pairing of G to G' is defined and the products $g_1 . g_2$ ($g_1, g_2 \in G$) all have order two.

The group $H^n(\pi, n; G)$ is isomorphic with the group of homomorphisms of $H_n(\pi, n)$ into G, and $H_n(\pi, n)$ is isomorphic with π. Hence $H^n(\pi, n; \pi)$ is isomorphic with the group of endomorphisms of π; let d^n be the element of $H^n(\pi, n; \pi)$ which acts as the identity endomorphism of π.

For $n > 2$, $\eta_0 : \pi_n(X) \rightarrow \pi_{n+1}(X)$ is a homomorphism with values in $_2(\pi_{n+1}(X))$ and therefore a commutative self-pairing of $\pi_n(X)$ to $\pi_{n+1}(X)$ can be found such that $\alpha . \alpha = \eta_0(\alpha)$. It then follows that $\alpha \beta$ always has order two and therefore $Sq_{n-2} : H^n(\pi_n(X), n; \pi_n(X)) \rightarrow H^{n+2}(\pi_n(X) n; \pi_{n+1}(X))$ is defined and is independent of the particular choice of the self-pairing. Using this pairing, we find

THEOREM 6. *If* $n > 2$, $Sq_{n-2}(d^n) = k^{n+2}$.

[1] Freudenthal, H., *Comp. Math.*, **5**, 299–314 (1937).

[2] Fox, R. H., *Bull. Am. Math. Soc.*, **49**, abstract 172 (1943).

[3] Hurewicz, W., *Proc. Akad. Amsterdam*, **39**, 117–126 (1936).

[4] Hopf, H., *Comm. Math. Helv.*, **17**, 307–326 (1945).

[5] These results depend on Pontrjagin's result that $\pi_{n+2}(S^n) = 0$ for $n \geqq 3$. A proof was outlined in *C. R. Acad. Sci. U.R.S.S.*, **19**, 361–363 (1938), but a complete proof has not yet appeared.

[6] Eilenberg, S., and MacLane, S., *Ann. Math.*, **46**, 489–509 (1945).

[7] Cf. Eilenberg, S., and MacLane, S., *Proc. Nat. Acad. Sci.*, **32**, 277–280 (1946) for the case $n = 1$.

[8] Eilenberg, S., and MacLane, S., *Ann. Math.*, **43**, 757–831 (1942).

[9] Steenrod, N. E., *Ibid.*, **48**, 290–320 (1947).

GROUP THEORETICAL DISCUSSION OF RELATIVISTIC WAVE EQUATIONS

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Introduction.[1]—The wave functions, ψ, describing the possible states of a quantum mechanical system form a linear vector space **V** which, in general, is infinite dimensional and on which a positive definite inner product (ϕ, ψ) is defined for any two wave functions ϕ and ψ (i.e., they form a Hilbert space). The inner product usually involves an integration over the whole configuration or momentum space and, for particles of higher spin, a summation over the spin indices.

If the wave functions in question refer to a free particle and satisfy relativistic wave equations, there exists a correspondence between the wave functions describing the same state in different Lorentz frames.

The transformations considered here form the group of all *inhomogeneous* Lorentz transformations (including translations of the origin in space and time). Let $\psi_{l'}$ and ψ_l be the wave functions of the same state in two Lorentz frames l' and l, respectively. Then $\psi_{l'} = U(L)\psi_l$, where $U(L)$ is a linear unitary operator which depends on the Lorentz transformation L leading from l to l'. By a proper normalization, U is determined by L up to a factor ± 1. (For all details the reader is referred to the paper of reference 2, hereafter quoted as (L).) Moreover, the operators U form a single- or double-valued representation of the inhomogeneous Lorentz group, i.e., for a succession of two Lorentz transformations L_1, L_2, we have

$$U(L_2L_1) = \pm U(L_2)U(L_1). \tag{1}$$

Since all Lorentz frames are equivalent for the description of our system, it follows that, together with ψ, $U(L)\psi$ is also a possible state viewed from the original Lorentz frame l. Thus, the vector space **V** contains, with every ψ, all transforms $U(L)\psi$, where L is any Lorentz transformation.

The operators U may also replace the wave equation of the system. In our discussion, we use the wave functions in the "Heisenberg" representation, so that a given ψ represents the system for all times, and may be chosen as the "Schrödinger" wave function at time 0 in a given Lorentz frame l. To find ψ_{t_0}, the Schrödinger function at time t_0, one must therefore transform to a frame l' for which $t' = t - t_0$, while all other coordinates remain unchanged. Then $\psi_{t_0} = U(L)\psi$, where L is the transformation leading from l to l'.

