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Realizations of the Unitary Representations of the Inhomogeneous Space-Time Groups I

General Structure

U. H. NIEDERER and L. O'RAIFEARTAIGH,

Zurich University

and

Dublin Institute for Advanced Studies.

Contents

Abstract

The aim of these two papers (I and II) is to try to give a unified systematic description of the different quantum mechanical realizations of the unitary representations of the inhomogeneous space-time groups.
Paper I, which is more general, deals with the interrelationships between the Mackey, the Wigner and the covariant realizations of induced representations, first for general groups, and then for semi-direct products. The Euclidean, Galilean and Poincaré groups are treated in some detail as examples.
Paper II gives a systematic treatment of the various covariant wavefunctions (Dirac, Bargmann-Wigner, etc.) that can be used to describe a single irreducible representation of the Poincaré group. It is self-contained and deals first with momentum space, and then with first and second quantized fields. In both papers the treatment is index-free.

Introduction

It is well-known that the irreducible representations of the inhomogeneous space-time groups (Euclidean, Galilean, Poincare groups) can be realized in several different ways in quantum mechanics, namely, on different kinds of states (or fields) such as Mackey, Wigner or covariant states, and also by different choices of each such kind of state, for example by choosing the covariant states to be Dirac or Fierz or Bargmann-Wigner states.
The aim of the present papers (I and II) is to try to unify and systemize the treatment of these various realizations of the group representations. The emphasis is on the logical procedure in constructing the realizations, and on the inter-relations between them. For example, in Paper I, it is emphasized that Wigner and covariant states can be introduced naturally without specializing to the case of semi-direct product groups, and that the little group concept follows naturally from the simpler demand that the inducing representation be a direct product. In paper II it is shown that the most important properties of covariant states can be obtained quite simply without specifying the covariant representation explicitly. Thus the papers deal with the logical, rather than the mathematical structure of the realizations of the groups. As a result there are few new mathematical results and there is no claim to complete mathematical rigour. On the other hand, previously known results are derived in a general, and hopefully transparent manner, and a reasonable level of rigour is attempted. For better or worse, the treatment in both papers is index-free.
The detailed contents of the two papers can be seen from the list of contents. Paper I is the more general and mathematical, and is concerned with the different *kinds* of realization (MACKEY, WIGNER, covariant) mentioned above. It starts from the abstract group representations, which are most simply and conveniently described by Mackey functions, and proceeds from there to the Wigner and covariant functions. It then specializes to the case of semi-direct products and shows how the concepts of orbit, little group and configuration-space present themselves in a natural manner. Finally, it specializes to the case when the invariant subgroup is abelian, and gives a proof of irreducibility and exhaustivity in that case. Attention is paid to the inner product at all stages, and at the end, the inhomogeneous spacetime groups are treated in some detail as examples.
Paper II is concerned with the different covariant realizations (DIRAC, FIERZ, JOOS-WEINBERG, etc.) of the Poincaré group, and, on account of the direct physical interest of this subject, paper II is made self-contained and has a separate introduction.

Part I.

Chapter I. General Groups

1. General Mackey Theory

Let G be a Lie group,[1]) and K a closed subgroup such that the right coset space $C = G/K$ admits a (right) invariant[2]) measure

$$d\mu(cg) = d\mu(c), \qquad c \in C, \quad g \in G \tag{1.1}$$

Then a continuous unitary representation $U(g)$ of G is induced [1] by a continuous unitary representation $D(k)$ of K according to the three equations:

transformation law: $\big(U(g')\big)(g) = f(gg'), \qquad g, g' \in G,$ (1.2)

subsidiary condition: $f(kg) = D(k)\, f(g), \quad k \in K, \quad g \in G,$ (1.3)

inner product: $(f_1, f_2) = \int d\mu(c) \big(f_1(g), f_2(g)\big)_D,$ (1.4)

where the (Mackey) functions $f(g)$ are vectors in the representation space of $D(k)$, and $(\quad,\quad)_D$ is the inner product on that space. The transformation law (1.2) is simply the standard law for linear representation by transitive action on functions over a group, and it is easy to verify that the action leaves the subsidiary condition and inner product invariant. The invariance of the inner product guarantees the unitarity of $U(g)$ and the purpose of the subsidiary condition is to ensure that the integrand in (1.4) depends only on c, i.e. to ensure that

$$\big(f_1(kg), f_2(kg)\big)_D = \big(f_1(g), f_2(g)\big)_D. \tag{1.5}$$

In general the irreducibility of the inducing representation is not sufficient to guarantee the irreducibility of the induced representation $U(g)$, but for a number of groups of interest, notably the space-time groups, it is.

2. Elimination of the Subsidiary Condition: Wigner Functions

It is sometimes convenient to eliminate the subsidiary condition (1.3) and there are two standard methods of doing this, each of which introduces some arbitrariness. The first method is to introduce (arbitrary) representative elements [2] for the cosets

$$\gamma(c) \equiv \gamma(g) = \gamma(kg) \in G, \qquad g \in G, k \in K, c \in C, \tag{2.1}$$

and then to define the new (Wigner) functions simply as

$$\omega(c) = f\big(\gamma(c)\big). \tag{2.2}$$

The $f(g)$ can then be recovered from the $\omega(c)$ by using the subsidiary condition

$$f(g) \equiv f\big(g\gamma^{-1}(g)\,\gamma(g)\big) = D\big(g\gamma^{-1}(g)\big)\, f\big(\gamma(g)\big) = D\big(g\gamma^{-1}(c)\big)\, \omega(c), \tag{2.3}$$

[1]) This part of the construction is valid for any separable, locally compact group, but for simplicity we confine ourselves to Lie groups.

[2]) More generally, quasi-invariant measure $d\mu(cg) = \varrho(c)\, d\mu(c)$. Such measures occur naturally for the dilation group for example.

since $g\gamma^{-1}(g) \in K$. The price we pay for eliminating the subsidiary condition by means of the Wigner functions is that the transformation law becomes more complicated. In fact from (2.2, 2.3 and 1.2) we have

$$\big(U(g)\omega\big)(c) = D\big(\varkappa(c, g)\big)\,\omega(cg), \tag{2.4}$$

$$\varkappa(c, g) = \gamma(c)\, g\gamma^{-1}(cg) \in K. \tag{2.5}$$

However since $D(k)$ is unitary the inner product remains the same,

$$(\omega_1, \omega_2) = \int d\mu(c)\,\big(\omega_1(c), \omega_2(c)\big)_D. \tag{2.6}$$

We call the operators $D\big(\varkappa(c, g)\big)$ Wigner rotations.
Note that if we choose the coset representative $\gamma(c)$ to be continuous in the neighbourhood of the 'origin' $c = \mathring{c} \equiv K$ and choose $\gamma(\mathring{c}) = e$ then from (2.5) we have

$$\varkappa(\mathring{c}, k) = k, \tag{2.7}$$

that is to say, at the origin the Wigner rotation for any element k of the subgroup K is k itself. If as happens in some cases of interest, G has the decomposition

$$G = KZK, \qquad Z \subset G, \tag{2.8}$$

for example the Euler angle decomposition [3] $u(\psi\theta\varphi) = u(oo\psi)\,u(o\theta o)\,u(oo\varphi)$ of $SU(2)$, which is unique up to elements of K that commute with Z, then equation (2.7) can be extended to all values of c. For then we can choose

$$\gamma(c) = \gamma(g) = h^{-1}zh, \quad \text{for} \quad g = h'zh, \quad h, h' \in K, \quad z \in Z, \tag{2.9}$$

and from (2.5) we then obtain

$$\varkappa(c, k) = (h^{-1}zh)\,k\big((hk)^{-1}\,z(hk)\big)^{-1} = k, \tag{2.10}$$

as required. For elements g of G which are not in the subgroup K the calculation of the Wigner angle depends on the group structure and hence has no general form.

