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Realizations of the Unitary Representations of the Inhomogeneous Space-Time Groups II

Covariant Realizations of the Poincaré Group

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Abstract

The aim of Part II of this paper is to try to give a unified, systematic description of the different covariant wave-functions (Dirac, Fierz, Bargmann-Wigner etc.) which can be used to carry the same physical representation of the Poincaré group. The procedure is based on the observation that the most important physical properties of the wave-functions, in particular the transformation law, the wave-equation, and the inner-product, depend only on the representation of the Lorentz group to which the wave-functions belong and the spin-projection. Accordingly, a formalism is set up which is valid for all (finite-dimensional) representations of the Lorentz group and all spin projections and so allows all the various wave-equations which are possible for a given mass and spin to be discussed collectively. The formalism is relatively simple and transparent (possibly even more simple and transparent than any of the special cases it embraces) and it allows the various conventional wave-functions and wave-equations to be easily identified as special cases. The topics covered in the discussion are listed in the table of contents, and Part II of the paper has been made self-contained so that it can be read independently of Part I.

Contents

Introduction

It is well-known [*1, 2, 3, 4*] that many different covariant wave-functions can be used to carry the same irreducible representations of the Poincaré group, standard examples being the DIRAC, FIERZ, BARGMANN-WIGNER and RARITA-SCHWINGER wave-equations [*3*]. The aim of Part II of this paper is to try to give a unified, systematic, representation-independent treatment [*5*] of the various possible wave-functions corresponding to a given mass ($m^2 \geqq 0$) and spin, as a full treatment of this kind does not seem to be available in any single place.

There are five chapters. Chapter V deals with the general Poincaré transformation law (including space-reversal) and the corresponding inner-product in momentum space. Chapters VI and VII deal with the massive ($m^2 > 0$) and massless [*6*] ($m^2 = 0$) cases respectively in much greater detail. It is in these two chapters that the main results of the paper are established, namely, that all the covariant wave-functions corresponding to a given mass and spin are characterized by just two quantities, namely $\mathscr{D}$ and Q, where $\mathscr{D}$ is the finite-dimensional representation of the Lorentz group to which the wave-function belongs, and Q is the spin projection, and that all the important properties of the wave-function, natably the transformation law, the wave-equation and the inner product, can be discussed without specifying $\mathscr{D}$ and Q. In the massless case, it is shown that an even stronger result holds, namely that the wave-function is characterized by $\mathscr{D}$ alone, because in that case the spin projection Q is fixed by the unitarity condition on the overall representation of the Poincaré group. The various standard wavefunctions and their equations (including the Weyl and Maxwell equations) are then identified as special cases of the general formalism.

In the last two chapters the discussion is transferred to configuration space, and the locality of the wave-equations and the inner product for general $\{\mathscr{D}, Q\}$ is discussed in detail. In particular, it is shown that a local inner-product is possible only for particles of non-zero mass and half-integer spin. The spin-statistics theorem for free quantized fields is established for general $\{\mathscr{D}, Q\}$ and the contrast between charge-conjugation in first- and second-quantization is exhibited explicitly.

Chapter V. Covariances in Momentum Space

17. Transformation Law and Inner Product

Following conventional practice [*1, 2, 3, 4*] we define a Poincaré covariant wave-function to be one which transforms according to

$$\big(U(a, \Lambda)\,\omega\big)\,(p) = e^{ip\cdot a}\,\mathscr{D}(\Lambda)\,\psi(\Lambda^{-1}p)\,, \tag{17.1}$$

where p is any vector on a single orbit $p^2 = \text{constant}$ ($p_0 \gtrless 0$ for $p^2 \geq 0$) in Minkowski-space, (a, Λ) is an element of the proper inhomogeneous Lorentz, or Poincaré, group

$\mathscr{P}_+^\uparrow$ and $\mathscr{D}(\Lambda)$ is any (continuous, bounded) representation of the homogeneous part $\mathscr{L}_+^\uparrow$ of $\mathscr{P}_+^\uparrow$. However, since two-valued representations of $\mathscr{P}_+^\uparrow$ are allowed physically, and these are true representations of the covering group $A_4 \wedge SL(2, C)$ of $\mathscr{P}_+^\uparrow$ where A_4 is the translation subgroup, and $\wedge$ denotes semi-direct product, it would be more exact to write (17.1) in the form

$$\big(U(a, s)\big)(p) = e^{ip\cdot a}\, \mathscr{D}(s)\, \psi\left(\Lambda^{-1}(s)\, p\right), \tag{17.2}$$

where $s \in SL(2, C)$ and $\Lambda(s)$ is the representative of s in $\mathscr{L}_+^\uparrow$. This is actually also the more convenient form to use and so it is the one which we shall adopt. The two-valued relationship between $s \in SL(2, C)$ and $\Lambda(s) \in \mathscr{L}_+^\uparrow$ is given by the well-known [7, 8] formula

$$s\, \sigma{\cdot}q\, s^\dagger = \sigma \cdot \Lambda(s) q, \qquad \sigma \cdot q = q_0 + \boldsymbol{\sigma} \cdot \boldsymbol{q}, \tag{17.3}$$

where q is any four-vector, σ are the Pauli matrices, and dagger denotes adjoint.
For the representation of $\mathscr{P}_+^\uparrow$ to be unitary it is not necessary that $\mathscr{D}(s)$ be unitary. It suffices that it be unitary when restricted to the little group $K \subset SL(2, C)$ of any fixed vector $\mathring{p}$ in the orbit of p, because in that case we have the relation

$$\mathscr{D}^\dagger(ks)\, \mathscr{D}(ks) = \mathscr{D}^\dagger(s)\, \mathscr{D}(s), \qquad k \in K. \tag{17.4}$$

This relation shows that $\mathscr{D}^\dagger(s)\, \mathscr{D}(s)$ depends only on p, and hence it allows the con struction of the positive-definite inner-product

$$(\psi_1, \psi_2) = \int d\mu(p)\, \psi_1{}^\dagger(p)\, \mathscr{D}^\dagger(s)\, \mathscr{D}(s)\, \psi_2(p), \tag{17.5}$$

where s is any element[1]) of $SL(2, C)$ such that $p = \Lambda^{-1}(s)\, \mathring{p}$ and $d\mu(p)$ is the invariant measure on the orbit $\left(\text{e.g. } d\mu(p) = d^3p/\omega \text{ for } p^2 \geqq 0,\ |p_0| = \omega\right)$. It is easy to verify that this positive-definite inner-product is invariant under the transformation (17.2).
Equations (17.1)—(17.5) are the same as those obtained from the general theory of induced representations in Part I and from the exhaustivity results obtained there it follows that all the nontrivial representations or $\mathscr{P}_+^\uparrow$ belonging to a given orbit can be expressed in the above form, which incidentally, is completely determined by the choice of $\mathscr{D}(s)$. However in this paper we shall confine ourselves to the orbits for which $p^2 \geqq 0$ and to finite-dimensional $\mathscr{D}(s)$. This will include *all* the representations on the orbits $p^2 > 0$ and those on the orbits $p^2 = 0$ which are induced by the (non-faithful) one-dimensional representations of the little group $K = E(2)$ for that case. These two classes of representations correspond to all the physical states which have so far been observed experimentally.
The fundamental property of finite-dimensional representations of $SL(2, C)$ which we shall use throughout is that (up to a similarity transformation) they can all be constructed from symmetrized direct products of the two fundamental representations s and $s^{\dagger-1}$, where dagger denotes adjoint [7]. Since the operations of taking the direct product and symmetrizing are invariant with respect to the adjoint we then have the relation

$$\mathscr{D}^\dagger(s) = \mathscr{D}(s^\dagger). \tag{17.6}$$

This relation, which is characteristic of finite-dimensional representations, will be used extensively in the sequel. For the moment we merely note that it can be used to reduce the inner-product (17.5) to

$$(\psi_1, \psi_2) = \int d\mu(p)\, \psi_1{}^\dagger(p)\, \mathscr{D}(\mathrm{s}^\dagger \mathrm{s})\, \psi_2(p), \qquad p = \Lambda^{-1}(\mathrm{s})\, \mathring{p}. \tag{17.7}$$

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[1]) Note that s is not necessarily a „standard boost", as defined in the last section of Part I.

Since the two fundamental representations s and $s^{\dagger-1}$ are not equivalent, it is clear that the representations $\mathscr{D}(s)$ and $\mathscr{D}^{\dagger-1}(s)$ are not equivalent in general. However, they may be equivalent, in which case we have

$$\mathscr{D}^{\dagger-1}(s) = \xi\mathscr{D}(s)\,\xi^{-1} \tag{17.8}$$

where ξ is a unitary matrix. It is then trivial to show that ξ^2 commutes with $\mathscr{D}(s)$, $\mathscr{D}^{\dagger-1}(s)$ and, of course, ξ itself. It follows that $\xi^2 = \alpha^2$, where α is a non-singular matrix which is a multiple of the identity on each irreducible subspace of the set of matrices $\{\mathscr{D}(s), \mathscr{D}^{\dagger-1}(s), \xi\}$ and hence that if we define η to be the matrix $\eta = \xi\alpha^{-1}$ we have

$$\mathscr{D}^{\dagger}(s)\,\eta\mathscr{D}(s) = \eta\,, \qquad \eta^2 = 1\,, \quad \eta^{\dagger} = \eta\,. \tag{17.9}$$

A representation satisfying (17.9) is said to be *pseudo-unitary* with pseudo-unitary metric η. Thus $\mathscr{D}(s)$ and $\mathscr{D}^{\dagger-1}(s)$ are equivalent if, and only if, $\mathscr{D}(s)$ is pseudo-unitary. We shall see in the next section that for reasons connected with the linear implementation of space-reversal (parity) a pseudo-unitary representation $\mathscr{D}(s)$ of $SL(2, C)$ is used in most cases of physical interest. Simple examples of such representations are the vector representation, for which η is the metric tensor, and the Dirac representation, for which η is the Dirac matrix β. Note that the pseudo-unitary metric is always indefinite, since otherwise $\mathscr{D}(s)$ would be equivalent to a unitary representation, in contradiction to the fact that $SL(2, C)$ has no non-trivial finite-dimensional unitary representations.

One of the most important questions concerning the representation (17.2) of the Poincaré group is its irreducibility. We shall see that this question is also of great physical interest because the subsidiary conditions which guarantee irreducibility are precisely the covariant wave-equations of relativistic physics. However before proceeding to discuss the question of irreducibility, we wish to amplify the above remarks on pseudo-unitary representations by extending the transformation law (17.2) to include space-reversal.

18. Inclusion of Space-Reversal (Parity)

To extend the Poincaré transformation (17.2) to include space-reversal we first recall the group relation between the space-reversal (parity) operator Π and the proper Poincaré operators $U(a, s)$, namely,

$$\Pi^{-1}U(a, s)\,\Pi = U(\hat{a}, \hat{s}) \tag{18.1}$$

where $\hat{a} = (a_0, -\boldsymbol{a})$ is the parity transform of $a = (a_0, \boldsymbol{a})$ and $\hat{s}$ is the parity transform of s. It is clear that since $\hat{a} = ga$ where g is the Minkowskian metric, we have

$$\hat{\Lambda}(s) = g\Lambda\,(s)\,g^{-1} \equiv \tilde{\Lambda}^{-1}(s)\,, \tag{18.2}$$

where tilde denotes transpose, but to determine $\hat{s}$ is slightly more complicated, and we proceed as follows: Taking the parity transform and the inverse of (17.3) we obtain

$$\hat{s}(\sigma\cdot\hat{a})\,\hat{s}^{\dagger} = (\sigma\cdot\hat{a}') \quad \text{and} \quad s^{\dagger-1}(\sigma\cdot a)^{-1}\,s^{-1} = (\sigma\cdot a')^{-1} \tag{18.3}$$

respectively, and hence, since

$$(\sigma\cdot\hat{a}) = (\sigma\cdot a)^{-1}, \tag{18.4}$$

we have

$$\hat{s} = s^{\dagger-1}, \tag{18.5}$$

up to a constant that is conventionally chosen to be positive. Note that since $\Lambda(s)$ is real (18.2) agrees with (18.5).