A classification of all unitary representations of the Lorentz group, i.e., of all solutions of (1), amounts, therefore, to a classification of all possible relativistic wave equations. Such a classification has been carried out in (L). Two representations $U(L)$ and $\tilde{U}(L) = VU(L)V^{-1}$, where V is a fixed unitary operator, are equivalent. If the system is described by wave functions ψ, the description by

$$\tilde{\psi}_l = V\psi_l \tag{2}$$

is isomorphic with respect to linear superposition, to forming the inner product of two wave functions, and also to the transition from one Lorentz frame to another. In fact, if $\psi_{l'} = U(L)\psi_l$, then $\tilde{\psi}_{l'} = V\psi_{l'} = \tilde{U}(L)\tilde{\psi}_l$. Thus, one obtains classes of equivalent wave equations. Finally, it is sufficient to determine the irreducible representations since any other may be built up from them.

Two descriptions which are equivalent according to (2) may be quite different in appearance. The best known example is the description of the electromagnetic field by the field strength and the four vector potential, respectively. It cannot be claimed either that equivalence in the sense of (2) implies equivalence in every physical aspect. Thus, two equivalent descriptions may lead to quite different expressions for the charge density or the energy density in configuration space (cf. Fierz,[3]) because (2) only implies global, but not local, equivalence of the wave functions. It should be emphasized, however, that any selection of one among the equivalent systems or the superposition of non-equivalent systems in any particular way involves an explicit or implicit assumption as to possible interactions, the positive character of densities, etc. Our analysis is necessarily restricted to free particles and does not lead to any assertions about possible interactions.

The present discussion is not based on any hypothesis about the structure of the wave equations provided that they be Lorentz invariant. In particular, it is not necessary to assume differential equations in configuration space. But it is a result of the analysis in (L) that every irreducible wave equation is equivalent (in the sense of (2)) to a system of differential equations. For the relation of the present point of view to other treatments of the subject see reference 11.

In the present note, we shall give, for every representation of (L), a differential equation the solutions of which transform according to that representation. We also will discuss in some detail the infinitesimal operators which generate the irreducible representations determined in (L), and we shall characterize these representations, and hence the covariant differential equations, by certain invariants constructed from the infinitesimal operators. This is of some interest, because the infinitesimal operators are closely related to dynamical variables of the system. L. Gårding[4] has recently shown that even in the infinite dimensional case one can rather freely operate with infinitesimal transformations. In particular, it immediately follows from his discussion (although it is not explicitly stated in his note) that the familiar commutation rules remain valid.

1. *The Infinitesimal Operators of the Lorentz Group.*—The metric tensor is assumed in the form $g_{44} = 1$, $g_{11} = g_{22} = g_{33} = -1$, $g_{kl} = 0$ $(k \neq l)$ and $g^{kl} = g_{kl}$. The scalar product of two four vectors a, b will be denoted by $\{a, b\} = a^k b_k$. Both c, the velocity of light, and $\hbar$, Planck's constant divided by 2π, are set equal to 1.

The Infinitesimal Operators p_k and M_{kl}. A translation in the x^k-direction is generated by p_k, a rotation in the $(x^k - x^l)$ plane by $M_{kl} = -M_{lk}$ $(k, l = 1, \ldots, 4)$. These operators are Hermitian, and the unitary operators U which represent the finite Lorentz transformation are obtained by exponentiation; thus $U = \exp(-i\alpha p_k)$ corresponds to a translation by the amount α in the direction x_k. Clearly, p_k are the four momenta of the system, and M_{23}, M_{31}, M_{12} the three components of the total angular momentum. The following commutation rules hold (where $[A, B] = AB - BA$)

$$[M_{kl}, M_{mn}] = i(g_{lm}M_{kn} - g_{km}M_{ln} + g_{kn}M_{lm} - g_{ln}M_{km}), \quad (3a)$$

$$[p_k, p_l] = 0 \qquad [M_{kl}, p_m] = i(g_{lm}p_k - g_{km}p_l). \quad (3b)$$

We now define four operators w_k by

$$(w^1, w^2, w^3, w^4) = (v_{234}, v_{314}, v_{124}, v_{321}), \quad (4a)$$

$$v_{klm} = p_kM_{lm} + p_lM_{mk} + p_mM_{kl} = M_{lm}p_k + M_{mk}p_l + M_{kl}p_m. \quad (4b)$$

Note that w_k is a "pseudo-vector," i.e., it is a vector only with respect to Lorentz transformations of determinant 1. By (3),

$$[M_{kl}, w_m] = i(g_{lm}w_k - g_{km}w_l) \qquad [p_k, w_l] = 0. \quad (5)$$

It follows from (3) and (5) that the two operators,

$$P = p^kp_k; \quad W = (1/6)v^{klm}v_{klm} = -w^kw_k, \quad (6)$$

commute with all the infinitesimal operators M_{kl} and p_k. Therefore, they have constant values (i.e., they are multiples of the unit operator) for every irreducible representation of the Lorentz group. (The familiar arguments which establish this for finite dimensional representations can be carried over to the infinite dimensional case. (Cf. V. Bargmann, reference 5, p. 602.))

W may also be written in the form

$$W = 1/2M_{kl}M^{kl}p_mp^m - M_{km}M^{lm}p^kp_l. \quad (7)$$

(This quantity was first introduced by W. Pauli, cf. Lubánski.[6]) The scalar product w^kp_k vanishes.

