Lecture 1: Newtonian mechanics

> I think that Isaac Newton is doing most of the driving right now.
> — Bill Anders, Apollo 8 mission

In this lecture we introduce the basic assumptions underlying newtonian mechanics, which deals with the motions of point particles in space. Being based on empirical evidence, these assumptions have a limited domain of validity and hence the laws derived from them are known to break down in the very small, the very large or the very fast. Nevertheless newtonian mechanics has a remarkably wide domain of applicability, encompassing for instance both apples falling on the surface of the Earth and planets orbiting stars. Historically it was also the first modern physical theory.

The universe according to Newton

The newtonian universe is $\mathbb{R} \times \mathbb{R}^3$, where $\mathbb{R}$ is TIME and $\mathbb{R}^3$ is a 3-dimensional euclidean space with the usual scalar ("dot") product,

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i \ , \tag{1}$$

for $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$.

> **Notation**
>
> This seems a good point to alert you to other notations you will come across. Our convention is that vectors are **boldfaced** in the printed notes, but underlined in the blackboard. In Physics, it is also common to write a_i for (the components of) the vector $\mathbf{a}$, and the scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$ is then written $a_i b_i$ with the convention that repeated indices are to be summed over all their values, in this case $i = 1, 2, 3$. Finally, an alternatetive notation for the "dot" product is $\mathbf{a} \cdot \mathbf{b}$, which explains the name.

A point $(t, \mathbf{a})$ in the universe is called an EVENT. Two events $(t, \mathbf{a})$ and $(t', \mathbf{b})$ are said to be SIMULTANEOUS if and only if $t = t'$. It makes sense to talk about the DISTANCE between simultaneous events $(t, \mathbf{a})$ and $(t, \mathbf{b})$, and this is given by

$$|\mathbf{a} - \mathbf{b}| = \sqrt{\langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle} \ . \tag{2}$$

Particle trajectories are given by WORLDLINES, which are graphs of functions $\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$; that is, subsets of the universe of the form

$$\{(t, \mathbf{x}(t)) \mid t \in \mathbb{R}\} \ . \tag{3}$$

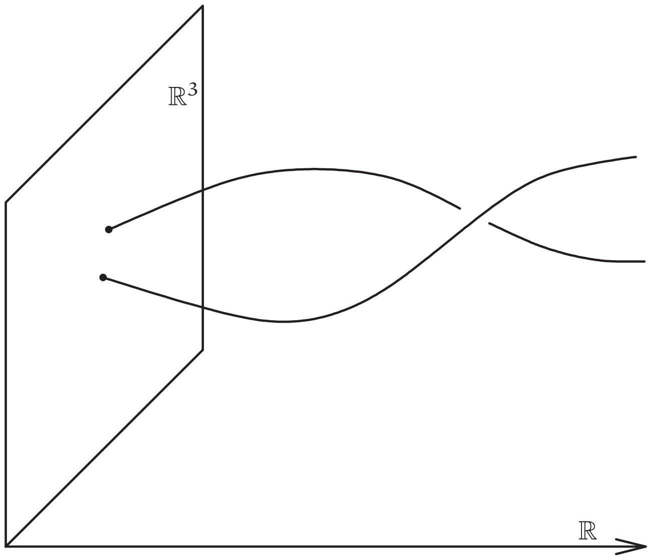

Figure 1: Two worldlines

We will assume that such functions x are continuously differentiable as many times as required. Figure 1 illustrates the worldlines of two particles.

Let $x : \mathbb{R} \to \mathbb{R}^3$ define the worldline of a particle. The first derivative (with respect to time) $\dot{x}$ is called the VELOCITY and the second derivative $\ddot{x}$ the ACCELERATION.