For the parity-operator Π we now write

$$(\Pi\psi)(p) = M\psi(\hat{p}), \tag{18.6}$$

where $\hat{p}$ is the parity transform of p, which is assumed to be

$$\hat{p} = (\omega, -\boldsymbol{p}) \tag{18.7}$$

and M is a matrix to be determined.

Note that once we have made the assumption (18.7) the operator Π must be unitary (as opposed to anti-unitary) because then the quantity $(a \cdot p)$ in (17.2) is a scalar with respect to Π and so the i in the exponential must retain its sign under Π. It can then also be shown that the phase of Π can be chosen so that

$$\Pi^2 = 1. \tag{18.8}$$

To determine the matrix M we use the compatibility of equations (18.6), (18.1) and (17.2). It is easy to see that a necessary and sufficient condition for compatibility is

$$M\mathscr{D}(s) = \mathscr{D}(\hat{s})\, M. \tag{18.9}$$

Using (17.6 and 18.5) this equation reduces to

$$M\mathscr{D}(s) = \mathscr{D}^{\dagger-1}(s)\, M. \tag{18.10}$$

But this is just the condition (17.9) for the pseudo-unitarity of $\mathscr{D}(s)$ with $M = \eta$. Hence we have result:
A necessary and sufficient condition for the linear implementation of parity is that the representation $\mathscr{D}(s)$ of $SL(2, C)$ be pseudo-unitary and that the transformation matrix M be the pseudo-unitary metric.
Thus the extension of the Poincaré transformation (17.2) to include parity is

$$(\Pi\psi)(p) = \eta\psi(\hat{p}), \qquad \hat{p} = (\omega, -\boldsymbol{p}), \tag{18.11}$$

where $\mathscr{D}(s)$ is pseudo-unitary and η is the pseudo-unitary metric.

Chapter VI. Massive Case in Momentum Space

19. Massive Inner Product

In this and the next two sections we treat the massive case ($p^2 > 0$) separately. One of the characteristic features of the massive case, which we shall discuss in this section, is that, independently of the choice of $\mathscr{D}(s)$, the inner-product

$$(\psi_1, \psi_2) = \int d\mu(p)\, \psi_1^\dagger(p)\, \mathscr{D}^\dagger(\mathrm{s})\, \mathscr{D}(\mathrm{s})\, \psi_2(p), \qquad p = \mathscr{D}^{-1}(\mathrm{s})\, \mathring{p}, \tag{19.1}$$

which is valid for all orbits, reduces to

$$(\psi_1, \psi_2) = \int \frac{d^3p}{\omega}\, \psi_1^\dagger(p)\, \mathscr{D}^{-1}\left(\frac{\sigma \cdot p}{m}\right) \psi_2(p), \tag{19.2}$$

for the massive case in general, and to

$$(\psi_1, \psi_2) = \int \frac{d^3p}{\omega}\, \psi_1^\dagger(p)\; \eta\psi_2(p), \tag{19.3}$$

when the massive wave-functions $\psi(p)$ are eigenstates of parity (as defined below). Note that $\mathscr{D}(\sigma \cdot p/m)$ makes sense since for $p^2 = m^2$ the matrix $\sigma \cdot p/m$ is unimodular. Note also that (19.2 and 19.3) exhibit explicitly the fact that the kernel $\mathscr{D}^+(\mathrm{s})\,\mathscr{D}(\mathrm{s})$ actually depends on s only through p.

To establish (19.2 and 19.3) the crucial property of the massive orbits which we shall use is that the fixed vector $\mathring{p}$ can be chosen to be $\mathring{p} = (m, o, o, o)$. To establish (19.2) we then set $q = p$ in (17.3) and bring s and $s^\dagger$ to the right hand side to obtain

$$\sigma \cdot p = \mathrm{s}^{-1}\big(\sigma \cdot \Lambda(\mathrm{s})\, p\big)\mathrm{s}^{\dagger-1} = \mathrm{s}^{-1}(\sigma \cdot \mathring{p})\, \mathrm{s}^{\dagger-1} = m\mathrm{s}^{-1}\mathrm{s}^{\dagger-1} = m(\mathrm{s}^\dagger\mathrm{s})^{-1}. \tag{19.4}$$

Using equation (17.6) to raise this equation to the level of the representation $\mathscr{D}(s)$ we obtain

$$\mathscr{D}(\mathrm{s})\,\mathscr{D}^\dagger(\mathrm{s}) = \mathscr{D}(\mathrm{s}^\dagger\mathrm{s}) = \mathscr{D}^{-1}\big((\mathrm{s}^\dagger\mathrm{s})^{-1}\big) = \mathscr{D}^{-1}\left(\frac{\sigma \cdot p}{m}\right), \tag{19.5}$$

which establishes (19.2) as required.

There is also a rather elegant way to prove that the inner-product (19.2) is invariant and positive, independently of (19.1). For by inserting the transformation (17.2) in (19.2) and using the invariance of the measure we see that a necessary and sufficient condition for the invariance of (19.2) is

$$\mathscr{D}^\dagger(s)\,\mathscr{D}^{-1}\left(\frac{\sigma \cdot \Lambda(s)\, p}{m}\right)\mathscr{D}(s) = \mathscr{D}^{-1}\left(\frac{\sigma \cdot p}{m}\right), \tag{19.6}$$

Writing this equation in the form

$$\mathscr{D}(s)\,\mathscr{D}\left(\frac{\sigma \cdot p}{m}\right)\mathscr{D}^+(s) = \mathscr{D}\left(\frac{\sigma \cdot \Lambda(s)p}{m}\right), \tag{19.7}$$

we see that it is just the relation (17.3) between $SL(2, C)$ and $\mathscr{L}_+^{\uparrow}$ raised to the level of the representation $\mathscr{D}(s)$. Since the two levels are equivalent on account of (17.6) we see that (19.6) is satisfied. Finally, by choosing $\Lambda(s)\; p = \Lambda(s)\; p = \mathring{p}$ in (19.6) we obtain

$$\mathscr{D}^\dagger(s)\,\mathscr{D}(s) = \mathscr{D}^{-1}\left(\frac{\sigma \cdot p}{m}\right), \tag{19.8}$$

which shows that $\mathscr{D}^{-1}(\sigma \cdot p/m)$ is positive.

We turn now to the proof of (19.3) in the case that the functions $\psi(p)$ are eigenstates of parity. To explain what we mean by eigenstates of parity, we let the fixed vector $\mathring{p}$ be (m, o, o, o) as before. Then from the parity transformation (18.11) we have

$$(\Pi\psi)\,(\mathring{p}) = \eta\psi(\hat{\mathring{p}}) = \eta\psi(\mathring{p}). \tag{19.9}$$

We therefore that the functions $\psi(p)$ are eigenfunctions of parity (with eigenvalue plus one for definiteness) if

$$\eta\psi(\mathring{p}) = \psi(\mathring{p}), \tag{19.10}$$

for all $\psi(p)$.

Let us now define the quantity

$$\varphi(\mathring{p}) \equiv \big(U(e, \mathrm{s})\,\psi\big)(\mathring{p}) = \mathscr{D}(\mathrm{s})\,\psi(p), \qquad p = \Lambda^{-1}(\mathrm{s})\,\mathring{p}. \tag{19.11}$$

Then (19.10) applies in particular to $\varphi(\mathring{p})$ and we have

$$\eta\varphi(\mathring{p}) = \varphi(\mathring{p}). \tag{19.12}$$

It follows that

$$\mathscr{D}(s)\,\psi(p) = \eta\mathscr{D}(s)\,\psi(p), \tag{19.13}$$

and hence that for the kernel $\mathscr{D}^\dagger(s)\,\mathscr{D}(s)$ in (19.1) we can make the substitution

$$\mathscr{D}^\dagger(s)\,\mathscr{D}(s) \to \mathscr{D}^\dagger(s)\,\eta\mathscr{D}(s) = \eta, \tag{19.14}$$

as required. Note that the inner-product (19.3) is positive definite in spite of the indefiniteness of η, since it is equal to the manifestly positive definite quantity (19.1). This is because of the subsidiary condition (19.13).

20. Irreducibility Condition and General Covariant Wave-Equation

We turn now to the question of the irreducibility of the representation of the Poincaré group $\mathscr{P}_+^\uparrow$ carried by the wave-function $\psi(p)$ in (17.2). As mentioned before, this question is of great physical interest because, as we shall see, the subsidiary conditions which guarantee irreducibility are precisely the covariant wave-equations of relativistic physics.

The first condition for the irreducibility of the representation $U(a, s)$ of $\mathscr{P}_+^\uparrow$ is, of course, the condition that we should be on a fixed orbit in momentum-space,

$$p^2 = m^2, \qquad \text{sign}\, p_0 \quad \text{definite}. \tag{20.1}$$

This condition we shall assume throughout and shall refer to it as the *mass* condition.

The more interesting condition is the *spin* condition. The starting point for this condition is the well-known result (see Part I of this paper for example) that the representation $U(a, s)$ will be irreducible on the single orbit (20.1) if, and only if, the 'rest-frame states' $\psi(\mathring{p})$ carry a single spin. That is to say, $U(a, s)$ will be irreducible if, and only if, all the components of $\psi(\mathring{p})$ vanish except those belonging to a single irreducible representation $D^j(u)$ of the little group $K = SU(2)$ of $\mathring{p}$ in the decomposition of $\mathscr{D}(s)$ with respect to the little group. To obtain a formal equation which expresses this condition we let Q denote the projection operator for the representation $D^j(u)$ in the decomposition of $\mathscr{D}(s)$. Then the condition is clearly equivalent to

$$Q\psi(\mathring{p}) = \psi(\mathring{p}). \tag{20.2}$$

In practice, for reasons that will be clear later, the representation $D^j(u)$ of the little group is chosen to be the highest spin representation of $SU(2)$ occurring in the decomposition of $\mathscr{D}(s)$. For example, for the vector representation of $SL(2, C)$, Q is taken to be the projector onto the threevector subspace, i.e.

$$Q = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \frac{1}{2}\,(1 - g), \tag{20.3}$$

where g is the metric tensor. If, because of parity restrictions, the highest spin representation is not unique, then Q is taken to project onto a combination of $D^j(u)$'s with definite parity. For example, for the Dirac representation of $SL(2, C)$, which contains $D^{1/2}(u)$ twice, we take

$$Q = \frac{1}{2}\,(I + \beta), \tag{20.4}$$

where β is the Dirac β-matrix. In the two examples just considered we have the relation

$$Q = \frac{1}{2}(I + \eta) \tag{20.5}$$

where η is the parity operator or pseudo-unitary metric, and hence the irreducibility condition (20.2) coincides with the positive parity condition (19.12). This is because in these two cases the eigenspaces of $(1 + \eta)/2$ are already irreducible with respect to the little group $SU(2)$ and so require no further reduction. In general, however, as we shall see later, if there is a positive parity condition, it is *contained* in (20.2) rather than equal to it. Thus in general the projection $(1 + \eta)/2$ contains the projection Q, a result that is expressed by the equation

$$Q = \eta Q = Q\eta, \tag{20.6}$$

which will be useful in the sequel.

We now wish to free the condition (20.2) from the dependence on $\mathring{p}$. For this purpose we take the quantity

$$\varphi(\mathring{p}) = \big(U(e, \mathrm{s})\,\psi\big)(\mathring{p}),$$

as in (19.11) and note that in particular this quantity satisfies (20.2), i.e.

$$Q\varphi(\mathring{p}) = \varphi(\mathring{p}). \tag{20.7}$$

Combining equations (20.7) and (19.11) we obtain a condition which is equivalent to (20.2) but is valid for all p, namely

$$Q(p)\psi(p) = \psi(p) \tag{20.8}$$

where

$$Q(p) = \mathscr{D}^{-1}(\mathrm{s})\,Q\mathscr{D}(\mathrm{s}), \qquad p = \Lambda^{-1}(\mathrm{s})\,\mathring{p}. \tag{20.9}$$

For example, in the two special cases just discussed, we obtain for $Q(p)\,\psi(p) = \psi(p)$,

$$\frac{1}{2}\left(I + \frac{\gamma \cdot p}{m}\right)\psi(p) = \psi(p), \qquad (g_\nu{}^\mu - p_\nu p^\mu/m^2)\,\psi_\mu(p) = \psi_\nu(p), \tag{20.10}$$

respectively, and these equations are clearly equivalent to the Dirac and Proca equations

$$(\gamma \cdot p - m)\,\psi(p) = 0, \qquad p^\mu\psi_\mu(p) = 0. \tag{20.11}$$

In writing down (20.9) we have anticipated the fact that $Q(p)$ depends only on p. To show this we note that on account of the $SU(2)$ invariance of Q we have

$$\mathscr{D}^{-1}(ks)\,Q\mathscr{D}(ks) = \mathscr{D}^{-1}(s)\,\mathscr{D}^{-1}(k)\,Q\mathscr{D}(k)\,\mathscr{D}(s) = \mathscr{D}^{-1}(s)\,Q\mathscr{D}(s), \qquad k \in K = SU(2), \tag{20.12}$$

which shows that the quantity $\mathscr{D}^{-1}(s)\,Q\mathscr{D}(s)$ is in one-to-one correspondence with the cosets $SL(2, C)/SU(2)$ in $SL(2, C)$. On the other hand from the definition of the little group $SU(2)$, the cosets, in turn, are in one-to-one correspondence with the points p in the orbit. Hence the quantity $\mathscr{D}^{-1}(\mathrm{s})\,Q\mathscr{D}(\mathrm{s})$ is a function of p only.