2. *Summary of the Results of* (*L*).—(*a*) For every irreducible representation the states ψ may be expressed as functions $\psi(p, \xi)$ of the momentum vector p and an auxiliary variable ξ which may assume a finite or an infinite number of values. The momenta p are either all zero, or they vary over the manifold $p^kp_k = P$, with a constant value P. We confine ourselves to the cases in which $p \neq 0$, and either $P > 0$ or $P = 0$, because the remaining cases are unlikely to have direct physical significance.[5]

(*b*) To every inhomogeneous Lorentz transformation $y^k = \lambda^k{}_lx^l + a^k$ (in vector form: $y = \Lambda x + a$) corresponds a unitary operator $U(L)$ defined by

$$U(L)\psi(p, \xi) = e^{-i\{a,p\}}Q(p, \Lambda)\psi(\Lambda^{-1}p, \xi), \quad (8)$$

where $Q(p,\Lambda)$ is a unitary operator which may depend on p but affects only the variable ξ. The inner product (ϕ, ψ) is obtained by an integration over the manifold $p^kp_k = P$ and by a summation or integration over the variable ξ.

(*c*) The subgroup of the homogeneous Lorentz transformations which keep a fixed momentum vector p_0 unchanged is called "little group."

(The little groups defined by different vectors p_0 are isomorphic.) The unitary operators $Q(p_0, \Lambda)$ (where $\Lambda p_0 = p_0$) form an irreducible representation of the little group and determine the irreducible representation $U(L)$ of the inhomogeneous Lorentz group.

In all cases the operators M_{kl} have the form $\underset{0}{M}_{kl} + S_{kl}$, where the

$$\underset{0}{M}_{kl} = i\left(p_k \frac{\partial}{\partial p^l} - p_l \frac{\partial}{\partial p^k}\right) = i(p_k g_{lj} - p_l g_{kj}) \frac{\partial}{\partial p_j} \tag{9}$$

act on the variables p and correspond to the orbital angular momenta, while the S_{kl} act on the variables ξ and correspond to the spin angular momenta. Both $\underset{0}{M}_{kl}$ and S_{kl} satisfy the commutation rules (3a). Since the $\underset{0}{M}_{kl}$ do not contribute to v_{klm} (cf. (4b)), we have

$$v_{klm} = p_k S_{lm} + p_l S_{mk} + p_m S_{kl}; \quad [S_{kl}, p_m] = 0, \tag{10}$$

or, introducing the three-dimensional vector operators,

$$\vec{S} = (S_{23}, S_{31}, S_{12}); \quad \vec{S}' = (S_{14}, S_{24}, S_{34}); \quad \vec{p} = (p^1, p^2, p^3); \tag{10a}$$
$$\vec{w} = (w^1, w^2, w^3); \quad w^4 = \vec{p}.\vec{S}; \quad \vec{w} = p^4\vec{S} - (\vec{p} \times \vec{S}').$$

Clearly, M_{kl} may also be replaced by S_{kl} in the expression (7) for W.

For a fixed momentum vector p_0 the operators w_k are the infinitesimal generators of the little group. Since $w^k p_k = 0$, only three of them are linearly independent.

3. *Classification of the Irreducible Representations.*—We now turn to a brief summary of the main results, including the characterization of the representations in terms of the operators p and w. A more detailed discussion will follow in the succeeding sections.

The classes found in (L) (§§ 7, 8) are these:

I. P_s. *Particles of finite mass and spin s.*—Here $P = m^2 > 0$. In the rest system of the particle, the momentum vector has only the one non-vanishing component $p^4 = \pm m$, hence, by (10a), $W = m^2 S^2$. The operator $P^{-1}W$ represents the square of the spin angular momentum, and has the value $s(s + 1)$ $(s = 0, {}^1/_2, 1, \ldots)$ for an irreducible representation. For a given momentum vector there are $2s + 1$ independent states. The representation $U(L)$ is single or double valued according to whether s is integral or half integral. The lowest cases $(s = 0, {}^1/_2, 1)$ correspond to the Klein-Gordon, Dirac and Proca equations, respectively.

II. 0_s. *Particles of zero rest mass and discrete spin.*—These representations may be considered limiting cases of the representations P_s for $m \to 0$. Then both P and W are equal to zero, and do not suffice to characterize these representations. For a given momentum vector, there exist 2 independent states if $s \neq 0$ (corresponding to two different states of polarization), and there is only one state if $s = 0$. Right and left circularly polarized states are described by the operator equations $w_k = sp_k$, and $w_k = -sp_k$, respectively, so that the representation 0_s is characterized by $P = 0$, $w_k w_l = s^2 p_k p_l$. The lowest cases ($s = 0, {}^1/_2, 1$) correspond to the scalar wave equation, the neutrino equation, and Maxwell's equations, respectively.

