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The Theory of Induced Representations in Field Theory

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ABSTRACT

These are some notes on Mackey's theory of induced representations. They are based on lectures at the ITP in Stony Brook in the Spring of 1987. They were intended as an exercise in the theory of homogeneous spaces (as principal bundles). They were never meant for distribution.

§1 Introduction

The theory of group representations is by no means a closed chapter in mathematics. Although for some large classes of groups all representations are more or less completely classified, this is not the case for all groups. The theory of induced representations is a method of obtaining representations of a topological group starting from a representation of a subgroup. The classic example and one of fundamental importance in physics is the Wigner construction of representations of the Poincaré group. Later Mackey systematized this construction and made it applicable to a large class of groups.

The method of induced representations appears geometrically very natural when expressed in the context of (homogeneous) vector bundles over a coset manifold. In fact, the representation of a group G induced from a representation of a subgroup H will be decomposed as a "direct integral" indexed by the elements of the space of cosets G/H. The induced representation of G will be carried by the completion of a suitable subspace of the space of sections through a given vector bundle over G/H.

In §2 we review the basic notions about coset manifolds emphasizing their connections with principal fibre bundles. In doing so we find it instructive to review the basic concepts associated with fibre bundles. The reader familiar with this should only scan this section to familiarize him/herself with the notation. In §3 we give the construction of induced representations of G carried by the space of sections through a homogeneous vector bundle or equivalently by functions defined on the group subject to an equivariance property. In §4 we discuss the unitarity of the induced representation and introduce the concept of a multiplier representation. In §5 we restrict ourselves to a special class of Lie groups (containing the spacetime groups) for which the method of induced representations yields all irreducible unitary representations and we shall give a method by which to obtain them. The method will be a generalization of Wigner's "little group" method. This usually gives us a representation in a different space than the one we would like to use and, moreover, covariance under H transformations is often not manifest. Hence in §6 we introduce the so-called covariant functions and present a method to relate induced representations on one coset — where the representations are irreducible — to another coset where the representations are more useful. This will be of fundamental importance since the constraints we must impose to guarantee irreducibility in the second coset are — in the case of the Poincaré group — nothing but the free field equations. Finally in §7 we look at some familiar examples in detail.

The following is a list of references — by no means complete — from which these notes have evolved:

[BR] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications.*

[H1] R. Hermann, *Lie Groups for Physicists.*

[H2] R. Hermann, *Fourier Analysis on Groups and Partial Wave Analysis.*

[Ma] G. W. Mackey, *Induced Representations of Groups and Quantum Mechanics.*

[NO] U. H. Niederer and L. O'Raifeartaigh, *Forschritte der Physik*, **22**, *111-158* (1974).

[NW] P. van Nieuwenhuizen and P. West, Chapter 10 of forthcoming book.

[We] S. Weinberg, *Quantum Theory of Massless Particles.*

§2 General facts about Coset Spaces

Given any group G and a subgroup H we have a natural equivalence relation between elements of G. We say that two elements g, g' are equivalent ($g \sim g'$) if $g' = g \cdot h$ for some $h \in H$. This partitions G into equivalence classes called (left) H cosets and the collection of all such cosets is denoted G/H. There is a canonical map $G \to G/H$ sending each group element to the coset containing it. We write this map as $g \mapsto [g]$. If G is a topological group, then all cosets are homeomorphic (as subspaces of G) and, indeed, if G is a Lie group and H a Lie subgroup then G/H can be given a differentiable structure such that the canonical map is smooth and furthermore $G \to G/H$ is a fibration of G by H. We now review some basic notions about fibre bundles. We shall, in view of the applications we have in mind, work in the differentiable category. All our spaces will be differentiable manifolds and all our maps will be smooth.

A fibre bundle is a generalization of the cartesian product. It consists of two differentiable manifolds E, B called the **total space** and **base** respectively with a smooth surjection $\pi\colon E \to B$ such that every point $p \in B$ has a neighbourhood U such that its preimage in E looks like a cartesian product. That is, there exists a diffeomorphism $\varphi_U\colon U \times F \xrightarrow{\cong} \pi^{-1}(U)$ giving local coordinates to E, where F is a topological space called the **typical fibre**. Hence locally E looks like $B \times F$ although generally this is not the case in the large. If this is the case then the bundle is called trivial. We often omit mention of the fibre when this is understood and refer to a fibre bundle simply as $E \xrightarrow{\pi} B$. Now cover B with a collection of trivializing neighbourhoods $\{U_i\}$ and let $p \in U_i \cap U_j$. Let φ_i and φ_j be the coordinate maps associated with U_i and U_j, respectively. Then the preimage in E of $U_i \cap U_j$ can be given coordinates in two ways: either by φ_i or by φ_j. The change of coordinates corresponds to a reparametrization of the fibre. More explicitly the map $\varphi_i^{-1} \circ \varphi_j$ mapping $(U_i \cap U_j) \times F$ to itself breaks up as $id \times g_{ij}$ where the $g_{ij}: U_i \cap U_j \to \mathrm{Diff} F$ are called the **transition functions**.

Generally the transition functions take values in a more manageable subgroup of Diff$F$ and we get different kinds of fibre bundles depending on this subgroup as well as on particular properties of F. For instance if F is a vector space (say, $\mathbb{V}$) and the transition functions take values in GL($\mathbb{V}$) — *i.e.*the linear structure of $\mathbb{V}$ is preserved — then the bundle is called a **vector bundle**. Now let F be a Lie group (say, H) and let there be a smooth free action of H on E on the right (for definiteness) which preserves the fibred structure — *i.e.*the action of H does not move a point of E away from its fibre — or, symbolically, denoting the right action of $h \in H$ by R_h, such that $\pi \circ R_h = \pi$. In this case we say that E is fibred by H and the base B is diffeomorphic to the space of orbits E/H. The transition functions then take values in H and the whole structure is called a principal fibre bundle. In particular when E is a Lie group G and the action of H is just right multiplication then the base can be identified with the space of left cosets G/H.

A way to give coordinates to the coset space G/H is to use the coordinates already present in G. The way to do this is to choose smoothly a representative for each coset. That is, for every coset $p \in G/H$ choose an element of $\sigma(p) \in G$ such that $[\sigma(p)] = p$. In fibre bundle language such a map is called a **section**. Given a bundle $E \xrightarrow{\pi} B$ any (smooth) map $\sigma: B \to E$ such that $\pi \circ \sigma = id$ is called a (smooth) section. Whereas a vector bundle has many sections (*e.g.*the **zero section** assigning to every point in the base the zero vector in the fibre) the existence of a global section in a principal fibre bundle is equivalent to triviality. Indeed we show that given a global section we can construct a diffeomorphism $E \cong B \times H$. Given any point $e \in E$ map it to $(\pi(e), h)$ where $e = R_h\left(\sigma\left(\pi(e)\right)\right)$. Because the action of H is transitive on each fibre and free such an h always exist and is unique. Moreover the map is smooth because the action of H and π are both smooth. Conversely a trivial principal fibre bundle admits a global section. In fact, a global section in this case is nothing but the graph of a smooth function $B \to H$.

We shall, for calculational and notational convenience, work with a global section. This, although true for the case of contractible bases such as Minkowski spacetime, is not always possible. However if the base contains a subspace M whose complement is of measure zero (with respect to a suitable measure on the base) and such that the bundle over M is trivial we can safely (*i.e.*for the purposes we have in mind) assume that the bundle is trivial. We shall see later how this arises.

Given a principal fibre bundle $E \xrightarrow{\pi} B$ with fibre H and a representation

$D\colon H \to \mathrm{GL}(\mathbb{V})$ of H we can construct a vector bundle with fibre $\mathbb{V}$ called an **associated vector bundle**. The construction runs as follows. Consider the cartesian product $E \times \mathbb{V}$ and define on it an equivalence relation as follows $(e, v) \sim (R_h(e), D(h^{-1}) \cdot v)$ for all $h \in H$. That this is an equivalence relation follows from the fact that the action of H on E is a right action. The collection of equivalence classes is denoted $E \times_D \mathbb{V}$. This forms the total space of the associated vector bundle $E \times_D \mathbb{V} \xrightarrow{\pi'} B$, where the projection is given by $\pi'([(e, v)]) = \pi(e)$. It is straight forward to show that this is indeed a vector bundle. The local product structure is inherited from that of the original principal bundle and it is easy to see that the transition functions are just the representation of the original transition functions. In the particular case where E is a Lie group G then the associated vector bundle is called a homogeneous vector bundle. Unless otherwise stated we take $\mathbb{V}$ to be a complex vector space throughout.

The sections through an associated vector bundle can be thought of as functions from the total space E of the principal bundle to the vector space $\mathbb{V}$ with the following equivariance condition. If $f\colon E \to \mathbb{V}$ then $f \circ R_h = D(h^{-1}) \circ f$ for all $h \in H$. In the context of induced representations these functions are known as **Mackey functions**. We shall denote by $\mathcal{M}$ the space of Mackey functions. It is possible to make $\mathcal{M}$ into a vector space by defining the relevant operations pointwise. That is, if $f_1, f_2 \in \mathcal{M}$ then we define $f_1 + f_2$ by $(f_1 + f_2)(g) = f_1(g) + f_2(g)$ and similiarly for scalar multiplication. The same is true for the space of sections of any vector bundle. Let Γ denote the space of sections of the homogeneous vector bundle $G \times_D \mathbb{V} \longrightarrow G/H$. We now exhibit an isomorphism between $\mathcal{M}$ and Γ. It should be remarked that although the proof given here is for the case of a homogeneous vector bundle the result is true for an arbitrary associated vector bundle as the reader can easily verify.

Given a Mackey function $f \in \mathcal{M}$ we define a section as follows:

$$\psi(p) \stackrel{\text{def}}{=} [(\sigma(p), f(\sigma(p)))] \qquad \forall p \in G/H, \tag{2.1}$$

where $p \mapsto \sigma(p)$ is a coset representative. It is easy to verify that it is well defined (*i.e.*independent of the coset representative), for if we choose another coset representative $p \mapsto \sigma'(p) = \sigma(p) \cdot h(p)$ for some $h(p) \in H$ then

$$\begin{aligned}
\psi'(p) &\stackrel{\text{def}}{=} \big[\big(\sigma'(p), f(\sigma'(p))\big)\big] \\
&= [(\sigma(p) \cdot h(p), f(\sigma(p) \cdot h(p)))] \\
&= \big[\big(\sigma(p) \cdot h(p), D(h(p)^{-1}) \cdot f(\sigma(p))\big)\big] \\
&= [(\sigma(p), f(\sigma(p)))]
\end{aligned}$$

$$=\psi(p) \quad .$$

Furthermore it is clear that $\pi \circ \sigma = id$, so $\psi \in \Gamma$.

Conversely given a section $\psi \in \Gamma$ and a coset representative $p \mapsto \sigma(p)$ we construct a Mackey function $\widetilde{\phi}$ as follows. Let $g \in G$ and write it as $\sigma(p) \cdot h$ where $p = [g]$. Then we define

$$\widetilde{\psi}(g) \stackrel{\text{def}}{=} D(h^{-1}) \cdot v_p \quad , \tag{2.2}$$

where $\psi(p) = [(\sigma(p), v_p)]$. Clearly $\widetilde{\psi}$ is equivariant by construction. Moreover it is again straight forward to show that the above defined Mackey function is independent of the choice of coset representative.

It is worth remarking that although we have assumed the existence of a global section $G/H \to G$ the above constructions are well defined even for non trivial bundles. The correspondence between $\mathcal{M}$ and Γ is defined locally and the proof just shown that it is independent of the choice of section can be reproduced *mutatis mutandis* to prove that the correspondence agrees in the intersection of trivializing neighbourhoods.

We remarked above that both spaces can be made into vector spaces by defining the relevant operations pointwise. In that case, the above correspondence can then be seen to be a vector space isomorphism.

In the particular case of a homogeneous vector bundle over G/H there is a natural action of G on the space of sections or equivalently on the Mackey functions. Let f be a Mackey function. Then we define $\widetilde{U}_g \cdot f \stackrel{\text{def}}{=} f \circ L_{g^{-1}}$ where L_g represents left multiplication by g. That this forms a representation follows from the fact that left multiplication realizes the group and that the space $\mathcal{M}$ of Mackey functions is a vector space. Then one verifies that $\widetilde{U}_g$ is linear with respect to this vector space structure and that $\widetilde{U}_{g_1 \cdot g_2} = \widetilde{U}_{g_1} \cdot \widetilde{U}_{g_2}$. This representation of G carried by $\mathcal{M}$ will induce a representation of G carried by the space Γ of sections and this will be what we call the representation of G induced by the representation D of H. It shall be topic of the ensuing sections, but before continuing with this topic it shall be convenient to develop some machinery to deal with calculations in coset spaces.

