Lecture 1
18 January 2012

Introduction

- **Topic:** Probability and random processes
- **Lectures:** AT3 on Wednesday and Friday at 12:10pm
- **Office hours:** by appointment (email) at JCMB 6321
- **Email:** j.m.figueroa@ed.ac.uk

The future was uncertain

[Graph showing stock market values before and after the Great Stock Market Crash of 1929]

The future is **still** uncertain

[Graph showing US house price trends with a forecast for Q2 2006]
The universe is fundamentally uncertain!

There is no god of Algebra, but...

there are gods of probability

- Fu Lu Shou
- Shichi Fukujin
- Lakshmi
- Tyche
- Fortuna

The mathematical study of Probability

Some notable names

- Gerolamo Cardano (1501-1576)
- Pierre de Fermat (1601-1665)
- Blaise Pascal (1623-1662)
- Christiaan Huygens (1629-1695)
- Jakob Bernoulli (1654-1705)
- Abraham de Moivre (1667-1754)
- Thomas Bayes (1702-1761)
- Pierre-Simon Laplace (1749-1827)
- Adrien-Marie Legendre (1752-1833)
- Andrei Markov (1866-1922)
- Andrei Kolmogorov (1903-1987)
- Claude Shannon (1916-2001)

What it is all about

Mathematical probability aims to formalise everyday sentences of the type:

“The chance of \( A \) is \( p \)”

where \( A \) is some “event” and \( p \) is some “measure” of the likelihood of occurrence of that event.

Example

“There is a 20% chance of snow.”
“There is 5% chance that the West Antarctic Ice Sheet will collapse in the next 200 years.”
“There is a low probability of Northern Rock having a liquidity problem.”
Trials and outcomes

This requires introducing some language.

**Definition/Notation**

By a trial (or an experiment) we mean any process which has a well-defined set $\Omega$ of outcomes. $\Omega$ is called the sample space.

**Example**

Tossing a coin: $\Omega = \{H, T\}$.

Tossing two coins: $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$.

**Example (Rolling a (6-sided) die)**

$\Omega = \{1, 2, 3, 4, 5, 6\}$.

"Events are what we assign a probability to"

**Definition**

An event $A$ is a subset of $\Omega$. We say that an event $A$ has (not) occurred if the outcome of the trial is (not) contained in $A$.

**Example**

Tossing a coin and getting a head: $A = \{H\}$.

Tossing two coins and getting at least one head: $A = \{(H, H), (H, T), (T, H)\}$.

Rolling a die and getting an even number: $A = \{2, 4, 6\}$.

Rolling two dice and getting a total of 5: $A = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

**Warning (for infinite $\Omega$)**

Not all subsets of $\Omega$ need be events!

The language of sets

Let us consider subsets of a set $\Omega$.

**Definition**

The complement of $A \subset \Omega$ is denoted $A^c \subset \Omega$:

$$\omega \in A^c \iff \omega \notin A$$

Clearly $(A^c)^c = A$.

**Example (The empty set)**

The complement of $\Omega$ is the empty set $\emptyset$:

$$\omega \notin \emptyset \quad \forall \omega \in \Omega$$

**Remark**

For all subsets $A$ of $\Omega$, $A \cup \emptyset = A$ and $A \cup \Omega = \Omega$. 
Definition

The \textbf{intersection} of $A$ and $B$ is denoted $A \cap B$:

$$\omega \in A \cap B \iff \omega \in A \text{ and } \omega \in B$$

If $A \cap B = \emptyset$ we say $A$ and $B$ are \textbf{disjoint}.

\begin{tikzpicture}
  \fill[orange,even odd rule,even color=red,odd color=white,fill opacity=0.7] (-1.5,0) circle (0.5) -- (0,1) arc (90:270:0.5) -- (-1.5,0);
  \draw (-1.5,0) circle (0.5);
  \draw (0,1) circle (0.5);
  \draw (-1.5,0) -- (0,1);
  \node at (-1,0) {$A \cap B$};
  \node at (-0.5,0.5) {$A$};
  \node at (0.8,0.5) {$B$};
\end{tikzpicture}

Remark

For all subsets $A$ of $\Omega$, $A \cap \emptyset = \emptyset$ and $A \cap \Omega = A$.

Distributivity identities

Union and intersection obey distributive properties.

\textbf{Theorem}

\textit{Let $(A_i)_{i \in I}$ be a family of subsets of $\Omega$ indexed by some index set $I$ and let $B \subset \Omega$. Then}

$$\bigcup_{i \in I} (B \cap A_i) = B \cap \bigcup_{i \in I} A_i$$

\textit{and}

$$\bigcap_{i \in I} (B \cup A_i) = B \cup \bigcap_{i \in I} A_i$$

\textbf{Proof.}

$$\omega \in \bigcap_{i \in I} (B \cup A_i) \iff \forall i \in I, \ \omega \in B \cup A_i$$

$$\iff \forall i \in I, \ \omega \in B \text{ or } \omega \in A_i$$

$$\iff \omega \in B \text{ or } \omega \in A_i \ \forall i \in I$$

$$\iff \omega \in B \text{ or } \omega \in \bigcap_{i \in I} A_i$$

$$\iff \omega \in B \cup \bigcap_{i \in I} A_i$$.

The other equality is proved similarly. \hfill \Box

De Morgan’s Theorem

Union and intersection are “dual” under complementation.