III. *$0(\Xi)$ and $0'(\Xi)$. Particles of zero rest mass and continuous spin.*— Here, $P = 0$, $W = \Xi^2$, where Ξ is a real positive number. For a given momentum vector there exist infinitely many different states of polarization, which may be described by a continuous variable. The representation $0(\Xi)$ is single valued, while $0'(\Xi)$ is double valued.

To construct these representations explicitly, we shall select, in each case, one among the equivalent sets of wave equations, define a Lorentz invariant inner product (ϕ, ψ), and prove the operator relations stated above. We shall operate in momentum space; this is particularly simple, because the momenta (but not the coördinates) are defined by the Lorentz group, as infinitesimal translations.

4. *The Class P_s.*—(*a*) $s = 0$. Here, the variable ξ assumes only one value and may therefore be omitted. Consequently, $Q(p, \Lambda) = 1$ (cf. reference 8), and for the little group the trivial one-dimensional representation is obtained. Hence, $S_{kl} = 0$, and $w_k = 0$. The wave equation reduces to $p^k p_k = m^2$; the inner product (ϕ, ψ) is determined by the norm (ψ, ψ) of a wave function,

$$(\psi, \psi) = \int |\psi(p)|^2 d\Omega, \text{ where } d\Omega = |p^4|^{-1} dp^1 dp^2 dp^3, \qquad (11)$$

the integral being extended over both sheets of the hyperboloid $p^k p_k = P = m^2$. The expression (11) is Lorentz invariant, because $d\Omega$ is an invariant volume element in momentum space. For the wave function in configuration space, one finds

$$\psi(x) = (2\pi)^{-3/2} \int e^{-i\{p,\, x\}} \psi(p) d\Omega, \qquad (12)$$

where x stands for x^1, x^2, x^3, x^4. It is well known that (ψ, ψ) cannot be simply expressed in configuration space, because for the Klein-Gordon equation the density is indefinite, and the integral over the density in configuration space coincides with (11) only if $\psi(p) = 0$ whenever $p^4 < 0$.

(*b*) $s = {}^1/_2 N$ with $N = 1, 2, 3, \ldots$. For particles of higher spin we use the equations first derived by Dirac[7] in the form essentially given in reference 8. We use for ξ the N four-valued variables $\zeta_1, \ldots, \zeta_N$ in which the wave function $\psi(p; \zeta_1, \ldots, \zeta_N)$ is symmetric. We define for every ζ_ν four-dimensional matrices γ_ν^k of the same nature as are used in Dirac's electron theory:

$$\gamma_\nu^k \gamma_\nu^l + \gamma_\nu^l \gamma_\nu^k = 2g^{kl}1 \quad (k, l = 1, 2, 3, 4). \qquad (13)$$

The γ with different lower indices ν commute. The γ^1, γ^2, γ^3 are skew Hermitian, γ^4 is Hermitian. The wave equations then are

$$\gamma_\nu{}^k p_k \psi = m\psi \quad (\nu = 1, 2, \ldots, N). \tag{14}$$

It follows from any of these equations in well known fashion that

$$g^{kl} p_k p_l \psi = p^k p_k \psi = m^2 \psi. \tag{14a}$$

The infinitesimal operators of displacement are the p, those of four-dimensional rotation the $M_{kl} = \underset{0}{M_{kl}} + S_{kl}$ with $\underset{0}{M_{kl}}$ of (9) and

$$S_{kl} = {}^1/_{2}i \sum_\nu \gamma_{\nu k}\gamma_{\nu l} \quad (k \neq l), \tag{15}$$

where the

$$\gamma_{\nu k} = g_{kl}\gamma_\nu{}^l \tag{15a}$$

satisfy the same relations (13) as do the $\gamma_\nu{}^k$.

The invariant scalar product is

$$(\dot{\psi}, \psi) = \int \left| \sum_\zeta \psi^* \gamma_1{}^4 \gamma_2{}^4 \ldots \gamma_N{}^4 \psi \right| d\Omega. \tag{16}$$

In fact, (16) is invariant both with respect to the operators $\underset{0}{M}$ and also with respect to the S. The latter condition means that

$$((1 + i\epsilon S_{kl})\psi, (1 + i\epsilon S_{kl})\psi) = (\psi, \psi),$$

up to terms with ϵ^2. This formula can be verified by observing that, if both k and l are space like S_{kl} is a Hermitian matrix and commutes with the product of the γ^4. If either k or l is 4, S_{kl} is skew Hermitian, but anticommutes with the product of the γ^4. It follows that (16) is invariant with respect to the proper Lorentz transformations. Its invariance with respect to reflections, etc., can also be shown.