3. Elimination of the Subsidiary Condition: Covariant Functions

The second method of eliminating the subsidiary condition (1.3) is to embed the inducing representation $D(k)$ in an (arbitrary) representation $\mathscr{D}(g)$ of G whose restriction $\mathscr{D}(k)$ to K is unitary, but which itself is not necessarily unitary or even faithful. At first sight there might seem to be little advantage in introducing one representation, $\mathscr{D}(g)$, of G in order to construct another, $U(g)$, but the point is that because the only requirement on $\mathscr{D}(g)$ is that $\mathscr{D}(k)$ be unitary and contain $D(k)$, $\mathscr{D}(g)$ may be much more trivial than $U(g)$. The classic example of this situation, which we shall discuss in chapter IV, is the construction of unitary representations $U(g)$ of the Poincaré group from finite-dimensional [4] representations $\mathscr{D}(g)$ of the Lorentz subgroup.
If one now induces with $\mathscr{D}(k)$ instead of $D(k) \subset \mathscr{D}(k)$ and defines the functions

$$\varphi(g) = \mathscr{D}^{-1}(g)\, f(g), \tag{3.1}$$

it is easy to verify that for the new functions the subsidiary condition reads

$$\varphi(kg) = \varphi(g), \qquad k \in K, \quad g \in G,$$

which means that the $\varphi(g)$ actually depend on c only. Hence we can eliminate the subsidiary condition by simply writing the $\varphi(g)$ explicitly as functions of c, i.e. by writing

$$\psi(c) = \varphi(g). \tag{3.2}$$

The other group properties of the $\psi(c)$ can easily be calculated from (3.1, 1.2 and 1.4) and are

transformation law: $$\big(U(g)\,\psi\big)(c) = \mathscr{D}(g)\,\psi(cg) \tag{3.3}$$

inner product: $$(\psi_1, \psi_2) = \int d\mu(c)\,\big(\mathscr{D}(g)\,\psi_1(c), \mathscr{D}(g)\,\psi_2(c)\big)_{\mathscr{D}}$$
$$= \int d\mu(c)\,\big(\psi_1(c), \mathscr{D}^\dagger(g)\,\mathscr{D}(g)\,\psi_2(c)\big)_{\mathscr{D}} \tag{3.4}$$

where the denotes adjoint in $\mathscr{D}$-space. Note that the integrand in (3.4) depends only on c, since

$$\mathscr{D}^\dagger(kg)\,\mathscr{D}(kg) = \mathscr{D}^\dagger(g)\,\mathscr{D}^\dagger(k)\,\mathscr{D}(k)\,\mathscr{D}(g) = \mathscr{D}^\dagger(g)\,\mathscr{D}(g) \tag{3.5}$$

on account of the unitarity of the restriction $\mathscr{D}(k)$. From (3.3) and (3.4) we see that the transformation law is much simpler than for the Wigner functions, but that, unless $\mathscr{D}(g)$ is unitary, the inner product is more complicated. However, there is a further, hidden, complication in the covariant case, namely that if $\mathscr{D}(k)$ contains unitary representations of K other than $D(k)$, then the unwanted representations must be eliminated. This requires a new kind of subsidiary condition which we shall discuss in the next section.

On account of the covariant form of the transformation law (3.3), we call the functions $\psi(c)$ covariant functions. Note that once we have embedded $D(k)$ in $\mathscr{D}(g)$ and induced with $\mathscr{D}(k)$, the representations of the 'rotations' $\gamma(g)\,g^{-1}$ in (2.3) and $\gamma(c)\,g\gamma^{-1}(cg)$ in (2.5)

Table 1

$U(g)$ is the unitary representation of G induced by the unitary representation $D(k)$ of the subgroup K. $g \in G$, $k \in K$, $c \in C = G/K$. $\gamma(c) \in G$ is a representative of the coset c. $\mathscr{D}(g)$ is an embedding representation of G for $D(k)$. $\mathscr{D}(c) \equiv \mathscr{D}(\gamma(c))$. $(\ ,\ )_D$ denotes inner products in the space of $D(k)$ and because $D(k)$ is unitary, the integrands in the inner products for $f(g)$ and $\psi(c)$ are functions of c only.

Functions	Mackey	Wigner	Covariant
Definition	$f(g) = \mathscr{D}(g)\,\psi(c)$ $= \mathscr{D}(g\gamma^{-1}(c))\,\omega(c)$	$\omega(c) = f(\gamma(c))$ $= \mathscr{D}(c)\,\psi(c)$	$\psi(c) = \mathscr{D}^{-1}(g)\,f(g)$ $= \mathscr{D}^{-1}(c)\,\omega(c)$
Transformation Law	$(U(g')\,f)\,(g)$ $= f(gg')$	$(U(g))\,(c)$ $= D(\gamma(c)\,g\gamma^{-1}(cg))\,\omega(cg)$	$(U(g)\psi)\,(c)$ $= \mathscr{D}(g)\,\psi(cg)$
Subsidiary Condition	$f(kg) = D(k)\,f(g)$		
Covariant Subsidiary Condition			$Q(c)\,\psi(c) = \psi(c)$
Inner Product	$\int d\mu(c)\,(f(g), f(g))_D$	$\int d\mu(c)\,(\omega(c), \omega(c))_D$	$\int d\mu(c)\,(\psi(c), \mathscr{D}^\dagger(g)\,\mathscr{D}(g)\,\psi(c))_{\mathscr{D}}$

can be split into $\mathscr{D}(\gamma(g))\,\mathscr{D}^{-1}(g)$ and $\mathscr{D}(\gamma(c))\,\mathscr{D}(g)\,\mathscr{D}^{-1}(\gamma(cg))$ respectively. Using this result we can obtain from (3.1) and (2.2) a direct relationship between the covariant and Wigner functions, namely

$$\omega(c) = \mathscr{D}(c)\,\psi(c). \tag{3.6}$$

The relationship between the Mackey, Wigner and covariant functions is summarized in table 1.

4. Covariant Subsidiary Condition

Let us suppose that the restriction $\mathscr{D}(k)$ of $\mathscr{D}(g)$ to K contains representations of K other than the required inducing representation $D(k)$, and let us suppose for simplicity, that $D(k)$ occurs discretely. Then we can eliminate the unwanted representations of K quite trivially for the Mackey functions by imposing the subsidiary condition

$$Qf(g) = f(g) \qquad \text{or} \qquad Qf(e) = f(e) \tag{4.1}$$

where Q is the projection operator for $D(k) \subset \mathscr{D}(k)$, e is the unit element in G, and the equivalence of the two relations follows from the transformation law (1.2). We wish to express the condition (4.1) in terms of covariant functions. For this purpose we note that from the definition (3.1), $f(e)$ and $\psi(\mathring{c})$ coincide,

$$\psi(\mathring{c}) = f(e), \qquad \mathring{c} \equiv K, \tag{4.2}$$

and hence the second equation in (4.1) is equivalent to the condition

$$Q\psi(\mathring{c}) = \psi(\mathring{c}), \tag{4.3}$$

for the covariant functions at the 'origin' $c = \mathring{c}$. To obtain the equivalent condition for general $\psi(c)$ we then use the covariant transformation law (3.3) and get

$$Q(c)\,\psi(c) = \psi(c), \tag{4.4}$$

where $Q(c)$ is the operator defined by

$$Q(c) = \mathscr{D}^{-1}(g)\,Q\mathscr{D}(g), \qquad g \in c, \tag{4.5}$$

and depends only on c because of the K-invariance of Q,

$$\begin{aligned} \mathscr{D}^{-1}(kg)\,Q\mathscr{D}(kg) &= \mathscr{D}^{-1}(g)\,\mathscr{D}^{-1}(k)\,Q\mathscr{D}(k)\,\mathscr{D}(g) \\ &= \mathscr{D}^{-1}(g)\,Q\mathscr{D}(g). \end{aligned} \tag{4.6}$$

Note that $Q(\mathring{c}) = Q$. It follows from (4.1) that $Q(c)$ must be a scalar with respect to G, and this property is displayed explicitly by the equation

$$\mathscr{D}^{-1}(g)\,Q(cg^{-1})\,\mathscr{D}(g) = Q(c), \tag{4.7}$$

which follows directly from the definition (4.5). A familiar example of $Q(c)$ is the DIRAC [5] operator $(2m)^{-1}\,(\gamma \cdot p + m)$. On account of the scalar property (4.7) we call (4.4) a covariant subsidiary condition.

Note that the covariant form of the induced representation is completely characterized by the pair of quantities $\{\mathscr{D}(g), Q\}$, where $\mathscr{D}(g)$ is the embedding representation of G, and Q is the projection operator defined above, since the transformation law (3.3), the inner product (3.4), and the covariant subsidiary condition (4.4) are completely determined by these two quantities. If $\mathscr{D}(k)$ contains the inducing representation $D(k)$ only once, as often happens, then the pair $\{\mathscr{D}(g), D(k)\}$ suffices for the characterization.