We are often interested in mechanical systems consisting of more that one particle. The CONFIGURATION SPACE of an n-particle system is the n-fold cartesian product

$$\underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{n} = \mathbb{R}^N , \qquad N = 3n .$$

The worldline of the ith particle is given by $x_i : \mathbb{R} \to \mathbb{R}^3$ and the n worldlines together define a curve $x : \mathbb{R} \to \mathbb{R}^N$ in the configuration space,

$$x(t) = (x_1(t), \ldots, x_n(t)) .$$

Newton's equation

The other basic assumption of newtonian mechanics is DETERMINACY, which means that the INITIAL STATE of a mechanical system, by which we mean the totality of the positions and velocities of all the particles at a given instant in time, uniquely determines the motion. In other words, $x(0)$ and $\dot{x}(0)$ determine $x(t)$ for all t, or at least for all t in some finite interval.

In particular, the acceleration is determined, so there must exist some relationship of the form

$$\boxed{\ddot{x} = \Phi(x, \dot{x}, t)} , \tag{4}$$

for some function $\Phi : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$. This second-order ordinary differential equation (ODE) is called NEWTON'S EQUATION. Solving a second-order ODE involves integrating twice, which gives rise to two constants of integration (per degree of freedom). These constants are then fixed by the initial conditions.

We will be dealing almost exclusively with functions Φ depending only on x and neither on $\dot{x}$ nor on t.

Done? ❑

Exercise 1.1. Let Φ depend only on x. Show that Newton's equation is invariant under time reversal; that is, show that if $x(t)$ solves the equation, so does $\bar{x}(t) := x(-t)$.

Example 1.1 (Particle in a force field). The version of Newton's equation (4) which describes the motion of a particle in the presence of a force field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is

$$\boxed{\mathbf{F}(x) = m\ddot{x}} \tag{5}$$

where m is the (INERTIAL) MASS of the particle.

Dimensional analysis

Physical quantities have dimension. The basic dimensions in these lectures are LENGTH (L), TIME (T) and MASS (M). For example, the position vector x has dimension of length, and we write this as $[x] = L$. Similarly, the time-derivative has dimension of inverse time, whence if a time-dependent quantity Q has dimension $[Q]$, then its time-derivative has dimension $[\dot{Q}] = [Q]T^{-1}$. It follows from this that the velocity and acceleration have dimensions $[\dot{x}] = LT^{-1}$ and $[\ddot{x}] = LT^{-2}$, respectively. Dimension is multiplicative in the sense that $[Q_1 Q_2] = [Q_1][Q_2]$, whence from Newton's equation (5) $[\mathbf{F}] = [m\ddot{x}] = MLT^{-2}$, where we have used that $[m] = M$, naturally. It is a very useful check of the correctness of a calculation that the result should have the expected dimension.

Done? ❑

Exercise 1.2. A particle of mass m is observed moving in a circular trajectory

$$x(t) = (R\cos\omega t, R\sin\omega t, 0) \ , \tag{6}$$

where R, ω are positive constants. What is the force acting on the particle?

Example 1.2 (The free particle). This is a particular case of the previous example, where $\mathbf{F} = 0$. Newton's equation (5) says that there is no acceleration, so that the velocity v is constant. Integrating a second time we obtain

$$x(t) = x_0 + tv \ , \tag{7}$$

where $x_0 = x(0)$ is the initial position. Given x_0 and v there is a unique solution $x(t)$ to Newton's equation with $\mathbf{F} = 0$ with $x(0) = x_0$ and $\dot{x}(0) = v$.

A standard trick allows us to turn Newton's equation (4) into an equivalent first-order ODE. The trick consists in introducing a new function $\boldsymbol{v} : \mathbb{R} \to \mathbb{R}^N$ together with the equation $\dot{\boldsymbol{x}} = \boldsymbol{v}$. Newton's equation is then

$$\ddot{\boldsymbol{x}} = \dot{\boldsymbol{v}} = \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{v}, t) .$$

In other words, in terms of the function $(\boldsymbol{x}, \boldsymbol{v}) : \mathbb{R} \to \mathbb{R}^{2N}$, Newton's equation becomes

$$\boxed{(\dot{\boldsymbol{x}}, \dot{\boldsymbol{v}}) = (\boldsymbol{v}, \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{v}, t)) .} \tag{8}$$

Done? ❑ **Exercise 1.3.** Show that equations (4) and (8) have exactly the same solutions.