\textbf{Theorem (De Morgan's)}

\textit{Let $(A_i)_{i \in I}$ be a family of subsets of $\Omega$ indexed by some index set $I$. Then}

$$\left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c \quad \text{and} \quad \left( \bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c$$

\textbf{Remark}

This shows that complementation together with either union or intersection is enough, since, e.g.,

$$A \cup B = (A^c \cap B^c)^c$$
Proof.

\[ \omega \in \left( \bigcup_{i \in I} A_i \right)^c \iff \omega \notin \bigcup_{i \in I} A_i \]
\[ \iff \omega \notin A_i \quad \forall i \in I \]
\[ \iff \omega \in A_i^c \quad \forall i \in I \]
\[ \iff \omega \in \bigcap_{i \in I} A_i^c . \]

The other equality if proved similarly. \( \square \)

**Definition**

The **difference** \( A \setminus B = A \cap B^c \) and the **symmetric difference** \( A \triangle B = (A \setminus B) \cup (B \setminus A) \).

**Remark**

Notice that \( A \setminus B = A \setminus (A \cap B) \).

**Probability/Set theory dictionary**

<table>
<thead>
<tr>
<th>Notation</th>
<th>Set-theoretic language</th>
<th>Probabilistic language</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega )</td>
<td>Universe</td>
<td>Sample space</td>
</tr>
<tr>
<td>( \omega \in \Omega )</td>
<td>member of ( \Omega )</td>
<td>outcome</td>
</tr>
<tr>
<td>( A \subseteq \Omega )</td>
<td>subset of ( \Omega )</td>
<td>some outcome in ( A ) occurs</td>
</tr>
<tr>
<td>( A^c )</td>
<td>complement of ( A )</td>
<td>no outcome in ( A ) occurs</td>
</tr>
<tr>
<td>( A \cap B )</td>
<td>intersection</td>
<td>Both ( A ) and ( B )</td>
</tr>
<tr>
<td>( A \cup B )</td>
<td>union</td>
<td>Either ( A ) or ( B ) (or both)</td>
</tr>
<tr>
<td>( A \setminus B )</td>
<td>difference</td>
<td>( A ), but not ( B )</td>
</tr>
<tr>
<td>( A \triangle B )</td>
<td>symmetric difference</td>
<td>Either ( A ) or ( B ), but not both</td>
</tr>
<tr>
<td>( \emptyset )</td>
<td>empty set</td>
<td><strong>impossible</strong></td>
</tr>
<tr>
<td>( \Omega )</td>
<td>whole universe</td>
<td><strong>certain</strong> event</td>
</tr>
</tbody>
</table>

**Which subsets can be events?**

- For finite \( \Omega \), any subset can be an event.
- For infinite \( \Omega \), it is not always sensible to allow all subsets to be events. (Trust me!)
- If \( A \) is an event, it seems reasonable that \( A^c \) is also an event.
- Similarly, if \( A \) and \( B \) are events, it seems reasonable that \( A \cup B \) and \( A \cap B \) should also be events.

In summary, the collection of events must be closed under complementation and pairwise union and intersection. By induction, it must also be closed under finite union and intersection: if \( A_1, \ldots, A_N \) are events, so should be \( A_1 \cap A_2 \cap \cdots \cap A_N \) and \( A_1 \cup A_2 \cup \cdots \cup A_N \).
The following example, shows that this is not enough.

Example

Alice and Bob play a game in which they toss a coin in turn. The winner is the first person to obtain H. Intuition says that the person who plays first has an advantage. We would like to quantify this intuition. Suppose Alice goes first. She wins if and only if the first H turns out after an odd number of tosses. Let \( \omega_i \) be the outcome \( TT\cdots T H \). Then the event that Alice wins is 

\[ A = \{ \omega_1, \omega_3, \omega_5, \ldots \} \]

which is a disjoint union of a countably infinite number of events. In order to compute the likelihood of Alice winning, it had better be the case that \( A \) is an event, so one demands that the family of events be closed under countably infinite unions; that is, if \( A_i \), for \( i = 1, 2, \ldots \), are events, then so is \( \bigcup_{i=1}^{\infty} A_i \).

\[ \sigma \text{-fields} \]

Definition

A family \( \mathcal{F} \) of subsets of \( \Omega \) is a \( \sigma \text{-field} \) if

1. \( \Omega \in \mathcal{F} \)
2. if \( A_1, A_2, \ldots \in \mathcal{F} \), then \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \)
3. if \( A \in \mathcal{F} \), then \( A^c \in \mathcal{F} \)

Remark

It follows from De Morgan's theorem that for a \( \sigma \text{-field} \) \( \mathcal{F} \), if \( A_1, A_2, \ldots \in \mathcal{F} \) then \( \bigcap_{i=1}^{\infty} A_i \in \mathcal{F} \). Also \( \Omega \in \mathcal{F} \), since \( \Omega = \emptyset^c \).

Finally, a \( \sigma \text{-field} \) is closed under (symmetric) difference.

Example

The smallest \( \sigma \text{-field} \) is \( \mathcal{F} = \{ \emptyset, \Omega \} \). The largest is the power set of \( \Omega \) (i.e., the collection of all subsets of \( \Omega \)).

Summary

- With any experiment or trial we associate a pair \( (\Omega, \mathcal{F}) \), where
  - \( \Omega \), the sample space, is the set of all possible outcomes of the experiment; and
  - \( \mathcal{F} \), the family of events, is a \( \sigma \text{-field} \) of subsets of \( \Omega \): a family of subset of \( \Omega \) containing the empty set and closed under complementation and countable unions.
- In the next lecture we will see how to enhance the pair \( (\Omega, \mathcal{F}) \) to a “probability space” by assigning a measure of likelihood (i.e., a “probability”) to the events in \( \mathcal{F} \).
- There is a dictionary between the languages of set theory and of probability. In particular, we will use the set-theoretic language freely.