \simeq \mathrm{diag}\,\underline{F} \to \mathrm{diag}\,\underline{\mathfrak{Y}} \simeq \mathfrak{Y}.$$

Example 6.50. If X is a dg-manifold (in the sense of Definition 1.24), then the functor $X : dg_+\mathcal{N}^\flat_R \to \mathrm{Set}$ given by $X(A) := \mathrm{Hom}((\mathrm{Spec}\, A_0, A), X)$ satisfies the conditions of [Pri10b, Theorem 3.16], so $\underline{X} : dg_+\mathcal{N}^\flat_R \to s\mathrm{Set}$ is a 0-truncated derived Artin (or equivalently DM) stack, i.e. a derived algebraic space.

However, the space of morphisms $\underline{X} \to F$ to a derived stack F, which would be complicated to calculate directly, is just equivalent to the space of morphisms $X \to F$ for the functor X above, so is given by the simplicial set $\mathbf{R}\Gamma(X^0, F(\mathscr{O}_X))$, which can be calculated via a Čech complex as in Example 5.30.(3).

See [Pri11b, §§3–6] for many more examples of 1-truncated derived Artin moduli stacks constructed from pre-representable groupoid-valued functors.

6.9 Addendum: derived hypergroupoids à la [Pri09]

6.9.1 Homotopy derived hypergroupoids

The definitions given in §6.1 for homotopy derived Artin and DM hypergroupoids are not the same as those of [Pri09, Pri11a] (which are cast to work in more general settings), but are equivalent. For want of a suitable reference, we now prove the equivalence of the two sets of definitions, but readers should regard this section as a glorified footnote.

As with simplicial affine schemes, we still have notions of matching objects $M_{\partial\Delta^m}(X)$ and partial matching objects $M_{\Lambda^{m,k}}(X)$ for simplicial derived affine schemes $X_\bullet$. Explicitly, $M_{\partial\Delta^m}(X)$ is the equaliser of a diagram

$$\prod_{0\le i\le m} X_{m-1} \rightrightarrows \prod_{0\le i<j\le m} X_{m-2},$$

and is characterised by the property that

$$\mathrm{Hom}_{d\mathrm{Aff}}(U, M_{\partial\Delta^m}(X)) \cong \mathrm{Hom}_{sd\mathrm{Aff}}(\partial\Delta^m \times U, X),$$

naturally in $U \in d\mathrm{Aff}$, while $M_{\Lambda^{m,k}}(X)$ is the equaliser of a diagram

$$\prod_{\substack{0\le i\le m\\ i\ne k}} X_{m-1} \rightrightarrows \prod_{\substack{0\le i<j\le m\\ i,j\ne k}} X_{m-2},$$

and is characterised by the property that

$$\mathrm{Hom}_{d\mathrm{Aff}}(U, M_{\Lambda^{m,k}}(X)) \cong \mathrm{Hom}_{sd\mathrm{Aff}}(\Lambda^{m,k} \times U, X),$$

naturally in $U \in d\mathrm{Aff}$.

In order to formulate the key definition from [Pri09], we now need to replace these limits with homotopy limits:

Definition 6.51. Define the *homotopy matching objects* and *homotopy partial matching objects*

$$\begin{aligned} M^h_{\partial\Delta^m}&\colon\ sd\mathrm{Aff} \to d\mathrm{Aff}\\ M^h_{\Lambda^{m,k}}&\colon\ sd\mathrm{Aff} \to d\mathrm{Aff} \end{aligned}$$

to be the right-derived functors of the matching and partial matching object functors $M_{\partial\Delta^m}$ and $M_{\Lambda^{m,k}}$, respectively.

Definition 6.52. We say that a morphism $f\colon\ X \to Y$ in $d\mathrm{Aff}$ is surjective if $\pi^0 f\colon\ \pi^0 X \to \pi^0 Y$ is a surjection of affine schemes.

Definition 6.53. Given $Y_\bullet \in sd\mathrm{Aff}$, a morphism $X_\bullet \to Y_\bullet$ in $sd\mathrm{Aff}$ is said to be a [Pri09]-*homotopy derived Artin (resp. DM) n-hypergroupoid* over $Y_\bullet$ if:

1. for all $m \ge 1$ and $0 \le k \le m$, the homotopy partial matching maps

   $$X_m \to M^h_{\Lambda^{m,k}}(X) \times^h_{M^h_{\Lambda^{m,k}}(Y)} Y_m$$

   are homotopy-smooth (resp. homotopy-étale) surjections;

2. for all $m > n$ and all $0 \leq k \leq m$, the homotopy partial matching maps

$$X_m \to M^h_{\Lambda^{m,k}}(X) \times^h_{M^h_{\Lambda^{m,k}}(Y)} Y_m$$

are weak equivalences.