The space of positions and velocities, here $\mathbb{R}^{2N}$, defines the STATE SPACE of the mechanical system. The pair of functions $(\boldsymbol{x}, \boldsymbol{v})$ defines a curve in the space of states, which, if it obeys (8), is called a PHYSICAL TRAJECTORY. This reformulation of Newton's equation makes contact with MAT-2-MAM, where it is proved that if $\boldsymbol{\Phi}$ is sufficiently differentiable, equation (8) has a unique solution for specified initial conditions $(\boldsymbol{x}(0), \boldsymbol{v}(0) = \dot{\boldsymbol{x}}(0))$, at least in some time interval. In other words, through every point in state space there passes a unique physical trajectory.

Example 1.3 (Galilean gravity). Consider dropping an apple of mass m from the Tower of Pisa. Empirical evidence suggests that the force of gravity points downwards, is constant and proportional to the mass. Letting $z(t)$ denote the height at time t, Newton's equation is then

$$m\ddot{z} = -mg , \tag{9}$$

where g is a constant with dimension $[g] = LT^{-2}$, and $g \approx 9.8\text{ms}^{-2}$ on the surface of the Earth. We can solve equation (9) by integrating twice

$$z(t) = z_0 + v_0 t - \tfrac{1}{2} g t^2 .$$

The relevant space of states is the right half-plane

$$\{(z, v) \mid z \geq 0\} \subset \mathbb{R}^2 , \tag{10}$$

and the physical trajectories are the parabolas given by

$$(z(t), v(t)) = \left(z_0 + v_0 t - \tfrac{1}{2} g t^2, v_0 - gt\right) . \tag{11}$$

Some of these trajectories are plotted in Figure 2. Notice that whatever the initial conditions (z_0, v_0) the apple always ends up on the floor. This is contrary to observation (e.g., rockets can break free of Earth's gravity) and indeed it is known that as the distance from the Earth increases, her gravitational pull weakens. This will be corrected in Newton's theory of gravity.

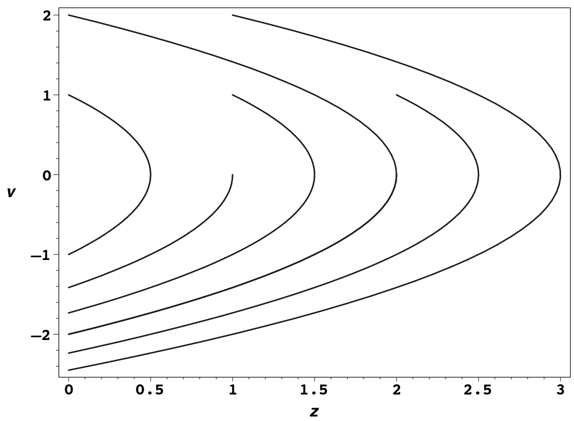

Figure 2: Physical trajectories of equation (9) in units where $g = 1$

The equivalence principle

The m in the RHS of equation (9) is called the (GRAVITATIONAL) MASS and it is an empirical fact (famously demonstrated by Galileo and later by Eötvös) that it is equal to the (inertial) mass appearing in the LHS. This equality is called the EQUIVALENCE PRINCIPLE: it hints at a geometric origin of gravity and is a cornerstone of Einstein's general theory of relativity.

Galilean transformations

This section contains more advanced material.

The identification of the newtonian universe with $\mathbb{R} \times \mathbb{R}^3$ requires a choice of origin in $\mathbb{R}^3$, which in the geocentric model of the universe was taken to be here on Earth; although before a consensus could be reached on the precise location, the model was abandoned in what is known as the copernican revolution. Mathematically this meant replacing $\mathbb{R} \times \mathbb{R}^3$ with four-dimensional AFFINE SPACE $\mathbb{A}^4$.

Roughly speaking, an affine space is a vector space who has forgotten where the origin is. This means that we cannot add points; although we can take differences of points; that is, parallel displacements. These displacements can be added and indeed do form a vector space. An explicit model for $\mathbb{A}^4$ is the affine hyperplane of $\mathbb{R}^5$ consisting of vectors whose 5th coordinate is equal to 1. The displacements are vectors in the hyperplane whose 5th coordinate is zero. (This fifth coordinate is of course *not* physical – it's an auxiliary construct to help us visualise an affine space.)