Consider the principal fibre bundle $G \longrightarrow G/H$ where the projection is the canonical map $g \mapsto [g]$. There is a natural action of G on the cosets G/H induced from left multiplication on G. That is, we define a map $\tau \colon G \to \mathrm{Diff}(G/H)$ such that $\tau_g \cdot [g'] = [g \cdot g']$. It is trivial to verify that $\tau_{g_1 \cdot g_2} = \tau_{g_1} \cdot \tau_{g_2}$. This action is obviously transitive so that any two points in G/H are connected via a G transformation. Now choose a coset representative (*i.e.*a section)

$p \mapsto \sigma(p)$. Then $\tau_g \cdot p = [g \cdot \sigma(p)]$ and, by definition, $\tau_g \cdot p = [\sigma(\tau_g \cdot p)]$ which allows us to conclude that $g \cdot \sigma(p) \equiv \sigma(\tau_g \cdot p) \bmod H$. We define a map $h: G/H \times G \to H$ by

$$g \cdot \sigma(p) = \sigma(\tau_g \cdot p) \cdot h(p, g) \qquad \forall p \in G/H\,,\ g \in G\,. \tag{2.3}$$

Because H is the isotropy subgroup of the identity coset p_o — *i.e.*$\tau_h \cdot p_o = p_o$ for all $h \in H$ —

$$h(p_o, h') = \mathrm{Ad}_{\sigma(p_o)^{-1}}(h') \qquad \forall h' \in H\,. \tag{2.4}$$

Defining our coset representative in such a way that $\sigma(p_o)$ is the identity — which we are always free to do performing, if necessary, a global H transformation — we find that $h(p_o, h') = h'$ for all $h' \in H$. Also from the definition we find that

$$h(p, g_1 \cdot g_2) = h(\tau_{g_2} \cdot p, g_1) \cdot h(p, g_2)\,. \tag{2.5}$$

This, together with the easily verifiable fact that $h(p_o, \sigma(p)) = \mathbf{1}$ yields $h(p_o, \sigma(p) \cdot h') = h'$.

We now consider in detail the following special case of which the Poincaré group forms part. Let G be the semidirect product $T \rtimes H$. That is, T is a normal subgroup of G and every element $g \in G$ can be (uniquely) written as a product $h \cdot t$ where $h \in H$ and $t \in T$. It then follows that as manifolds G/H and T are diffeomorphic. The correspondence is the following: Let $g = h \cdot t$ be any element of G, then $[g] = [h \cdot t] = [Ad_h(t) \cdot h] = [Ad_h(t)]$ and we define the map $[h \cdot t] \mapsto Ad_h(t)$. This map is easily seen to be a diffeomorphism. A few straightforward calculations yield the following properties, where $t \in T$, $h' \in H$ and where we have chosen a coset representative in $T \subset G$:

$$h(p, t) = \mathbf{1} \tag{2.6}$$

$$\sigma(\tau_t \cdot p) = t \cdot \sigma(p) \tag{2.7}$$

$$h(p, h') = h' \tag{2.8}$$

$$\sigma(\tau_{h'} \cdot p) = Ad_{h'}(\sigma(p))\,. \tag{2.9}$$

These equations will be very useful in the sequel.

We saw in the last section how there was a natural action of G on the vector space $\mathcal{M}$ of Mackey functions. We also saw that there was a vector space isomorphism between $\mathcal{M}$ and the vector space Γ of sections through the homogeneous vector bundle. This isomorphism will induce a representation of G on Γ which we now construct.

Fix a choice of coset representative and let us denote the corresponding isomorphism $\Gamma \xrightarrow{\simeq} \mathcal{M}$ by $\psi \mapsto \widetilde{\psi}$. We denote by $\widetilde{U}$ the action of G on $\mathcal{M}$. We induce an action U of G on Γ by demanding commutativity of the following diagram:

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathcal{M} \\ \big\downarrow{\scriptstyle U_g} & & \big\downarrow{\scriptstyle \widetilde{U}_g} \\ \Gamma & \longrightarrow & \mathcal{M} \end{array}$$

That is,

$$\widetilde{U_g \cdot \psi} = \widetilde{U}_g \cdot \widetilde{\psi} \qquad (\forall g \in G)\,. \tag{3.1}$$

Explicitly,

$$\begin{aligned} (U_g \cdot \psi)\,(p) &= \Big[\Big(\sigma(p), \widetilde{U_g \cdot \psi}(\sigma(p))\Big)\Big] \\ &= \Big[\Big(\sigma(p), \widetilde{U}_g \cdot \widetilde{\psi}(\sigma(p))\Big)\Big] \\ &= \Big[\Big(\sigma(p), \widetilde{\psi}(g^{-1}\sigma(p))\Big)\Big] \end{aligned}$$

Recalling that $g^{-1} \cdot \sigma(p) = \sigma(\tau_{g^{-1}} \cdot p) \cdot h(p, g^{-1})$ and using the equivariance of the Mackey functions we find that $\widetilde{\psi}(g^{-1} \cdot \sigma(p)) = D(h(p, g^{-1})^{-1}) \cdot \widetilde{\psi}(\sigma(\tau_{g^{-1}} \cdot p)$ and thus

$$(U_g \cdot \psi)\,(p) = \Big[\Big(\sigma(p), D(h(p, g^{-1})^{-1}) \cdot \widetilde{\psi}\sigma(\tau_{g^{-1}} \cdot p)\Big)\Big]\,. \tag{3.2}$$

Fixing a global coset representative (under the assumptions of §2) we may identify the fiber at p of $G \times_D \mathbb{V}$ with $\mathbb{V}$ itself, although this identification, depending on the choice of coset representative, is not canonical. Hence if $\psi(p) = [(\sigma(p), v_p)]$ then we may write $\psi(p) = v_p$ with a little abuse of notation.

In this case the relation between Γ and $\mathcal{M}$ is even more explicit. If $\psi \mapsto \widetilde{\psi}$ then we see that $\psi = \widetilde{\psi} \circ \sigma$. We can then rewrite the action of G on Γ as

$$(U_g \cdot \psi)(p) = D(h(p, g^{-1})^{-1}) \cdot \psi(\tau_{g^{-1}} \cdot p). \tag{3.3}$$

We may rewrite this equation in yet another way. Using equation (2.5) we get

$$D(h(p, (g_1 g_2)^{-1})^{-1}) = D(h(p, {g_1}^{-1})^{-1} \cdot h(\tau_{{g_1}^{-1}} \cdot p, {g_2}^{-1})^{-1}), \tag{3.4}$$

which after letting $g_1 = g$ and $g_2 = g^{-1}$ gives

$$D(h(p, g^{-1})^{-1}) = D(h(\tau_{g^{-1}} \cdot p, g)). \tag{3.5}$$

This allows us to write the action of G as

$$(U_g \cdot \psi)(p) = D(h(\tau_{g^{-1}} \cdot p, g)) \cdot \psi(\tau_{g^{-1}} \cdot p). \tag{3.6}$$

Although a direct verification of the fact that $U_{g_1 \cdot g_2} = U_{g_1} \cdot U_{g_2}$ is possible, this follows trivially from the fact that $\widetilde{U}$ is a representation and the commutativity of the diagram above (3.1).

To familiarize ourselves with this representation let us look at some special cases of the above formula. Let $p = p_o$, the identity coset, and let $\sigma(p_o) = \mathbf{1}$ and let $h \in H$. Then a short calculation shows that

$$(U_h \cdot \psi)(p_o) = D(h) \cdot \psi(p_o), \tag{3.7}$$

where we have used that H is the isotropy subgroup at p_o. This shows that at the identity coset we recover the original representation of H from which we induced. (Had we chosen another coset representative it is easily verifiable that we would have obtained another representation of H equivalent to D.) Now let $g = \sigma(p)^{-1}$. Then we see that

$$(U_g \cdot \psi)(p_o) = \psi(p). \tag{3.8}$$

In the special case of $G = T \rtimes H$ we find after short calculations that

$$(U_t \cdot \psi)(p) = \psi(\tau_{t^{-1}} \cdot p) \qquad (\forall t \in T), \tag{3.9}$$

$$(U_h \cdot \psi)(p) = D(h) \cdot \psi(\tau_{h^{-1}} \cdot p) \qquad (\forall h \in H), \tag{3.10}$$

where $\tau_{h^{-1}} \cdot p = [\mathrm{Ad}_{h^{-1}}(\sigma(p))]$ and $\tau_{t^{-1}} \cdot p = \left[t^{-1} \cdot \sigma(p)\right]$. Since any element $g \in G$ can be written uniquely as a product $t \cdot h$ we find that

$$(U_g \cdot \psi)(p) = D(h) \cdot \psi(\tau_{g^{-1}} \cdot p) \qquad (\forall h \in H), \tag{3.11}$$

where $\tau_{g^{-1}} \cdot p = [\mathrm{Ad}_{h^{-1}}(\sigma(\tau_{t^{-1}} p))]$.

In the familiar case where G is the Poincaré group and H is the Lorentz group and choosing as coset representative $x \mapsto \exp(x \cdot T)$ we find that for $g = \exp(a \cdot T) \cdot \exp(\Lambda \cdot J)$ the transformation properties of a field $\psi(x)$ is

$$(U_g \cdot \psi)(x) = D(\Lambda) \cdot \psi(\mathrm{Ad}^*_\Lambda(x-a)), \tag{3.12}$$

where abusing the notation we have put $D(\exp(\Lambda \cdot J)) = D(\Lambda)$ and the same for Ad^* which, of course, is the (co-adjoint) representation of H on the dual to the translation subalgebra. We recognize in the above equation the transformation properties of the classical fields on spacetime. (A useful exercise for the reader is to repeat all of the constructions for the case of right cosets $G \backslash H$.)

Before discussing unitarity of induced representations there is one point that must be addressed. Does our construction depend in an essential way on the choice of coset representative? As expected it does not. Indeed we show that starting from two different coset representatives the induced representations are equivalent. That is, we show that there exists a section thorough the bundle $\mathrm{Aut}(G \times_D \mathbb{V}) \longrightarrow G/H$ which intertwines between the two induced representations. Let $p \mapsto \sigma(p)$ and $p \mapsto \sigma'(p)$ be two different coset representatives. Then we must have that $\sigma'(p) = \sigma(p) \cdot \widetilde{h}(p)$ for some $\widetilde{h}(p)$. Given a Mackey function $\widetilde{\psi}$ we get two different sections $p \mapsto \psi(p)$ and $p \mapsto \psi'(p)$ depending on which coset representative we use. The different induced representations will be denoted by U and U', respectively. The relationship between the two sections is the following:

$$\begin{aligned}
\psi'(p) =& \widetilde{\psi}(\sigma'(p)) \\
=& \widetilde{\psi}(\sigma(p) \cdot \widetilde{h}(p)) \\
=& D(\widetilde{h}(p)^{-1}) \cdot \widetilde{\psi}(\sigma(p)) \\
=& D(\widetilde{h}(p)^{-1}) \cdot \psi(p) .
\end{aligned}$$

Hence we define an element $S \in \mathrm{Aut}(\Gamma)$ as follows

$$\psi'(p) = (S \cdot \psi)(p) . \tag{3.13}$$

Now,

$$\begin{aligned}
(U'_g \cdot \psi')(p) =& (\widetilde{U}_g \cdot \widetilde{\psi}(\sigma'(p)) \\
=& \widetilde{\psi}(g^{-1} \cdot \sigma(p) \cdot \widetilde{h}(p)) \\
=& D(\widetilde{h}(p)^{-1}) \cdot \widetilde{\psi}(g^{-1} \cdot \sigma(p)) \\
=& D(\widetilde{h}(p)^{-1}) \cdot (\widetilde{U}_g \cdot \widetilde{\psi})(\sigma(p)) \\
=& D(\widetilde{h}(p)^{-1}) \cdot (U_g \cdot \psi)(p)
\end{aligned}$$

$$= (S \cdot U_g \cdot \psi)(p) \, .$$

Putting this together with the previous equation we obtain

$$\left(U'_g \cdot S \cdot \psi\right)(p) = (S \cdot U_g \cdot \psi)(p) \qquad (\forall p \in G/H \, , g \in G \, , \psi \in \Gamma) \tag{3.14}$$

Therefore the two representations are equivalent. Furthermore they are unitarily equivalent with respect to the inner product exhibited in the next section.

§4 Unitarity of Induced Representations

In the previous section we obtained a representation of a group G on the space of sections of a homogeneous vector bundle. We will in fact restrict ourselves to those sections which have finite norm with respect to the inner product exhibited below in equation (4.2). In this section we discuss the unitarity of these representations. We call a representation **unitary** if it preserves a positive definite inner product. If the inner product is not positive definite but it is preserved by the action of the group then we say that the representation is **pseudo-unitary**.