The morphism $X_\bullet \to Y_\bullet$ is then said to be homotopy-smooth (resp. homotopy-étale, resp. surjective) if $X_0 \to Y_0$ is homotopy-smooth (resp. homotopy-étale, resp. surjective).

Definition 6.54. Given $Y_\bullet \in sd\mathrm{Aff}$, a morphism $X_\bullet \to Y_\bullet$ in $sd\mathrm{Aff}$ is said to be a [Pri09]-*homotopy trivial derived Artin (resp. DM) n-hypergroupoid* over $Y_\bullet$ if and only if:

1. for each m, the homotopy matching map

$$X_m \to M^h_{\partial\Delta^m}(X) \times^h_{M^h_{\partial\Delta^m}(Y)} Y_m$$

is a homotopy-smooth (resp. homotopy-étale) surjection;

2. for all $m \geq n$, the homotopy matching maps

$$X_m \to M^h_{\partial\Delta^m}(X) \times^h_{M^h_{\partial\Delta^m}(Y)} Y_m$$

are weak equivalences.

We now have the following consistency check:

Lemma 6.55. *A [Pri09]-homotopy (trivial) derived Artin (resp. DM) n-hypergroupoid is precisely the same as a homotopy (trivial) derived Artin (resp. DM) n-hypergroupoid in the sense of §6.1.*

Proof. If $f\colon X_\bullet \to Y_\bullet$ is a [Pri09]-homotopy derived Artin n-hypergroupoid, then as in the proof of [Pri09, Theorem 4.7], the morphisms $(f, \partial_i)\colon X_m \to Y_m \times^h_{\partial_i Y_{m-1}} X_{m-1}$ are all homotopy-smooth for all $m > 0$ and all i, since Δ^{m-1} and Δ^m are contractible. In particular, those morphisms are strong, satisfying the second condition of Definitions 6.5, 6.8; the first is automatic.

For the converse, we start by using the following observation † (as in the end of the proof of [TV04, Lemma 2.2.2.8]): that a morphism $W \to Z$ in $d\mathrm{Aff}$ is strong if and only if the map $\pi^0 W \to W \times^h_Z \pi^0 Z$ is a weak equivalence. Thus the second condition of Definition 6.5 can be rephrased as saying that whenever $f\colon X_\bullet \to Y_\bullet$ is a homotopy derived Artin n-hypergroupoid, the map

$$g\colon \pi^0 X_\bullet \to X_\bullet \times^h_{Y_\bullet} \pi^0 Y_\bullet$$

is homotopy-Cartesian (the derived analogue of the notion in Example 4.33.3). But this is equivalent to saying that g is a [Pri09]-homotopy derived 0-hypergroupoid[79]. In particular, the homotopy partial maps of g are all quasi-isomorphisms, so the homotopy partial matching maps of f are all strong, by †. Combined with the first condition, this completes the proof for Definition 6.5.

Likewise, the second condition of Definition 6.8 says that g is a levelwise equivalence whenever f is a homotopy trivial derived Artin n-hypergroupoid, which is equivalent to saying that all the homotopy matching maps of g are quasi-isomorphisms, and hence that the homotopy matching maps of f are all strong. $\square$

[79] We do not specify Artin or DM, as 0-hypergroupoids are independent of the notion of covering.

6.9.2 Derived hypergroupoids

We now introduce an equivalent, but much more restrictive, model for homotopy derived hypergroupoids, which is especially useful when describing morphisms, but can be unwieldy to construct.

By [Hov99, Theorem 5.2.5], there is a model structure (the *Reedy model structure*) on $s d\mathrm{Aff}$ in which a map $X \to Y$ is a weak equivalence if it is a quasi-isomorphism in each level $X_m \xrightarrow{\sim} Y_m$, a cofibration if it is a cofibration in each level, and a fibration if the matching maps

$$X_m \to Y_m \times_{M_{\partial\Delta^m}(Y)} M_{\partial\Delta^m}(X)$$

are fibrations for all $m \geq 0$.