5. Pseudo-Unitary Embedding Representations

A simplification occurs in the inner product for covariant functions when $\mathscr{D}(g)$ is pseudo-unitary,

$$\mathscr{D}^\dagger(g)\, \eta \mathscr{D}(g) = \eta, \qquad \eta^2 = 1, \qquad \eta^\dagger = \eta. \tag{5.1}$$

provided that the inducing representation $D(k) \subset \mathscr{D}(k)$ is irreducible (or, more generally, belongs to a definite eigenspace of η). For then from the subsidiary condition (4.1) for the Mackey states we see that

$$\eta f(g) = \varepsilon f(g) \tag{5.2}$$

where $\varepsilon = \pm 1$, and hence from the definition of $\psi(c)$ in (3.1, 3.2) we have

$$\mathscr{D}^\dagger(g)\, \mathscr{D}(g)\, \psi(c) = \mathscr{D}^\dagger(g)\, f(g) = \varepsilon \mathscr{D}^\dagger(g)\, \eta f(g) = \varepsilon \mathscr{D}^\dagger(g)\, \eta \mathscr{D}(g)\, \psi(c) = \varepsilon \eta \psi(c), \tag{5.3}$$

and inserting this result in the inner product (3.4) we obtain the simplification

$$(\psi_1, \psi_2) = \varepsilon \int d\mu(c) \left(\psi_1(c), \eta \psi_2(c)\right). \tag{5.4}$$

Relevant examples of pseudo-unitary $\mathscr{D}(g)$ are many of the finite-dimensional representations of the classical noncompact [6] groups $SU(n, m)$, $SO(n, m)$, $Sp(n, m)$.

Chapter II. Semi-Direct Products

6. Reduction of the Semi-Direct Product Formalism

We wish to consider induced representations or semi-direct products [7] of the form $A \wedge G$ and subgroups $A \wedge K$ where G and K are as in the last chapter, and we have the same A for both the group and the subgroup. Because A is the same for the group and the subgroup the coset space is the same as before,

$$C = A \wedge G / A \wedge K = G/K, \tag{6.1}$$

and in this section we establish a lemma which shows how this circumstance can be used to simplify the procedure.

Lemma: To induce a representation of $A \wedge G$ with a unitary representation $D(a, k)$ of $A \wedge K$, it is necessary and sufficient to supplement eq. (1.2—1.4) for inducing a representation of G with a unitary representation $D(e, k)$ of K, with the „translation" law

$$\left(U(a, e) f\right)(g) = D(a_g, e)\, f(g), \tag{6.2}$$

where a_g is the element of A into which a is transformed by the action of g in $A \wedge G$.

Proof: We assume the usual semi-direct product transformation law

$$(a, g)\,(a', g') = (a a'_g, g g'). \tag{6.3}$$

Then if for the moment we let the G and K of Chapter I be the present $A \wedge G$ and $A \wedge K$ respectively, the rules (1.2—1.4) for inducing a representation of $A \wedge G$ by the representation $D(a, k)$ of $A \wedge K$ are clearly

transformation law: $$\big(U(a', g')f\big)(a, g) = f(aa'_g, gg'), \tag{6.4}$$

subsidiary condition: $$f\big((b, k)(a, g)\big) = D(b, k) f(a, g), \qquad (b, k) \in A \wedge K, \tag{6.5}$$

nner product: $$(f_1, f_2) = \int d_\mu(c) \big(f_1(a, g), f_2(a, g)\big)_D, \tag{6.6}$$

and what we have to show is that these three equations are equivalent to (1.2—1.4) plus (6.2).
First, we assume (6.4—6.6). Then as a special case of (6.5) we have

$$f(a, g) = f\big((a, e)(e, g)\big) = D(a, e) f(e, g), \tag{6.7}$$

and hence if we set $a = b = e$ throughout (6.4—6.6), defining $f(e, g) = f(g)$, $U(e, g) = U(g)$, $D(e, k) = D(k)$ we obtain at once (1.2—1.4) from (6.4—6.6). Furthermore by setting $g' = a = e$ in (6.4) we obtain

$$\big(U(a', e) f\big)(e, g) = f(a'_g, g) = D(a'_g, e) f(e, g) \tag{6.8}$$

which is just equation (6.2).

To establish the converse we assume (6.2) and (1.2—1.4) with G and K now as in this chapter. Then we define $U(g) = U(e, g)$, $D(k) = D(e, k)$ and

$$f(a, g) = D(a, e) f(g). \tag{6.9}$$

We then have

$$\begin{aligned}\big(U(a', g') f\big)(a, g) &= D(a, e)\big(U(a', g') f\big)(g) \text{ from (6.9)}\\ &= D(a, e)\big(U(a', e)\, U(e, g') f\big)(g)\\ &= D(a, e)\, D(a'_g, e)\big(U(e, g') f\big)(g) \text{ from the translation law (6.2)}\\ &= D(aa'_g, e) f(gg') = f(aa'_g, gg'), \text{ again from (6.9)}\end{aligned}$$

for the transformation law,

$$\begin{aligned}f\big((b, k)(a, g)\big) &= f(ba_k, kg) = D(ba_k, e) f(kg)\\ &= D(ba_k, e)\, D(e, k) f(g) = D(b, k)\, D(a, e) f(g) = D(b, k) f(a, g),\end{aligned}$$

for the subsidiary condition, and

$$\big(f_1(a, g), f_2(a, g)\big)_D = \big(D(a, e) f_1(g), D(a, e) f_2(g)\big)_D = \big(f_1(g), f_2(g)\big)_D,$$

for the inner product. These equations are equivalent to (6.4—6.6) as required.
The advantage of the lemma is that the entire discussion of the last chapter can be applied without change when we extend G and K to $A \wedge G$ and $A \wedge K$ respectively. Note that the induction which was uniquely determined by the pair $\{K, D(k)\}$ for G, is now uniquely determined by the pair $\{K, D(a, k)\}$.

7. Direct Product Inducing Representations and the Little Group Concept

In many cases of interest, notably the space-time groups, it is sufficient to consider only those representations $D(a, k)$ of the inducing group $A \wedge K$ which are direct products of the form

$$D(a, k) = \chi(a) \otimes D(k), \tag{7.1}$$

where $\chi(a)$ and $D(k)$ are unitary representations of A and K respectively. The compatibility of the direct product (7.1) with the semi-direct product structure of $A \wedge K$ itself imposes, however, a condition on (7.1), namely, that $\chi(a)$ should not be affected by the action of K on A, i.e. that up to equivalence

$$\chi(a) = \chi(a_k), \qquad a \in A, \qquad k \in K. \tag{7.2}$$

Equation (7.2) can also be obtained formally by combining (7.1) with the group transformation law (6.3). The condition (7.2) is a relationship between the subgroup K and the representation $\chi(a)$ of A, and, if $\chi(a)$ is chosen first, then K must be chosen so that it leaves $\chi(a)$ invariant. If K is the *maximal* closed subgroup of G leaving $\chi(a)$ invariant, it is called the *little group* [2] of $\chi(a)$. This is the case of greatest interest and henceforth we shall assume that the inducing subgroup K of G is the little group of $\chi(a)$. Note that since K is then uniquely determined by $\chi(a)$, the induction with $\{A \wedge K, \chi(a) \otimes D(k)\}$ is uniquely determined by $\chi(a) \otimes D(k)$. Note also how the simple idea of direct product representation for the inducing subgroup $A \wedge K$ leads immediately to the less simple, but very useful, concept of the little group. Examples of equation (7.2) will be given in sections (10), (14), (15), (16).

8. Orbits

Equation (7.2) suggests the study of $\chi(a_g)$ for g *not* in the little group. In that case $\chi(a_g)$ is not equivalent to $\chi(a)$, but since $\chi(a_g b_g) = \chi\big((ab)_g\big)$ it is still a representation, and we denote it by

$$\chi_g(a) = \chi(a_g). \tag{8.1}$$

From (8.1, 7.2) and the maximality of K, it is clear that $\chi_{g_1}(a)$ and $\chi_{g_2}(a)$ will be equal if, and only if, g_1 and g_2 belong to the same coset $c \in C = G/K$, and hence that it would be more appropriate to label the χ 's according to

$$\chi_c(a) = \chi(a_g), \qquad g \in c \in C = G/K. \tag{8.2}$$

The representation $\chi_c(a)$ generated from $\chi(a)$ by (8.2) are said to form the *orbit* [1, 7] of $\chi(a)$.