First we show that an inner product always[1] exist with respect to which the representation obtained in §2 is unitary. Fix a coset representative $p \mapsto \sigma(p)$ throughout. Let ϕ and ψ be elements of Γ. Then we define the following inner product

$$(\phi, \psi) = \int_G d\mu(g) \left\langle \widetilde{\phi}(g), \widetilde{\psi}(g) \right\rangle \, , \tag{4.1}$$

where $\widetilde{\phi}, \widetilde{\psi}$ are the respective Mackey functions, $d\mu$ is the left-invariant measure on G, and $\langle , \rangle$ is *any* positive definite inner product on $\mathbb{V}$. To show that U is unitary with respect to this inner product we compute

$$\begin{aligned}
(U_{g_o} \cdot \phi, U_{g_o} \cdot \psi) &= \int_G d\mu(g) \left\langle \left(\widetilde{U_{g_o} \cdot \phi}\right)(g), \left(\widetilde{U_{g_o} \cdot \psi}\right)(g) \right\rangle \\
&= \int_G d\mu(g) \left\langle \left(\widetilde{U}_{g_o} \cdot \widetilde{\phi}\right)(g), \left(\widetilde{U}_{g_o} \cdot \widetilde{\psi}\right)(g) \right\rangle \\
&= \int_G d\mu(g) \left\langle \widetilde{\phi}(g_o^{-1} \cdot g), \widetilde{\psi}(g_o^{-1} \cdot g) \right\rangle \\
&= \int_G d\mu(g_o \cdot g') \left\langle \widetilde{\phi}(g'), \widetilde{\psi}(g') \right\rangle \\
&= (\phi, \psi) \, ,
\end{aligned}$$

where in the last step we have used the left-invariance of the measure.

[1] As long as the group has a left-invariant measure.

We are interested however in an inner product defined on G/H. In the case of the Poincaré group this amounts to having the inner product defined on Minkowski space rather than on the group manifold. We notice that if $\mathbb{V}$ admits a positive definite inner product with respect to which the inducing representation, D, is unitary, then the integrand $\left\langle \widetilde{\phi}(g), \widetilde{\psi}(g) \right\rangle$ defines a function in G/H. We see this as follows. Any point $g \in G$ can be written as $\sigma([g]) \cdot \eta(g)$ for some $\eta(g) \in H$. Then

$$\begin{aligned}
\left\langle \widetilde{\phi}(g), \widetilde{\psi}(g) \right\rangle &= \left\langle \widetilde{\phi}(\sigma([g]) \cdot \eta(g)), \widetilde{\psi}(\sigma([g]) \cdot \eta(g)) \right\rangle \\
&= \left\langle D(\eta(g)^{-1}) \cdot \widetilde{\phi}(\sigma([g])), D(\eta(g)^{-1} \cdot \widetilde{\psi}(\sigma([g])) \right\rangle \\
&= \langle \phi([g]), \psi([g]) \rangle \ ,
\end{aligned}$$

where in the last step we have used the unitarity of D and the isomorphism between Γ and $\mathcal{M}$ induced by a section σ. Thus we can define an inner product on Γ as follows:

$$(\phi, \psi) = \int_{G/H} d\nu(p) \, \langle \phi(p), \psi(p) \rangle \ . \tag{4.2}$$

This inner product defines a norm on Γ as usual

$$\|\phi\| \stackrel{\text{def}}{=} \sqrt{(\phi, \phi)} \, ,$$

for all $\phi \in \Gamma$. We will consider the subspace $\mathcal{H}$ of Γ consisting of those elements of Γ with finite norm. We shall call $\mathcal{H}$ the space of **integrable** sections. For a trivial bundle we can see that $\mathcal{H} \cong \mathcal{L}^2(G/H, \nu) \otimes \mathbb{V}$.

Because of the unitarity of D with respect to $\langle , \rangle$ we see that the representation U is unitary with respect to this inner product if the measure is invariant under the action of G, *i.e.*$d\nu(p) = d\nu(\tau_g \cdot p)$ for all $g \in G$. Explicitly,

$$\begin{aligned}
(U_g \cdot \phi, U_g \cdot \psi) &= \int_{G/H} d\nu(p) \left\langle D(h(\tau_{g^{-1}} \cdot p, g)) \cdot \phi(\tau_{g^{-1}} \cdot p), D(h(\tau_{g^{-1}} \cdot p, g)) \cdot \psi(\tau_{g^{-1}} \cdot p) \right\rangle \\
&= \int_{G/H} d\nu(\tau_g \cdot p') \left\langle \phi(p'), \psi(p') \right\rangle \\
&= (\phi, \psi) \ ,
\end{aligned}$$

where we have used the invariance of the measure to get unitarity.

But what if the measure on G/H is not invariant under the action of G? In this case we must redefine our representation to allow for a multiplicative factor that cancels the Jacobian term arising from the non-invariance of the measure. This leads us to the concept of **multiplier representations**. For definiteness we take the inner product on $\mathbb{V}$ to be linear in the second factor and conjugate linear in the first.

Assume that the measure $d\nu$ is not G-invariant but **quasi-invariant** — that is, that the measure $d\nu(p)$ and its translates $d\nu(\tau_g \cdot p)$ have the same sets of measure zero[2]. Then by the Radon-Nikodym theorem there exists a positive function J on G/H, which will depend on the group element by which we are translating, such that for all $g \in G$

$$d\nu(\tau_g \cdot p) = J(g,p)\, d\nu(p)\,. \tag{4.3}$$

Because τ is a G-action — *i.e.*$\tau_{g_1 \cdot g_2} = \tau_{g_1} \cdot \tau_{g_2}$ — J satisfies the following condition

$$J(g_1 \cdot g_2, p) = J(g_1, \tau_{g_2} \cdot p)\, J(g_2, p)\,. \tag{4.4}$$

We now wish to redefine the action of G on $\mathcal{M}$ — and hence on Γ — in such a way that we recover unitarity without imposing invariance of the measure. For $g_o \in G$ and $\widetilde{\psi} \in \mathcal{M}$ define

$$\left(\widetilde{U}_{g_o} \cdot \widetilde{\psi}\right)(g) = \widetilde{m}(g_o, g)\, \widetilde{\psi}(g_o^{-1} g)\,, \tag{4.5}$$

where $\widetilde{m}\colon G \times G \to \mathbb{C}$ is called a multiplier. For $\widetilde{U}_{g_o} \cdot \widetilde{\psi}$ to be again a Mackey function $\widetilde{m}(g_o, g)$ must only depend on the coset $[g]$. To see this notice that $\left(\widetilde{U}_{g_o} \cdot \widetilde{\psi}\right)(g \cdot h)$ is both equal to $D(h^{-1})\left(\widetilde{U}_{g_o} \cdot \widetilde{\psi}\right)(g)$ — because it is a Mackey function — and to $\widetilde{m}(g_o, g \cdot h)\, \widetilde{\psi}(g_o^{-1} \cdot g \cdot h)$ by definition of $\widetilde{U}$. Applying the same two facts but in the opposite order we find that $\widetilde{m}(g_o, g) = \widetilde{m}(g_o, g \cdot h)$ for all $h \in H$. Hence $\widetilde{m}$ defines a function $m\colon G \times G/H \to \mathbb{C}$ via $m(g_o, [g]) = \widetilde{m}(g_o, g)$. Hence we can induce a representation on Γ which acts as follows

$$(U_g \cdot \psi)(p) = m(g,p)\, D(h(\tau_{g^{-1}} \cdot p))\, \psi(\tau_{g^{-1}} \cdot p)\,. \tag{4.6}$$

For U to be a representation m must satisfy a further property as can be trivially checked

$$m(g_1 \cdot g_2, p) = m(g_1, p)\, m(g_2, \tau_{g_1^{-1}} \cdot p)\,. \tag{4.7}$$

Now we find out what m must be in order for U to be a unitary representation with respect to the inner product defined in equation (4.2). A brief

[2] Mackey has shown that for G a separable locally compact topological group and H a closed subgroup, the coset space G/H admits a quasi-invariant measure.

calculation using the sesqui-linearity of the inner product in $\mathbb{V}$ shows that

$$(U_g \cdot \phi, U_g \cdot \psi) = \int_{G/H} d\nu(p)\, J(g,p)\, |m(g, \tau_g \cdot p)|^2\, \langle \phi(p), \psi(p) \rangle \ .$$

This allows us to conclude that U is unitary with respect to this inner product if and only if

$$|m(g, \tau_g \cdot p)| = J(g,p)^{-\frac{1}{2}} \tag{4.8}$$

almost everywhere. We must check, however, that this equality is consistent with equations (4.4) and (4.7). A quick calculation — which is left to the reader — shows that this is indeed the case.

Notice that m is defined up to an arbitary phase consistent with equation (4.7). Different choices lead, however, to equivalent representations and thus we choose the phase in such a way that m is real valued

$$m(g,p) = J(g, \tau_{g^{-1}} \cdot p)^{-\frac{1}{2}} \ . \tag{4.9}$$

From now on we will assume, for notational convenience, that the measure on G/H is invariant. None of the results depend on this fact, but only on unitarity which we have shown to hold.

§5 Irreducible Induced Representations: the Little Group Method

We are now in a position where we can construct unitary infinite dimensional representations of many Lie groups starting from a unitary rerpresentation of a subgroup. However, in general, these representations will not be irreducible. Consider, for instance, the representation of the Poincaré group induced from the trivial representation of the Lorentz group. Because the representation is trivial, the sections are just graphs of functions from Minkowski space to, say, the real line. This is clearly not irreducible because in an irreducible representation the Casimir operators act as multiples of the identity. For the Poincaré group the non-trivial Casimir in this case is the mass operator T^2. Choosing coordinates for Minkowski space induced from those in the Poincaré group via the following choice of coset representative $x \mapsto \sigma(x) = \exp x \cdot T$, we see that the translation subalgebra is realized as differential operators $T_i \mapsto \frac{\partial}{\partial x^i}$ and thus the Casimir T^2 is realized as the D'Alembertian $\Box$. For this to be a multiple of the identity the functions must satisfy the Klein-Gordon equation $\Box \phi = k\, \phi$, for some constant k, which clearly is a non-trivial constraint not obeyed by arbitrary functions.

This does not mean that the method of induced representations does not work for the Poincaré group, but it does mean that we must choose a different subgroup from which to induce. The method we describe in this section is a generalization of Wigner's little group method due to Bargmann. We restrict our attention to the special case of Lie groups which are the semidirect product of a semisimple group with an abelian group. That is, $G = T \rtimes S$, where S is semisimple and T is abelian.

Choose an irreducible representation α of T. Because T is abelian all irreducible representations are one-dimensional (Schur's Lemma) and α is just a character. Moreover we assume that the representation is unitary so that the image of $\alpha: T \to \mathbb{C}$ lies in the unit circle. We define the following subset of S denoted by $L(\alpha)$:

$$l \in L(\alpha) \Leftrightarrow \alpha(t) = \alpha(l \cdot t \cdot l^{-1}) \qquad \forall t \in T\,. \tag{5.1}$$

Clearly $L(\alpha)$ is a subgroup of S. It is called the "little group" of α. We now choose an irreducible representation $\beta: L(\alpha) \to \mathrm{Aut}(\mathbb{W})$ and form the representation $D: T \rtimes L(\alpha) \to \mathrm{Aut}(\mathbb{W} \otimes \mathbb{C})$ defined as follows. Let $h \in L(\alpha) \rtimes T$. Then it may be written uniquely as $l \cdot t$ where $l \in L(\alpha)$ and $t \in T$. Then for $w \in \mathbb{W}$ and $z \in \mathbb{C}$ we define

$$D(h)(w \otimes z) = (\beta(l) \cdot w) \otimes (\alpha(t) \cdot z)\,. \tag{5.2}$$

That this forms a representation follows from the definition of $L(\alpha)$. To check this let $h' \in T \rtimes L(\alpha)$ de decomposed as $l' \cdot t'$. Then

$$\begin{aligned}
D(h \cdot h')(w \otimes z) =& D(l \cdot t \cdot l' \cdot t')(w \otimes z)\\
=& D(l \cdot l' \cdot l'^{-1} \cdot t \cdot l' \cdot t')(w \otimes z)\\
=& (\beta(l \cdot l') \cdot w) \otimes (\alpha(\mathrm{Ad}_{l'^{-1}}(t) \cdot t') \cdot z)\\
=& (\beta(l) \cdot \beta(l') \cdot w) \otimes (\alpha(\mathrm{Ad}_{l'^{-1}}(t)) \cdot \alpha(t') \cdot z)\\
=& (\beta(l) \cdot \beta(l') \cdot w) \otimes (\alpha(t) \cdot \alpha(t') \cdot z)\\
=& D(l \cdot t)((\beta(l') \cdot w) \otimes (\alpha(t') \cdot z))\\
=& D(l \cdot t) D(l' \cdot t')(w \otimes z)\\
=& D(h) D(h')(w \otimes z)\,.
\end{aligned}$$

In fact it is easy to verify that D is really a representation of $L(\alpha) \times T$ since $L(\alpha)$ and T commute in this representation.