The next step is to show that the irreducibility condition (20.8) is a covariant wave-equation, that is to say that, on making the identification $p_\mu = i\partial_\mu$ it becomes a covariant differential equation of finite order. For this, it clearly suffices to show that $Q(p)$ is a scalar and that it is a polynomial in $p = (\omega, \boldsymbol{p})$, as is manifestly the case in the two examples in (20.10). In general the scalarity of $Q(p)$ is expressed by the equation

$$\mathscr{D}^{-1}(s)\, Q(p)\, \mathscr{D}(s) = Q\big(\Lambda^{-1}(s)\, p\big), \tag{20.13}$$

and this equation follows directly from the definition (20.9) of $Q(p)$. The polynomiality of $Q(p)$ then follows from the scalarity of $Q(p)$ and the finite-dimensionality of $\mathscr{D}(s)$, because in a finite-dimensional space the rank of the possible tensor coefficients for the powers of p in $Q(p)$ is bounded.

We now make the claim that (20.8) is actually the most general covariant wave-equation corresponding to a given (nozero) mass and spin, in the sense that it contains any other such equation [*3*, *4*, *9*] as a special case. In fact, this is clear from the construction of (20.8) in which we used nothing but the covariance of the transformation law and the irreducibility. However to verify the statement explicitly and to show how the various special wave-equations can be identified as special cases of (20.8) in practice, we identify in the next section the Bargmann-Wigner, Rarita-Schwinger, Fierz, and Joos-Weinberg equations.

Finally we note that since (20.8) contains all the wave-equations corresponding to a given mass and spin as special cases, and since (20.8) and its inner-product (19.2, 19.3) are determined completely by the pair of quantities $\mathscr{D}(s)$ and Q, where $\mathscr{D}(s)$ is the representation of $SL(2, C)$ according to which $\psi(p)$ transforms, and Q is the projection on to the required spin representation of the little group, the pair of quantities $\{\mathscr{D}(s), Q\}$ provide a complete systematic characterization of all the wave-equations corresponding to a given (nonzero) mass and spin. That is to say, to each pair $\{\mathscr{D}(s), Q\}$ corresponds a unique wave-equation, and conversely.

In the special case that $\mathscr{D}(s)$ is irreducible, it contains each irreducible representation $D^j(u)$ of $SU(2)$ at most once, and so in that case the pair $\{\mathscr{D}(s), j\}$ suffice for the characterization. Similarly, if $\mathscr{D}(s)$ is irreducible up to parity, then $\{\mathscr{D}(s), j, \varepsilon\}$ where ε is the sign of the parity, suffices. Finally, if, in addition, and as often happens in practice, it is understood that j is the highest spin occurring in $\mathscr{D}(s)$ and ε is positive, then $\mathscr{D}(s)$ alone suffices.

21. Identification of Conventional Wave-Equations

In table 2 we display the Dirac, Proca, Symmetric-Tensor, Rarita-Schwinger, Bargmann-Wigner, Fierz, and Joos-Weinberg equations [*3*]. Before identifying them as special cases of the general wave-equation (20.8) we recall a few properties (7) of the finite-dimensional representations of $SL(2, C)$. These are that they are labelled according to the notation $\mathscr{D}^{(m,n)}(s)$, where m and n are the spin labels of the corresponding unitary irreducible representations of $SU(2) \otimes SU(2)$, that $\mathscr{D}^{(m,n)}(s)$ contains each irreducible representation $D^{(j)}(u)$ of $SU(2) \subset SL(2, C)$ once for $j = m + n,\ m + n - 1,\ m + n - 2, \dots |m - n|$, and that the parity transform $\mathscr{D}^{\dagger -1(m,n)}(s)$ of $\mathscr{D}^{(m,n)}(s)$ is $\mathscr{D}^{(n,m)}(s)$.

We now verify that all the conventional wave-equations in table 2 are special cases of the general wave-equation (20.8) by displaying the wave-operator $Q(p)$ in each case and then showing how the conventional form of the wave-equation is derived from (20.8) in the form

$$\psi(p) = Q(p)\, \psi(p). \tag{21.1}$$

The $Q(p)$ are displayed in the table for all cases. In the Dirac and Proca cases, the conventional form of the equations has already been derived in (20.10) and (20.11), but as they are prototypes for more general cases, it is worth while to note here that they follow from (21.1) by multiplying to the left by the operators $(\gamma \cdot p - m)$ and p^μ, respectively,

$$(\gamma \cdot p - m)\,\psi(p) = \frac{1}{2m}(\gamma \cdot p - m)(\gamma \cdot p + m)\,\psi(p) = \frac{1}{2m}(p^2 - m^2)\,\psi(p) = 0, \tag{21.2a}$$

$$p^\mu \psi_\mu(p) = p^\mu \left(g_\mu{}^\nu - \frac{1}{m^2} p_\mu p^\nu\right)\psi_\nu(p) = \left(1 - \frac{p^2}{m^2}\right) p^\nu \psi_\nu(p) = 0. \tag{21.2b}$$

The $Q(p)$ for the next three cases (symmetric tensor, Rarita-Schwinger and Bargmann-Wigner) are simply direct products of the Dirac and Proca $Q(p)$'s, and the corresponding coventional wave-equations are obtained by multiplying (21.1) with the displayed $Q(p)$'s to the left by $(\gamma \cdot p - m)$ and p^μ as before. There is, however, one new feature to note, namely that each factor $(\gamma \cdot p - m)$ and p^μ can be multiplied *separately*, so that we get a set of equations rather than a single equation. For example,

$$\begin{aligned}\left(\overset{(1)}{\gamma} \cdot p - m\right)\psi(p) &= (2m)^{-2j}\left(\overset{(1)}{\gamma} \cdot p - m\right)\left(\overset{(1)}{\gamma} \cdot p + m\right)\prod_{r=2}^{2j}\left(\overset{(r)}{\gamma} \cdot p + m\right)\psi(p)\\ &= (2m)^{-2j}(p^2 - m^2)\prod_{r=2}^{2j}\left(\overset{(r)}{\gamma} \cdot p + m\right)\psi(p) = 0,\end{aligned} \tag{21.3}$$

in the Bargmann-Wigner case. More generally, whenever we have

$$\mathscr{D} = \prod_r \otimes \mathscr{D}_r, \qquad Q = \prod_r \otimes Q_r, \tag{21.4}$$

then

$$\big(1 - Q_r(p)\big)\,\psi(p) = \big(1 - Q_r(p)\big)\Big(\prod_s \otimes Q_s(p)\,\psi(p)\Big) = 0, \tag{21.5}$$

and hence we have

$$Q_r(p)\,\psi(p) = \psi(p), \tag{21.6}$$

for each individual $Q_r(p)$.

By the "restricted" Fierz case we mean that we have included only those Fierz representations which do not overlap with the other representations we have discussed, namely only the representations $\mathscr{D}^{(m,n)}(s)$. These do not allow the linear implementation of parity unless $m = n$. For definiteness, we have chosen $m \geqq n$, and it is to be noted that the summation extends only over the first n lower indices. The conventional Fierz equations are obtained by multiplying (21.1) to the left by $(\sigma \cdot p/m)^{b_r}_{\dot a_r}$ since

$$(\sigma \cdot p)^b_{\dot a}\left(\delta^{b'}_b \delta^{\dot a}_{\dot a'} - \left(\frac{\sigma \cdot p}{m}\right)^{\dot a}_b \left(\frac{\sigma \cdot p}{m}\right)^{b'}_{\dot a'}\right) = (\sigma \cdot p)^{b'}_{\dot a'} - \frac{p^2}{m}\left(\frac{\sigma \cdot p}{m}\right)^{b'}_{\dot a'} = 0. \tag{21.7}$$

Finally the Joos-Weinberg case is particularly interesting and simple because the projection operator Q is just $(1 + \eta)/2$ where η is the parity operator, or pseudo-unitary metric, in that case and hence we have

$$\begin{aligned}\psi(p) &= Q(p)\,\psi(p) = \frac{1}{2}\mathscr{D}^{-1}(\mathrm{s})\,[1 + \eta]\,\mathscr{D}(\mathrm{s})\,\psi(p) = \frac{1}{2}\,[1 + \eta\mathscr{D}^\dagger(\mathrm{s})\,\mathscr{D}(\mathrm{s})]\,\psi(p)\\ &= \frac{1}{2}\left[1 + \eta\mathscr{D}\left(\frac{\sigma \cdot p}{m}\right)\right]\psi(p) = \frac{1}{2}\begin{bmatrix} 1 & \mathscr{D}^{(oj)}\left(\frac{\sigma \cdot p}{m}\right) \\ \mathscr{D}^{(jo)}\left(\frac{\sigma \cdot p}{m}\right) & 1\end{bmatrix}\psi(p),\end{aligned} \tag{21.8}$$

which is just the Weinberg [3] equation in closed form.

Note that in all cases the spin is the highest spin available in $\mathscr{D}(s)$. This is not accidental, for were it not the case the wave-functions would simply re-arrange themselves so that it was. For example, if we used the vector field $\psi_\mu(p)$ to describe spin zero, $Q(p)$ would be the *complement* $p_\mu p^\nu m^2$ of the Proca operator, and hence the wave-equation (21.1) would be

$$\psi_\mu(p) = p_\mu p^\nu \psi_\nu(p)/m^2, \tag{21.9}$$

and this equation effectively reduces the vector field $\psi_\mu(p)$ to the scalar field $p^\nu\psi_\nu(p)$. Note also the frequent appearance of the parity projections $(1+\beta)/2$ and $(1-g)/2$ in the table. As discussed earlier their appearance stems from the fact in order to implement parity linearly, we must use representations of the form $\mathscr{D}^{(mn)}(s) \oplus \mathscr{D}^{(nm)}(s)$ and then to obtain definite parity we must put in a parity projection.

Finally we should perhaps emphasize that in all cases we have assumed the mass condition $p^2 = m^2$ as a separate condition, that is to say, in all cases we are on the mass-shell from the beginning. In particular the wave-operator $Q(p)$ depends on the mass in a very definite way. In fact, since the most general solution of the equation $p = \Lambda^{-1}(\mathrm{s})\,\mathring{p}$, where $\mathring{p} = (m, o, o, o)$, is

$$\mathrm{s} = e^{\varkappa(\sigma\cdot\boldsymbol{p})}u, \qquad \varkappa = \frac{-1}{2|\boldsymbol{p}|}\sinh^{-1}\left(\frac{|\boldsymbol{p}|}{m}\right), \qquad u \in SU(2), \tag{21.10}$$

$Q(p)$ takes the explicit form

$$Q(p) = e^{-\varkappa(\boldsymbol{K}\cdot\boldsymbol{p})}\,Q e^{\varkappa(\boldsymbol{K}\cdot\boldsymbol{p})}, \tag{21.11}$$

where the $\boldsymbol{K}$ are the generators of the accelerations in $SL(2, C)$. It may happen, as in the Dirac case, that the mass condition is an automatic consequence of the wave-equation with the operator (21.11), in which case we do not have to impose it as a separate condition. Or it may happen, as for the Proca equation $p_\nu p^\mu \psi_\mu = m^2\psi_\nu$, that the mass condition is not an automatic consequence of the wave-equation with $Q(p)$ as in (21.11), but that it may be incorporated into the wave-equation, e.g. by writing the Proca equation in the form

$$(m^2 g_\nu{}^\mu + p_\nu p_\mu)\,\psi_\mu(p) = p^2\psi_\nu(p). \tag{21.12}$$

Or it may happen that the mass condition can not be incorporated in the wave-equation in any reasonable way, as is known to be the case for the Weinberg equation for $J \geqq 1$. An interesting problem is to investigate the exact conditions under which the mass condition can be derived from the wave equation, and more generally, to investigate what happens off the mass-shell, but we shall not investigate this problem here.