The importance of the orbits is that the representations $\chi_c(a)$ in the same orbit have the same little group K, and the representation of $A \wedge G$ induced by $\{A \wedge K, \chi_c(a) \otimes D(k)\}$ for fixed $D(k)$ are equivalent. To see this we note first from (7.2) that if k is in the little group of $\chi(a)$, then gkg^{-1}, $g \in c$ is in the little group of $\chi_c(a)$, so that the little groups are the same up to conjugation. Second, we note that the Mackey equations (1.2—1.4) for the induction of G with $\{K, D(k)\}$ are independent of c. Finally from (8.2) we see that the supplementary transformation law (6.2) can be written in the present case as

$$\big(U(a, e)\, f\big)\,(g) = \chi_c(a)\, f(g), \qquad g \in c, \tag{8.3}$$

and so is independent of the initial point $\chi(a)$ in the orbit. Thus for the direct product case $A \wedge G$ the induction with $\{A \wedge K, \chi(a) \otimes D(k)\}$ is completely determined by the orbits and the $D(k)$.
Note that (8.3) allows the immediate transfer of the supplementary transformation law to the Wigner and covariant functions, which depend only on c. For example, for the covariant states we have

$$\big(U(a, e)\, \psi\big)\,(c) = \chi_c(a)\, \psi(c)\,. \tag{8.4}$$

9. Configuration Space Functions

The existence of orbits in the case when the inducing representation is a direct product affords us a method of expressing the induced representations as transformations on functions over the invariant subgraph A as follows: We define the functions of A by the generalized Fourier transform

$$\varphi(a) = \int d\mu(c)\, \chi_c^{-1}(a)\, \psi(c) \tag{9.1}$$

for any orbit $\chi_c(a)$, where the $\psi(c)$ are the covariant functions of sect. 3. Then since we are inducing with direct product $\chi(a) \otimes D(k)$ we also have the direct product $\chi(a) \otimes \mathscr{D}(g)$, where $\mathscr{D}(g)$ is the embedding representation, and hence

$$\begin{aligned}\big(U(a, g)\, \varphi\big)\,(x) &= \int d\mu(c)\, \chi_c^{-1}(x)\, \mathscr{D}(g)\, \chi_c(a)\, \psi(cg)\\ &= \mathscr{D}(g) \int d\mu(c)\, \chi_c^{-1}(a^{-1}x)\, \psi(cg)\\ &= \mathscr{D}(g) \int d\mu(c)\, \chi_{cg^{-1}}^{-1}(a^{-1}x)\, \psi(c)\\ &= \mathscr{D}(g) \int d\mu(c)\, \chi_c^{-1}\big((a^{-1}x)_g\big)\, \psi(c)\\ &= \mathscr{D}(g)\, \varphi\big((a^{-1}x)_{g^{-1}}\big)\,,\end{aligned} \tag{9.2}$$

is the transformation law for the $\varphi(x)$. In the case of the Poincaré group it reduces to the well-known transformation law [8]

$$\big(U(a, \Lambda)\varphi\big)\,(x) = \varphi\big(\Lambda^{-1}(x - a)\big) \tag{9.3}$$

for fields in configuration space, and for this reason we call the functions $\varphi(a)$ configuration space functions. In general the inner product is not simple or local in configuration space, a point which we shall consider in detail for the Poincare group in Part II.

Chapter III. Abelian Invariant Subgroups A

10. Orbits for Abelian A

We specialize now to the case when the invariant subgroup A in the semi-direct product $A \wedge G$ is abelian. In this case the preceding formalism is particularly relevant, because as we shall see, we then have the remarkable result [1] that the representations induced by $\{A \wedge K, \chi(a) \otimes D(k)\}$, where K is the little group of $\chi(a)$ are both *irreducible* and *exahustive* (the latter subject to a mild technical condition which is certainly satisfied for the ordinary space-time groups). We shall establish these two properties in the following sections, and prepare the ground here by investigating the orbits for abelian A. For abelian A the unitary irreducible representations $\chi(a)$ in $\chi(a) \otimes D(k)$ are necessarily one-dimensional and of the form

$$\chi(a) = e^{i\mathring{p}\cdot a}, \qquad \mathring{p}\cdot a = \sum_{r=1}^{n} \mathring{p}_r a_r \tag{10.1}$$

where a_r are the parameters of A and the vector $\mathring{p}_r$ which characterizes the representation $\chi(a)$ is any set of n real numbers, i.e. any vector in Euclidean space E_n. Furthermore, since A is a vector space with group multiplication equal to vector summation $ab = a + b$, the action of G on A is linear

$$a_g = \Lambda(g)\, a\,, \tag{10.2}$$

where $\Lambda(g)$ is an $n \times n$ matrix representation of G. Since in the present case

$$\mathring{p}\cdot a_g = \mathring{p}\cdot \Lambda(g)a = \tilde{\Lambda}(g)\mathring{p}\cdot a \tag{10.3}$$

where $\tilde{\Lambda}(g)$ is the transpose of $\Lambda(g)$, we see that the orbital equation $\chi_c(a) = \chi(a_g)$ reduces to

$$p_c = \tilde{\Lambda}(g)\,\mathring{p}. \tag{10.4}$$

Thus for abelian A the abstract orbits reduce to the geometrical orbits $\tilde{\Lambda}(g)\mathring{p}$ (hence the name orbit). In particular the little group condition $\chi(a) = \chi(a_k)$ reduces to

$$\mathring{p} = \tilde{\Lambda}(k)\,\mathring{p}\,, \qquad k \in K. \tag{10.5}$$

so that the little group is just the stability group of the vector $\mathring{p}$ in E_n.
Since the relationship between the coset space $C = G/K$ and the orbit $\tilde{\Lambda}(g)\mathring{p}$ is one-one, it is conventional to parametrize the cosets by the points p on the geometrical orbits rather than the other way around and hence it is more conventional to write (10.4) as

$$p = \tilde{\Lambda}(g)\,\mathring{p}\,, \tag{10.6}$$

and to replace the coset variable c by p throughout. This convention we shall adopt henceforth.
Note that in the abelian case the orbits are completely determined by the action of G on A, i.e. by the group structure $A \wedge G$ itself. Since we saw in section 8 that the induction with $\{A \wedge K, \chi(a) \otimes D(k)\}$ is completely determined by the orbits and the $D(k)$ we see now that it is completely determined by the action of G on A and the $D(k)$.

11. Irreducibility of the Induced Representations for Abelian A

From (1.4) we see that the Hilbert space for the induced representation $U(a, g)$ can be written as the direct product

$$\mathscr{H} = \mathscr{L}_2(p) \otimes \mathscr{H}_D \tag{11.1}$$

where $\mathscr{L}_2(p)$ has the measure $d\mu(p) \equiv d\mu(c)$ and $\mathscr{H}_D$ is the Hilbert space for $D(k)$. Now let X be any bounded operator on $\mathscr{H}$ which commutes with $U(a, g)$. Then it will commute in particular with $U(a, e)$, and since $U(a, e)$ acts in $\mathscr{H}$ according to

$$\big(U(a, e)\, f\big)\,(g) = e^{ipa}\, f(g)\,, \qquad p = \tilde{\Lambda}(g)\,\mathring{p}\,, \tag{11.2}$$

it follows from the n-dimensional spectral theorem (SNAG theorem [9]) that X must reduce to a bounded measurable operator valued function $X(p)$ of p on $\mathscr{H}_D$. That is to say, on $\mathscr{H}_D$,

$$Xf(g) = X(p)\, f(g)\,, \qquad p = \tilde{\Lambda}(g)\, \mathring{p}\,. \tag{11.3}$$

But then since $f(kg)$ belongs to the same coset as $f(g)$, we also have $Xf(kg) = X(p)\, f(kg)$, and comparing this equation with the subsidiary condition (1.3) and (11.3) we have

$$[X(p), D(k)] = 0\,, \qquad \text{on} \qquad \mathscr{H}_D\,. \tag{11.4}$$

It follows that if $D(k)$ is irreducible $X(p)$ is a multiple of the identity for each value of p. Finally, the compatibility of (11.3) with the transformation law (1.2) requires that

$$X(p) = X\big(\tilde{\Lambda}(g)\, p\big) \tag{11.5}$$

which shows that $X(p)$ is the same multiple of the identity for each point on the same orbit. Thus for a fixed orbit and irreducible $D(k)$ any bounded operator X which commutes with $U(a, g)$ is a multiple of the identity on $\mathscr{H}$, and the induced representation is irreducible as required.