We now assume that β is unitary so that the represetation D is also unitary and we induce a representation of G starting from D. It will be carried by the "integrable" sections through a homogeneous complex vector bundle over G/H where H is now $T \rtimes L(\alpha)$. We know that the induced representation is unitary. We want to prove that it is also irreducible. But first we need to discuss some properties of the coset space G/H.

The first thing we notice is that $G/H \cong S/L(\alpha)$. To see this notice that any element $g \in G$ can be uniquely decomposed as $s \cdot t$ where $s \in S$ and $t \in T$. Then,

$$[g]_H = [s \cdot t]_H = [s]_{L(\alpha)} \,. \tag{5.3}$$

Thus the desired map is $[g]_H \mapsto [s]_{L(\alpha)}$. It is easy to see that this map is well defined and that it is a diffeomorphism. A natural consequence of this is that τ restricted to T is trivial. Indeed, letting $t \in T$ and $g \in G$,

$$\tau_t \cdot [g] = [t \cdot g] = [g \cdot g^{-1} \cdot t \cdot g] = [g]\,, \tag{5.4}$$

since T is normal in G.

Hence let U denote the (unitary) induced representation of G on the Hilbert space $\mathcal{H}$ of integrable sections and consider the restriction of U to the abelian subgroup T. Let $t = \exp\, a \cdot T \in T$ and let z_o denote the indentity coset. Then

$$(U_t \cdot \psi)(z_o) = D(t) \cdot \psi(z_o) = \alpha(t) \cdot \psi(z_o)\,, \tag{5.5}$$

where

$$\begin{aligned} \alpha(t) =& \alpha(\exp\, a \cdot T) \\ =& \exp\, a \cdot \alpha_*(T) \\ \equiv& e^{i\, a \cdot p_o}\,, \end{aligned} \tag{5.6}$$

which defines p_o. Now, a short calculation yields

$$(U_t \cdot \psi)(z) = e^{i\, a \cdot \Lambda(\sigma(z)^{-1}) \cdot p_o} \cdot \psi(z)\,, \tag{5.7}$$

where $\Lambda(s)$ is the adjoint representation of $s \in S$ in the abelian subalgebra of T. Let's define $p \equiv \Lambda(\sigma(z)^{-1}) \cdot p_o$. Then we have an identification between the points z on the coset $S/L(\alpha)$ and the orbit of p_o under the adjoint action of S. Moreover, because p_o is a fixed point of this orbit, the identification is one to one. It is then obvious that each representation of T occuring in U determines and is determined by a point p in the adjoint orbit of p_o. Hence all the different representations of T occuring in U are indexed by the points in $S/L(\alpha)$. To make this identification more explicit we shall denote the points in $S/L(\alpha)$ by the representation of T they determine. Thus $z_o \leftrightarrow p_o$ and $z \leftrightarrow p$ in the previous case. Thus for $t = \exp\, a \cdot T$,

$$(U_t \cdot \psi)(p) = e^{i\, a \cdot p}\, \psi(p)\,. \tag{5.8}$$

There is a more functional-analytical way to look at this identification. It is the Mackey direct integral decomposition of a representation. It is a generalization of the direct sum in a sense to be made precise below. Let $\mathcal{H}$ be a Hilbert space and Σ be some parameter space. Then in some cases $\mathcal{H}$ may be decomposed as

$$\mathcal{H} = \int_\Sigma d\nu(p)\, \mathcal{H}_{(p)} \qquad p \in \Sigma\,, \tag{5.9}$$

where ν is some measure in Σ. When ν is a discrete measure then one recovers the direct sum. Perhaps the simplest example is the Fourier decomposition of a function in $\mathcal{L}^2(\mathbb{R},\mu)$ where μ is the Lebesgue measure. Then $g \in \mathcal{L}^2(\mathbb{R},\mu)$ may be decomposed as

$$g(x) = \int_{\mathbb{R}} e^{i\,x\cdot p}\, \widetilde{g}(p)\, d\mu(p)\,, \tag{5.10}$$

where $\widetilde{g}$ is also in $\mathcal{L}^2(\mathbb{R},\mu)$. Thinking of $\mathbb{R}$ as a group under addition we see that $\mathcal{L}^2(\mathbb{R},\mu)$ carries the regular representation. Then $x \mapsto e^{i\,x\cdot p}$ is a character and the Fourier decomposition is a decomposition of the regular representation into irreducibles. A similar situation arises in our case. Let $\mathcal{H}$ be the Hilbert space of "integrable" sections. Then we may break up $\mathcal{H}$ into the direct integral

$$\mathcal{H} = \int_{S/L(\alpha)} d\nu(p)\, \mathcal{H}_{(p)}\,, \tag{5.11}$$

where on $\mathcal{H}_{(p)}$, U_t is $e^{i\,a\cdot p}$ times the identity. In this case the decomposition is nothing but the spectral theorem for a commuting family of self-adjoint operators. The argument is basically the following. The representation U is unitary in $\mathcal{H}$ and in particular its restriction to T is unitary. Now $U(T)$ consists of $\dim T$-families of commuting unitary operators. Each family gives rise to a self-adjoint operator which is, up to a factor of i, the representation U_* induced on the algebra of T. The spectral theorem for a commuting family of self-adjoint operators offers a decomposition of $\mathcal{H}$ as a direct integral over the spectrum, which we have shown to be in one to one correspondence with $S/L(\alpha)$.

Now we show that this representation is irreducible. The idea of the proof is very simple: We prove that any (bounded) linear operator commuting with the representation U of G is necessarily a multiple of the identity. This implies that the representation is irreducible for otherwise the projection onto any invariant subspace is a bounded linear operator commuting with the representation.

Let $\mathcal{O}$ be a bounded linear operator in $\mathcal{H}$ which commutes with $U(G)$. Because it commutes with $U(T)$ the SNAG (Stone-Naimark-Ambrose-Godement) theorem says that $\mathcal{O}$ has a decomposition compatible with the direct integral decomposition of $\mathcal{H}$ such that if $\psi \in \mathcal{H}$ then

$$(\mathcal{O} \cdot \psi)(p) = \mathcal{O}(p) \cdot \psi(p)\,, \tag{5.12}$$

where $\mathcal{O}(p)$ is an endomorphism of $\mathcal{H}_{(p)}$. In particular, now, $\mathcal{O}$ commutes with $U_{\sigma(p)^{-1}}$ so that

$$\begin{aligned} \mathcal{O}(p) \cdot \psi(p) =& (\mathcal{O} \cdot \psi)(p) \\ =& (U_{\sigma(p)^{-1}} \cdot \mathcal{O} \cdot \psi)(p_o) \\ =& (\mathcal{O} \cdot U_{\sigma(p)^{-1}} \cdot \psi)(p_o) \\ =& \mathcal{O}(p_o) \cdot \psi(p)\,. \end{aligned} \tag{5.13}$$

That is, $\mathcal{O}(p) = \mathcal{O}(p_o) \equiv \mathcal{O}_o$ for all p. (Notice that this argument is still valid if we introduce multipliers.) Now recall that at p_o the induced representation U is just the inducing representation. In particular, $U(L(\alpha))$ at the identity coset is just $\beta(L(\alpha))$. Hence $\mathcal{O}_o$ commutes with $\beta(L(\alpha))$. Because β is irreducible, $\mathcal{O}_o$ must be a multiple of the identity (Schur's Lemma) so that $\mathcal{O}$ is a multiple of the identity.

Let us see how this process works for the case of the Poincaré group. We choose a representation of the translations, *i.e.*we choose a momentum, p_o. The little group will be the subgroup of the Lorentz group leaving that momentum fixed and the coset will be the Lorentz orbit of p_o. In this particular case there exists a bilinear form which is invariant under the adjoint action of the Lorentz group — namely the Minkowski metric η — which means that $p_o^2 \equiv \eta(p_o, p_o)$ is a constant of the orbit. Thus we see that the cosets we are looking at are the mass hyperboloids and the mass shell condition is determined by the choice of coset. In the case that $p_o^2 = m^2 > 0$ (positive mass squared) we can choose $p_o^t = (m, 0, 0, 0)$, *i.e.*the rest frame. Because no boost will preserve the rest frame, we see that the little group is formed by the rotations and hence is isomorphic to SO_3. Different points in the coset $SO_{3,1}/SO_3$ correspond to different frames related by boosts.

A question that must be asked is whether two representations of T which are in the same adjoint S-orbit produce equivalent induced representations of G. As expected they do. The proof runs as follows.

PROOF GOES HERE

Finally we can show that for the special case of a Lie group $G = T \rtimes S$ such as the one we've been dealing with in this section *all* irreducible unitary representations are obtained this way — *i.e.*they are induced representations.

PROOF GOES HERE

In the previous section we exhibited a method to construct arbitrary irreducible representations of a special class of Lie groups, namely those which are the semidirect product of a semisimple group and an abelian group. These representations are carried by the integrable sections of a complex homogeneous vector bundle over G/H where $G = T \rtimes S$ and $H = T \rtimes L$ for some $L \subset S$. These representations, despite being irreducible, suffer two drawbacks: they are not covariant under S transformations and they are defined on a coset space which is not the most useful. An example may help to understand this. Let G be the Poincaré group and consider the representation induced from the representation D of $T_4 \rtimes SO_3$ where, if $t = \exp\, a \cdot T$,

$$D(t) = e^{i\, a \cdot p_o}\,, \tag{6.1}$$

where $p_o^t = (m, 0, 0, 0)$ and D restricted to SO_3 is the vector (three dimensional) representation. This representation will be carried by vector valued functions [3] $p \mapsto \widetilde{A}_i(p)$ such that $\widetilde{A}_i(p)$ transforms according to the vector representation of SO_3. We recognize this representation of the Poincaré group as that corresponding to a massive vector particle. However $\widetilde{A}_i$ is not Lorentz covariant (A_μ would be) and is not defined on the configuration space of the massive vector particle (*i.e.*on Minkowski space) but on the mass hyperboloid, unlike A_μ. Briefly, in this section we show how to go from $\widetilde{A}_i(p)$ to $A_\mu(x)$.

Notice that we pay a price for Lorentz covariance: A_μ has more degrees of freedom than $\widetilde{A}_i$. To see this notice that A_μ is in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group which under SO_3 breaks into $\mathbf{1} \oplus \mathbf{0}$, *i.e.*a vector and a scalar, where the vector part is $\widetilde{A}_i$. Hence in order to get an irreducible representation we must keep only the vector part. To achieve this we introduce projectors which will turn out to be polynomials of the mass-shell momenta and hence, via mass-shell Fourier transform, differential operators in Minkowski space. These projections, together with the mass shell condition, will become the free field equations. We will return to this particular example later on but first we discuss the general set-up.

Let $G = T \rtimes S$, $L \subset S$ and $D = \beta \otimes \alpha$ be an irreducible unitary representation of $T \rtimes L$ such that L and T commute under D. Moreover, let L be maximal in this respect. We remark, parenthetically, that this is enough to characterize L as $L(\alpha)$. (*Proof:* We have seen in §5 that if $L = L(\alpha)$ this is

[3] The upper sheet of the mass hyperboloid is contractible and hence all fibre bundles defined on it are trivial. Therefore sections through a vector bundle are just graphs of vector valued functions.

obeyed. Conversely let L and T commute in the representation $D = \beta \otimes \alpha$. Then, $D(l \cdot t) = \beta(l) \otimes \alpha(t) = D(t \cdot l) = D(l \cdot l^{-1} \cdot t \cdot l) = \beta(l) \otimes \alpha(l^{-1} \cdot t \cdot l)$ so that $\alpha(t) = \alpha(l^{-1} \cdot t \cdot l)$ for all $t \in T$. Thus $L \in L(\alpha)$, but because it is maximal, it is $L(\alpha)$.) Assume further that β extends to a representation of S, although we don't require that the extension be unitary nor irreducible, since, in general, this won't be the case. We shall denote the extension by β as well, since no confusion should arise. Then D also extends to a representation of G which we also denote by D. This representation of G will not be unitary nor irreducible.