Example 6.56. Derived affine schemes U are almost never Reedy fibrant when regarded as objects of $s d\mathrm{Aff}$ with constant simplicial structure, because then we would have $M_{\partial\Delta^1}(U) \cong U \times U$, with the matching map being the diagonal map $U \to U \times U$, which can only be a fibration if it is an isomorphism.

For instance, the diagonal map $\mathbb{A}^1 \to \mathbb{A}^2$ corresponds to $k[x, y] \to k[x, y]/(x-y) \cong k[x]$, which is not quasi-free,[80] so $\mathbb{A}^1$ is not Reedy fibrant, despite being the prototypical fibrant derived affine scheme.

In fact, the homotopy matching objects $M^h_{\partial\Delta^n}(U)$ (see §6.9.1) are given by higher derived loop spaces $U^{hS^{n-1}}$, and in particular $M^h_{\partial\Delta^2}(U) \simeq \mathcal{L}U$ for the derived loop space $\mathcal{L}$ of Definition 3.13.

For Reedy fibrant simplicial derived affine schemes, the matching and partial matching objects are already *homotopy* matching and partial matching objects, leading to the following strictified analogues of Definitions 6.53, 6.54:

Definition 6.57. Given $Y_\bullet \in s d\mathrm{Aff}$, define a derived Artin (resp. DM) n-hypergroupoid over $Y_\bullet$ to be a morphism $X_\bullet \to Y_\bullet$ in $s d\mathrm{Aff}$, satisfying the following:

1. $X \to Y$ is a Reedy fibration.

2. for each $m \geq 1$ and $0 \leq k \leq m$, the partial matching map

   $$X_m \to M_{\Lambda^{m,k}}(X) \times_{M_{\Lambda^{m,k}}(Y)} Y_m$$

   is a homotopy-smooth (resp. homotopy-étale) surjection in $d\mathrm{Aff}$;

3. for all $m > n$ and all $0 \leq k \leq m$, the partial matching maps

   $$X_m \to M_{\Lambda^{m,k}}(X) \times_{M_{\Lambda^{m,k}}(Y)} Y_m$$

   are trivial fibrations in $d\mathrm{Aff}$.

Definition 6.58. A trivial derived Artin (resp. DM) n-hypergroupoid $X_\bullet \to Y_\bullet$ is a morphism in $s d\mathrm{Aff}$ satisfying the following:

1. for each m, the matching map

   $$X_m \to M_{\partial\Delta^m}(X) \times^h_{M_{\partial\Delta^m}(Y)} Y_m$$

   is a fibration and a homotopy-smooth (resp. homotopy-étale) surjection in $d\mathrm{Aff}$;

[80] The smallest Reedy fibrant replacement of the affine line $\mathbb{A}^1$ is given in level n by $\mathrm{Spec}\,\mathrm{Symm}(\bar{C}_\bullet(\Delta^n))$ in the $dg_+\mathrm{Alg}$ setting, where $\bar{C}_\bullet(\Delta^n)$ denotes normalised chains on the n-simplex, and by a similar construction with unnormalised chains in the $s\mathrm{Alg}$ setting.

2. for all $m \geq n$, the matching maps

$$X_m \to M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

are trivial fibrations.

Since a model structure comes with fibrant replacement, the following is an immediate consequence of Reedy fibrant replacement combined with Lemma 6.55:

Lemma 6.59. *A map $X_\bullet \to Y_\bullet$ is a homotopy derived Artin n-hypergroupoid if and only if its Reedy fibrant replacement $\hat{X} \to Y$ is a derived Artin n-hypergroupoid.*

A map $X_\bullet \to Y_\bullet$ is a homotopy trivial derived Artin n-hypergroupoid if and only if its Reedy fibrant replacement $\hat{X}_\bullet \to Y_\bullet$ is a trivial derived Artin n-hypergroupoid.

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[dr] & \hat{X} \arrow[d, "\text{Reedy fibration}"] \\ & Y \end{tikzcd}$$

Theorem 6.11 now has the following refinement (its original form as in [Pri11a, Theorem 5.11]).