The absolute sign in (16) is necessary to make it positive definite. We now shall give (16) a new form which is based on the set of identities

$$(p_4)^\nu \gamma_\nu{}^4 \ldots \gamma_2{}^4 \gamma_1{}^4 \psi = m^\nu \psi + A_\nu \psi, \tag{17}$$

where A_ν is a skew Hermitian matrix involving only the first ν of the γ^k (and the p). We can prove (17) best by induction: applying $p_4\gamma_{\nu+1}{}^4$ to (17) gives, by means of (14),

$$\begin{aligned}(p_4)^{\nu+1}\gamma_{\nu+1}{}^4\gamma_\nu{}^4 \ldots \gamma_2{}^4\gamma_1{}^4\psi &= m^\nu p_4\gamma_{\nu+1}{}^4\psi + p_4\gamma_{\nu+1}{}^4 A_\nu\psi \\ &= m^{\nu+1}\psi + (-m^\nu p_k \gamma_{\nu+1}{}^k + p_4\gamma_{\nu+1}{}^4 A_\nu)\psi \quad (k = 1, 2, 3).\end{aligned} \tag{17a}$$

The last bracket is $A_{\nu+1}$: it is skew Hermitian and involves only the first $\nu + 1$ of the γ so that (17) is established by induction. Setting $\nu = N$ in (17), multiplying with ψ and summing over the ζ yields

$$p_4^N \sum_{\zeta} \psi^* \gamma_1^4 \gamma_2^4 \ldots \gamma_N^4 \psi = m^N \sum_{\zeta} |\psi|^2 + \sum_{\zeta} \psi^* A_N \psi. \quad (17b)$$

Because of the skew Hermitian nature of A_N, the last term is imaginary. Since the two other terms of (17b) are real, they must be equal. As a result, we can write for (16) also

$$(\psi, \psi) = \int |m/p_4|^N \sum_{\zeta} |\psi|^2 d\Omega. \quad (18)$$

At the same time, (18) permits us to give another form to the scalar product,

$$(\psi, \psi) = \int |p_4|^{-N-1} \sum_{\zeta} |\psi|^2 dp_1 dp_2 dp_3, \quad (18a)$$

which differs from (18) or (16) by the positive constant m^{-N}. It may be worth noting here that the absolute signs in (16), and in the definition (11) of $d\Omega$ (or in (18a)), can be omitted in case of an odd N. This makes it possible to define a simple positive definite scalar product in coördinate space by means of (12). In particular, for $N = 1$, (16) (or (18a)) equals the integral of $|\psi|^2$ over ordinary space. In case of even N (integer spin s) no simple positive definite scalar product can be defined in coördinate space.

It is now established that the solutions of (14) form a Lorentz invariant set in which a positive definite scalar product (16) or (18a) can be defined. We shall now determine the representation of §2 to which the solutions belong and will also calculate the invariants P and W.

In order to define a little group, we choose as momentum p_0 with the components 0, 0, 0, m. The little group then becomes the group of rotations in ordinary space. If we assume that the γ^4 are diagonal, with diagonal elements 1, 1, -1, -1, equation (14) shows that only those components of ψ can be different from zero which correspond to the first two rows of γ_ν. There are 2^N such components, the rest of the 4^N components of ψ must vanish. Even these components will not be independent: as a result of the symmetry of the ψ in the ζ, all components of ψ will be equal in which the same number κ of the N indices ζ correspond to the first row of the γ, the $N - \kappa$ other indices to the second row. Since κ can assume any of the values between 0 and N, there are $N + 1$ such components. If $p_4 = -m$, the same considerations will hold, except that the last two rows of γ will play the rôle which the first two rows play in case of $p_4 = m$.

In order to determine the transformation properties of these $N + 1 = 2s + 1$ independent components under the elements of the little group, we note that the space like $\underset{0}{M}$ give zero if applied to ψ with a purely time like $p = p_0$. We need only to calculate, therefore, the effect of the S_{kl};

on ψ. Since, in particular, $^1/_2 i\gamma_1\gamma_2$ commutes with γ^4, but is not identical with it, we can assume that it is diagonal and has the diagonal elements $^1/_2$, $-^1/_2$, $^1/_2$, $-^1/_2$. If the sum of such $^1/_2 i\gamma_1\gamma_2$ is applied to the component of ψ in which κ of the ζ_ν correspond to the first row, $N - \kappa$ to the second row, this component will be multiplied by $^1/_2\kappa - {}^1/_2(N - \kappa) = \kappa - s$. Since κ runs from 0 to $N = 2s$, the $M_{12}\psi = S_{12}\psi$ will run from $-s\psi$ to $s\psi$. Hence the representation of the little group in question is $D^{(s)}$, as was postulated.

Because of (10a), W becomes $m^2(S_{23}^2 + S_{31}^2 + S_{12}^2)$ or, since the S_{23}, S_{31}, S_{12} are the infinitesimal operators of $D^{(s)}$, we have $W = m^2s(s + 1)$ as given[9] in §3. The value of P is m^2 because of (14a).

5. *The Class O_s.*—(*a*) $s = 0$. The corresponding discussion in the preceding section may be literally applied to this case, with the exception that $m = 0$ and that the integral (11) is to be extended over the light-cone.