Table 2

Name	Spin	Wave-equation	$\mathscr{D}$	Q	$Q(p)$
Vector	1	$p^{\mu}\psi_{\mu}(p) = 0$	$\mathscr{D}^{(\frac{1}{2}\frac{1}{2})}$	$\frac{1}{2}(1-g)$	$g_{\nu}{}^{\mu} - \frac{p^{\mu}p_{\nu}}{m^2}$
Symmetric Tensor	$2j$	$p^{\mu_1}\psi_{\mu_1\mu_2\ldots\mu_{2j}}(p) = 0$	$\mathscr{D}^{(jj)}$	$\prod_{r=1}^{2j} \frac{1}{2}(1-g_r)$	$\prod_{r=1}^{2j}\left(g_{\nu_r}^{\mu_r} - \frac{p^{\mu_r}p_{\nu_r}}{m^2}\right)$
Dirac	$\frac{1}{2}$	$(\gamma \cdot p - m)\,\psi(p) = 0$	$\Delta = \mathscr{D}^{(\frac{1}{2}0)} \oplus \mathscr{D}^{(0\frac{1}{2})}$	$\frac{1}{2}(1+\beta)$	$\frac{1}{2m}(\gamma \cdot p + m)$
Rarita-Schwinger	$2j + \frac{1}{2}$	$(\gamma \cdot p - m)\,\psi_{\mu_1\ldots\mu_{2j}} = 0$ $p^{\mu_1}\psi_{\mu_1\ldots\mu_{2j}} = 0$	$\Delta \,\overline{\oplus}\, \mathscr{D}^{(jj)}$	$\frac{1}{2}(1+\beta)\prod_{r=1}^{2j}\frac{1}{2}(1-g_r)$	$\frac{1}{2m}(\gamma \cdot p + m)\prod_{r=1}^{2j}\left(g_{\nu_r}^{\mu_r} - \frac{p^{\mu_r}p_{\nu_r}}{m^2}\right)$
Bargmann-Wigner	j	$(\gamma \cdot p - m)_{\alpha_1\beta_1}\psi_{\beta_1\ldots\beta_{2j}} = 0$	$\prod_{r=1}^{2j} \oplus \Delta_r$	$\prod_{r=1}^{2j}\frac{1}{2}(1+\beta_r)$	$\prod_{r=1}^{2j}\frac{1}{2m}\left(\overset{(r)}{\gamma} \cdot p + m\right)$
Restricted Fierz	$m + n$ $(m \geqq n)$	$(\sigma \cdot p)_{\dot{a}_1}^{b_1}\,\psi_{b_1\ldots b_{2m}}^{\dot{a}_1\ldots\dot{a}_{2n}} = 0$	$\mathscr{D}^{(m,n)}$	$\prod_{r=1}^{2n} \delta_{b_r}^{b_r'}\,\delta_{\dot{a}_r'}^{\dot{a}_r} - \delta_{b_r}^{\dot{a}_r}\,\delta_{\dot{a}_r'}^{b_r'}$	$\prod_{r=1}^{2n} \delta_{b_r}^{b_r'}\,\delta_{\dot{a}_r'}^{\dot{a}_r} - \left(\frac{\sigma \cdot p}{m}\right)_{b_r}^{\dot{a}_r}\left(\frac{\sigma \cdot p}{m}\right)_{\dot{a}_r'}^{b_r'}$
Joos-Weinberg	j	$(\sigma \cdot p)_{\dot{a}_1}^{b_1} \ldots (\sigma \cdot p)_{\dot{a}_{2j}}^{b_{2j}}\,\psi^{\dot{a}_1\ldots\dot{a}_{2j}} = \psi^{b_1\ldots b_{2j}}$	$\mathscr{D}^{(j0)} \oplus \mathscr{D}^{(0j)}$	$\frac{1}{2}(1+\eta)$	$\frac{1}{2}\left(1 + \eta\mathscr{D}\left(\frac{\sigma \cdot p}{m}\right)\right)$

Two-component indices are denoted by $a, \dot{a}, b, \dot{b}$, Dirac indices by indices $\alpha, \beta = 1, 2, 3, 4$, and Lorentz indices by $\mu = 0, 1, 2, 3$. All spinors and tensors shown are assumed to be totally symmetric, and in the case of tensors also traceless. The matrices g and β are the pseudo-unitary metric and the Dirac β respectively. In the Rarita-Schwinger case, the symbol $\overline{\otimes}$ means that the unwanted $2j - 1$ representations of $SL(2, C)$ have been removed by the p-independent invariant contraction $\gamma^{\mu_1}\psi_{\mu_1\ldots\mu_{2j}} = 0$. In the restricted Fierz case we have included only those Fierz functions which do not overlap with any of the other functions. These belong to the representations $\mathscr{D}^{(mn)}(s)$ of $SL(2, C)$, which are irreducible with respect to $SL(2, C)$, but not with respect to $SU(2)$ unless m or n is zero, and which do not allow the linear implementation of parity unless $m = n$.

Chapter VII. Massless Case in Momentum-Space

22. Unitarity, Irreducibility and Covariant Wave-equations

The results of section 17 for the transformation law and inner-product apply equally well to the massless case. However the results of sections 19 and 20 on the form of the inner product and the covariant wave-equations must be modified to take account of the fact that the little group is the two-dimensional Euclidean group $E(2)$, which unlike $SU(2)$, is neither semi-simple nor compact.
In the massless case, the invariant measure on the orbit is still d^3p/ω, where now $\omega = |\boldsymbol{p}|$, and the fixed vector may be taken to be $\mathring{p} = (1, 0, 0, 1)$. The greatest difference, however, concerns unitary, because, on account of the non-compactness of $E(2)$, the restriction $\mathscr{D}(k)$ of the finite-dimensional representations $\mathscr{D}(s)$ of $SL(2, C)$ to $E(2)$ are not unitary. In fact, they take the form

$$\mathscr{D}(k) = T(\alpha)\, D(\varphi), \qquad k = (\alpha, \varphi) \in \tilde{E}(2) = A_2 \wedge U(1), \tag{22.1}$$

where $T(\alpha)$ is a finite-dimensional representation of the translation subgroup A_2 of $E(2)$, and is unitary only on those states on which it is trivial. It follows that in the massless case a *unitarity* condition

$$T(\alpha)\, \psi(\mathring{p}) = \psi(\mathring{p}), \tag{22.2}$$

must be imposed on the wave-functions. The consequences of this condition are, perhaps, best expressed by means of the following lemma.
Lemma: The unitarity condition (22.2) is equivalent to either of the two wave-equations

$$(\boldsymbol{L} \cdot \boldsymbol{p})\, \psi(p) = \omega(M - N)\, \psi(p), \tag{22.3a}$$

$$i(\boldsymbol{K} \cdot \boldsymbol{p})\, \psi(p) = \omega(M + N)\, \psi(p), \tag{22.3b}$$

for all $\psi(p)$, where $(\boldsymbol{L}, \boldsymbol{K})$ are the conventional generators of $SL(2, C)$ and (M, N) are invariants that to reduce (m, n) on each irreducible part $\mathscr{D}^{(m,n)}(s)$ of $\mathscr{D}(s)$.

Proof: The natural proof the of lemma is via the Lie algebra. Let $\boldsymbol{E} = (E_1, E_2) = (L_1 - K_2, L_2 + K_1)$ be the generators of $T(\alpha)$ and $\boldsymbol{A} = (\boldsymbol{L} + i\boldsymbol{K})/2$ and $\boldsymbol{B} = (\boldsymbol{L} - i\boldsymbol{K})/2$ the generators of the representation of $SU(2) \otimes SU(2)$ associated with $\mathscr{D}(s)$. Then the unitarity condition (22.2) is equivalent to

$$\boldsymbol{E}\psi(\mathring{p}) = 0, \tag{22.4}$$

and since $E_+ = A_+$ and $E_- = B_-$, where $X_\pm = X_1 \pm iX_2$ for $X = A,\ B,\ E$, (22.4) in turn is equivalent to

$$A_+\psi(\mathring{p}) = B_-\psi(\mathring{p}) = 0. \tag{22.5}$$

But equation (22.5) means that $\psi(\mathring{p})$ belongs to the highest weight of $\boldsymbol{A}$ and the lowest weight of B in each irreducible part $\mathscr{D}^{(mn)}(s)$ of $\mathscr{D}(s)$. Hence from the standard results for the representations of $SU(2)$, (22.5) is equivalent to

$$A_3\psi(\mathring{p}) = M\psi(\mathring{p}), \qquad B_3\psi(\mathring{p}) = -N\psi(\mathring{p}), \tag{22.6}$$

and boosting these two equations to general p and taking appropriate linear combinations, we obtain (22.3).

It remains to show that either of the equations (22.3) implies the other. The point is that since either equation holds for *all* $\psi(p)$ and $\boldsymbol{E}$ is a generator of the little group of $\mathring{p}$, when we set $p = \mathring{p}$ in (22.3a) say, we obtain not only

$$L_3\psi(\mathring{p}) = (M - N)\,\psi(\mathring{p}) \tag{22.7}$$

but also

$$L_3\boldsymbol{E}\psi(\mathring{p}) = (M - N)\,\boldsymbol{E}\psi(\mathring{p}), \tag{22.7}$$

and since $[L_3, E_a] = i\varepsilon_{ab}E_b$, $a, b = 1, 2$ these two equations imply (22.4), which, as we have just seen, is equivalent to the unitarity condition. Similarly for (22.3b).
Note that equations (22.3) can also be written in the covariant form

$$W_0\psi(p) = p_0(M - N)\,\psi(p), \qquad V_0\psi(p) = -ip_0(M + N)\,\psi(p), \tag{22.8}$$

where W_μ is the Pauli-Lubanski operator $\varepsilon_{\mu\lambda\sigma\tau}M^{\lambda\sigma}p^\tau/2$ and V_μ is the vector $M_{\mu\nu}p^\nu$ where $M_{\mu\nu} = (\boldsymbol{L}, \boldsymbol{K})$ are the generators of $\mathscr{D}(s)$.
Not also that for each irreducible $\mathscr{D}(s)$ equations (22.3) imply that for each p, all the components of $\psi(p)$ vanish except one, and that this one has helicity $m - n$. Thus in the massless case, when $\mathscr{D}(s)$ is irreducible, the unitarity of $U(a, s)$ implies its irreducibility, and the equations (22.3) are the complete set of covariant wave-equations.

23. Massless Inner-Product

Returning now to general $\mathscr{D}(s)$, we show that equation (22.3b) can be used to reduce the inner-product to the simple form

$$(\psi_1, \psi_2) = \int \frac{d^3p}{\omega}\, \psi_1{}^\dagger(p)\, \omega^{-2(M+N)}\psi_2(p)\,. \tag{23.1}$$

To prove this we boost $\mathring{p}$ to p with the Lorentz transformation

$$\Lambda^{-1}(\mathrm{s}) = R(\boldsymbol{p})\,Z(\chi), \tag{23.2}$$

where $Z(\chi)$ is a Lorentz transformation along the z-axis such that

$$Z(\chi)\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}\omega\\0\\0\\\omega\end{pmatrix}, \qquad \omega = e^\chi, \tag{23.3}$$

and $R(\boldsymbol{p})$ is a rotation such that

$$R(\boldsymbol{p})\begin{pmatrix}\omega\\0\\0\\\omega\end{pmatrix} = \begin{pmatrix}\omega\\p_1\\p_2\\p_3\end{pmatrix}. \tag{23.4}$$

Then we have

$$\mathscr{D}^\dagger(\mathrm{s})\,\mathscr{D}(\mathrm{s}) = \mathscr{D}(\boldsymbol{p})\,\mathscr{D}^{-2}(\chi)\,\mathscr{D}^{-1}(\boldsymbol{p}) = \mathscr{D}(\boldsymbol{p})\,e^{-2i\chi K_3}\,\mathscr{D}^{-1}(\boldsymbol{p}) = e^{-2i\chi(\boldsymbol{K}\cdot\boldsymbol{p})/\omega}, \tag{23.5}$$

and hence from (22.3b)

$$\mathscr{D}^\dagger(\mathrm{s})\,\mathscr{D}(\mathrm{s}) = e^{-2\chi(M+N)} = \omega^{-2(M+N)}, \qquad \text{on } \psi(p)\,. \tag{23.6}$$

Inserting this result in (17.5) we obtain (23.1) as required.