Theorem 6.60. *The homotopy category of strongly quasi-compact $(n-1)$-geometric derived Artin stacks is given by taking the full subcategory of $sd\mathrm{Aff}$ consisting of derived Artin n-hypergroupoids $X_\bullet$, and formally inverting the trivial relative Artin n-hypergroupoids $X_\bullet \to Y_\bullet$.*

In fact, a model for the ∞-category of strongly quasi-compact $(n-1)$-geometric derived Artin stacks is given by the relative category $(\mathcal{C}, \mathcal{W})$ with $\mathcal{C}$ the full subcategory of $sd\mathrm{Aff}$ on derived Artin n-hypergroupoids $X_\bullet$ and $\mathcal{W}$ the subcategory of trivial relative derived Artin n-hypergroupoids $X_\bullet \to Y_\bullet$.

The same results hold true if we substitute "Deligne–Mumford" for "Artin" throughout.

In particular, this means we obtain the simplicial category of such derived stacks by simplicial localisation of derived n-hypergroupoids at the class of trivial relative derived n-hypergroupoids.

We can now give a direct proof of one of the ingredients we saw within the representability theorems:

Corollary 6.61. *Every derived n-geometric Artin stack $F\colon d\mathrm{Alg}_R \to s\mathrm{Set}$ is homotopy-homogeneous.*

Proof. We need to show that for maps $A \to B \leftarrow C$ in $d\mathrm{Alg}_R$, with $A \twoheadrightarrow B$ a surjection with nilpotent kernel, we have

$$F(A \times_B C) \xrightarrow{\sim} FA \times^h_{FB} FC.$$

For a derived Artin $(n+1)$-hypergroupoid X, this is an immediate consequence of the infinitesimal smoothness criterion, because $X(A) \to X(B)$ is then a Kan fibration, so $X(A) \times_{X(B)} X(C) \simeq X(A) \times^h_{X(B)} X(C)$, while we also have an isomorphism $X(A \times_B C) \cong X(A) \times_{X(B)} X(C)$ for any $X \in sd\mathrm{Aff}$. The result passes to hypersheafifications because étale morphisms lift nilpotent extensions uniquely. □

6.9.3 Explicit morphism spaces

Definition 5.14 and Theorem 5.15 now adapt in the obvious way to give a description of the derived mapping spaces $\mathbf{R}\mathrm{Map}(X^\sharp, Y^\sharp)$:

Definition 6.62. Define the simplicial Hom functor on simplicial derived affine schemes by letting $\underline{\mathrm{Hom}}_{sd\mathrm{Aff}}(X, Y)$ be the simplicial set given by

$$\underline{\mathrm{Hom}}_{sd\mathrm{Aff}}(X, Y)_n := \mathrm{Hom}_{sd\mathrm{Aff}}(\Delta^n \times X, Y),$$

where $(X \times \Delta^n)_i$ is given by the coproduct of Δ^n_i copies of X_i.

There then exist derived n-Artin and n-DM universal covers, defined similarly to Definition 5.13. Then every derived n-DM universal cover is also a derived n-Artin universal cover, and as in Definition 5.14:

Definition 6.63. Given a derived Artin n-hypergroupoid Y and $X \in sd\mathrm{Aff}$, we define

$$\underline{\mathrm{Hom}}^\sharp_{sd\mathrm{Aff}}(X, Y) := \varinjlim \underline{\mathrm{Hom}}_{sd\mathrm{Aff}}(\tilde{X}(i), Y),$$

where the colimit runs over the objects $\tilde{X}(i)$ of any n-Artin universal cover $\tilde{X} \to X$.

The following is a case of [Pri09, Corollary 4.10]:

Theorem 6.64. *If $X \in sd\mathrm{Aff}$ and Y is a derived Artin n-hypergroupoid, then the derived* Hom *functor on the associated hypersheaves (a.k.a. derived n-stacks) $X^\sharp, Y^\sharp$ is given (up to weak equivalence) by*

$$\mathbf{R}\mathrm{Map}(X^\sharp, Y^\sharp) \simeq \underline{\mathrm{Hom}}^\sharp_{sd\mathrm{Aff}}(X, Y).$$

In particular, this means the functor $Y^\sharp \colon (d\mathrm{Aff})^{\mathrm{op}} \to s\mathrm{Set}$ is given by $\underline{\mathrm{Hom}}^\sharp_{sd\mathrm{Aff}}(-, Y)$.

Warning 6.65. Beware that the truncation formulae of §5.2.2 do not have derived analogues, following Warning 6.10. Also note that Theorem 6.64 cannot be relaxed by taking Y to be a homotopy derived Artin n-hypergroupoid.

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