(*b*) The wave equations can be obtained by setting $m = 0$ in (14). The infinitesimal operators continue to be given by (9) and (15). The scalar product must be defined by (18a) because (16) vanishes for all ψ. The invariance of this scalar product follows from the invariance of (18a) for finite mass because, except at $p_1 = p_2 = p_3 = 0$, the wave function is continuous in m.

The essential difference between finite and zero mass is that, in the latter case, not only the infinitesimal operators but also the wave equation are invariant under any one of the operators $\Gamma_\nu = i\gamma_\nu{}^1\gamma_\nu{}^2\gamma_\nu{}^3\gamma_\nu{}^4$. As a result, for $m = 0$, the linear manifold defined by (14) can be decomposed into invariant manifolds by giving definite values to the Γ_ν. In particular, we shall be concerned henceforth with the manifold defined by (14) and

$$\Gamma_\nu\psi = \psi \qquad (\nu = 1, 2, \ldots, N); \tag{19a}$$

and with the other one for which

$$\Gamma_\nu\psi = -\psi \qquad (\nu = 1, 2, \ldots, N) \tag{19b}$$

holds. Both manifolds are invariant under proper Lorentz transformations but go over into each other by reflections: they correspond physically to right and left circular polarization.[10]

Let us now again choose a particular momentum vector p_0 in order to define the little group. The covariant components of p_0 shall be 0, 0, 1, 1. The wave equations (14) then can be written, after multiplication with $\gamma_\nu{}^3$, in the form

$$\gamma_\nu{}^3\gamma_\nu{}^4\psi = \psi \qquad (\nu = 1, 2, \ldots, N). \tag{20}$$

It is now advantageous to assume that the $\gamma^3\gamma^4$ are diagonal, their diagonal elements being 1, 1, -1, -1. Equation (20) then expresses the fact that ψ for the p_0 in question is different from zero only if all ζ have values corresponding to the first two rows of the γ. Since the Γ commute with the $\gamma^4\gamma^3$ but are not identical with them, they may be also assumed to be diagonal, with diagonal elements $1, -1, 1, -1$. Hence, in the manifold defined by (20) and (19a) all components of ψ vanish (for $p = p_0$) unless all ζ have values corresponding to the first rows of the γ: the manifold (20), (19a) is one dimensional for given p. The same holds for the manifold defined by (20), (19b) except that in this case $\psi(p_0;\ \zeta_1, \ldots, \zeta_N)$ differs from zero only if all ζ have values corresponding to the second row of the γ. For given momentum, ψ has only two independent components.

The infinitesimal operators of the little group are M_{12}, $M_{13} - M_{14}$, $M_{23} - M_{24}$ which leave p_0 invariant. The corresponding $\underset{0}{M}$ give again zero if applied to ψ at $p = p_0$. The S corresponding to the second of the above operators (cf. (15), (15a)) is a sum of matrices $\frac{1}{2}i(\gamma_\nu{}^1\gamma_\nu{}^3 + \gamma_\nu{}^1\gamma_\nu{}^4)$. It vanishes if applied to our ψ as can be seen by applying $\gamma_\nu{}^1\gamma_\nu{}^3$ to (20). The same holds for $M_{23} - M_{24}$. On the other hand, $\frac{1}{2}i\gamma_\nu{}^1\gamma_\nu{}^2$ gives $\frac{1}{2}\psi$ if applied to the ψ of (20), (19a), and gives $-\frac{1}{2}\psi$ if applied to the ψ of (20), (19b). One sees this most easily by applying $\frac{1}{2}i\gamma_\nu{}^1\gamma_\nu{}^2$ to (20) and making use of (19). As a result, $M_{12}\psi = \pm\frac{1}{2}N\psi = \pm s\psi$ for the two manifolds in question: these indeed belong to the representation 0_s of the inhomogeneous Lorentz group.

The value of the invariant P is zero. The above also involves a calculation of the w for the ψ at $p = p_0$: we have $w^3\psi = M_{12}\psi = \pm s\psi$, $w^1\psi = (M_{42} + M_{23})\psi = 0$, $w^2\psi = (M_{31} + M_{14})\psi = 0$, $-w^4\psi = M_{12}\psi = \pm s\psi$. It follows that the value of the second invariant $W = -(w^4)^2 + (w^1)^2 + (w^2)^2 + (w^3)^2$ is also zero for all the manifolds 0_s; these cannot be characterized by P and W. However, these manifolds can be characterized by the equation $P = 0$ with the additional set

$$w_k = sp_k \text{ and } w_k = -sp_k, \tag{21}$$

the $+$ applying to (19a), the $-$ to (19b). Both these equations are invariant with respect to proper Lorentz transformations. If reflections are to be included, one can combine them into $w_kw_l = s^2p_kp_l$.

