Problem Sheet 1

Problem 1.1. Let $(\mathbb{A}^4, \eta)$ denote four-dimensional Minkowski spacetime, with metric

$$\eta = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \ .$$

1. Check that the the affine transformations:

$$x^\mu \mapsto A^\mu{}_\nu x^\nu + a^\mu \ , \tag{1}$$

   where a^μ is a constant 4-vector and $A^\mu{}_\nu$ is a 4×4 matrix which obeys

$$A^t \eta A = \eta \qquad \text{or, equivalently,} \qquad A^\mu{}_\rho A^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} \ , \tag{2}$$

   define isometries of Minkowski spacetime. (For extra credit: show that all isometries are of this type.)

2. Check that such transformations define a group, called the **Poincaré group**.

   The subgroup which fixes the point with coordinates $x^\mu = 0$ is called the **Lorentz group**, denoted $O(3,1)$.

3. Show that the subgroup which fixes any other point in $\mathbb{A}^4$ is isomorphic to the Lorentz group.

4. Show that the Lorentz group has four connected components. Let $SO_0(3,1)$ denote the connected component containing the identity.

5. Show that there is a covering homomorphism $SL(2,\mathbb{C}) \to SO_0(3,1)$ as follows:

   (a) Let E denote the space of 2×2 hermitian matrices. It is a real four-dimensional vector space. Show that the determinant defines an indefinite quadratic form which can be identified with (minus) the Minkowski norm in $\mathbb{R}^4$ via

$$x^\mu \mapsto \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \ . \tag{3}$$

   (b) Let $SL(2,\mathbb{C})$ act on E via $a \cdot X = aXa^\dagger$, where $a \in SL(2,\mathbb{C})$ and $X \in E$. Show that this defines a homomorphism $SL(2,\mathbb{C}) \to O(3,1)$ and work out explicitly the image of a matrix

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C}) \tag{4}$$

   in $O(3,1)$.

   (c) Show that $SL(2,\mathbb{C})$ is connected and hence that $SL(2,\mathbb{C}) \to SO_0(3,1)$ is a surjective homomorphism with kernel the finite group of order 2 consisting of $\{\pm \mathbf{1}\}$ with $\mathbf{1}$ the identity matrix.

Problem 1.2. Let $\phi : \mathbb{A}^4 \to \mathbb{R}$ be a scalar field in four-dimensional Minkowski spacetime and consider the action functional

$$S = \int d^4x \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2 - V(\phi)\right) \ , \tag{5}$$

where $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$ and $\partial_\mu\phi = \frac{\partial\phi}{\partial x^\mu}$.

1. Work out the Euler–Lagrange equation obtained by extremising S. It is (at least when the potential is absent) the **Klein–Gordon equation**.
2. Show that S is invariant under the Poincaré group, and deduce that the Poincaré group transforms solutions of the Klein–Gordon equation into solutions. Show that in the absence of the potential $V(\phi)$, the space of solutions of the Klein–Gordon equation is a vector space. It is a representation of the Poincaré group.
3. What are the conserved quantities associated (via Noether's theorem) to the Poincaré symmetry?

Problem 1.3. Let G be a matrix group (e.g., $SL(2,\mathbb{C})$). By a **curve in** G we shall mean a differentiable map $c : (-\varepsilon, \varepsilon) \to G$, sending t to $c(t)$, such that $c(0) = \mathbf{1}$. The space

$$\mathfrak{g} = \{c'(0) | c \text{ a curve in } G\} \tag{6}$$

of velocities at the identity of curves in G is called the **Lie algebra** of G.

1. Show that $\mathfrak{g}$ is a real vector space and show that if $X, Y \in \mathfrak{g}$ then so is $[X, Y] := XY - YX$.
2. For each group $G = SL(2,\mathbb{C})$ and $SO(3,1)$ do the following:
   (a) determine the Lie algebra $\mathfrak{g}$ and the real dimension $\dim_{\mathbb{R}} \mathfrak{g}$;
   (b) prove that if $X \in \mathfrak{g}$, then for all $t \in \mathbb{R}$, $\exp(tX) \in G$, where exp denotes the matrix exponential, defined by the power series
   $$\exp(X) = \mathbf{1} + X + \tfrac{1}{2}X^2 + \tfrac{1}{3!}X^3 + \cdot = \sum_n \frac{1}{n!}X^n \ .$$
   (c) exhibit a basis (X_i) for $\mathfrak{g}$ and write down the **structure constants** $[X_i, X_j] = \sum_k f_{ij}{}^k X_k$;
   (d) calculate the **Killing form** $\kappa_{ij} := \sum_{k,\ell} f_{ik}{}^\ell f_{j\ell}{}^k$ and show that is nondegenerate;
   (e) check that $f_{ijk} := \sum_\ell f_{ij}{}^\ell \kappa_{\ell k}$ is totally antisymmetric;

   Let $U\mathfrak{g}$ be the unital associative algebra generated by *abstract symbols* X_i subject to the relations
   $$X_iX_j - X_jX_i = \sum_k f_{ij}{}^k X_k \ . \tag{7}$$
   It is called the **universal enveloping algebra** of $\mathfrak{g}$.

(f) Let κ^{ij} denote the inverse of the Killing form and let $c := \sum_{i,j} \kappa^{ij} X_i X_j \in \mathfrak{U}\mathfrak{g}$ denote the **quadratic casimir** of $\mathfrak{g}$. Show that the following identities hold in $\mathfrak{U}\mathfrak{g}$:

$$X_i c = c X_i \qquad \text{for all } i. \tag{8}$$

(g) Now thinking of the X_i as the basis elements for $\mathfrak{g}$, work out the matrix c and check that it is a multiple of the identity.

3. Show that $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(3,1)$ are isomorphic as Lie algebras.

Problem 1.4. Let $\mathfrak{P}$ stand for the Lie algebra of the Poincaré group.

1. Show that the vector fields (on $\mathbb{A}^4$) given by

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \qquad \text{and} \qquad P_\mu = \partial_\mu \ , \tag{9}$$

where $x_\mu = \eta_{\mu\nu} x^\nu$, define an action of $\mathfrak{P}$ on the space of differentiable functions in $\mathbb{A}^4$. Show that

$$\begin{aligned} [L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho} \\ [L_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\ [P_\mu, P_\nu] &= 0 \ . \end{aligned} \tag{10}$$

2. Compute the Killing form of $\mathfrak{P}$ relative to this basis. Is it nondegenerate?

3. Let $\mathfrak{U}\mathfrak{P}$ be the universal enveloping algebra of $\mathfrak{P}$. Let $P^2 := \eta^{\mu\nu} P_\mu P_\nu$ and

$$W^2 = \eta_{\mu\nu} W^\mu W^\nu \qquad \text{where} \qquad W^\mu = \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} L_{\rho\sigma} P_\nu \ . \tag{11}$$

Show that $P_\mu W^\mu = 0$ and show that $P^2, W^2 \in \mathfrak{U}\mathfrak{P}$ commute in $\mathfrak{U}\mathfrak{P}$ with $L_{\mu\nu}$ and P_μ.