12. Exhaustivity of the Induced Representations for Abelian A

Let $U(a, g)$ be any continuous unitary representation of $A \wedge G$. Since A is abelian, according to the n-dimensional spectral theorem [9] the most general continuous unitary representation of the restriction $U(a, e)$ of $U(a, s)$ to A is of the form

$$\big(\omega, U(a, e)\, \omega\big)_{\mathscr{H}} = \int d\nu(p)\, e^{ip\cdot a}\big(\omega(p), \omega(p)\big)_{\mathscr{H}(p)} \tag{12.1}$$

where $p \in E_n$, $\nu(p)$ is a positive measure in E_n, and the $\mathscr{H}(p)$ are (not necessarily identical) Hilbert spaces.

The first step is to show that the measure $\nu(p)$ is quasi-invariant with respect to G. For this purpose we take the expectation value of the semi-direct product condition

$$U(a, e) = U^{\dagger}(e, g)\, U\big(\Lambda(g) a, e\big)\, U(e, g) \tag{12.1}$$

with respect to ω, and obtain from (12.1) the relation

$$\int d\nu(p)\; e^{ip\cdot a}\big(\omega(p),\, \omega(p)\big)_{(p)} = \int d\nu(p)\; e^{ip\cdot\Lambda a}\big(\omega_g(p),\, \omega_g(p)\big)_{\mathscr{H}(p)}\,,\;\; \omega_g(p) \equiv \big(U(e,\, g)\;\; \omega\big)\, (p)\,,$$

which, on writing $p.\, \Lambda a$ in the form $\tilde{\Lambda} p \cdot a$ and integrating over any sufficiently smooth function $\tilde{\varrho}(a)$ with Fourier transform $\varrho(p)$, takes the form

$$\int d\nu(p)\, \varrho(p)\, \big(\omega(p), \omega(p)\big)_{\mathscr{H}(p)} = \int d\nu(p)\, \varrho\big(\tilde{\Lambda} p\big)\, \big(\omega_g(p), \omega_g(p)\big)_{\mathscr{H}(p)}\,. \tag{12.3}$$

Then choosing $\varrho(p)$ and $\big(\omega(p), \omega(p)\big)_{(p)}$ to be the characteristic functions of any interval I of E_n, we have from (12.3) the relation

$$\nu(I) = \int_{I'} d\nu(p)\, \big(\omega_g(p), \omega_g(p)\big)_{\mathscr{H}(p)}\,, \tag{12.4}$$

where I' is the $\tilde{\Lambda}$-transform of I, *and* $\omega_g(p)$ *is square-integrable.* It follows that $\nu(I') = 0$ implies $\nu(I) = 0$. Similarly $\nu(I) = 0$ implies $\nu(I') = 0$. Thus $\nu(p)$ is quasi-G-invariant, as required.

From the RADON-NIKODYM theorem [10] we then have

$$d\nu\big(\tilde{\Lambda}(g)\, p\big) = \varrho_g(p)\, d\nu(p)\,, \tag{12.5}$$

where $\varrho_g(p)$ is a positive function. For simplicity, and because there is no essential difficulty in extending the remaining arguments to the general case, we shall assume henceforth that $\varrho_g(p)$ is unity, i.e. that the measure is G-invariant. Feeding (12.5) with $\varrho_g(p) = 1$ back into (12.3) and changing the variable $\tilde{\Lambda} p$ to p on the right hand side, we obtain the relation

$$\big(\omega(p), \omega(p)\big)_{\mathscr{H}(p)} = \big(\omega_g(\tilde{\Lambda}^{-1}p), \omega_g(\tilde{\Lambda}^{-1}p)\big)_{\mathscr{H}(p)} \tag{12.6}$$

almost everywhere.

The next step is to partition E_n into orbits with respect to $\tilde{\Lambda}$, and it is at this point that we need the mild technical condition (1) mentioned earlier. The condition is simply that the orbits of non-zero ν-measure exhaust E_n. (The possibility that is excluded by this condition is the existence of so-called strictly ergodic subspaces of E_n, i.e. subspaces which are themselves of nonzero ν-measure but which consist of orbits of zero ν-measure[1]). It is clear that once the orbits of nonzero ν-measure exhaust E_n, it is sufficient to prove induction on each orbit. For the space-time groups, for which E_n is momentum space and the action of $\tilde{\Lambda}$ the rotations and accelerations therein, the only invariant measures are the conventional physical ones, and for these measures the orbits certainly exhaust E_n.

The final step in the proof is to map the well-behaved functions $\omega(p)$ on the orbits into functions[3]) $f(g)$ on G by the relation

$$f(g) = \big(U(g)\,\omega\big)\,(\mathring{p}) \equiv \omega_g(\mathring{p}) \tag{12.7}$$

where $\mathring{p}$ is any fixed point in an orbit, and to show that the $f(g)$ are Mackey functions, i.e. satisfy the Mackey equations (1.2, 1.3, 1.4).

First, under the group transformation g' we have

$$f(g) \to f'(g) = \big(U(g)\,\omega_{g'}\big)\,(\mathring{p}) = \big(U(g)\,U(g')\,\omega\big)\,(\mathring{p}) = \big(U(gg')\,\omega\big)\,(\mathring{p}) = f(gg'), \tag{12.8}$$

so that (1.2) is satisfied.

Second, from (12.6) and (12.7) we have

$$\big(f(g), f(g)\big)_{\mathscr{H}(p)} = \big(\omega(p), \omega(p)\big)_{\mathscr{H}(p)}, \qquad p = \tilde{\Lambda}\mathring{p} \tag{12.9}$$

which shows that $\big(f(g), f(g)\big)_{\mathscr{H}(p)}$ is measurable, and

$$\int d\nu(p)\,\big(f(g), f(g)\big)_{\mathscr{H}(p)} < \infty, \tag{12.10}$$

so that (1.4) is satisfied.

Finally to establish (1.3) we consider the transformations $f(g) \to f(kg)$ where k is an element of the little group K of $\mathring{p}$. From (12.7) these transformations are linear, and from (12.9) they satisfy the relation

$$\big(f(kg), f(kg)\big)_{\mathscr{H}(p)} = \big(f(g), f(g)\big)_{\mathscr{H}(p)}, \tag{12.11}$$

that is to say, they preserve the inner products on $\mathscr{H}(p)$. It follows that they form unitary representations $D(k, p)$ of K on $\mathscr{H}(p)$,

$$f(kg) = D(k, p)\,f(g). \tag{12.12}$$

But the compatibility of (12.12) and the transformation law (12.7) requires that the $D(k, p)$ be independent of p. Thus all the $D(k, p)$ form the *same* unitary representation $D(k, p) = D(k)$ of K, and (1.3) is satisfied as required.

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[3]) We assume the existence of a dense set of functions $\omega(p)$ such that $\big(U(g)\,\omega\big)\,(p)$ exists pointwise for all p in the orbit and $g \in G$. For connected Lie groups, such a set is provided by any $U(g)$-invariant dense set S of well-behaved vectors for $U(g)$ (e.g. differentiable vectors [*11a*]). Since the coset space G/K corresponding to a given orbit can be parametrized [*11b*] by a set of (at least piecewise) well-behaved coset representatives $\gamma(p) \in G$. Then S is a set of well-behaved vectors for $U(\gamma(p))$, and hence is a set of well-behaved functions of $\gamma(p)$ and, since $\Lambda(\gamma(p))$ is entire, of $p = \Lambda(\gamma(p)\,)\mathring{p}$.

It remains to show that if the induced representation $U(a, g)$ is continuous, then the inducing representation $D(k)$ must also be continuous. To show this, we note that if g is any element of a compact subspace of G, $g^{-1}kg \to e$ uniformly for $k \to e$, and hence if $f(g)$ is any function of compact support,

$$D(k)\, f(g) = f(kg) = f\big(g(g^{-1}kg)\big) \to f(g)\,, \tag{12.13}$$

for $k \to e$.