Given a section $\psi \in \Gamma$ we define a **covariant section** as follows: $\psi'(p) = D(\sigma(p)) \cdot \psi(p)$, where $p \mapsto \sigma(p)$ is a coset representative (in the previous example of the massive vector particle, $\sigma(p)$ is a boost necessary to get from the rest frame p_o to p.) We denote by Γ_c the vector space of covariant sections. We define an action U' of G on Γ_c demanding commutativity of the following diagram, from which it is obvious that U' is a representation:

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma_c \\ \Big\downarrow{\scriptstyle U_g} & & \Big\downarrow{\scriptstyle U'_g} \\ \Gamma & \longrightarrow & \Gamma_c \end{array}$$

That is,

$$U'_g \cdot \psi' = (U_g \cdot \psi)' \,. \tag{6.2}$$

Explicitly, and forgetting about multipliers,

$$\begin{aligned}(U'_g \cdot \psi)(p) =&(U_g \cdot \psi)'(p) \\ =&D(\sigma(p)) \cdot (U_g \cdot \psi)(p) \\ =&D(\sigma(p)) \cdot D(h(\tau_{g^{-1}} \cdot p, g)) \cdot \psi(\tau_{g^{-1}} \cdot p) \\ =&D(\sigma(p) \cdot h(\tau_{g^{-1}} \cdot p, g)) \cdot \psi(\tau_{g^{-1}} \cdot p) \\ =&D(g \cdot \sigma(\tau_{g^{-1}} \cdot p)) \cdot \psi(\tau_{g^{-1}} \cdot p) \\ =&D(g) \cdot \psi'(\tau_{g^{-1}} \cdot p) \,,\end{aligned} \tag{6.3}$$

where in the fifth line we have used equation (2.3) which defines $h(p, g)$. We see that, indeed, the ψ' are covariant.

Next we define an inner product on Γ_c such that U' is unitary. We take the obvious one: $(\psi', \phi') = (\psi, \phi)$, that is,

$$(\phi', \psi') = \int_{S/L} d\nu(p) \, \left\langle D(\sigma(p)^{-1} \cdot \psi'(p), D(\sigma(p)^{-1}) \cdot \phi'(p) \right\rangle \quad . \tag{6.4}$$

Because U is unitary, U' is also. Indeed,

$$\left(U'_g \cdot \psi', U'_g \cdot \phi'\right) = \left((U_g \cdot \psi)', (U_g \cdot \phi)'\right)$$

$$
\begin{aligned}
&= (U_g \cdot \psi, U_g \cdot \phi) \\
&= (\psi, \phi) \\
&= \left(\psi', \phi'\right) \, . \qquad (6.5)
\end{aligned}
$$

Recall that having fixed a coset representative $p \mapsto \sigma(p)$ the relation between the Mackey functions $\mathcal{M}$ and the sections Γ was given smply by $\psi(p) = \widetilde{\psi}(\sigma(p))$. Now we ask the obvious question. Given a covariant section what is the object in the group analogous to the Mackey functions? Using the definitions at hand the answer falls right out. Notice that $\psi'(p) = D(\sigma(p)) \cdot \widetilde{\psi}(\sigma(p))$. This clearly suggests introducing the following functions on the group. Let $\widetilde{\psi} \in \mathcal{M}$ and define

$$
\widetilde{\psi}'(g) \stackrel{\text{def}}{=} D(g) \cdot \widetilde{\psi}(g) \, . \qquad (6.6)
$$

These functions are clearly not Mackey functions. In fact they are honest functions on the coset since they are constant along the fibers. Indeed

$$
\begin{aligned}
\widetilde{\psi}'(g \cdot h) =& D(g \cdot h) \cdot \widetilde{\psi}(g \cdot h) \\
=& D(g \cdot h) \cdot D(h^{-1}) \cdot \widetilde{\psi}(g) \\
=& D(g) \cdot \widetilde{\psi}(g) \\
=& \widetilde{\psi}'(g) \, . \qquad (6.7)
\end{aligned}
$$

In fact they are the pull-back via the canonical map $G \to G/H$ of the covariant sections. We shall denote the vector space of these covariant "Mackey functions" as $\mathcal{M}_c$. We define an action of G on $\mathcal{M}_c$ in the usual way. Namely, we demand commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathcal{M} & \longrightarrow & \mathcal{M}_c \\
\Big\downarrow \widetilde{U}_g & & \Big\downarrow \widetilde{U}'_g \\
\mathcal{M} & \longrightarrow & \mathcal{M}_c
\end{array}
$$

for all $g \in G$. That is,

$$
\begin{aligned}
(\widetilde{U}'_g \cdot \widetilde{\psi}')(g') =& (\widetilde{U} \cdot \widetilde{\psi})'(g') \\
=& D(g') \cdot (\widetilde{U}_g \cdot \widetilde{\psi})(g') \\
=& D(g') \cdot \widetilde{\psi}(g^{-1} \cdot g') \\
=& D(g') \cdot D(g'^{-1} \cdot g) \cdot \widetilde{\psi}'(g^{-1} \cdot g') \\
=& D(g) \cdot \widetilde{\psi}'(g^{-1} \cdot g') \, . \qquad (6.8)
\end{aligned}
$$

We have, thus, a unitary covariant representation of G but we have sacrificed irreducibility. It seems, therefore, that we are only slightly better off than at the end of §2. Irreducibility, however, is easily recovered. The representation β of S when restricted to L will, in general, reduce — assuming that the representation is fully reducible, which is always the case if L is semisimple (Weyl's theorem) — into a direct sum of irreducible subrepresentations. We must therefore construct projectors onto whichever irreducible factor we are interested in. This will turn out to be easier starting with the covariant Mackey functions just introduced.

Let $\mathbb{V}$ be a representation space of G affording the representation $D = \beta \otimes \alpha$ and let's consider it as a representation space of $H = T \rtimes L$. Let's assume that $\mathbb{V}$ splits into irreducible H-subspaces

$$\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \cdots . \tag{6.9}$$

Let P be a projector onto $\mathbb{V}_1$, say. That is, $P^2 = P$, $\mathrm{im}\, P = \mathbb{V}_1$, $\ker P = {\mathbb{V}_1}^c$ and P commutes with $D(H)$. Notice that because α is just a character, the non-trivial restriction is for P to commute with $\beta(L)$. Let us consider the subspace of Mackey functions obeying

$$P \cdot \widetilde{\psi}(g) = \widetilde{\psi}(g) \qquad \forall g \in G . \tag{6.10}$$

That is, $\widetilde{\psi}: G \to \mathbb{V}_1$. And let us define an endomorphism $\widetilde{\mathbb{P}}$ of $\mathcal{M}$ by

$$(\widetilde{\mathbb{P}} \cdot \widetilde{\psi})(g) = P \cdot \widetilde{\psi}(g) . \tag{6.11}$$

It is worth remarking that $\widetilde{\mathbb{P}} \in \mathrm{End}(\mathcal{M})$ is a non-trivial constraint. It is essentially due to the fact that P commutes with $D(H)$. Indeed, notice that for $\widetilde{\mathbb{P}} \cdot \widetilde{\psi}$ to be again a Mackey function

$$\begin{aligned}(\widetilde{\mathbb{P}} \cdot \widetilde{\psi})(g \cdot h) =& D(h^{-1}) \cdot (\widetilde{\mathbb{P}} \cdot \widetilde{\psi})(g)\\ =& D(h^{-1}) \cdot P \cdot \widetilde{\psi}(g) ,\end{aligned}$$

whereas from the definition we have that

$$\begin{aligned}(\widetilde{\mathbb{P}} \cdot \widetilde{\psi})(g \cdot h) =& P \cdot \widetilde{\psi}(g \cdot h)\\ =& P \cdot D(h^{-1}) \cdot \widetilde{\psi}(g) .\end{aligned}$$

Clearly both expressions agree when P and $D(H)$ commute. Notice also that $\widetilde{\mathbb{P}}^2 = \widetilde{\mathbb{P}}$ so that $\widetilde{\mathbb{P}}$ is also a projection. We can thus consider the subspace $\mathrm{im}\, \widetilde{\mathbb{P}} \subset \mathcal{M}$. This subspace is easily seen to be G-invariant, since G acts by left translations on the Mackey functions. Because $\mathbb{V}_1$ is an irreducible H-space, we see that $\mathrm{im}\, \widetilde{\mathbb{P}}$ is an irreducible G-space.

Let us now construct a projector on the covariant Mackey functions. We define the endomorphism $\widetilde{\mathbb{P}}'$ by the usual commutativity argument:

$$\widetilde{\mathbb{P}}' \cdot \widetilde{\psi}' = (\widetilde{\mathbb{P}} \cdot \widetilde{\psi})' \,. \tag{6.12}$$

Explicitly,

$$(\widetilde{\mathbb{P}}' \cdot \widetilde{\psi}')(g) = D(g) \cdot P \cdot D(g^{-1}) \cdot \widetilde{\psi}'(g) \,. \tag{6.13}$$

Again $\widetilde{\mathbb{P}}' \in \mathrm{End}(\mathcal{M}_c)$ because P commutes with $D(H)$ and moreover it is again a projection, *i.e.*$\widetilde{\mathbb{P}}'^2 = \widetilde{\mathbb{P}}'$. This then defines for us a projection in the space of covariant sections as follows:

$$\begin{aligned}(\mathbb{P} \cdot \psi')(p) =& D(\sigma(p)) \cdot P \cdot D(\sigma(p)^{-1}) \cdot \psi'(p) \\ \stackrel{\text{def}}{=}& \mathbb{P}(p) \cdot \psi'(p) \,.\end{aligned} \tag{6.14}$$

Notice that $\mathbb{P}(p_o)$ is just the original projection P. It is easy to verify that this is well defined — *i.e.*independent of the choice of coset representative — and that it commutes with the G-action in Γ_c — since this was true at the level of Mackey functions and all our constructions have been equivariant.

Summarising, to obtain irreducibility we just impose the following constraint on the covariant sections:

$$\mathbb{P}(p) \cdot \psi'(p) = \psi'(p) \,. \tag{6.15}$$

As an example let's consider the vector representation of SO_3 embedded in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathbf{R} & \\ 0 & & & \end{pmatrix} ,$$

where $\mathbf{R}$ is a matrix in the vector representation of SO_3. Thus we see that the $(\frac{1}{2}, \frac{1}{2})$ breaks up into a scalar and a vector of SO_3. The prejection onto the vector representation is clearly seen to be

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Because P is $\mathbb{P}(p_o)$ we need to write this as a function of p_o, which we chose to be $p_o^t = (m, 0, 0, 0)$. We clearly see that

$$\mathbb{P}(p_o) = \mathbf{1} - \frac{p_o \otimes p_o^\flat}{p_o^2} \,, \tag{6.16}$$

where $\mathbf{1}$ is the identity transformation, $p_o^\flat$ is the dual to the vector p_o associated to the Minkowski metric — *i.e.*$p_o^\flat(p) = \eta(p_o, p)$ — and p_o^2 is just $\eta(p_o, p_o)$ where

η is, of course, the Minkowski metric. Because $p_o^2 = m^2$ is a constant of the orbit we can write the projection as

$$\mathbb{P}(p_o) = \mathbf{1} - \frac{p_o \otimes p_o^\flat}{m^2}\,, \tag{6.17}$$

which trivially yields

$$\mathbb{P}(p) = \mathbf{1} - \frac{p \otimes p^\flat}{m^2}\,. \tag{6.18}$$

Hence to obtain an irreducible representation of the Poincaré group corresponding to a massive vector particle we take the non-covariant section $\widetilde{A}_i$ on the mass hyperboloid, we extend it to a covariant section $\widetilde{A}_\mu$ and, in order to get rid of the extra degrees of freedom we had to introduce to exhibit Lorentz covariance, we impose the following constraint

$$\mathbb{P}(p)_\mu{}^\nu\, \widetilde{A}_\nu(p) = \widetilde{A}_\mu(p)\,, \tag{6.19}$$

which translates simply into

$$p^\mu\, \widetilde{A}_\mu(p) = 0\,. \tag{6.20}$$

Still we have the covariant irreducible representation carried by sections in a coset space which is not the most useful. That is, we have $\widetilde{A}_\mu(p)$ when what we really want is $A_\mu(x)$. In this particular case we know what to do: we perform a mass shell Fourier transform. We now suitably generalize this to arbitrary $G = T \rtimes S$.