AFFINE TRANSFORMATIONS (of $\mathbb{A}^4$) are those invertible linear transformations of $\mathbb{R}^5$ which take this hyperplane to itself. In terms of 5×5 matrices they take the form:

$$\begin{pmatrix} L & \mathbf{a} \\ 0 & 1 \end{pmatrix} , \qquad \mathbf{a} \in \mathbb{R}^4, \quad L : \mathbb{R}^4 \to \mathbb{R}^4 , \tag{12}$$

where L is invertible. Such matrices are themselves invertible

$$\begin{pmatrix} L & \mathbf{a} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} L^{-1} & -L^{-1}a \\ 0 & 1 \end{pmatrix} \tag{13}$$

and are closed under multiplication

$$\begin{pmatrix} L & \mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M & \mathbf{b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} LM & L\mathbf{b} + \mathbf{a} \\ 0 & 1 \end{pmatrix} . \tag{14}$$

In other words, affine transformations form a group, called the AFFINE GROUP (of $\mathbb{A}^4$). This group acts on vectors $(\mathbf{y}, 1) \in \mathbb{R}^5$ as

$$\begin{pmatrix} L & \mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} = \begin{pmatrix} L\mathbf{y} + \mathbf{a} \\ 1 \end{pmatrix} . \tag{15}$$

To make contact with the newtonian universe, let points in $\mathbb{A}^4$ have coordinates $(t, \mathbf{x}, 1)$. The TIME INTERVAL between two events $(t, \mathbf{x}, 1)$ and $(t', \mathbf{x}', 1)$ is defined to be $t - t'$. Recall that the distance between two simultaneous events $(t, \mathbf{x}, 1)$ and $(t, \mathbf{x}', 1)$ is defined by $|\mathbf{x} - \mathbf{x}'|$. An affine transformation is GALILEAN if it preserves time intervals and the distances between simultaneous events.

Done? ❑ **Exercise 1.4.**✰ Show that a galilean transformation takes the form

$$\begin{pmatrix} 1 & 0 & s \\ \mathbf{v} & R & \mathbf{s} \\ 0 & 0 & 1 \end{pmatrix} , \tag{16}$$

with $\mathbf{s}, \mathbf{v} \in \mathbb{R}^3$, $s \in \mathbb{R}$ and $R : \mathbb{R}^3 \to \mathbb{R}^3$ an orthogonal transformation. Show that such matrices form a subgroup of the affine group. This is called the GALILEAN GROUP.

A general galilean transformation is the product of three simpler galilean transformations:

$$\begin{pmatrix} 1 & 0 & s \\ \mathbf{v} & R & \mathbf{s} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & s \\ 0 & I & \mathbf{s} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{v} & I & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

where I is the 3×3 identity matrix.

Each of these three simpler transformations has a different physical interpretation, which we can glean by their action on an event $(t, \mathbf{x})$. For the first type,

$$\begin{pmatrix} 1 & 0 & s \\ 0 & I & \mathbf{s} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} t + s \\ \mathbf{x} + \mathbf{s} \\ 1 \end{pmatrix} , \tag{17}$$

which we recognise as a change of origin. For the second type,

$$\begin{pmatrix} 1 & 0 & 0 \\ \boldsymbol{v} & I & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ \boldsymbol{x} \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ \boldsymbol{x} + t\boldsymbol{v} \\ 1 \end{pmatrix} , \tag{18}$$

which we recognise as moving with uniform velocity $\boldsymbol{v}$. Finally, the third type is simply a reorientation of the axes:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ \boldsymbol{x} \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ R\boldsymbol{x} \\ 1 \end{pmatrix} . \tag{19}$$

The principle of GALILEAN RELATIVITY posits the existence of INERTIAL COORDINATES where, basically, free particles move in straight lines.

Done? ❑ **Exercise 1.5.** Let $\{(t, \boldsymbol{x}_0 + \boldsymbol{v}_0 t)\}$ be the worldline of a free particle. Show that its image under a galilean transformation again takes this form, whence conclude that galilean transformations take inertial coordinates to inertial coordinates.