Chapter 3

Integral Transforms

This part of the course introduces two extremely powerful methods to solving differential equations: the Fourier and the Laplace transforms. Beside its practical use, the Fourier transform is also of fundamental importance in quantum mechanics, providing the correspondence between the position and momentum representations of the Heisenberg commutation relations.

An integral transform is useful if it allows one to turn a complicated problem into a simpler one. The transforms we will be studying in this part of the course are mostly useful to solve differential and, to a lesser extent, integral equations. The idea behind a transform is very simple. To be definite suppose that we want to solve a differential equation, with unknown function f. One first applies the transform to the differential equation to turn it into an equation one can solve easily: often an algebraic equation for the transform F of f. One then solves this equation for F and finally applies the inverse transform to find f. This circle (or square!) of ideas can be represented diagrammatically as follows:

$$\begin{tikzcd}
\boxed{\text{algebraic equation for } F} \arrow[r] & \boxed{\text{solution: } F} \arrow[d, "\text{inverse transform}"] \\
\boxed{\text{differential equation for } f} \arrow[u, "\text{transform}"'] \arrow[r, dashed] & \boxed{\text{solution: } f}
\end{tikzcd}$$

We would like to follow the dashed line, but this is often very difficult.

Therefore we follow the solid line instead: it may seem a longer path, but it has the advantage of being straightforward. After all, what is the purpose of developing formalism if not to reduce the solution of complicated problems to a set of simple rules which even a machine could follow?

We will start by reviewing Fourier series in the context of one particular example: the vibrating string. This will have the added benefit of introducing the method of separation of variables in order to solve partial differential equations. In the limit as the vibrating string becomes infinitely long, the Fourier series naturally gives rise to the Fourier integral transform, which we will apply to find steady-state solutions to differential equations. In particular we will apply this to the one-dimensional wave equation. In order to deal with transient solutions of differential equations, we will introduce the Laplace transform. This will then be applied, among other problems, to the solution of initial value problems.

3.1 Fourier series

In this section we will discuss the Fourier expansion of periodic functions of a real variable. As a practical application, we start with the study of the vibrating string, where the Fourier series makes a natural appearance.

3.1.1 The vibrating string

Consider a string of length L which is clamped at both ends. Let x denote the position along the string: such that the two ends of the string are at $x = 0$ and $x = L$, respectively. The string has tension T and a uniform mass density μ, and it is allowed to vibrate. If we think of the string as being composed of an infinite number of infinitesimal masses, we model the vibrations by a function $\psi(x,t)$ which describes the vertical displacement at time t of the mass at position x. It can be shown that for small vertical displacements, $\psi(x,t)$ obeys the following equation:

$$T\,\frac{\partial^2\psi(x,t)}{\partial x^2} = \mu\,\frac{\partial^2\psi(x,t)}{\partial t^2}\ ,$$

which can be recognised as the **one-dimensional wave equation**

$$\boxed{\frac{\partial^2}{\partial x^2}\psi(x,t) = \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\psi(x,t)\ ,} \tag{3.1}$$

where $c = \sqrt{T/\mu}$ is the wave velocity. This is a partial differential equation which needs for its solution to be supplemented by boundary conditions for

x and initial conditions for t. Because the string is clamped at both ends, the boundary conditions are

$$\psi(0,t) = \psi(L,t) = 0 \;, \qquad \text{for all } t. \tag{3.2}$$

As initial conditions we specify that at $t = 0$,

$$\left.\frac{\partial\psi(x,t)}{\partial t}\right|_{t=0} = 0 \qquad \text{and} \qquad \psi(x,0) = f(x) \;, \qquad \text{for all } x, \tag{3.3}$$

where f is a continuous function which, for consistency with the boundary conditions (3.2), must satisfy $f(0) = f(L) = 0$. In other words, the string is released from rest from an initial shape given by the function f.

This is not the only type of initial conditions that could be imposed. For example, in the case of, say, a piano string, it would be much more sensible to consider an initial condition in which the string is horizontal so that $\psi(x,0) = 0$, but such that it is given a blow at $t = 0$, which means that $\frac{\partial\psi(x,t)}{\partial t}|_{t=0} = g(x)$ for some function g. More generally still, we could consider mixed initial conditions in which $\psi(x,0) = f(x)$ and $\frac{\partial\psi(x,t)}{\partial t}|_{t=0} = g(x)$. These different initial conditions can be analysed in roughly the same way.