6. *The Class $O(\Xi)$.*—Here, the auxiliary variable is a space like four vector ξ of length l, orthogonal to p. The scalar function $\psi(p, \xi)$ is determined by the equations[11]

$$g^{kl}p_kp_l\psi = 0;\ g^{kl}p_k\xi_l\psi = 0;\ g^{kl}\xi_k\xi_l\psi = -\psi, \tag{22}$$

$$p_k\partial\psi/\partial\xi_k = -i\Xi\psi, \tag{22a}$$

with a real positive constant Ξ. By (22a), for every real number ρ,

$$\psi(p, \xi + \rho p) = e^{-i\rho\Xi}\psi(p, \xi). \tag{23}$$

The infinitesimal operators of displacement are the p_k, those for rotations are the $\underset{0}{M}$ of (9) plus the

$$S_{kl} = i\left(\xi_k \frac{\partial}{\partial \xi^l} - \xi_l \frac{\partial}{\partial \xi^k}\right) = i(\xi_k g_{lj} - \xi_l g_{kj}) \frac{\partial}{\partial \xi_j}. \qquad (24)$$

In order to find the invariant scalar product, we introduce, for every vector p on the light cone, two real space like vectors $u^{(1)}(p)$ and $u^{(2)}(p)$ of length one, orthogonal to p and to each other, so that

$$\{u^{(r)}(p), p\} = 0, \ \{u^{(r)}(p), u^{(s)}(p)\} = -\delta_{rs} \quad (r, s = 1, 2). \qquad (25)$$

Then ξ is a linear combination of p, $u^{(1)}(p)$, $u^{(2)}(p)$,

$$\xi = \alpha p + \beta_1 u^{(1)}(p) + \beta_2 u^{(2)}(p), \qquad (26)$$

where α and the β are real. $\{\xi, \xi\} = -1$ implies $\beta_1^2 + \beta_2^2 = 1$, hence $\beta_1 + i\,\beta_2 = e^{i\tau}$ with a suitable real angle τ. $\psi\,(p, \xi)$ is therefore a function of p, α, τ,

$$\psi(p, \xi) = \phi(p, \alpha, \tau). \qquad (27)$$

The choice of the $u^{(r)}(p)$ is, of course, not unique. Let $v^{(r)}(p)$ be another system of vectors which satisfy (25). They may be expressed in the form (26), i.e.,

$$v^{(r)}(p) = \kappa_r p + \sum_s \lambda_{sr} u^{(s)}(p) \qquad (r, s = 1, 2).$$

By (25), the matrix λ_{sr} is orthogonal. In terms of the $v^{(r)}$, $\xi = \alpha' p + \sum_r \beta'_r v^{(r)}(p)$, where $\beta'_r = \sum_s \lambda_{sr}\beta_s$. In particular

$$\beta'_1 + i\beta'_2 = e^{i\tau'}; \quad \tau' = \pm(\tau + \lambda) \qquad (28)$$

λ depending on the λ_{sr}. By (23), $|\phi(p, \alpha, \tau)| = |\phi(p, 0, \tau)|$, and we define the norm of ψ by

$$(\psi, \psi) = \int |\phi(p, 0, \tau)|^2 d\Omega d\tau. \qquad (29)$$

This expression is independent of the choice of the $u^{(r)}$. In fact, let $\phi(p, \alpha, \tau) = \phi'(p, \alpha', \tau')$ where the primed variables refer to another set $v^{(r)}$. Then $|\phi(p, 0, \tau)| = |\phi'(p, \alpha', \tau')| = |\phi(p, 0, \tau')|$, and $|d\tau'/d\tau| = 1$. To prove the Lorentz invariance of (29) we proceed as follows: If a homogeneous Lorentz transformation maps p on $\Lambda^{-1}p$, and ξ on $\Lambda^{-1}\xi$, we may, in particular, choose the $u^{(r)}(p)$ in the new system to be the transforms of the original ones; then the coefficients α, β_1, β_2 in (26), and hence τ remain unchanged, and the integral (29) is invariant.

If we choose as the basic vector again p_0 with the components 0, 0, 1, 1 the infinitesimal operators of the little group are again M_{12}, $M_{13} - M_{14}$.

$M_{23} - M_{24}$. The $\underset{0}{M}$ parts of these give zero for $p = p_0$, the S parts of the latter two are

$$S_{13} - S_{14} = -i\xi_1 \left(\frac{\partial}{\partial \xi_3} + \frac{\partial}{\partial \xi_4}\right) + i(\xi_3 - \xi_4) \frac{\partial}{\partial \xi_1}, \qquad (30a)$$

$$S_{23} - S_{24} = -i\xi_2 \left(\frac{\partial}{\partial \xi_3} + \frac{\partial}{\partial \xi_4}\right) + i(\xi_3 - \xi_4) \frac{\partial}{\partial \xi_2}. \qquad (30b)$$

Because of (22a), the first term gives, if applied to ψ at $p = p_0$ just $\Xi\xi_1\psi$ and $\Xi\xi_2\psi$, respectively. The second terms vanish because of the second equation of (22). Hence ψ is not invariant under the "displacements" $M_{13} - M_{14}$ and $M_{23} - M_{24}$ in ξ space, and the sum of the squares of the "momenta" is $(\xi_1^2 + \xi_2^2)\, \Xi^2 = \Xi^2$ because of the last equation of (22). This is also the value of W, while $P = 0$.