We are interested in going from sections of a homogeneous vector bundle over a coset space G/H to similar objects over another coset space G/H'. Because both coset spaces have the group G in common it is natural to make this transition at the level of Mackey functions. In doing so we will show a (formal) method to obtain Mackey functions from arbitrary functions on the group. So let $\widetilde{\psi}\colon G \to \mathbb{V}$ be any function whatsoever.[4] Define then

$$\widetilde{\phi}(g) = \int_S d\mu(s)\, D(s) \cdot \widetilde{\psi}(g \cdot s) \quad , \tag{6.21}$$

where $d\mu$ is a left invariant measure in S. This function satisfies

$$\widetilde{\psi}(g \cdot s_o) = \int_S d\mu(s)\, D(s) \cdot \widetilde{\psi}(g \cdot s_o \cdot s)$$

[4] Subject to the condition that the integral below makes sense, since the group S may in principle be non-compact.

$$
\begin{aligned}
&= \int_S d\mu(s_o \cdot s)\, D(s_o^{-1}) \cdot D(s_o \cdot s) \cdot \widetilde{\psi}(g \cdot s_o \cdot s) \\
&= D(s_o^{-1}) \cdot \widetilde{\phi}(g)\,. \qquad (6.22)
\end{aligned}
$$

That is, $\widetilde{\phi}$ is a Mackey function associated to a section of a homogeneous vector bundle over G/S. In particular we can apply this to a Mackey function associated to a section of a homogeneous vector bundle and obtain a Mackey function associated to a section of a different homogeneous vector bundle. Let's apply this to a Mackey function $\widetilde{\psi}$ on $S/L(\alpha)$, and let the representation D of $T \rtimes L(\alpha)$ extend to a representation of G which we also called D. Notice that by equivariance the integrand $D(s) \cdot \widetilde{\psi}(g \cdot s)$, when thought of as a function in S, defines a function on the coset $S/L(\alpha)$. To see this let $s = \sigma([s]) \cdot \lambda(s)$ where $[s] \to \sigma([s])$ is a coset representative of $S/L(\alpha)$ and $\lambda(s) \in L(\alpha)$. Then,

$$
\begin{aligned}
D(s) \cdot \widetilde{\psi}(g \cdot s) &= D(\sigma([s]) \cdot \lambda(s)) \cdot \widetilde{\psi}(g \cdot \sigma([s]) \cdot \lambda(s)) \\
&= D(\sigma([s])) \cdot D(\lambda(s)) \cdot D(\lambda(s)^{-1}) \cdot \widetilde{\psi}(g \cdot \sigma([s])) \\
&= D(\sigma([s])) \cdot \widetilde{\psi}(g \cdot \sigma([s])) \qquad (6.23)
\end{aligned}
$$

which also shows that it does not depend on the coset representative. Hence we can write the transformation as

$$
\widetilde{\phi}(g) = \int_{S/L(\alpha)} d\nu(p)\, D(\sigma(p)) \cdot \widetilde{\psi}(g \cdot \sigma(p))\,. \qquad (6.24)
$$

Now let $x \to \zeta(x) \in T$ be a coset representative for $x \in G/S$. Then

$$
\begin{aligned}
\phi(x) \equiv \widetilde{\phi}(\zeta(x)) &= \int_{S/L(\alpha)} d\nu(p)\, D(\sigma(p)) \cdot \widetilde{\psi}(\zeta(x) \cdot \sigma(p)) \\
&= \int_{S/L(\alpha)} d\nu(p)\, D(\sigma(p)) \cdot \widetilde{\psi}(\sigma(p) \cdot \sigma(p)^{-1} \cdot \zeta(x) \cdot \sigma(p)) \\
&= \int_{S/L(\alpha)} d\nu(p)\, D(\sigma(p)) \cdot D(\mathrm{Ad}_{\sigma(p)^{-1}}\, \zeta(x)^{-1}) \cdot \widetilde{\psi}(\sigma(p)) \\
&= \int_{S/L(\alpha)} d\nu(p)\, \alpha(\mathrm{Ad}_{\sigma(p)^{-1}}\, \zeta(x)^{-1})\, \psi'(p) \qquad (6.25)
\end{aligned}
$$

where we have used the definition of the covariant section ψ' and the fact that D restricted to T is just the character α. In particular in the case that $\zeta(x) = \exp x \cdot T$ we have that

$$
\alpha(\mathrm{Ad}_{\sigma(p)^{-1}} \exp\left(-x \cdot T\right)) = \alpha(\exp\left(-x \cdot \Lambda(\sigma(p)^{-1}) \cdot T\right)
$$

$$
\begin{aligned}
&= \exp\left(-i\, x \cdot \Lambda(\sigma(p)^{-1}) \cdot p_o\right) \\
&= \exp\left(-i\, x \cdot p\right) .
\end{aligned}
\tag{6.26}
$$

And the transformation reduces to the familiar case of the mass-shell Fourier transform

$$
\phi(x) = \int_{S/L(\alpha)} d\nu(p)\, e^{-i\, x \cdot p}\, \psi'(p) . \tag{6.27}
$$

Thus we see how via this correspondence the irreducibility constraints furnished by the projection operators on $S/L(\alpha)$ become differential equations on G/S. For instance, let ψ' be a covariant section satisfying (6.15). Then we have

$$
\begin{aligned}
\phi(x) = &\int_{S/L(\alpha)} d\nu(p)\, e^{-i\, x \cdot p}\, \mathbb{P}(p) \cdot \widetilde{\psi}(p) \\
= &\mathbb{P}(i\frac{\partial}{\partial x})\phi(x) \ ,
\end{aligned}
\tag{6.28}
$$

where we have integrated by parts. Hence (6.15) becomes

$$
\left(\mathbb{P}(i\frac{\partial}{\partial x}) - \mathbf{1}\right) \cdot \phi(x) = 0 \ . \tag{6.29}
$$

Moreover because the coset $S/L(\alpha)$ can be identified with (part of the) quadric surface $p^2 = k$, we obtain after integrating by parts the function $(p^2 - k)\ \widetilde{\psi}(p)$ — which vanishes identically — the mass-shell condition

$$
(\Box + k)\ \phi(x) = 0 \ . \tag{6.30}
$$

In ther particular case of the Poincaré group we identify k with the mass squared m^2 of a physical particle. Hence we see that the mass-shell condition is given by the choice of little group (*i.e.*of coset) whereas the extra field equations come from irreducibility constraints. For the case of the massive vector field we find that it obeys both the Klein-Gordon equation

$$
\left(\Box + m^2\right)\ A_\mu(x) = 0
$$

and the irreducibility contraint (6.20) which in Minkowski space reads

$$
\partial \cdot A(x) = 0 \ .
$$

In the following section we shall look in detail at the case of the Poincaré group and we shall obtain explicitly the equations of motion for the cases $m^2 > 0$ and $m^2 = 0$ corresponding to spin (helicity in the massless case) $0, \frac{1}{2}, 1$. But before doing that let us briefly mention an important point in our discussion of the inner product in the space of covariant sections.

Imagine there exists an operator Π in the space of covariant sections which acts by pointwise matrix multiplication. That is,

$$(\Pi \cdot \psi')(p) = \Pi \cdot \psi'(p) \ , \tag{6.31}$$

and which obeys the following condition

$$D(\sigma(p)^{-1})^\dagger \cdot \Pi \cdot D(\sigma(p)^{-1}) = \Pi \ , \tag{6.32}$$

where † denotes the adjoint with respect to the pointwise $T \rtimes L(\alpha)$-invariant inner product $\langle , \rangle$. The we can define a different scalar product for the covariant sections as follows

$$(\phi', \psi') = \int_{S/L(\alpha)} d\nu(p) \ \big\langle \Pi \cdot \phi'(p) \, , \, \psi'(p) \big\rangle \ . \tag{6.33}$$

It is easy to verify that this is invariant under the action of G. The usefulness of this inner product is that it is simpler than the one given in (6.4) and moreover it coincides with it when Π is the taken to be the parity operator in the representation β of S (if one exists) and when we take the sections at the origin to be parity eigenstates. We shall present some examples of these inner products in the next (and last) section.

§7 EXAMPLES: CLASSICAL FIELDS IN MINKOWSI SPACE

In this section we consider in detail the construction of unitary irreducible representations of the Poincaré group as "fields" on Minkowski space. Actually we find it convenient, since we will be dealing with spinors, to consider the double cover of the Poincaré group $\mathcal{P} = T_4 \rtimes SL_2\mathbb{C}$ since the Lorentz group does not admit spinorial representations. Appendix A contains some facts we shall need about $SL_2\mathbb{C}$.

To obtain irreducible unitary representations we first choose a representation α of the translation subgroup T_4, or equivalently a 4-momentum p_o. Then the coset on which the representations will live is the adjoint orbit of p_o. Because T_4 is abelian it leaves p_o fixed and hence the coset is just the adjoint $SL_2\mathbb{C}$ orbit $\mathcal{O}_{p_o} = \{Ad_*(s)\, p_o \mid s \in SL_2\mathbb{C}\}$. Denoting by L the subgroup of $SL_2\mathbb{C}$ which leaves p_o fixed we see that, by definition, $\mathcal{O}_{p_o} \cong SL_2\mathbb{C}/L$. Because of the $SL_2\mathbb{C}$ invariance of the Minkowki metric all points $p \in \mathcal{O}_{p_o}$ will satisfy $p^2 =$ constant. We will consider two cases of all the possible ones:

(a) $p^2 > 0, p_0 > 0$

(b) $p^2 = 0, p_0 > 0$

since they seem to be the physically relevant ones. Another interesting case might be the "tachyonic" representations with $p^2 < 0$, but we shall not discuss them here.

We already remarked in §6 that the little group L for the case $p^2 > 0$ is $SO_3 \subset SO_{3,1}$ or, in our case, $SU_2 \subset SL_2\mathbb{C}$. We find it instructive, however, to find the little group for both cases simultaneously by finding the little group of a momentum p_o corresponding to a particle of mass m moving along the z-axis. We then get a Lie group whose Lie algebra obeys a commutation law which depends explicitly on the parameter m. For any $m \neq 0$ we see that the algebra is isomorphic to su_2, whereas for $m = 0$ we obtain the Lie algebra of the group of Euclidean motions in two dimensions.

We work in a spinor basis where a real 4-vector is a hermitian 2×2 matrix. That is, $p = p_\mu \sigma^\mu$ where σ^0 is the identity matrix of rank 2 and σ^i are the Pauli matrices which we take to be

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Hence if $p_\mu^t = (p_0, p_1, p_2, p_3)$ we represent it as the matrix

$$p = \begin{pmatrix} p_0 + p_3 & p_1 - i\,p_2 \\ p_1 + i\,p_2 & p_0 - p_3 \end{pmatrix} \equiv \begin{pmatrix} p_+ & \overline{p}_\perp \\ p_\perp & p_- \end{pmatrix} \tag{7.1}$$

which defines $p_\pm$ and $p_\perp$ and where the bar denotes complex conjugation. Let $g \in SL_2\mathbb{C}$, that is,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc = 1 .$$

Then in the $(\frac{1}{2}, \frac{1}{2})$ representation we have that the action of g on p can be written as

$$D^{(\frac{1}{2},\frac{1}{2})}(g) \cdot p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p_+ & \overline{p}_\perp \\ p_\perp & p_- \end{pmatrix} \cdot \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} . \tag{7.2}$$

For a particle of mass m moving along the z-axis, $p_\perp = 0$ and $p_+ p_- = m^2$. Define $\rho = p_-/p_+$. Notice that ρ is finite for all values of m since p_+ never vanishes and that it depends on m implicitly. For the case $m = 0$ we see that $\rho = 0$ whereas for $m \neq 0$ the value of $\rho = 1$ corresponds to the rest frame.

Without loss of generality we may choose $p_+ = 1$ and then the condition for g to be in the little group L is that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \cdot \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} .$$

After some straight-forward algebra one finds that the little group is given by

$$L = \left\{ \begin{pmatrix} a & b \\ -\rho\bar{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C} \, , |a|^2 + \rho |b|^2 = 1 \right\} . \tag{7.3}$$

For elements near the identity, $a = 1 + \varepsilon$ where $|\varepsilon| \ll 1$ and $|b| \ll 1$. To first order we see that the condition $|a|^2 + \rho|b|^2 = 1$ implies

$$1 + \varepsilon + \overline{\varepsilon} + O(|b|^2, |\varepsilon|^2) = 1$$

or that ε is pure imaginary. Hence letting $b = \Re b + i\Im b$ and $\varepsilon = i\Im\varepsilon$ we can expand an arbitrary element of L around the identity to obtain

$$\begin{aligned} g \simeq & \mathbf{1} + \Im\varepsilon \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \Re b \begin{pmatrix} 0 & -1 \\ -\rho & 0 \end{pmatrix} + \Im b \begin{pmatrix} 0 & i \\ i\rho & 0 \end{pmatrix} \\ = & \mathbf{1} + 2\Im\varepsilon \, L_3 + 2\Re b \, L_1 + 2\Im b \, L_2 \, , \end{aligned}$$

which defines L_i. The $\{L_i\}$ obey the following Lie algebra

$$[L_3, L_1] = L_2 \qquad [L_3, L_2] = -L_1 \qquad [L_1, L_2] = \rho L_3 \, . \tag{7.4}$$

Notice that the algebra depends explicitly on the parameter ρ.

In particular, for $\rho \neq 0$ we can rescale our generators such that they obey an su_2 algebra. Explicitly, let $\hat{L}_i = \rho^{-\frac{1}{2}} L_i$ for $i = 1, 2$ and $\hat{L}_3 = L_3$. Then it's easily verified that the $\hat{L}_i$ obey $[\hat{L}_i, \hat{L}_j] = \epsilon_{ijk}\hat{L}_k$. For $\rho = 0$, however, we have the semidirect sum of two abelian algebras. We will see that this is in fact the Lie algebra of the two dimensional Euclidean group.