A simple check on the sign of the exponent in (23.5) and (23.6) is to note that if, as we have assumed, $\sigma \cdot \mathring{p} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, then $s(\chi^{-1}) = \begin{pmatrix} \omega^{-1/2} & 0 \\ 0 & \omega^{1/2} \end{pmatrix}$ while $T(\alpha) = \begin{pmatrix} 1 & e^{\alpha} \\ 0 & 1 \end{pmatrix}$ and hence $\psi(\mathring{p}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus $s^{\dagger}(\chi^{-1})\, s(\chi^{-1}) = s^2(\chi^{-1}) = \omega$ on $\psi(p)$, which agrees with (23.6) for $M = N = 1/2$.

24. Identification of the Weyl and Maxwell Equations

In the massive case we saw that the various conventional wave-equations could be identified as special cases of the general wave-equation (20.8) by specifying the pair of quantities. $\{\mathscr{D}, Q\}$. From the lemma of section 22, it follows that in the massless case (for irreducible $\mathscr{D}(s)$ at any rate) we need only specify $\mathscr{D}$ since Q is automatically determined by the unitarity condition. It follows that in the massless case the conventional wave-equations should be special cases of the equations (22.3), determined completely by the choice of representation $\mathscr{D}(s)$. We now verify that this is indeed the case for the Weyl and Maxwell equations.

First, letting $\mathscr{D}(s) = \mathscr{D}^{(1/2\,0)}(s)$, we have $\boldsymbol{L} = \boldsymbol{\sigma}/2$ and so (22.3a) reduces to

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p})\, \psi(p) = \omega \psi(p) \tag{24.1}$$

which is just the Weyl equation. Second, letting $\mathscr{D}(s) = \mathscr{D}^{(10)}(s) \oplus \mathscr{D}^{(01)}(s)$ and $\psi = \varphi \oplus \xi$ we have $i(L_a)_{bc} = \varepsilon_{abc}$, $a, b, c = 1, 2, 3$ on both irreducible parts of $\mathscr{D}(s)$ and hence $(22.3a)$ reduces to

$$\boldsymbol{p} \times \boldsymbol{\varphi}(\boldsymbol{p}) = -i\omega \boldsymbol{\varphi}(p), \qquad \boldsymbol{p} \times \boldsymbol{\xi}(p) = i\omega \boldsymbol{\xi}(p). \tag{24.2}$$

Letting $\boldsymbol{\varphi} = \boldsymbol{H} + i\boldsymbol{E}$, $\boldsymbol{\xi} = \boldsymbol{H} - i\boldsymbol{E}$, and noting that (24.2) implies also $\boldsymbol{p} \cdot \boldsymbol{\varphi} = \boldsymbol{p} \cdot \boldsymbol{\xi} = 0$ we obtain

$$\begin{aligned} &\boldsymbol{p} \times \boldsymbol{E} = \omega \boldsymbol{H}, && \boldsymbol{p} \cdot \boldsymbol{H} = 0, \\ &\boldsymbol{p} \cdot \boldsymbol{E} = 0, && \boldsymbol{p} \times \boldsymbol{H} = -\omega \boldsymbol{E}, \end{aligned} \tag{24.3}$$

which are just Maxwell's equations in vacuo.

25. The Vector-Potential

It might be asked where the vector-potential description of the electromagnetic field enters the present formalism. The answer is that it enters only by relaxing the unitarity condition (22.4). However before relaxing this condition, it is important to note that it cannot be relaxed arbitrarily for the following reason: For any representation $U(a, \Lambda)$ of the Poincaré group, unitary of otherwise, we have

$$\mathscr{D}(k)\, \psi(\mathring{p}) = \big(U(e, k)\, \varphi\big) \big(\Lambda(k)\, \mathring{p}\big) = \big(U(e, k)\, \psi\big)\, (\mathring{p}) = \psi'(\mathring{p}), \tag{25.1}$$

for $k \in E(2)$. This equation implies that $\psi(\mathring{p})$ must belong to an invariant subspace of $E(2)$. It follows that, so long as we demand convariance, we can relax (22.4) only to the extent that $\psi(p)$ should still belong to *some* representation of $E(2)$. Now for a vector field $\psi_\mu(p)$ there are only two proper $E(2)$-invariant subspaces, namely,

$$\left.\begin{aligned} \psi_0(\mathring{p}) - \psi_3(\mathring{p}) &= 0, \\ \psi_a(\mathring{p}) &= 0, \quad a = 1, 2. \end{aligned}\right\} \quad \text{and} \quad \psi_0(\mathring{p}) - \psi_3(\mathring{p}) = 0 \tag{25.2}$$

The first subspace satisfies the unitarity condition (22.4) and corresponds to a zero-helicity representation of $\mathscr{P}_+^\uparrow$. The second subspace is the one which corresponds to the conventional vector-potential. Note that it may be characterized covariantly by the equation

$$p^\mu \psi_\mu(p) = 0, \tag{25.3}$$

and that it contains the helicities ± 1, and 0.

Since the major difference between unitary and non-unitary representations is in the inner-product, let us now consider the inner-product for the field (25.3). Because the transformation law (17.2) for $\psi(p)$ is formally the same for all orbits, the inner-product

$$(\psi, \varphi) = -\int \frac{d^3p}{\omega} \psi^\mu(p)\, \varphi_\mu(p), \tag{25.4}$$

is manifestly invariant for both the massive and massless cases. However in the massive case it is also positive definite, whereas in the massless case we know in advance that this cannot be the case, since otherwise $\psi(p)$ would carry a unitary representation of $P_+^\uparrow$ in contradiction to the relaxation of the unitarity condition (22.4). The question is: how does the positive-definiteness break down? The answer is that whereas in the massive case (25.3) implies that $\psi(p)$ is spacelike and hence that $(\psi, \psi) = 0$ implies $\psi = 0$, in the massless case (25.3) allows $\psi(p)$ to have a component parallel to p and hence $(\psi, \psi) = 0$ implies only $\psi^\mu(p)\, \psi_\mu(p) = 0$ which, in turn, implies only

$$\psi_\mu(p) = p_\mu \psi(p). \tag{25.5}$$

In other words, in the massless case the inner-product is only positive *semi*-definite. In particular, if we define the norm $\|\psi\|$ to be $(\psi, \psi)^{1/2}$ in the usual way, we have

$$\|\psi' - \psi\| = 0 \Rightarrow \psi_\mu{}'(p) = \psi_\mu(p) + p_\mu \psi(p), \tag{25.6}$$

where ψ is a scalar, i.e. if the norm of the difference of two fields is zero, the fields are not necessarily equal, but may differ by a gauge-transformation.

Note that for the vector-potential the positive-definite inner-product (17.5) does not exist, much less equal (25.4), because when the unitarity condition (22.4) is violated, the proof that the kernel $\mathscr{D}^\dagger(s)\, \mathscr{D}(s)$ depends only on p breaks down. Similarly when, as in (25.3), $\psi(p)$ is not an eigenstate of parity, the proof that $\mathscr{D}^\dagger(s)\, \mathscr{D}(s) = \eta$ on $\psi(p)$ breaks down.

Thus from the group-theoretical point of view, the introduction of the vector-potential is at the expense of the definiteness of the inner-product, and also the definiteness of the spin ($s = 0, 1$) and the parity. These defects can be removed by choosing a special gauge such as the radiation gauge $(\psi_0(p) = 0)$, but such a choice of gauge violates the covariant transformation law (17.2).

Chapter VIII. Poincaré Configuration Space

26. Transformation Law, Doubling of the Orbit and Γ_5

The conventional transformation law [2, 12] for wave-functions in configuration space corresponding to (17.2) is

$$\big(U(a, s)\, \psi\big)(x) = \mathscr{D}(s)\, \psi\big(\Lambda^{-1}(s)\,(x - a)\big), \tag{26.1}$$

and it is easy to see that this can be obtained from the conventional transformation law in momentum space (17.2) for $\psi(p)$ by defining $\psi(x)$ to be

$$\psi(x) = \int d\mu(p)\, e^{-ip\cdot x}\psi(p). \tag{26.2}$$

For the 'physical' orbits $d\mu(p) = d^3p/\omega$, $\omega = (\boldsymbol{p}^2 + m^2)^{1/2}$ and so $\psi(x)$ is not the true Fourier transform of $\psi(p)$ except in the nonrelativistic limit $\omega \to m$. However, the orbital condition $p^2 = m^2$ and the irreducibility condition or wave-equation $Q(p)\,\psi(p) = \psi(p)$ clearly transform into the local conditions

$$(\Box + m^2)\,(x) = 0, \qquad \Box = \partial^\mu\partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial x\mu},$$
$$Q(i\partial)\,\psi(x) = \psi(x), \tag{26.3}$$

for $\psi(x)$. The orbital condition $p_0 \gtrless 0$, on the other hand, has no local counterpart. Hence to preserve locality one usually relaxes this condition, and allows both $p_0 > 0$ and $p_0 < 0$ in going to configuration space. That is to say, one usually writes instead of (26.2) the equation

$$\varphi(x) = \int \frac{d^3p}{\omega}\,\{e^{-ip\cdot x}\psi(\boldsymbol{p}) + e^{ip\cdot x}\chi(\boldsymbol{p})\}, \qquad p = (\omega, \boldsymbol{p}), \qquad \omega > 0 \tag{26.4}$$

where $\psi(\boldsymbol{p})$ is, as before, the wave-function on $p_0 > 0$, and $\chi(\boldsymbol{p})$ is a wave-function on $p^2 = m^2$, $p_0 < 0$. (There is no immediate conflict with the positivity of the energy in writing (26.4), since one can identify the positive and negative frequency parts with emission and absorption, respectively, as is normally done for the classical electromagnetic field.) Henceforth p will denote $p = (\omega, \boldsymbol{p})$, where $\omega > 0$.

If we now require that the wave-function $\psi(x)$ satisfy the transformation law (26.1) and the local conditions (26.3) we find that $\psi(\boldsymbol{p})$ behaves as before, but that $\chi(\boldsymbol{p})$ must have the transformation law

$$\big(U(a, s)\big)\,\chi(\boldsymbol{p}) = e^{-ip\cdot a}\mathscr{D}(s)\,\chi\big(\Lambda^{-1}(\mathbf{s})\,\boldsymbol{p}\big), \tag{26.5}$$

and must satisfy the wave-equation

$$Q(-p)\,\chi(\boldsymbol{p}) = \chi(\boldsymbol{p}). \tag{26.6}$$

Thus the locality of the wave-equation $Q(i\partial)\,\varphi(x) = \varphi(x)$ imposes a condition on the choice of the projection operator Q for $\chi(\boldsymbol{p})$ i.e. it must be the same as that for $\psi(\boldsymbol{p})$ with $p \to -p$.

We now establish a lemma which relates the invariant operators $Q(p)$ and $Q(-p)$ algebraically.

Lemma: For any scalar polynomial $Q(p)$ in the space of $\mathscr{D}(s)$, there exists a Lorentz invariant matrix Γ_5 (a generalization of the Dirac γ_5) such that

$$\Gamma_5^{-1}\,Q(p)\,\Gamma_5 = Q(-p). \tag{26.7}$$

Further, if the representation $\mathscr{D}(s)$ is pseudo-unitary, then

$$\Gamma_5^{-1}\,\eta\Gamma_5 = (-1)^{2J}\eta, \tag{26.8}$$

where η is the pseudo-unitary metric and J is the spin-operator.