Chapter IV. Space-Time Groups

13. General

The space-time groups we consider are the connected Euclidean [9, 12] group $\mathscr{E}(3)_+ = A_3 \wedge SO(3)_+$, the connected Galilean [9, 12] group $\mathscr{G}_+^\uparrow = A_4 \wedge \mathscr{E}(3)_+$ and the connected Poincaré group [9] $\mathscr{P}_+^\uparrow = A_4 \wedge SO(3, 1)_+^\uparrow$, where A_n is the translation group in n dimensions. However, since, from the physical point of view, projective unitary as well as true unitary representations of these groups are allowed [13], it will be more convenient to treat not the above groups themselves but groups whose true unitary representations are the projective representations of these groups. For $\mathscr{E}(3)_+$ and $\mathscr{P}_+^\uparrow$ the groups in question are well-known [2, 13] to be the covering groups $\tilde{\mathscr{E}}(3)_+ = A_3 \wedge SU(2)$ and $\tilde{\mathscr{P}}_+^\uparrow = A_4 \wedge SL(2, C)$ respectively, the correspondence being two-to-one. For $\mathscr{G}_+^\uparrow$ the group in question is more complicated [9, 12], namely $\tilde{\mathscr{G}}_+^\uparrow = A_5 \wedge \tilde{\mathscr{E}}(3)_+$ where in addition to the usual two-to-one covering obtained by replacing $\mathscr{E}(3)_+$ by $\tilde{\mathscr{E}}(3)_+$, the translation group is increased to five dimensions by the introduction of a new parameter a_5 so that the group action [9, 12] on A_5 is

$$A(\boldsymbol{v}, u) \begin{pmatrix} a_0 \\ \boldsymbol{a} \\ a_5 \end{pmatrix} = \begin{pmatrix} a_0 \\ R(u)\,\boldsymbol{a} + \boldsymbol{v} a_0 \\ a_5 + \boldsymbol{v} \cdot R(u)\,\boldsymbol{a} + \dfrac{1}{2}\,\boldsymbol{v}^2 a_0 \end{pmatrix}, \tag{13.1}$$

where $(a_0\, \boldsymbol{a}\, a_5)$ are the five parameters of A_5, and $(\boldsymbol{v}, u)$ are the parameters of $\tilde{\mathscr{E}}(3)_+ = A_3 \wedge SU(2)$, u being the Euler angles [3] for $SU(2)$, and $\boldsymbol{v}$ the (commuting) accelerations in A_3.

The groups $\tilde{\mathscr{E}}(3)_+$, $\tilde{\mathscr{G}}_+^\uparrow$ and $\tilde{\mathscr{P}}_+^\uparrow$ are all connected Lie groups of the form $A \wedge G$ where A is abelian. Hence the analyses of the preceding chapters applies to these groups, and we shall now use these groups to illustrate the analyses. We proceed as follows:

1) From the action of G on A, we find the action of $\tilde{\Lambda}(g)$ on the character space E_n, and hence determine the orbits and their little groups.
2) We write down the Mackey equations.
3) We write down the corresponding (a) Wigner and (b) covariant states.

In all cases we combine the transformation laws for A and G separately into a single law for $A \wedge G$ according to

$$U(a, e)\, U(e, g) = U(a, g)\,. \tag{13.2}$$

14. The Euclidean Group

1) We parametrize $\tilde{\mathscr{E}}(3)_+ = A_3 \wedge SU(2)$ according to $(\boldsymbol{a}, u)$ where $u = (\psi, \theta, \varphi)$ are the Euler angles [3]. The action of $SU(2)$ on A_3 is the ordinary $SO(3)_+$ action and hence its action on the character space is the transpose (or, equivalently, inverse) $SO(3)_+$ action.

It follows that the orbits are the spheres $p_1^2 + p_2^2 + p_3^2 = \text{constant}$ in E_3, and that the little groups are $SU(2)$ itself in the trivial case $p^2 = 0$ and $U(1)$ otherwise.
2) In the trivial case $p^2 = 0$ the induced representations are simply the representations of $SU(2)$. In the nontrivial case the Mackey equations are clearly

$$\begin{aligned} \big(U(\boldsymbol{a}, u')\, f\big)\,(u) &= e^{i\boldsymbol{p}\cdot\boldsymbol{a}}\, f(uu') \\ f([\psi]\, u) &= e^{i\lambda\psi}\, f(u)\,, \qquad [\psi] \equiv (\psi, 0, 0) \\ (f_1, f_2) &= \int d(\cos\theta)\, d\varphi f^*{}_1\,(u)\, f_2(u)\,, \end{aligned} \tag{14.1}$$

where the fixed vector $\mathring{\boldsymbol{p}}$ is $\mathring{\boldsymbol{p}} = p(001)$, λ is the character of the one-dimensional representation of the little group $K = U(1)$, and $\boldsymbol{p} = \tilde{R}(u)\, \mathring{\boldsymbol{p}} = R^{-1}(u)\, \mathring{\boldsymbol{p}} = p(\sin\theta \sin\varphi, -\sin\theta\cos\varphi, \cos\theta)$.
3a) The corresponding Wigner equations are

$$\begin{aligned} \big(U(\boldsymbol{a}, u)\, \omega\big)\,(\boldsymbol{p}) &= e^{i\boldsymbol{p}\cdot\boldsymbol{a}}\, e^{i\lambda\xi}\, \omega\big(R^{-1}(u)\, \boldsymbol{p}\big)\,, \\ (\omega_1, \omega_2) &= \int d(\cos\theta)\, d\varphi\, \omega_1{}^*(p)\, \omega_2(p)\,, \end{aligned} \tag{14.2}$$

where ξ is the Wigner rotation given by

$$(\xi, 0, 0) = U(\boldsymbol{p})\, UU^{-1}\big(R^{-1}(u)\, p\big)\,, \tag{14.3}$$

$u(\boldsymbol{p})$ being the coset representative for $SU(2)/U(1)$. Since $SU(2)$ has the decomposition KZK discussed in section 2 we can choose $u(\boldsymbol{p})$ in the manner described in that section, and in the present case that is easily seen to be $u(\boldsymbol{p}) = u(\varphi, -\theta, -\varphi)$ where (θ, φ) are the polar angles of $\boldsymbol{p}$. A simple calculation then shows that ξ is given by

$$\cos(\xi - \varphi + \varphi') = \cos(\alpha - \varphi)\cos(\gamma + \varphi') + \sin(\alpha - \varphi)\sin(\gamma + \varphi')\cos\beta\,, \tag{14.4}$$

where $u = (\alpha, \beta, \gamma)$ and (θ', φ') are the polar angles of $R^{-1}(u)\,\boldsymbol{p}$.
3b) Since the only representations of $SU(2)$ in which the representation $\exp\,(i\lambda\psi)$ of $U(1)$ can be embedded are the full unitary representations $D^j(u)$, the covariant procedure is somewhat redundant for $\xi(3)_+$. However, the covariant equations are quite simple, namely,

$$\begin{aligned} \big(U(\boldsymbol{a}, u)\, \psi\big)\,(\boldsymbol{p}) &= e^{i\boldsymbol{p}\cdot\boldsymbol{a}}\, D^j(u)\, \psi\,\big(R^{-1}(u)\, \boldsymbol{p}\big) \\ (\psi_1, \psi_2) &= \int d(\cos\theta)\, d\varphi\, \psi_1{}^\dagger(\boldsymbol{p})\, \psi_2(\boldsymbol{p})\,, \end{aligned} \tag{14.5}$$

where the dagger denotes adjoint in the $(2j + 1)$-dimensional space of $D^j(u)$. To find the covariant subsidiary condition which eliminates the $2j$ redundant components of $\psi(\boldsymbol{p})$ we note that since the generator $\boldsymbol{\Sigma}_3$ of $U(1)$ has the diagonal form diag $(j, j - 1, \ldots -j)$ the simplest way to express the covariant subsidiary condition is to replace the strict projective relation $Q\psi(\mathring{\boldsymbol{p}}) = \psi(\mathring{\boldsymbol{p}})$ of the general theory with

$$\boldsymbol{\Sigma}_3\psi(\mathring{\boldsymbol{p}}) = \lambda\psi(\mathring{\boldsymbol{p}}) \tag{14.6}$$

and then the covariant subsidiary condition is clearly

$$(\boldsymbol{\Sigma}\cdot\boldsymbol{p})\, \psi(\boldsymbol{p}) = \lambda p \psi(\boldsymbol{p}) \tag{14.7}$$

where $\boldsymbol{\Sigma}$ are the generators of $SU(2)$ and $p = |\boldsymbol{p}|$. The configuration space functions are clearly

$$\varphi(x) = \int d\,(\cos\theta)\, d\varphi\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\, \psi(\boldsymbol{p})\,, \qquad p^2 = \text{constant}. \tag{14.8}$$