We have identified the Lie algebras corresponding to the little groups but we have not identified the little groups themselves. For $\rho \neq 0$ we have that the little group is given by (7.3). We claim that this subgroup is isomorphic to SU_2. There are many ways to see this. First it is obviuos that when $\rho = 1$ — *i.e.*the particle is in its rest frame — L is precisely the "canonical" SU_2 inside $SL_2\mathbb{C}$, *i.e.*the special unitary matrices. And it suffices to notice that we can always boost to the rest frame. More explicitly, we can exhibit an inner automorphism between the matrices described by (7.3) and the special unitary matrices given by conjugation with the $SL_2\mathbb{C}$ element

$$\begin{pmatrix} \rho^{1/4} & 0 \\ 0 & \rho^{-1/4} \end{pmatrix} ,$$

which corresponds precisely to the boost required to go to the particle's rest frame.

For $\rho = 0$ — $i.e. m^2 = 0$ — we have that the little group is

$$L = \left\{ \begin{pmatrix} a & b \\ 0 & \overline{a} \end{pmatrix} \Big| a, b \in \mathbb{C} \, , |a|^2 = 1 \right\} , \tag{7.5}$$

which can be parametrized as follows

$$L = \left\{ \begin{pmatrix} e^{i\varphi} & z \\ 0 & e^{-i\varphi} \end{pmatrix} \Big| z \in \mathbb{C}, \varphi \in [0, 2\pi] \right\} .$$

Let's write this as a semidirect product. Recall that a semidirect product $A \rtimes B$ of two groups A and B, where — only to simplify the discussion — we take A to be abelian written additively and B arbitrary, is, setwise, the Cartesian product of the two groups but with the following multiplication law:

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 + b_1 \cdot a_2, b_1 \, b_2) \; ,$$

where the action of B on A is realized — in $A \rtimes B$ — as conjugation, since A is a normal subgroup. It is trivial to verify that with the parametrization given above the group is not a semidirect product. However we can reparametrize it as follows

$$(z, \varphi) = \begin{pmatrix} e^{i\varphi} & z e^{-i\varphi} \\ 0 & e^{-i\varphi} \end{pmatrix} , \tag{7.6}$$

where now the multiplication law is that of a semidirect product

$$(z_1, \varphi_1) \cdot (z_2, \varphi_2) = (z_1 + e^{2i\varphi_1} z_2, \varphi_1 + \varphi_2) \, .$$

Now we want to identify this group with the two dimensional Euclidean motions. This group consists of rotations and translations. Thinking of the two dimensional Euclidean space as the complex plane we can think of a vector as a complex number. The translations act via addition of complex numbers and the rotations act by multiplication by a phase, $i.e.$under a rotation by an angle θ a vector w is mapped to $e^{i\theta} w$ whereas under a translation by a vector z the same vector is mapped to $w + z$. Hence under the action of an element

(z,θ) of the Euclidean group a vector w is mapped to $e^{i\theta}w+z$. Iterating this we see that acting with the element $(z_1,\theta_1)\cdot(z_2,\theta_2)$ is tantamount to acting with the element $(z_1+e^{i\theta_1}z_2,\theta_1+\theta_2)$. Because this representation is faithful this is the multiplication law in the Euclidean group. We notice that it is not quite the multiplication law that we have obtained above unless we reparametrize the group again. We define the element (z,θ) as the matrix

$$\begin{pmatrix} e^{i\frac{1}{2}\theta} & e^{-i\frac{1}{2}\theta}z \\ 0 & e^{-i\frac{1}{2}\theta} \end{pmatrix} ,$$

where still $z\in\mathbb{C}$ but now $\theta\in[0,4\pi]$. Hence we see that our group is the double cover of the Euclidean group $\widetilde{E}_2 = T_2 \rtimes Spin_2$, where $Spin_2$ is the double cover of SO_2.

Having found the little groups for the two cases we are interested in we are now in a position to analyse various irreducible representations of the group $\mathcal{P}$.

Massive Representations

We start by choosing a finite dimensional irreducible representation of $T_4 \rtimes SU_2$. We then induce representation of $\mathcal{P}$ which will be carried by fields in the mass hyperboloid $SL_2\mathbb{C}/SU_2$. We then covariantize the representation by embedding the representation of SU_2 in a suitable representation of $SL_2\mathbb{C}$. This representation will, in general, reduce under SU_2 and we will need to introduce projector to extract the irreducible factor we are interested in. These projectors together with the mass shell condition are the free field equations. Finite dimensional irreducible representations of SU_2 are characterized by the spin. We now treat spin $0,\frac{1}{2}$ and 1 systematically. We shall choose the simpler embeddings possible, although other embeddings are always allowed.

Spin 0

The trivial representation of SU_2 embedds (trivially) in the trivial $(0,0)$ representation of $SL_2\mathbb{C}$. Hence no prejector is needed and the only field equation is the Klein-Gordon equation

$$(\Box + m^2)\,\phi = 0\ .$$

We also saw in §6 how there is an SU_2 scalar in the $(\frac{1}{2},\frac{1}{2})$ representation of $SL_2\mathbb{C}$. The projector in this case is just the complement of the projector needed to extract the **1** representation of SU_2. This projector was given in §6 by (6.18). Hence we see that a massive scalar field may be realized by a 4-vector obeying the following equations

$$\Box A_\mu = -\, m^2\, A_\mu$$
$$\partial_\mu(\partial \cdot A) = -\, m^2\, A_\mu \ .$$

Taking a derivative of the second equation and using the fact that partial derivatives commute we find that

$$\partial_\mu\, A_\nu = \partial_\nu\, A_\mu \ ,$$

from which follows, given that Minkowski space is contractible, that

$$A_\mu = \partial_\mu\, \phi \ ,$$

for some scalar field ϕ, which is seen to satisfy the following

$$\partial_\mu(\Box + m^2)\, \phi = 0 \ ,$$

or, equivalently,

$$(\Box + m^2)\, \phi = \text{constant} \ .$$

However ϕ was defined up to a constant, since shifting ϕ by a constant does not change A_μ. Hence we may use this freedom to make the constant in the above equation vanish, and thus we are left with a scalar field of mass squared m^2 as we expected. This example serves the purpose of remarking that different fields may be used to describe the same physical content, at least at the classical level. It is known in a large class of theories that the S-matrix is independent of the interpolating fields, although anomalies may differ depending on the definition of anomalies one uses.

Spin $\frac{1}{2}$

The $\frac{1}{2}$ representation of SU_2 can be embedded in several irreducible representations of $SL_2\mathbb{C}$. Among the simplest are the $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$ and their direct sum $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The first two representations are still irreducible under SU_2 so that no projector is needed and there is only the mass shell condition. Hence we get a 2-spinor obeying the Klein-Gordon equation

$$(\Box + m^2)\, \psi = 0 \ .$$

The third representation offers many possibilities. First of all the $\frac{1}{2}$ of SU_2 could be contained entirely in one of the two summands. In that case the projectors are the usual chirality projectors $\frac{1}{2}(\mathbf{1} \pm \gamma_5)$. Because they commute with the generators of $SL_2\mathbb{C}$ they don't change from point to point and hence they don't yield any differential equation. Hence in this case the field equations are just

$$\begin{aligned} \Box\,\psi &= -\,m^2\,\psi \\ \gamma_5\,\psi &= \pm\,\psi\ . \end{aligned}$$

This is just the previous case since we have not mixed the two chiralities *i.e.*we have used only one of the direct summands in $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. A more interesting equation results if we embedd in a way that mixes chiralities. Consider the following projectors $\frac{1}{2}(\mathbf{1}\pm\gamma_0)$. These commute with the action of SU_2 and have rank 2, hence they project onto a two dimensional SU_2 invariant subspace: a copy of the spin $\frac{1}{2}$ representation of SU_2. In order to make this projector $SL_2\mathbb{C}$ covariant we must rewrite it as a polynomial in p_o. Since $p_o^t = (m,0,0,0)$ we see that $\gamma_0 = (1/m)\gamma\cdot p_o$. Noticing this, we can rewrite the projector at an arbitrary point p as $\frac{1}{2}(\mathbf{1}\pm(1/m)\not{p})$, where, as usual, $\not{p} = \gamma\cdot p$. This irreducibility condition then yields the massive Dirac equation

$$\not{p}\psi = \pm m\psi\ .$$

In this particular case we see that the mass shell condition is implied by the irreducibility constraint. Hence, without being redundant, the only field equation is

$$(i\not{\partial} \pm m)\,\psi = 0\ .$$

The reader may wonder whether this is the only equation that we obtain in this case since it is far from obvious — and, in fact, it is false — that the above projector is the most general one. In fact, up to a global field redefinition, the Dirac equation is the only equation in this embedding which mixes chiralities. The messy details are shown in Appendix B.

Spin 1

We can embedd the $\mathbf{1}$ representation of SU_2 inside the $(\frac{1}{2},\frac{1}{2})$ of $SL_2\mathbb{C}$ which reduces under as SU_2 as $\mathbf{1}\oplus\mathbf{0}$. As we saw in §6 the field equations are

$$\begin{aligned} \Box\,A_\mu &= -\,m^2\,A_\mu \\ \partial\cdot A &= 0\ . \end{aligned}$$

Massless Representations

This case is a bit more involved but much more interesting. First of all we must characterize the unitary finite-dimensional irreducible representations of $\widetilde{E}_2$. Because of the structure of $\widetilde{E}_2$, finite dimensional representations must be trivial when restricted to T_2. This is a general fact whenever there is an abelian normal subgroup. The reason is the following. Let $D\colon \widetilde{E}_2 \to \operatorname{Aut}\mathbb{W}$ be a finite dimensional (unitary) representation and let $\psi \in \mathbb{W}$ be such that

$$\begin{aligned} D(\exp\, x \cdot L)\,\psi =& e^{x\cdot D_*(L)}\,\psi \\ \equiv& e^{ix\cdot l}\,\psi\ , \end{aligned}$$

where $\{L\}$ generate the translation ideal. Then consider the vector $D(\exp\,\theta J)\,\psi$ where J generates the rotation subalgebra. Then

$$\begin{aligned} D(\exp\, x \cdot L)\, D(\exp\,\theta J)\,\psi =& D(\exp\,\theta J)\, D(\exp\,\mathrm{Ad}_*(-\theta)(x\cdot L))\,\psi \\ =& e^{ix\cdot\Lambda(-\theta)\cdot l}\, D(\exp\,\theta J)\,\psi\ , \end{aligned}$$

where Λ is the rotation matrix acting on the two-vector l. Hence if the representation of T_2 characterized by the vector l appears then so do all the representations of T_2 characterized by the vectors in the adjoint orbit of l under $Spin_2$, which is the circle of radius $|l|$ in the complex plane. This set is uncountable unless $l = 0$. Therefore any finite dimensional (unitary) representation of $\widetilde{E}_2$ is necessarily trivial when restricted to the translation subgroup and thus it is just a representation of $Spin_2$. This being an abelian compact group implies that its unitary finite-dimensional representations are one dimensional and are characterized by a number j such that

$$D(\exp\,\theta J) = e^{i\theta j}\,. \tag{7.7}$$

Since $\theta = 0$ and $\theta = 4\pi$ denote the same group element we must have that

$$e^{i4\pi j} = 1\,,$$

from which we conclude that $j \in \frac{1}{2}\mathbb{Z}$. This number j is called the **helicity**. Notice that half-integral helicity is allowed because we are working with the covering group $Spin_2$. Had we studied E_2 which contains only SO_2 we would have obtained only integral helicities. This is exactly analogous to the fact that SU_2 and not SO_3 has half-integral spin representations.

It turns out that, unlike the massive case, the irreducible representations of $SL_2\mathbb{C}$ in which the helicity j representation of $\widetilde{E}_2$ can be embedded are very restricted. I shall sketch the details surrounding this restriction. I will be using results derived in Appendix A, so the interested reader is encouraged to look at them now.