Proof: Since $\mathscr{D}(s)$ is finite-dimensional and $Q(p)$ is a polynomial in $p = (\omega, \boldsymbol{p})$, the invariance condition

$$\mathscr{D}^{-1}(s)\, Q(p)\, \mathscr{D}(s) = Q\big(\Lambda^{-1}(s)\, p\big), \tag{26.9}$$

of (20.13) can be analytically continued to any complex values of the parameters of $SL(2, C)$. Continuing it to one of the two values corresponding to $\Lambda(s) = -1$ $\big((1, -1)$ or $(-1, 1)$ in $SL(2, C) \otimes SL(2, C)\big)$ and letting Γ_5 be the value of $\mathscr{D}(s)$ at that point, we obtain (26.7). The Lorentz invariance of Γ_5 then follows by comparing (26.7) and (26.9). Finally, equation (26.8) is obtained by noting that

$$\mathscr{D}^{-1}(\mathrm{s})\, \eta \mathscr{D}(\mathrm{s}) = \eta \mathscr{D}^{\dagger}(\mathrm{s})\, \mathscr{D}(\mathrm{s}) = \eta \mathscr{D}^{-1}\left(\frac{\sigma \cdot p}{m}\right), \qquad p = \Lambda^{-1}(\mathrm{s})\, \mathring{p}, \tag{26.10}$$

where $\mathscr{D}(\sigma \cdot p/m)$ is a polynomial, and hence that by letting $\mathscr{D}(\mathrm{s}) = \Gamma_5$ and $\mathring{p} = (m, 0, 0, 0)$ we obtain

$$\Gamma_5^{-1} \eta \Gamma_5 = \eta \mathscr{D}(-1) = \eta(-1)^{2J} \tag{26.11}$$

as required.

Note that the conjugation equation (26.7) applies only to scalar operators $Q(p)$ i.e. those satisfying (26.9). In particular it does not apply to the operators $\boldsymbol{L} \cdot \boldsymbol{p}$ and $\boldsymbol{K} \cdot \boldsymbol{p}$ which are used in the wave-equations for massless functions, since these are not scalar but, as we have seen in (22.8), time-components of vectors. However, the conjugation is unnecessary in that case anyway, since the massless wave-equations (22.3) are linear in $p = (\omega, p)$ and hence are the same on both branches of the hyperboloid (light-cone).

27. Invariants in Configuration Space

Since each branch of the hyperboloid $p^2 = m^2$ in (26.4) carries an independent representation of the Poincaré group, there are two independent invariants which can be formed from the field $\varphi(x)$. Since $\psi(\boldsymbol{p})$ and $\chi(\boldsymbol{p})$ both belong to the same representation $\mathscr{D}(s)$ of $SL(2, C)$, the invariants can be written in the form

$$(\varphi_1, \varphi_2)_\varepsilon = \int \frac{d^3p}{\omega} \{\psi_1{}^\dagger(\boldsymbol{p})\, K(p)\, \psi_2(\boldsymbol{p}) + \varepsilon \chi_1{}^\dagger(\boldsymbol{p})\, K(p)\, \chi_2(\boldsymbol{p})\}, \quad \varepsilon = \pm 1 \tag{27.1}$$

where $K(p)$ is the kernel for an irreducible orbit constructed in sections 19 and 23, i.e.

$$K(p) = \mathscr{D}^{-1}\left(\frac{\sigma \cdot p}{m}\right), \qquad K(p) = \omega^{-2(M+N)} \tag{27.2}$$

for the massive and massless cases respectively. Only the invariant with $\varepsilon = 1$ is positive definite, of course, but nevertheless it is interest to keep both $\varepsilon = 1$ and $\varepsilon = -1$. The Hilbert space with the inner product defined by (27.1) with $\varepsilon = 1$ will be denoted by $\mathscr{H}$. It carries the direct sum of the representations of the Poincaré group on the positive and negative frequency orbits.

We now wish to write (27.1) in terms of the configuration space functions $\varphi(x)$. For this purpose we invert (27.1) by taking the time-derivative and writing

$$\psi(\boldsymbol{p}) = \frac{1}{2(2\pi)^3} \int d^3x e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \{\omega \varphi(x) + i\dot{\varphi}(x)\}, \tag{27.3}$$

$$\chi(\boldsymbol{p}) = \frac{1}{2(2\pi)^3} \int d^3x e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \{\omega \varphi(x) - i\dot{\varphi}(x)\}. \tag{27.4}$$

Substituting these formulae into (27.1) it is then a simple matter to show that

$$(\varphi_1, \varphi_2)_\varepsilon = \int d^3(xy)\, \Phi_1^\dagger(x)\, \mathscr{K}_\varepsilon(x-y)\, \Phi_2(y), \tag{27.5}$$

where

$$\Phi(x) = \begin{pmatrix} \varphi(x) \\ i\dot{\varphi}(x) \end{pmatrix}, \tag{27.6}$$

and $\mathscr{K}_\varepsilon(x-y)$ is the three-dimensional Fourier transform of the kernel

$$\mathscr{K}_\varepsilon(p) = \frac{1}{2\omega} \begin{pmatrix} \omega^2 \Delta_+ & \omega\Delta_- \\ \omega\Delta_- & \Delta_+ \end{pmatrix}, \tag{27.7}$$

where

$$\Delta_\pm = K(p) \pm \varepsilon K(\hat{p}), \qquad \hat{p} = (\omega, -\boldsymbol{p}). \tag{27.8}$$

In particular, in the mass-zero case we have

$$\mathscr{K}_\varepsilon(p) = \omega^{-2(M+N)-1} \begin{bmatrix} \omega^2(1+\varepsilon) & \omega(1-\varepsilon) \\ \omega(1-\varepsilon) & (1+\varepsilon) \end{bmatrix}, \quad m^2 = 0. \tag{27.9}$$

The most interesting question concerning $\mathscr{K}_\varepsilon(x-y)$ is whether it is *local*, that is, whether $\mathscr{K}_\varepsilon(p)$ is polynomial in the *spatial* part $\boldsymbol{p}$ of p.

In the massless case this is never so unless $M = N = 0$, since for any other values of M and N (which are both positive) we see from (27.9) that $\mathscr{K}_\varepsilon(p)$ is not a polynomial in ω, let alone $\boldsymbol{p}$. Even for $M = N = 0$ we see from (27.9) that $\mathscr{K}_\varepsilon(q)$ will be polynomial in $\boldsymbol{p}$ (actually a constant) if, and only if, $\varepsilon = -1$.

In the massive case, we see from (27.2) that $\mathscr{K}_\varepsilon(p)$ is at any rate a polynomial in $(\omega, \boldsymbol{p})$. Since $\omega^2 = \boldsymbol{p}^2 + m^2$, it follows that it will be a polynomial in $\boldsymbol{p}$ if, and only if, it is even in ω. Now from (27.2) we have

$$K(-p) = \mathscr{D}^{-1}(-1)\, \mathscr{D}^{-1}\left(\frac{\sigma \cdot p}{m}\right) = (-1)^{2J} K(p). \tag{27.10}$$

Hence from (27.8)

$$\Delta_\pm(-\omega, \boldsymbol{p}) = \pm\varepsilon(-1)^{2J} \Delta_\pm(\omega, \boldsymbol{p}), \tag{27.11}$$

and so from (27.7)

$$\mathscr{K}_\varepsilon(-\omega, \boldsymbol{p}) = -\varepsilon(-1)^{2J} \mathscr{K}_\varepsilon(\omega, p). \tag{27.12}$$

Thus the condition for $\mathscr{K}_\varepsilon(p)$ to be even in ω, and hence polynomial in $\boldsymbol{p}$, is

$$\varepsilon = (-1)^{2J+1}. \tag{27.13}$$

Equation (27.13) is therefore the necessary and sufficient condition for locality in the massive case. It shows that for half-integral spin the invariant (27.1) can be both local and positive ($\varepsilon = 1$), while for integral spin it can be either local ($\varepsilon = -1$) or positive ($\varepsilon = 1$), but not both.

If parity is implemented linearly, that is to say if the representation $\mathscr{D}(s)$ is pseudo-unitary and the rest-frame states $\psi(\mathring{p})$ and $\chi(\mathring{p})$ are eigenstates of the parity operator η then the formalism for the massive case simplifies considerably. To see this we recall first of all from section 19 that under these circumstances we can make the substitution

$$\mathscr{D}^{-1}\left(\frac{\sigma \cdot p}{m}\right) \to \eta \qquad \text{on} \qquad \psi(\boldsymbol{p}), \tag{27.14}$$

for the function $\psi(\boldsymbol{p})$ on a single orbit. However we cannot immediately make this substitution on two orbits simultaneously because, as we shall see now, the parity assignments on the two orbits are correlated. To see this we adopt the assignment $+1$ on the positive orbit for definiteness. Then from (20.6) we have

$$\eta Q_+ = Q_+ . \tag{27.15}$$

But from (26.7) and (26.11) we see that if we take the Γ_5 transform of this equation we obtain

$$\eta Q_- = (-1)^{2J} Q_- . \tag{27.16}$$

Thus the parity assignment of the negative orbit is $(-1)^{2J}$ and the appropriate substitution to accompany (27.14) is

$$\mathscr{D}^1\left(\frac{\sigma \cdot p}{m}\right) \to (-1)^{2J}\,\eta \quad \text{on} \quad \chi(\boldsymbol{p}) . \tag{27.17}$$

Making the substitutions (27.14) and (27.17) one easily sees that

$$\Lambda_\pm \to \eta \pm \varepsilon(-1)^{2J}\,\eta , \tag{27.18}$$

and hence

$$\mathscr{K}_\varepsilon(p) \to \eta \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad \mathscr{K}_\varepsilon(p) \to \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{27.19}$$

for $\varepsilon = \pm(-1)^{2J}$ respectively. In particular, for $\varepsilon = (-1)^{2J+1}$ we have

$$(\varphi_1, \varphi_2) = \frac{i}{(2\pi)^3}\int d^3x\,\{\varphi_1{}^\dagger(x)\,\eta\dot\varphi_2(x) - \dot\varphi_1{}^\dagger(x)\,\eta\varphi_2(x)\} , \tag{27.20}$$

which is just the space integral of the time-component of the (dispersive) conserved vector current

$$j_\mu(x) = \frac{i}{(2\pi)^3}\left\{\varphi_1{}^\dagger(x)\,\eta\partial_\mu\varphi_2(x) - \left(\partial_\mu\varphi_1{}^\dagger(x)\right)\eta\varphi_2(x)\right\} . \tag{27.21}$$

We can summarize the results of this section by saying that for the massless case the invariants are strictly nonlocal except for the special case $M = N = 0$, $\varepsilon = -1$, while for the massive case the invariant with $\varepsilon = (-1)^{2J+1}$ is local and that with $\varepsilon = (-1)^{2J}$ is non-local. Since only the invariant with ε positive qualifies as an inner-product, it follows that we have a local innerproduct only when $\varepsilon = (-1)^{2J+1} = 1$, i.e. only when the spin is half-integral. We also see that the formulae for the massive case simplify if the states are eigenstates of parity.

Chapter IX. Second Quantization for Massive Fields

28. Second Quantization and Spin-Statistics Theorem

We have just seen that even with the addition of negative energy orbits one cannot construct an invariant positive local inner-product for integral spin. Similarly it can be shown that one cannot construct a positive energy density for half-integral spin [*13*]. These difficulties are resolved in second quantization by divorcing the states from the fields, so that the one-particle states have positive energies and the positive but nonlocal inner-products of section 19 while the fields do not have these properties, but are local in the sense that they satisfy causal commutation or anti-commutation relations of the from

$$[\boldsymbol{\varphi}(x), \boldsymbol{\varphi}^*(y)] = S(x, y) \tag{28.1}$$

where $S(x, y)$ is assumed to be a c-number in second-quantized (Fock) space. Henceforth star will denote adjoint in Fock space and complex conjugation in spinor space, while dagger will denote adjoint in spinor space as before. Second quantized fields will be written in boldface.