15. Galilean Group

1) From the action (13.1) of $\tilde{\mathscr{E}}(3)_+$ on A_5, we see at once that the action of $\tilde{\Lambda}(v, u)$ on the character space $p = (E, \boldsymbol{p}, m)$ is

$$\tilde{\Lambda}(\boldsymbol{v}, u)\begin{pmatrix}E\\ \boldsymbol{p}\\ m\end{pmatrix} = \begin{pmatrix}E - \boldsymbol{p}\cdot\boldsymbol{v} + \frac{1}{2}\,mv^2\\ R^{-1}(u)\,(\boldsymbol{p} - m\boldsymbol{v})\\ m\end{pmatrix} \tag{15.1}$$

and hence that the orbits are $m =$ constant, $p^2 - 2mE =$ constant. We neglect the trivial orbits $p^2 = m = 0$, $E =$ constant. On the others E is identified with energy, $\boldsymbol{p}$ with three-momentum and m with mass, and only the massless orbits $m = 0$ correspond to true (non-projective) unitary representations of the Galilean group $A_4 \wedge \mathscr{E}(3)_+$. The little groups for the orbits $m \neq 0$ and $m = 0$ are $SU(2)$ and $E(2)$ respectively.

In this section we shall treat only the massive case $m \neq 0$. The massless case $m = 0$ is less interesting physically and more complicated mathematically, but for completeness it is treated in the appendix.

Massive Case, $m \neq 0$.

2) The Mackey equations are

$$\begin{aligned}\big(U(a, \boldsymbol{v}, u')\,f\big)\,(\boldsymbol{v}, u) &= e^{ip\cdot a}f\big(\boldsymbol{v} + R(u)\,\boldsymbol{v}', uu'\big),\; p\cdot a = Ea_0 - \boldsymbol{p}\cdot\boldsymbol{a} + ma_5,\\ f\big((e, u')\,(\boldsymbol{v}, u)\big) &= D^j(u')\,f\big((\boldsymbol{v}, u)\big),\\ (f_1, f_2) &= \int d^3p f_1 + (\boldsymbol{v}, u)\,f_2(\boldsymbol{v}, u),\end{aligned} \tag{15.2}$$

where $D^j(u)$ is the inducing representation $SU(2)$, and dagger denotes adjoint in the corresponding $(2j + 1)$-dimensional space.

3) The most characteristic feature of the massive Galilean case is that if we choose the coset representatives of $\tilde{E}(3)/SU(2)$ to be the simplest possible, namely $\big(\boldsymbol{v}(\boldsymbol{p}), 0\big)$ where $\boldsymbol{v}(\boldsymbol{p}) = -\boldsymbol{p}/m$, then the Wigner and covariant functions coincide and have simple transformation properties. This is because the semi-direct product structure of $\tilde{\mathscr{E}}(3)_+$ allows us to regard the inducing representations of $SU(2)$ as (non-faithful) embedding representations of $\tilde{\mathscr{E}}(3)_+$, and then from sections 2 and 3 we have $\omega(\boldsymbol{p}) = \psi(\boldsymbol{p})$, with

$$\begin{aligned}\big(U(a, \boldsymbol{v}, u)\,\omega\big)\,(\boldsymbol{p}) &= e^{ip\cdot a}\,D^j(u)\,\omega\big(R^{-1}(u)\,(\boldsymbol{p} - m\boldsymbol{v})\big),\\ (\omega_1, \omega_2) &= \int d^3p\,\omega_1{}^\dagger(\boldsymbol{p})\,\omega_2(\boldsymbol{p}),\end{aligned} \tag{15.3}$$

and no covariant subsidiary condition.

To find the projective representations $V(g)$ of $\mathscr{G}(3)_+$ from the true representations $U(g)$ of $\tilde{\mathscr{G}}(3)_+$, we simply put $a_5 = 0$ in (15.2) and (15.3). The operators $V(g) = U(a_0, \boldsymbol{a}, 0, \boldsymbol{v}, u)$ then satisfy the relation

$$V(g')V(g) = e^{im\sigma(g',g)}V(g'g), \qquad \sigma(g', g) = \big(\Lambda(g')\,a\big)_5 - a_5 = \frac{V'^2}{2}\,a_0 + \boldsymbol{v}'\cdot R(u)\,\boldsymbol{a}. \tag{15.4}$$

The representations for different values of the constant $p^2 - 2mE$ are inequivalent for $\tilde{\mathscr{G}}(3)_+$ but are ray-equivalent for $\mathscr{G}(3)_+$, and hence without loss of physical generality we may set $E = p^2/2m$. The configurationspace functions are then defined by

$$\varphi(t, \boldsymbol{x}) = \int d^3p e^{-i\left(\frac{\boldsymbol{p}^2}{2m}t - \boldsymbol{p}\cdot\boldsymbol{x}\right)}\psi(p), \tag{15.5}$$

and they have the transformation property

$$\big(V(g)\,\varphi\big)\,(t, \boldsymbol{x}) = e^{-im\delta} D^j(u)\,\varphi\Big(t - a_0, R^{-1}(u)\,\big(\boldsymbol{x} - \boldsymbol{v}(t - a_0) - \boldsymbol{a}\big)\Big),$$

$$\delta = \frac{\boldsymbol{v}^2}{2}\,(t - a_0) - \boldsymbol{v}\cdot(\boldsymbol{x} - \boldsymbol{a}). \qquad (15.6)$$

16. The Poincaré Group

1) The action of $SL(2, C)$ on A_4 in the Poincaré case is the ordinary Lorentz action $\Lambda(s)\,a$, where $\Lambda(s)$ is the Lorentz transformation corresponding to $s \in SL(2, C)$. Hence the action in orbit space is $\tilde{\Lambda}(s)$, and if we take p to be contravariant so that $p \cdot a$ denotes Minkowski inner product, it is $g\tilde{\Lambda}(s)\,g^{-1} = \Lambda^{-1}(s)$, where g is the metric tensor. Hence the orbits are $(p^2 > 0,\ p_0 \gtrless 0)$ $(p^2 = 0,\ p_0 \gtrless 0)$, $p^2 < 0$ and $p = 0$, and the corresponding little groups are $SU(2)$, $\tilde{E}(2)$, $SU(1,1)$ and $SL(2, C)$ respectively. In general p is identified with the energy-momentum vector, and those orbits $p^2 \geqq 0$ for which the energy p_0 is definite are called 'physical'. For irreducible representations they may be identified with stable particles. The other orbits also enter into physics, but less directly.

2) Leaving aside the trivial case $p = 0$, the Mackey equations for the Poincaré group are

$$\big(U(a, s')\,f\big)\,(s) = e^{ip\cdot a} f(ss')$$

$$f(ks) = D(k)\,f(s) \qquad (16.1)$$

$$(f_1, f_2) = \int d\mu(p)\,\big(f_1(s), f_2(s)\big)_D,$$

where $s \in SL(2, C)$, $k \in K = SU(2)$, $\tilde{E}(2)$, $SU(1,1)$, D is the inducing representation, $p = \Lambda^{-1}(s)\,\mathring{p}$ where $\mathring{p}$ is a fixed vector in the orbit, and $d\mu(p) = d^3p/|p_0|$ for $p^2 \geqq 0$ and $dp_0\,dp_1\,dp_2/|p_3|$ for $p^2 < 0$.