Recall from Appendix A that $sl_2\mathbb{C}$ has a basis in which it has the structure of $su_2 \oplus su_2$, although both sets of su_2 generators are not independent, being related by conjugation. In terms of these generators the generators of e_2 are

$$\begin{aligned} J =& S_3 + \overline{S}_3 \\ L_1 =& (1/2)\,(S_- + \overline{S}_+) \\ L_2 =& (i/2)\,(S_- - \overline{S}_+) \ . \end{aligned}$$

Because

$$\begin{aligned} D_*(J) =& i\, j \\ D_*(L_1) =& 0 \\ D_*(L_2) =& 0 \ , \end{aligned}$$

we obtain

$$D_*(S_3 + \overline{S}_3) = i\, j$$

and after taking $\mathbb{C}$-linear combinations — which we can do since the space carrying the representation is a complex vector space —

$$\begin{aligned} D_*(S_-) =& 0 \\ D_*(\overline{S}_+) =& 0 \ . \end{aligned}$$

These last two conditions are very restrictive. Let's try to embed the 1-dimensional helicity j representation of e_2 inside the (j_1, j_2) representation of $sl_2\mathbb{C}$. Then the last two equations tell us that it is the lowest weight vector of the spin j_1 representation of the su_2 algebra spanned by the $\{S_i\}$ — since it is annihilated by S_- — and it is the highest weight vector of the spin j_2 representation of the su_2 algebra spanned by the $\{\overline{S}_i\}$ — since it is annihilated by $\overline{S}_+$. This also tells us that on this 1-dimensional subspace S_3 has eigenvalue $-i\, j_1$ and $\overline{S}_3$ has eigenvalue $i\, j_2$. Therefore the following relation must hold

$$j_2 - j_1 = j \ .$$

This restricts the possible irreducible representations of $sl_2\mathbb{C}$ in which we can embbed the helicity j representation of e_2. In particular we notice the striking fact that in order to describe a particle of helicity ± 1 we cannot use the $(\frac{1}{2}, \frac{1}{2})$ representation. The smallest representations we can use for this particle are the $(0, 1)$, $(1, 0)$ representations. This corresponds to using the $\vec{E}, \vec{B}$ fields — or, rather, their self-dual and anti-self-dual combinations — instead of the gauge potential A_μ. Of course this is not too surprising if one notices that one of the equations of motion will be the massless Klein-Gordon equation, which is not obeyed by the gauge potential unless we choose a particular gauge, namely $\partial \cdot A =$ constant. We now investigate the helicity 0, $\pm\frac{1}{2}$ and ± 1 cases in detail.

Helicity 0

We can embedd this representation in any irreducible $sl_2\mathbb{C}$ representation of the form (j, j) for any non-negative integer j. If we choose to embbed this representation in the $(0, 0)$ representation of $sl_2\mathbb{C}$ we have no prejector and the only equation of motion is the mass shell condition, *i.e.*the massless Klein-Gordon equation

$$\Box\phi = 0 \ .$$

To be continued...

Appendix A General Facts about $SL_2\mathbb{C}$

In this appendix we collect the results we use in §7 concerning the group $SL_2\mathbb{C}$. This groups plays an important role because it is the universal covering group of the Lorentz group $SO_{3,1}$. The correspondence is easily seen as follows. Using the Pauli matrices we can make a vector space isomorphism between Minkowski space and the space of 2×2 hermitian matrices. This is shown in §7. It is a trivial calculation to verify that if the 4-vector x is mapped onto the hermitian matrix $\tilde{x}$ then the norm $\|x\|$ of x is just the determinant $\det\tilde{x}$.

Obviously not yet finished!

Appendix B Essential Uniqueness of the Dirac Equation

In this appendix we consider the question which arose in §7 concerning the uniqueness of the Dirac equation in the particular embedding of the spin $\frac{1}{2}$ representation of su_2 into the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of $sl_2\mathbb{C}$. We first find the most general projector which commutes with the the action of su_2. That is, if we let $\mathbb{V}$ denote the four dimensional vector which carries the aforementioned representation of $sl_2\mathbb{C}$, then we are looking for an endomorphism $\mathbb{P}$ of $\mathbb{V}$ which commutes with the generators of su_2 and which obeys $\mathbb{P}^2 = \mathbb{P}$. The task of finding such a $\mathbb{P}$ is simplified significantly due to the fact that the Dirac matrices $\{\mathbf{1}, \gamma_\mu, \gamma_5, \gamma_5\gamma_\mu, \gamma_{\mu\nu}\}$ form a basis for the space of endomorphisms. The su_2 generators are given by γ_{ij} where i, j are spatial indices. It is easy to see that for a matrix to commute with γ_{ij} it must not contain any spatial indices: clearly anything with one spatial index will transform as a vector whereas anything with two spatial indices is necessarily antisymmetric and hence is one of the generators. Thus all possible projectors are linear combinations of $\{\mathbf{1}, \gamma_0, \gamma_5, \gamma_0\gamma_5\}$. Hence let $\mathbb{P}$ be the following linear combination

$$\mathbb{P} = a\,\mathbf{1} + b\,\gamma_5 + c\,\gamma_0 + d\,\gamma_0\,\gamma_5\ . \tag{B.1}$$

Now $\mathbb{P}^2 = \mathbb{P}$ yields the following constraints on the coefficients

$$\begin{aligned} 2ab =& b \\ 2ac =& c \\ 2ad =& d \\ a^2 + b^2 + c^2 - d^2 =& a\ , \end{aligned}$$

which give rise to two different cases

(1) At least one among $\{b, c, d\}$ is non-zero. Then clearly $a = \frac{1}{2}$ and

$$b^2 + c^2 = d^2 + \frac{1}{4}\ . \tag{B.2}$$

(2) $b = c = d = 0$ and $a = 0$ or $a = 1$. This is the trivial case in which the projector is either the zero endomorphism or the identity. Neither case is interesting since they do not project onto su_2-irreducible subspaces.

Hence we stick with case (1) and we have that the most general projector is given by

$$\mathbb{P} = \tfrac{1}{2}\left(\mathbf{1} + \breve{b}\,\gamma_5 + \breve{c}\,\gamma_0 + \breve{d}\,\gamma_0\,\gamma_5\right)\ , \tag{B.3}$$

where

$$\breve{b}^2 + \breve{c}^2 = \breve{d}^2 + 1 \tag{B.4}$$

and where the $\breve{}$ indicates multiplication by 2. We have already seen the cases $\breve{c} = \breve{d} = 0$, $\breve{b} = \pm 1$ corresponding to the chirality projectors and $\breve{b} = \breve{d} = 0$, $\breve{c} = \pm 1$ corresponding to the Dirac equation.

Notice that because $p_o^t = (m, 0, 0, 0)$, γ_0 is nothing but $\frac{1}{m}\not{p}_o$ and hence we may rewrite $\mathbb{P} = \mathbb{P}(p_o)$ as follows

$$\mathbb{P}(p_o) = \tfrac{1}{2}\left(\mathbf{1} + \breve{b}\,\gamma_5 + \frac{\breve{c}}{m}\not{p}_o + \frac{\breve{d}}{m}\not{p}_o\,\gamma_5\right)\ ,$$

which immediately allows us to write the covariant projector

$$\mathbb{P}(p) = \tfrac{1}{2}\left(\mathbf{1} + \breve{b}\,\gamma_5 + \frac{\breve{c}}{m}\not{p} + \frac{\breve{d}}{m}\not{p}\,\gamma_5\right)\ . \tag{B.5}$$

Imposing the irreducibility constraint $\mathbb{P}(p)\,\psi(p) = \psi(p)$ we obtain the following equation

$$m\,(\mathbf{1}-\breve{b}\,\gamma_5)\,\psi(p)=\not p\,(\breve{c}+\breve{d}\,\gamma_5)\,\psi(p)\ . \tag{B.6}$$

Multiplying both sides by $\not p/m$ and noticing that $\not p^2=m^2$ we have

$$(\mathbf{1}+\breve{b}\,\gamma_5)\,\not p\,\psi(p)=m\,(\breve{c}+\breve{d}\,\gamma_5)\,\psi(p)\ . \tag{B.7}$$

Two cases present themselves to us now:

(1) $\breve{b}^2=1$. In this case $(\mathbf{1}+\breve{b}\,\gamma_5)$ is not invertible so that we cannot find a Dirac-type equation expressing $\not p\,\psi$ as a p-independent operator. We call this a pre-Dirac equation.

(2) $(\mathbf{1}+\breve{b}\,\gamma_5)$ is invertible so that we find a pre-Dirac equation.

We now show that in case (1) we recover the uninteresting chirality projectors whereas in case (2) we can redefine ψ globally — *i.e.*independent of p — in such a way that the new field does obey the usual Dirac equation. We now treat each case independently. It turns out that the analysis is simplified if we break the first case into subcases:

(1a) $\breve{c}=0$. In this case $\breve{d}=$ since for $\breve{b}^2=1$ (B.4) reads $\breve{c}^2=\breve{d}^2$. Thus we have that the projector becomes the chirality projector.

(1b) $\breve{c}\neq 0$. Then let $\epsilon=\breve{d}/\breve{c}$. Clearly from (B.4), $\epsilon^2=1$. Two possibilities occur

(1b.i) $\epsilon=-\breve{b}$. In this case (B.7) becomes

$$(\mathbf{1}+\breve{b}\,\gamma_5)\,\not p\,\psi(p)=m\,\breve{c}\,(\mathbf{1}-\breve{b}\,\gamma_5)\,\psi \tag{B.8}$$

from which we obtain, after acting on both sides with $\mathbf{1}-\breve{b}\,\gamma_5$,

$$0=m\,\breve{c}\,(\mathbf{1}-\breve{b}\,\gamma_5)^2\,\psi(p)\ ,$$

or, equivalently, the chirality projector $\gamma_5\,\psi(p)=\pm\,\psi(p)$.

(1b.ii) $\epsilon=\breve{b}$. In this case the projector (B.5) can be written as follows

$$\mathbb{P}(p)=\tfrac{1}{2}\,(\mathbf{1}+\breve{b}\,\gamma_5)\,(\mathbf{1}+\frac{\breve{c}}{m}\not p)\ ,$$

from which it follows that $(\mathbf{1}-\breve{b}\,\gamma_5)\,\mathbb{P}(p)=0$. Hence if $\mathbb{P}(p)\,\psi(p)=\psi(p)$, then $(\mathbf{1}-\breve{b}\,\gamma_5)\,\psi(p)=0$, which is nothing but the chirality projectors.

In summary, case (1) produces only the chirality projectors. Case (2) is more interesting. Now we can invert $\mathbf{1} + \breve{b}\,\gamma_5$ so that we get the following equation

$$\not{p}\,\psi(p) = \frac{m}{1-\breve{b}^2}\,(\mathbf{1} - \breve{b}\,\gamma_5)\,(\breve{c} + \breve{d}\,\gamma_5)\,\psi(p)\ . \tag{B.9}$$

This equation is of the general form

$$\not{p}\,\psi(p) = m(A + B\,\gamma_5)\,\psi(p)\,. \tag{B.10}$$

Notice that for this equation to be compatible with the mass shell condition $\not{p}^2 = m^2$ A and B must satisfy $A^2 - B^2 = 1$. The reader can verify that this indeed holds in our case. We will show that there always exists an automorphism S of $\mathbb{V}$ such that

$$\not{p}\,(S\,\psi(p)) = \pm m\,(S\,\psi(p))\ .$$

That is, $S\,\psi$ satisfies the Dirac equation. Without lack of generality let $S = \mathbf{1} + \beta\,\gamma_5$. Then if $S\,\psi$ is to satisfy the Dirac equation the following must hold

$$(\mathbf{1} - \beta\,\gamma_5)\,\not{p}\,\psi(p) = \pm m\,(\mathbf{1} + \beta\,\gamma_5)\,\psi(p)\ , \tag{B.11}$$

which, upon using the pre-Dirac equation for ψ, becomes

$$(\mathbf{1} - \beta\,\gamma_5)\,(A + B\,\gamma_5)\,\psi(p) = \pm\,(\mathbf{1} + \beta\,\gamma_5)\,\psi(p)\,. \tag{B.12}$$

Matching coefficients yields (assuming that $B \neq 0$ or $A^2 \neq 1$)

$$\begin{aligned} A - \beta\,B &= \pm 1 \qquad &\Rightarrow \qquad \beta &= \frac{A \mp 1}{B} \\ B - \beta\,A &= \pm\,\beta \qquad &\Rightarrow \qquad \beta &= \frac{B}{A \pm 1}\ . \end{aligned}$$

For both solutions to be consistent the following must hold

$$(A\,\pm\,1)\,(A\,\mp\,1) = B^2 \qquad \Rightarrow \qquad A^2 - B^2 = 1\ ,$$

which is precisely the condition for the pre-Dirac equation to be consistent with the mass shell condition. The trasformation can easily checked to be invertible if it exists. That is, if $B \neq 0$, then $\det S \neq 0$. What about the case $B = 0$? In that case we already have the Dirac equation so that there is no need to perform the transformation.

In conclusion, the only equation of motion — up to a global field redefinition — which mixes chiralities is the Dirac equation. A brief final remark is in order. We may impose extra constraints on the projector. For instance if we impose that it commute with the parity operator — which for this representation can be chosen to be γ_0 — we see immediately that $\breve{b}$ and $\breve{c}$ must vanish, yielding the Dirac equation right from the beginning.