We wish to show in this section that our formalism allows the unique determination of $S(x, y)$ and the establishment of the spin-statistics theorem for free fields in a simple transparent manner. To show this we first note that the transformation properties and wave-equations (irreducibility conditions) for the field $\boldsymbol{\varphi}(x)$ reflect themselves in terms of $S(x, y)$ as follows:

$$\big(U(a, s)\boldsymbol{\varphi}\big)(x) = \mathscr{D}(s)\, \boldsymbol{\varphi}\big(\Lambda^{-1}(s)\,(x-a)\big) \to S(x, y) = \mathscr{D}(s)\, S\big(\Lambda^{-1}(x-a), \Lambda^{-1}(y-a)\big)\, \mathscr{D}^\dagger(s), \tag{28.2a}$$

$$(\Box + m^2)\, \boldsymbol{\varphi}(x) = 0 \to (\Box_x + m^2)\, S(x, y) = 0, \tag{28.2b}$$

$$Q(i\partial)\, \boldsymbol{\varphi}(x) = \boldsymbol{\varphi}(x) \to Q(i\partial_x)\, S(x, y) = S(x, y)\, Q^\dagger\big(i\overleftarrow{\partial}_y\big) = S(x, y). \tag{28.2c}$$

From the translational part of (28.2a) it follows that

$$S(x, y) = S(x-y), \tag{28.3}$$

and from the mass-condition (28.2b) that $S(x-y)$ has the representation

$$S(x-y) = \int \frac{d^3p}{\omega} \left\{e^{-ip(x-y)}\, S_+(\boldsymbol{p}) + e^{ip(x-y)}\, S_-(\boldsymbol{p})\right\}, \qquad \omega = |p_0|. \tag{28.4}$$

From the homogeneous part of (28.2a) we then see that

$$S_\pm(\boldsymbol{p}) = \mathscr{D}(s) S_\pm\big(\Lambda^{-1}(s)\, \boldsymbol{p}\big)\, \mathscr{D}^\dagger(s), \tag{28.5}$$

from which we have

$$S_\pm(\boldsymbol{p}) = \mathscr{D}^{-1}(\mathrm{s})\, S_\pm \mathscr{D}^{\dagger-1}(\mathrm{s}), \qquad S_\pm \equiv S_\pm(0), \quad p = \Lambda^{-1}(\mathrm{s})\, \mathring{p}, \tag{28.6}$$

and

$$S_\pm = \mathscr{D}^{-1}(k)\, S_\pm \mathscr{D}(k), \qquad k \in SU(2). \tag{28.7}$$

Equation (28.6) shows that $S_\pm(\boldsymbol{p})$ is completely determined by $S_\pm(0)$, its value at $\boldsymbol{p} = 0$, and equation (28.7) shows that $S_\pm(0)$ is invariant with respect to the little group $SU(2)$. Let us finally apply the spin condition (28.2c). We obtain

$$Q(\pm p)\, S_\pm(\boldsymbol{p})\, Q^\dagger(\pm p) = S_\pm(\boldsymbol{p}), \tag{28.8}$$

and hence

$$Q_\pm S_\pm Q_\pm = S_\pm. \qquad Q_\pm = Q(\pm p), \boldsymbol{p} = 0. \tag{28.9}$$

But since $Q_\pm$ project into *single irreducible* representations of $SU(2)$ equations (28.7) and (28.9) together imply

$$S_+ = \lambda Q_+ \qquad \text{and} \qquad S_- = \mu Q_- \tag{28.10}$$

where λ and μ are constants. It follows that

$$S_+(\boldsymbol{p}) = \lambda \mathscr{D}^{-1}(\mathrm{s})\, Q_+ \mathscr{D}^{\dagger -1}(\mathrm{s}) = \lambda\big(\mathscr{D}^{-1}(\mathrm{s})\, Q^+ \mathscr{D}(\mathrm{s})\big)\,\big(\mathscr{D}^{-1}(\mathrm{s})\,\mathscr{D}^{\dagger -1}(\mathrm{s})\big) = \lambda Q(p)\,\mathscr{D}\left(\frac{\sigma\cdot p}{m}\right). \tag{28.11}$$

Similarly,

$$S_-(\boldsymbol{p}) = \mu \mathscr{D}^{-1}(\mathrm{s})\, Q_- \mathscr{D}^{\dagger -1}(\mathrm{s}) = \mu Q(-p)\,\mathscr{D}\left(\frac{\sigma\cdot p}{m}\right). \tag{28.12}$$

Thus $S(x)$ has the form

$$S(x) = \int \frac{d^3p}{\omega}\, \mathscr{D}^{-1}(\mathrm{s})\left\{e^{-ip\cdot x}\lambda Q_+ + e^{ip\cdot x}\mu Q_-\right\}\mathscr{D}^{\dagger -1}(\mathrm{s}), \qquad p = \Lambda^{-1}(\mathrm{s})\,\mathring{p}, \tag{28.13a}$$

$$= \int \frac{d^3p}{\omega}\left\{e^{-ip\cdot x}\lambda Q(p) + e^{ip\cdot x}\mu Q(-p)\right\}\mathscr{D}\left(\frac{\sigma\cdot p}{m}\right), \qquad p_0 = \omega > 0, \tag{28.13b}$$

$$= Q(i\partial)\,\mathscr{D}\left(\frac{i\sigma\cdot\partial}{m}\right)\int\frac{d^3p}{\omega}\left\{\lambda e^{-ip\cdot x} + (-1)^{2J}\mu e^{ip\cdot x}\right\}, \tag{28.13c}$$

which is unique up to the constants λ and μ. Equation (28.13) represents the maximum information that we can obtain from covariance and irreducibility.

To proceed further we add another condition, namely causality. This is the condition that $S(x)$ should vanish outside the light-cone, and, as we shall see, it determines μ in terms of λ and the spin. To apply the causality condition we write (28.4) in the form

$$S(x) = \int d^3p e^{ip\cdot x}\left\{\frac{e^{-i\omega t}}{\omega}\, S_+(\boldsymbol{p}) + \frac{e^{i\omega t}}{\omega}\, S_-(-\boldsymbol{p})\right\}, \qquad \mathring{p} = (\omega, -\boldsymbol{p}) \tag{28.14}$$

It is then easy to see that $S(x)$ will vanish outside the light-cone if, and only if, the expression inside the bracket is even in ω, that is to say, if, and only if,

$$S_+(\boldsymbol{p}) + S_-(-\boldsymbol{p}) = -[S_+(\boldsymbol{p}) + S\ (-\boldsymbol{p})]', \tag{28.15}$$

where S'_- denotes the value of $S_\pm$ as given by (28.11) and (28.12) for $\omega \to -\omega$. But since $\omega \to -\omega$ is the same as $p \to -\mathring{p}$, we have

$$\mu S'_+(\boldsymbol{p}) = \mu\lambda Q(-\mathring{p})\,\mathscr{D}\left(\frac{-\sigma\cdot\mathring{p}}{m}\right) = \mu\lambda(-1)^{2J} Q(-\mathring{p})\,\mathscr{D}\left(\frac{\sigma\cdot\mathring{p}}{m}\right) = \lambda(-1)^{2J}\, S_-(-\boldsymbol{p}). \tag{28.16}$$

Hence (28.15) is equivalent to

$$[\lambda + (-1)^{2J}\mu]\left[\frac{S_+(\boldsymbol{p})}{\lambda} + (-1)^{2J}\,\frac{S_-(-\boldsymbol{p})}{\mu}\right] = 0. \tag{28.17}$$

Suppose now that the expression in the bracket on the right were zero. Then it is easy to see from (28.11) and (18.12) that we would have

$$\mathscr{D}^{-1}(\mathrm{s})\, Q_+ \mathscr{D}^{-1\dagger}(\mathrm{s}) = (-1)^{2J+1}\,\mathscr{D}^{\dagger}(\mathrm{s})\, Q_- \mathscr{D}(\mathrm{s}), \qquad p = \Lambda^{-1}(\mathrm{s})\,\mathring{p}, \tag{28.18}$$

and expanding this equation up to first order in the group parameters

$$Q_- = (-1)^{2J+1} Q_+, \qquad \boldsymbol{K} Q_+ + Q_+ \boldsymbol{K} = 0, \tag{28.19}$$

where $\boldsymbol{K}$ are the generators of accelerations in $\mathscr{D}(s)$. Now since Q_+ is a projection operator it is easy to see that the second equation in (28.19) implies that Q_+ commutes with $\boldsymbol{K}$

and that $\boldsymbol{K} = 0$ on the eigenspace $Q_+ = 1$. But since the only representation of $SL(2, C)$ with $\boldsymbol{K} = 0$ is the trivial one, this implies that $J = 0$. But then from the first equation in (28.19) we would have $Q_+ = -Q_-$, which is impossible since Q_+ and Q_- are both projection operators. Thus equation (28.19) is not possible, and so the term in the second bracket in (28.17) does not vanish. It follows that the necessary and sufficient condition for causality is

$$\lambda + (-1)^{2J}\mu = 0. \tag{28.20}$$

The expression (28.13) for $S(x)$ then reduces to

$$S(x) = \lambda \int \frac{d^3p}{\omega}\, \mathscr{D}^{-1}(\mathrm{s}) \left\{e^{-ip\cdot x} Q_+ + (-1)^{2J+1} e^{ip\cdot x} Q_-\right\} \mathscr{D}^{\dagger-1}(\mathrm{s}), \qquad p = \Lambda^{-1}(\mathrm{s})\, \mathring{p}, \tag{28.21 a}$$

$$= \lambda \int \frac{d^3p}{\omega} \left\{e^{-ip\cdot x} Q(p) + (-1)^{2J+1} e^{ip\cdot x} Q(-p)\right\} \mathscr{D}\left(\frac{\sigma\cdot p}{m}\right), \qquad p_0 = \omega > 0, \tag{28.21 b}$$

$$= \lambda Q(\partial)\, \mathscr{D}\left(\frac{i\sigma\cdot\partial}{m}\right) \int \frac{d^3p}{\omega} \left\{e^{-ip\cdot x} - e^{ip\cdot x}\right\}, \tag{28.21 c}$$

$$= \lambda Q(i\partial)\, \mathscr{D}\left(\frac{i\sigma\cdot\partial}{m}\right) \Delta(x), \tag{28.21 d}$$

where $\Delta(x)$ is the well-known commutator function for scalar fields. Equation (28.21) represents the maximum information on $S(x)$ which can be obtained from covariance, irreducibility and causality.

To proceed further we have to impose one more condition, namely, that of positive energy. This condition, as we shall see, fixes the sign of λ and gives us the spin-statistics theorem. To apply the positive energy condition we note that since $\boldsymbol{\varphi}(x)$ has an expansion similar to (28.4) for $S(x)$, it makes sense to talk about the positive and negative frequency parts $\boldsymbol{\varphi}^{(\pm)}(x)$ of $\boldsymbol{\varphi}(x)$ and to write them in the form

$$\boldsymbol{\varphi}^{(+)}(x) = \int \frac{d^3p}{\omega}\, e^{-ip\cdot x} \boldsymbol{\varphi}^{(+)}(\boldsymbol{p}), \qquad \boldsymbol{\varphi}^{(-)}(x) = \int \frac{d^3p}{\omega}\, e^{ip\cdot x} \boldsymbol{\varphi}^{(-)}(\boldsymbol{p}). \tag{28.22}$$

Then, on account of the translational invariance of $S(x, y)$, i.e. the dependence of $S(x, y)$ on $S(x - y)$ only, the commutation or anti-commutation relations (28.1) split into

$$[\boldsymbol{\varphi}^{(+)}(x), \quad \boldsymbol{\varphi}^{(+)*}(y)] = S^{(+)}(x - y), \quad [\boldsymbol{\varphi}^{(-)}(x), \boldsymbol{\varphi}^{(-)*}(y)] = S^{(-)}(x - y), \tag{28.23}$$

or equivalently

$$[\boldsymbol{\varphi}^{(+)}(\boldsymbol{p}), \boldsymbol{\varphi}^{(+)*}(\boldsymbol{q})] = \lambda\omega\delta^3(\boldsymbol{p} - \boldsymbol{q})\, Q(p)\, \mathscr{D}\left(\frac{\sigma\cdot p}{m}\right), \quad [\boldsymbol{\varphi}^{(-)}(\boldsymbol{p}), \boldsymbol{\varphi}^{(-)*}(\boldsymbol{q})]$$
$$= (-1)^{2J+1} \lambda\omega\delta^3(\boldsymbol{p} - \boldsymbol{q})\, Q(-p)\, \mathscr{D}\left(\frac{\sigma\cdot p}{m}\right), \tag{28.24}$$

with the other two combinations completely commuting or anti-commuting. The point now is that for positive energy we must have

$$\boldsymbol{\varphi}^{(+)} |0\rangle = 0, \qquad \boldsymbol{\varphi}^{(-)*} |0\rangle = 0 \tag{28.25}$$

where $|0\rangle$ is the vacuum, since otherwise these would be states of negative energy, i.e. negative values of $E = i\partial_t$.

