

# Mathematical Techniques III

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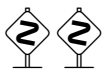
Version of December 5, 2004

# Preface

This is the first draft of the Lecture Notes for **Mathematical Techniques III** (PHY 317), a course offered in the Physics Department of Queen Mary and Westfield College (University of London). These notes are loosely based on pre-existing notes by Professor John Charap. The notes contain all that is said in Lecture and sometimes more. The extra bits are typeset in smaller font and are adorned with one or two “dangerous bend” signs as in the next paragraphs.



Most paragraphs like this fill gaps in the main presentation (e.g., proofs, mathematical remarks, . . .). They contain material which, although necessary for the logical coherence of the presentation, may be skipped at a first reading or ignored by the less mathematically inclined student who is not interested in proofs, . . . They are not an essential part of the course, although I believe they are an essential part of the topic.



Most paragraphs like this contain material which is generally more advanced than the rest of the lectures, but which I personally find interesting and have found useful at one time or other. They are not an essential part of the course, but I have included them in the hope that some of you might find them interesting enough to make the detour.

Some remarks about notation. Terms which are being defined for the first time appear in **bold sans-serif** type. Although the notation will be introduced as we go, here is a summary of the main notational conventions:

- $\mathbb{R}$  and  $\mathbb{C}$  stand for the sets of real and complex numbers, respectively;
- vector spaces, subspaces, . . . are denoted by so-called “blackboard bold” uppercase Latin letters:  $\mathbb{V}$ ,  $\mathbb{W}$ , . . . ;
- abstract vectors are denoted by bold lowercase Latin letters:  $\mathbf{v}$ ,  $\mathbf{w}$ , . . . ;
- linear maps are denoted by uppercase Latin letters  $A$ ,  $B$ , . . . , except for the identity map which is denoted  $\mathbf{1}$ .
- column vectors are denoted by sans-serif lowercase Latin letters:  $\mathbf{v}$ ,  $\mathbf{w}$ , . . . ;

- matrices are denoted by sans-serif uppercase Latin letters:  $A, B, \dots$ .  
The identity matrix will be denoted  $I$ .

The notes are not yet complete: in particular many of the asides are still to be completed, and the introductions have to be rewritten in light of what they are meant to introduce: they were written in advance in most cases. Many diagrams are missing, and many more examples and applications need to be added. The next stage in the development of the notes will consist in some changes in the visual layout, to break the monotony of the present style, and to make the exercises and the problems an integral part of the notes. The solutions, of course, will be available separately.

# Contents

<b>1</b>	<b>Linear Algebra</b>	<b>4</b>
1.1	Vector spaces . . . . .	4
1.1.1	Displacements in the plane . . . . .	5
1.1.2	Displacements in the plane (revisited) . . . . .	6
1.1.3	Abstract vector spaces . . . . .	8
1.1.4	Vector subspaces . . . . .	10
1.1.5	Linear independence . . . . .	11
1.1.6	Bases . . . . .	13
1.2	Linear maps . . . . .	15
1.2.1	Linear maps . . . . .	15
1.2.2	Composition of linear maps . . . . .	18
1.2.3	Linear transformations . . . . .	18
1.2.4	The vector space of linear maps . . . . .	21
1.2.5	Matrices . . . . .	23
1.2.6	Change of basis . . . . .	28
1.2.7	Matrix invariants . . . . .	30
1.3	Inner products . . . . .	33
1.3.1	Norms and inner products . . . . .	33
1.3.2	The Cauchy–Schwartz and triangle inequalities . . . . .	37
1.3.3	Orthonormal bases and Gram–Schmidt . . . . .	38
1.3.4	The adjoint of a linear transformation . . . . .	40
1.3.5	Complex vector spaces . . . . .	43
1.3.6	Hermitian inner products . . . . .	45
1.4	The eigenvalue problem and applications . . . . .	50
1.4.1	Eigenvectors and eigenvalues . . . . .	50
1.4.2	Diagonalisability . . . . .	52
1.4.3	Spectral theorem for hermitian transformations . . . . .	55
1.4.4	Application: quadratic forms . . . . .	62
1.4.5	Application: normal modes . . . . .	64
1.4.6	Application: near equilibrium dynamics . . . . .	68

<b>2</b>	<b>Complex Analysis</b>	<b>73</b>
2.1	Analytic functions . . . . .	73
2.1.1	The complex plane . . . . .	73
2.1.2	Complex-valued functions . . . . .	76
2.1.3	Differentiability and analyticity . . . . .	77
2.1.4	Polynomials and rational functions . . . . .	82
2.1.5	The complex exponential and related functions . . . . .	85
2.1.6	The complex logarithm . . . . .	88
2.1.7	Complex powers . . . . .	93
2.2	Complex integration . . . . .	96
2.2.1	Complex integrals . . . . .	97
2.2.2	Contour integrals . . . . .	98
2.2.3	Independence of path . . . . .	104
2.2.4	Cauchy's Integral Theorem . . . . .	109
2.2.5	Cauchy's Integral Formula . . . . .	116
2.2.6	Liouville's Theorem and its applications . . . . .	124
2.3	Series expansions for analytic functions . . . . .	126
2.3.1	Sequences and Series . . . . .	127
2.3.2	Taylor series . . . . .	134
2.3.3	Power series . . . . .	141
2.3.4	Laurent series . . . . .	145
2.3.5	Zeros and Singularities . . . . .	151
2.4	The residue calculus and its applications . . . . .	156
2.4.1	The Cauchy Residue Theorem . . . . .	156
2.4.2	Application: trigonometric integrals . . . . .	161
2.4.3	Application: improper integrals . . . . .	163
2.4.4	Application: improper integrals with poles . . . . .	173
2.4.5	Application: infinite series . . . . .	177
<b>3</b>	<b>Integral Transforms</b>	<b>185</b>
3.1	Fourier series . . . . .	186
3.1.1	The vibrating string . . . . .	186
3.1.2	The Fourier series of a periodic function . . . . .	192
3.1.3	Some properties of the Fourier series . . . . .	198
3.1.4	Application: steady-state response . . . . .	202
3.2	The Fourier transform . . . . .	206
3.2.1	The Fourier integral . . . . .	206
3.2.2	Some properties of the Fourier transform . . . . .	210
3.2.3	Application: one-dimensional wave equation . . . . .	213
3.2.4	Application: steady-state response . . . . .	215
3.3	The Laplace transform . . . . .	217

3.3.1	The Heaviside $D$ -calculus . . . . .	217
3.3.2	The Laplace transform . . . . .	220
3.3.3	Basic properties of the Laplace transform . . . . .	224
3.3.4	Application: stability and the damped oscillator . . . . .	227
3.3.5	Application: convolution and the tautochrone . . . . .	232
3.3.6	The Gamma and Zeta functions . . . . .	239