7. *The class* 0′ (Ξ).—Since the discussion of this last case follows the pattern of the preceding section we confine ourselves to stating the main results. We introduce, in addition to the vector ξ, a discrete spin variable ζ which can assume four values. The wave equations become

$$\gamma^k p_k \psi = 0; \quad g^{kl} p_k \xi_l \psi = 0; \quad g^{kl} \xi_k \xi_l \psi = -\psi. \qquad (31)$$

$$p_k \partial\psi/\partial\xi_k = -i\Xi\psi \qquad (31a)$$

The parameters α and τ are introduced as before. The norm is given by

$$(\psi, \psi) = \int p_4^{-2} \sum |\phi(p, 0, \tau)|^2 dp_1 dp_2 dp_3 d\tau. \qquad (32)$$

(Cf. (18a) and (29).) Again $W\psi = \Xi^2\psi$, $P\psi = 0$.

It may be remarked that the scalar product has a simple positive definite form in coördinate space for these equations.[11]

[1] All the essential results of the present paper were obtained by the two authors independently, but they decided to publish them jointly.

[2] Wigner, E. P., *Ann. Math.*, **40**, 149–204 (1939).

[3] Fierz, M., *Helv. Phys. Acta*, **XII**, 3–37 (1939).

[4] Gårding, L., *Proc. Nat. Acad. Sci.*, **33**, 331–332 (1947).

[5] Gelfand, L., and Neumark, M., *J. Phys.* (*USSR*), **X**, 93–94 (1946); Harish-Chandra, *Proc. Roy. Soc.* (London), **A**, **189**, 372–401 (1947); and Bargmann, V., *Ann. Math.*, **48**, 568–640 (1947), have determined the representations of the homogeneous Lorentz group. These are representations also of the inhomogeneous Lorentz group. In the quantum mechanical interpretation, however, all the states of the corresponding particles are invariant under translations and, in particular, independent of time. It in very unlikely that these representations have immediate physical significance. Is addition, the third paper contains a determination of those representations for which the momentum vectors are space like. These are not considered in the present article as they also are unlikely to have a simple physical interpretation.

[6] Lubánski, J. K., *Physica*, **IX**, 310–324 (1942).

[7] Dirac, P. A. M., *Proc. Roy. Soc.*, **A, 155,** 447–459 (1936).

[8] The literature on relativistic wave equations is very extensive. Besides the papers quoted in reference 11, we only mention the book by de Broglie, L., *Théorie générale des particules à spin* (Paris, 1943), and the following articles which give a systematic account of the subject: Pauli, W., *Rev. Mod. Phys.*, **13,** 203–232 (1941); Bhabha, H. J., *Rev. Mod. Phys.*, **17,** 203–209 (1945); Kramers, H. A., Belinfante, F. J., and Lubânski, J. K., *Physica*, **VIII,** 597–627 (1941). In this paper, the sum of (14) over all ν was postulated; (14a) then has to be added as an independent equation (except for $N = 1$). Reference 11 uses these equations in the form given by Kramers, Belinfante and Lubânski.

[9] One may derive this result in a more elegant way, without specializing the coördinate system. For the sake of brevity, we omit this derivation.

[10] de Wet, J. S., *Phys. Rev.*, **58,** 236–242 (1940), in particular, p. 242.

[11] Wigner, E. P., *Z. Physik*, (1947).

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*STEREOSCOPIC ACUITY FOR VARIOUS LEVELS OF ILLUMINATION**

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Several experiments have demonstrated that the threshold for stereoscopic vision is influenced by certain important variables (see the review by Graham[1]), but little attention has been paid to the systematic exploration of parameters (e.g., intensity and wave-length) which are known to be important for other visual functions.[2] The present report gives data on one of those variables, intensity of "white" light, as it influences the threshold for stereoscopic vision.

Apparatus.—Two 300-watt Mazda bulbs are used as light sources, one for each eye. The light sources are fastened to a movable wooden stand which may be placed in either of two positions, thus allowing for a small range of intensity variation. Additional adjustment of intensity may be achieved by inserting filters of various densities in a holder adjacent to the light source for each eye.

The two filter holders are attached to a pair of metal funnels, 4 inches in diameter and $3^3/_4$ inches in length; the funnels in turn are fastened to the outer wall of the dark room in which the subject sits.

A piece of opal glass, fastened to the inner wall of the dark room and in front of the funnels, diffuses the light from the two bulbs. A piece of masonite, containing two holes of $3^1/_2$ inch diameter, is mounted in front of the opal glass. These holes expose two photographic plates which are fitted into slots in the masonite and on which the reticles of the two visual fields are photographed. Both test fields contain three vertical reticle marks, each with a width of 20 minutes and a height of two degrees of visual angle. The reticle marks are equidistant from one another at a