3) In this section and chapter we shall confine our attention to the Poincaré Wigner states [*14, 13*], leaving the important covariant and configuration-space states [*14, 13, 2*] to Part II of the paper. The Wigner equations corresponding to (16.1) are

$$\begin{aligned}\big(U(a, s)\,\omega\big)\,(p) &= e^{ip\cdot a}\, D\big(\gamma(p)\,s\gamma^{-1}(\Lambda^{-1}(s)\,p\big)\,\omega\big(\Lambda^{-1}(s)\,p\big),\\ (\omega_1, \omega_2) &= \int d\mu(p)\,\big(\omega_1(p), \omega_2(p)\big)_D\end{aligned} \qquad (16.2)$$

where $\gamma(p)$ are the coset representatives which we shall now discuss for $p^2 \geqq 0$.
Since $SL(2, C)/\big(SU(2)$ and $\tilde{E}(2)\big)$ are in one-one correspondence with $SO(3,1)^{\uparrow}_{+}/SO(3)_{+}$ and $E(2)\big)$ respectively, it is usual to take coset representatives $\mathscr{L}(p)$ in $SO(3,1)^{\uparrow}_{+}$ instead of $\gamma(p)$ in $SL(2, C)$. In the massive case $p^2 > 0$ the little group is $K = SU(2)$ and so the wellkown decomposition [*14*]

$$\Lambda = R(\alpha\beta\gamma)\,Z(\chi)\,R(\psi\theta\varphi) \qquad (16.3)$$

of $SO(3,1)^{\uparrow}_{+}$, where $Z(\chi)$ is a pure Lorentz transformation along the positive third axis with hyperbolic angle χ, is a decomposition of the kind discussed in section 2. Hence we can make [*15*] the choice of $\mathscr{L}(p)$ discussed in that section, namely,

$$\mathscr{L}^{-1}(p) = R(\varphi, \theta, -\varphi)\,Z(\chi)\,R^{-1}(\varphi, \theta, -\varphi), \qquad p = \mathscr{L}^{-1}(p)\,\mathring{p}, \qquad (16.4)$$

where (θ, φ) are the polar angles of $\boldsymbol{p}$ and $\cosh \chi = p_0/m$, and, as shown in that section, the Wigner rotations corresponding to physical rotations $R(\alpha, \beta, \gamma)$ are the rotations themselves.

Another choice of $\mathscr{L}(p)$, due to Jacob and Wick [16], which is not well-defined at the origin $p = 0$, but which is very useful for describing scattering processes is

$$\mathscr{L}^{-1}(p) = R(\varphi, \theta, -\varphi)\, Z(\chi). \tag{16.5}$$

In this case the Wigner angle corresponding to rotations is easily seen to reduce to

$$\begin{aligned} &Z^{-1}(\chi)\, R^{-1}(\varphi, \theta, -\varphi)\, R(\alpha, \beta, \gamma)\, R(\varphi', \theta', -\varphi')\, Z(\chi) \\ &\quad = Z^{-1}(\chi)\, R(\xi, 0, 0) Z(\chi) = R(\xi, 0, 0), \end{aligned} \tag{16.6}$$

where ξ is the Wigner angle for $SU(2)/U(1)$ calculated in section 14. In particular for coplanar processes, for which $R(\alpha, \beta, \gamma)$ is restricted to $R(0, \beta, 0)$, the Wigner angles corresponding to rotations reduce to unity. For both (16.4) and (16.5) the Wigner angles corresponding to general Lorentz transformations are quite complicated to calculate, but have been treated in detail in the literature [17].

In the massless case $p^2 = 0$, the little group is $E(2)$ and there is no decomposition $SO(3,1)_+^\uparrow = E(2) Z E(2)$ corresponding to (16.3). However, there is a decomposition $SO(3,1)_+^\uparrow = E(2) Z SU(2)$ (which is just the Iwasawa decomposition [18] if we pull the rotations in $E(2)$ through the Z to $SU(2)$). This decomposition allows only the second of the two choices (16.4), (16.5) of $\mathscr{L}(p)$ above (Jacob and Wick choice) to be made, and for $p^2 = 0$ this is the conventional choice of $\mathscr{L}(p)$.

Appendix

For completeness we treat here the representations of the Galilean group in the massless case $m = 0$.

2) The Mackey equations are

$$\begin{aligned} \big(U(a, \boldsymbol{v}', u')\, f\big)\, (\boldsymbol{v}, u) &= e^{ip\cdot a} f\big(\boldsymbol{v} + R(u)\, \boldsymbol{v}', uu'\big), \qquad p \cdot a = E a_0 - \boldsymbol{p} \cdot \boldsymbol{a} \\ f\big((v_\perp[\psi])\, (\boldsymbol{v}, u)\big) &= D(v_\perp[\psi])\, f(\boldsymbol{v}, u), \\ (f_1, f_2) &= \int dE d(\cos\theta)\, d\varphi \big(f_1(\boldsymbol{v}, u), f_2(\boldsymbol{v}, u)\big)_D, \end{aligned} \tag{A.1}$$

where $v_\perp = (v_1, v_2, o)$ (the transverse part of $\boldsymbol{v}$) and $[\psi] = (\psi, o, o)$ are the parameters of the little group $\tilde{E}(2)$, the fixed vector in the orbit having been chosen to be $\mathring{p} = (E_0 = o, o, o, p)$, and $D(v_\perp, [\psi])$ is the inducing representation of $\tilde{E}(2)$.

3a) For the Wigner states, the obvious coset representatives are $\gamma(p) = \big(v_3(p), u(\boldsymbol{p})\big)$ where $v_3(p) = (o, o, -p/m)$ are the longitudinal accelerations and $u(p)$ are the coset representatives for $SU(2)/U(1)$ already discussed for the Euclidean group. With this choice of $\gamma(p)$ the Wigner equations are

$$\begin{aligned} \big(U(a, \boldsymbol{v}, u)\, \omega\big)\, (E, \boldsymbol{p}) &= D\Big(\big(R(p)\, \boldsymbol{v}\big)_\perp [\xi]\Big)\, \omega(E - \boldsymbol{p} \cdot \boldsymbol{v}, R(u)\, \boldsymbol{p})\, e^{ip\cdot a} \\ (\omega_1, \omega_2) &= \int dE d(\cos\theta)\, d\varphi \big(\omega_1(E, \boldsymbol{p}), \omega_2(E, \boldsymbol{p})\big)_D, \end{aligned} \tag{A.2}$$

where D is the inducing representation of $\tilde{E}(2)$, $\big(R(p)\, \boldsymbol{v}\big)_\perp$ is the transver separt of $R(p)\, \boldsymbol{v}$ and ξ is the Wigner rotation for $SU(2)/U(1)$ given in equations (14.3) and (14.4). If the inducing representation D is trivial on the translation subgroup of $\tilde{E}(2)$, then it reduces to the one-dimensional representation $\exp(i\lambda\xi)$.

For the covariant states, the simplest representations of $\tilde{E}(3)_+$ in which the faithful unitary representations of $\tilde{E}(2)$ can be embedded are the faithful unitary irreducible representations of $\tilde{E}(3)_+$ discussed in the last section, and as these are highly reducible on $\tilde{E}(2)$, the covariant functions are again somewhat redundant. However, they take a simple and interesting form, namely,

$$\begin{aligned} \big(U(a, \boldsymbol{v}, u)\, \psi\big)\, (E, \boldsymbol{p}, \boldsymbol{q}) &= D^j(u)\, \psi\big(E - \boldsymbol{v}\cdot\boldsymbol{p}, R^{-1}(u)\,\boldsymbol{p}, R^{-1}(u)\boldsymbol{q}\big)\, e^{i(Ea_0 - \boldsymbol{p}\cdot\boldsymbol{a})}\, e^{i\boldsymbol{v}\cdot\boldsymbol{q}}, \\ (\psi_1, \psi_2) &= \int dEd\, (\cos\theta)\, d\varphi\, d(\cos\beta)\, d\alpha\, \psi^+(E, \boldsymbol{p}, \boldsymbol{q})\, \psi(E, \boldsymbol{p}, \boldsymbol{q}), \end{aligned} \tag{A.3}$$

where we have changed the $\boldsymbol{p}$ of the $\tilde{E}(3)$ representation of section 14 to $\boldsymbol{q}$ to avoid confusion with $p = (E, \boldsymbol{p})$, $p^2 = \text{constant}$, $q^2 = \text{constant}$, (θ, φ) and (β, α) are the polar angles of $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, and dagger denotes adjoint in the $(2j + 1)$-dimensional space of $D^j(u)$. The covariant subsidiary condition is easily seen to be

$$(\boldsymbol{\Sigma}\cdot\boldsymbol{p})\, \psi(E, \boldsymbol{p}, \boldsymbol{q}) = \lambda p \psi(E, \boldsymbol{p}, \boldsymbol{q}) \tag{A.4}$$

in analogy with the Euclidean case, together with the condition $\boldsymbol{p}\cdot\boldsymbol{q} = \text{constant}$. The special, but important, case in which the inducing representation D is trivial on the translation subgroup of $\tilde{E}(2)$ is obtained by setting $\boldsymbol{q} \equiv 0$ in the above formalism.

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