From (28.24) and (28.25) we then have

$$\lambda m\delta^3(0)Q_+ = \langle 0|\,[\boldsymbol{\varphi}^{(+)}(0), \boldsymbol{\varphi}^{(+)*}(0)]\,|0\rangle = \langle 0|\,\boldsymbol{\varphi}^{(+)}(0)\,\boldsymbol{\varphi}^{(+)*}(0)\,|0\rangle = \|\boldsymbol{\varphi}^{(+)*}(0)\,|0\rangle\|^2, \tag{28.26a}$$

$$\begin{aligned}(-1)^{2J+1}\lambda m\delta^3(0)\,Q_- &= \langle 0|\,[\boldsymbol{\varphi}^{(-)}(0), \boldsymbol{\varphi}^{(-)*}(0)]|0\rangle = \pm\langle 0|\,\boldsymbol{\varphi}^{(-)*}(0)\,\boldsymbol{\varphi}^{(-)}(0)\,|0\rangle \\ &= \pm\|\boldsymbol{\varphi}^{(-)}(0)\,|0\rangle\|^2.\end{aligned} \tag{28.26b}$$

Since $Q_\pm$ are positive it follows from these two equations that $\lambda > 0$, and that the commutator or anti-commutator must be chosen in (28.1) according as $(-1)^{2J} = 1$. The second result stated is the spin-statistics theorem. The first shows that by suitably normalizing the field $\varphi(x)$, λ can be set equal to unity. Thus the final form of the commutation or anti-commutation relation (28.1) is given by (28.3) and (28.21) with $\lambda = 1$ and the understanding that the commutator and anti-commutator are to be used for integral and half-integral spin respectively.
Note that if parity is linearly implemented, so that we can replace $\mathscr{D}^{-1}(\pm\sigma\cdot p/m)$ by η on the positive and negative frequency parts respectively, then we can replace $\mathscr{D}^{-1}(i\sigma\cdot\partial\,/m)$ by η and the expression (28.21d) simplifies to

$$S(x, y) = Q(i\partial)\,\eta\Delta(x-y), \tag{28.27}$$

an equation which is familiar [*14*] in the Dirac case.

29. Charge-Conjugation in First and Second Quantization

In this section we wish to introduce the charge-conjugation operator, which has the property that it interchanges the positive and negative frequency parts of the field. The charge-conjugation operator also has the property that it is unitary in second-quantization but anti-unitary in first, and we wish to show in particular how this situation comes about, and under what assumptions it is true.
Let us consider first the first-quantized fields $\psi(\boldsymbol{p})$ and $\chi(\boldsymbol{p})$. These transform according to the inequivalent, complex conjugate representations $\exp(\pm\, ip\cdot a)$ of the translation, and therefore of the Poincaré, group, but according to the same representation $\mathscr{D}(s)$ of $SL(2, C)$. On the other hand, $\psi(\boldsymbol{p})$ and $\chi^*(\boldsymbol{p})$ transform according to the same representation of the Poincaré group, but according to the complex conjugate representations $\mathscr{D}(s)$ and $(\mathscr{D}(s))^*$ of $SL(2, C)$. However, in contrast to the representations $\exp(\pm\, ip\cdot a)$ of the translation group, the representations $\mathscr{D}(s)$ and $(\mathscr{D}(s))^*$ of $SL(2, C)$ are not necessarily inequivalent. In fact, it is well-known [7] that $(\mathscr{D}(s))^*$ is equivalent to $\mathscr{D}^{\dagger-1}(s)$, i.e. that $(\mathscr{D}(s))^* = C^{-1}(\mathscr{D}^{\dagger-1}(s))\,C$, where C is the so-called charge-conjugation matrix, and we already know that $\mathscr{D}^\dagger(s)$ is equivalent to $\mathscr{D}(s)$ if (and only if) $\mathscr{D}(s)$ is pseudo-unitary. It follows that for pseudo-unitary representations $(\mathscr{D}(s))^*$ is equivalent to $\mathscr{D}(s)$ and

$$(\mathscr{D}(s))^* = (\eta C)^{-1}\,\mathscr{D}(s)\,(\eta C). \tag{29.1}$$

Combining all these remarks it can be seen at once that for pseudo-unitary representations of $SL(2, C)$, $\psi(\boldsymbol{p})$ and $\eta C\chi^*(\boldsymbol{p})$ transform in the same way with respect to *both* the Poincaré group and $SL(2, C)$. However, they satisfy slightly different wave-equations, namely,

$$Q(p)\,\psi(\boldsymbol{p}) = \psi(\boldsymbol{p}), \qquad Q(-p)\,\eta C\chi^*(\boldsymbol{p}) = \eta C\chi^*(\boldsymbol{p}), \tag{29.2}$$

respectively. On the other hand, from section 26, $Q(-\boldsymbol{p}) = \Gamma_5^{-1}Q(p)\,\Gamma_5$. It follows that the pair of quantities $\psi(\boldsymbol{p})$ and $\eta\Gamma_5C\chi^*(\boldsymbol{p})$ satisfy the same transformation laws *and* the same wave-equation. Similarly for the pair of quantities $\chi(\boldsymbol{p})$ and $\eta\Gamma_5 C\psi^*(\boldsymbol{p})$. It is therefore natural to define the charge-conjugation operator to be

$$(\mathscr{C}\psi)\,(\boldsymbol{p}) = \eta\Gamma_5 C\chi^*(\boldsymbol{p}), \qquad (\mathscr{C}\chi)\,(\boldsymbol{p}) = \eta\Gamma_5 C\psi^*(\boldsymbol{p}), \tag{29.3}$$

(up to a constant). For the same reasons, in the second-quantized case we define it to be

$$(\mathscr{C}\boldsymbol{\varphi}^{(+)})\,(\boldsymbol{p}) = \eta\Gamma_5 C\boldsymbol{\varphi}^{(-)*}(\boldsymbol{p}), \qquad (\mathscr{C}\boldsymbol{\varphi}^{(-)})\,(\boldsymbol{p}) = \eta\Gamma_5 C\boldsymbol{\varphi}^{(+)*}(\boldsymbol{p}). \tag{29.4}$$

Let us now discuss the unitarity or anti-unitarity of $\mathscr{C}$ and $\mathscr{C}$. To do this satisfactorily we must first define the inner-product. In the first-quantized case this is easy since the inner-product has already been defined by (27.1) with $\varepsilon = +1$, and hence is

$$(\varphi_1, \varphi_2) = \int \frac{d^3p}{\omega}\left\{\varphi_1{}^\dagger(\boldsymbol{p})\,\mathscr{D}^{-1}\left(\frac{\sigma\cdot p}{m}\right)\varphi_2(\boldsymbol{p}) + \chi_1{}^\dagger(p)\,\mathscr{D}^{-1}\left(\frac{\sigma\cdot p}{m}\right)\chi_2(\boldsymbol{p})\right\}. \tag{29.5}$$

It is trivial to verify that $\mathscr{C}$ as defined in (29.3) is anti-unitary with respect to the inner-product (29.5).

In the second-quantized case the situation is more complicated. First we have to take into account that since the fields $\boldsymbol{\varphi}^{(\pm)}(\boldsymbol{p})$ must satisfy the commutation relations (28.1) they are only *base*-fields and must be smeared in order to obtain a sensible inner-product. The natural way to smear them is to use as test functions the first quantized fields which transform contragrediently to the $\varphi^{(\pm)}(p)$ i.e. according to the contragredient representation $U(\acute{a}, \acute{s})$ of $\mathscr{P}_+^\uparrow$. Secondly in the second-quantized case we must distinguish between the smeared *fields*

$$\boldsymbol{\varphi}(\psi, \chi) = \int \frac{d^3p}{\omega}\,\{\psi^*(\hat{\boldsymbol{p}})\cdot\boldsymbol{\varphi}^{(+)}(\boldsymbol{p}) + \chi^*(\hat{\boldsymbol{p}})\cdot\varphi^{(-)}(\boldsymbol{p})\}, \tag{29.6}$$

and the smeared *one-particle states*

$$|\varphi\rangle = \int \frac{d^3p}{\omega}\,\{\psi(\hat{\boldsymbol{p}})\cdot\boldsymbol{\varphi}^{(+)*}(\boldsymbol{p}) + \chi^*(\hat{\boldsymbol{p}})\cdot\varphi^{(-)}(\boldsymbol{p})\}\;|0\rangle. \tag{29.7}$$

The fields transform in the manifestly covariant manner (26.1) but do not have positive definite energy, while the one-particle states have positive definite energy but do not transform in this simple manner. The point now is that in the second-quantized case the inner-product is formed with the one-particle states and not with the fields. Using the definition (29.7) of the one-particle states, and the commutation relations (28.24) we see that the inner-product is of the form

$$\langle\varphi_1, \varphi_2\rangle = \int \frac{d^3p}{\omega}\left\{\psi_1{}^\dagger(\hat{\boldsymbol{p}})\,Q(p)\,\mathscr{D}^{-1}\left(\frac{\sigma\cdot p}{m}\right)\psi_2(\hat{\boldsymbol{p}}) + \chi_2{}^\dagger(\hat{\boldsymbol{p}})\,Q(-p)\,\mathscr{D}^{-1}\left(\frac{\sigma\cdot p}{m}\right)\chi_1(\hat{\boldsymbol{p}})\right\}. \tag{29.8}$$

Using the identity

$$Q(\pm\hat{p})\,\mathscr{D}\left(\frac{\sigma\cdot\hat{p}}{m}\right) = \mathscr{D}^\dagger(s)\,Q_\pm\mathscr{D}(s) = \mathscr{D}^{-1}\left(\frac{\sigma\cdot p}{m}\right)Q(\pm p), \tag{29.9}$$

and absorbing the wave-operators $Q(\pm p)$ in the test-functions on the right, the inner-product (29.8) reduces to

$$\langle\varphi_1|\varphi_2\rangle = \int \frac{d^3p}{\omega}\left\{\psi_1{}^\dagger(\boldsymbol{p})\,\mathscr{D}^{-1}\left(\frac{\sigma\cdot p}{m}\right)\psi_2(\boldsymbol{p}) + \chi_2{}^\dagger(\boldsymbol{p})\,\mathscr{D}^{-1}\left(\frac{\sigma\cdot p}{m}\right)\chi_1(\boldsymbol{p})\right\}, \tag{29.10}$$

and since $\mathscr{D}^{-1}(\sigma \cdot p/m)$ is hermitian this can finally be written in the form

$$\langle \varphi_1 | \varphi_2 \rangle = \int \frac{d^3p}{\omega} \left\{ \psi_1^\dagger(\boldsymbol{p})\, \mathscr{D}^{-1} \left(\frac{\sigma \cdot p}{m} \right) \psi_2(\boldsymbol{p}) + \left(\chi_1^\dagger(\boldsymbol{p})\, \mathscr{D}^{-1} \left(\frac{\sigma \cdot p}{m} \right) \chi_2(\boldsymbol{p}) \right)^* \right\}. \tag{29.11}$$

If we now compare the first- and second-quantized inner-products (29.5) and (29.11) we see that the first term in each is the same but the second terms are complex conjugates of one another. On account of this difference, the charge conjugation operator $\mathscr{C}$ of (29.3), which is anti-unitary with respect to (29.5) becomes unitary with respect to (29.11). Thus the change from anti-unitarity to unitarity on going from first- to second-quantization lies in the change of inner-product. From the construction it is clear that the change in inner-product comes about because in first-quantization the inner-product is formed with the fields, which transform according to (26.1) but do not have definite energy, whereas in second-quantization it is formed with the one-particle states, which do not transform according to (26.1) but have definite energy.

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