We will solve the wave equation by the method of **separation of variables**. This consists of choosing as an Ansatz for $\psi(x,t)$ the product of two functions, one depending only on x and the other only on t: $\psi(x,t) = u(x)\,v(t)$. We do not actually expect the solution to be of this form; but because, as we will review below, the equation is linear and one can use the principle of superposition to construct the desired solution out of decomposable solutions of this type. At any rate, inserting this Ansatz into (3.1), we have

$$u''(x)\,v(t) = \frac{1}{c^2}\,u(x)v''(t) \;,$$

where we are using primes to denote derivatives with respect to the variable on which the function depends: $u'(x) = du/dx$ and $v'(t) = dv/dt$. We now divide both sides of the equation by $u(x)\,v(t)$, and obtain

$$\frac{u''(x)}{u(x)} = \frac{1}{c^2}\,\frac{v''(t)}{v(t)} \;.$$

Now comes the reason that this method works, so pay close attention. Notice that the right-hand side does not depend on x, and that the left-hand side does not depend on t. Since they are equal, both sides have to be equal to a constant which, with some foresight, we choose to call $-\lambda^2$, as it will be a negative number in the case of interest. The equation therefore breaks up into two ordinary differential equations:

$$u''(x) = -\lambda^2\,u(x) \qquad \text{and} \qquad v''(t) = -\lambda^2\,c^2\,v(t) \;.$$

The boundary conditions say that $u(0) = u(L) = 0$.

Let us consider the first equation. It has three types of solutions depending on whether λ is nonzero real, nonzero imaginary or zero. (Notice that $-\lambda^2$ has to be real, so that these are the only possibilities.) If $\lambda = 0$, then the solution is $u(x) = a + b\,x$. The boundary condition $u(0) = 0$ means that $a = 0$, but the boundary condition $u(L) = 0$ then means that $b = 0$, whence $u(x) = 0$ for all x. Clearly this is a very uninteresting solution. Let us consider λ imaginary. Then the solution is now $a\exp(|\lambda|\,x) + b\exp(-|\lambda|\,x)$. Again the boundary conditions force $a = b = 0$. Therefore we are left with the possibility of λ real. Then the solution is

$$u(x) = a\cos\lambda x + b\sin\lambda x \ .$$

The boundary condition $u(0) = 0$ forces $a = 0$. Finally the boundary condition $u(L) = 0$ implies that

$$\sin\lambda L = 0 \qquad \Longrightarrow \qquad \lambda = \frac{n\,\pi}{L} \qquad \text{for } n \text{ an integer.}$$

Actually $n = 0$ is an uninteresting solution, and because of the fact that the sine is an odd function, negative values of n give rise to the same solution (up to a sign) as positive values of n. In other words, all nontrivial distinct solution are given (up to a constant multiple) by

$$u_n(x) \equiv \sin\lambda_n\,x \ , \quad \text{with } \lambda_n = \frac{n\,\pi}{L} \quad \text{and where } n = 1, 2, 3, \cdots. \tag{3.4}$$

Let us now solve for $v(t)$. Its equation is

$$v''(t) = -\lambda^2 c^2 v(t) \ ,$$

whence

$$v(t) = a\,\cos\lambda ct + b\,\sin\lambda ct \ .$$

The first of the two initial conditions (3.3) says that $v'(0) = 0$ whence $b = 0$. Therefore for any positive integer n, the function

$$\psi_n(x,t) = \sin\lambda_n x\,\cos\lambda_n ct \ , \qquad \text{with } \lambda_n = \frac{n\pi}{L} \ ,$$

satisfies the wave equation (3.1) subject to the boundary conditions (3.2) and to the first of the initial conditions (3.3).

Now notice something important: the wave equation (3.1) is **linear**; that is, if $\psi(x,t)$ and $\phi(x,t)$ are solutions of the wave equation, so is any linear combination $\alpha\,\psi(x,t) + \beta\,\phi(x,t)$ where α and β are constants.

Clearly then, any linear combination of the $\psi_n(x,t)$ will also be a solution. In other words, the most general solution subject to the boundary conditions (3.2) and the first of the initial conditions in (3.3) is given by a linear combination

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x \, \cos \lambda_n ct \ .$$

Of course, this expression is formal as it stands: it is an infinite sum which does not necessarily make sense, unless we chose the coefficients $\{b_n\}$ in such a way that the series converges, and that the convergence is such that we can differentiate the series termwise at least twice.

We can now finally impose the second of the initial conditions (3.3):

$$\psi(x,0) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x = f(x) \ . \tag{3.5}$$

At first sight this seems hopeless: can any function $f(x)$ be represented as a series of this form? The Bernoullis, who were the first to get this far, thought that this was not the case and that in some sense the solution was only valid for special kinds of functions for which such a series expansion is possible. It took Euler to realise that, in a certain sense, all functions $f(x)$ with $f(0) = f(L) = 0$, can be expanded in this way. He did this by showing how the coefficients $\{b_n\}$ are determined by the function $f(x)$.

To do so let us argue as follows. Let n and m be positive integers and consider the functions $u_n(x)$ and $u_m(x)$ defined in (3.4). These functions satisfy the differential equations:

$$u_n''(x) = -\lambda_n^2 \, u_n(x) \qquad \text{and} \qquad u_m''(x) = -\lambda_m^2 \, u_m(x) \ .$$

Let us multiply the first equation by $u_m(x)$ and the second equation by $u_n(x)$ and subtract one from the other to obtain

$$u_n''(x) \, u_m(x) - u_n(x) \, u_m''(x) = (\lambda_m^2 - \lambda_n^2) \, u_n(x) \, u_m(x) \ .$$

We notice that the left-hand side of the equation is a total derivative

$$u_n''(x) \, u_m(x) - u_n(x) \, u_m''(x) = \left(u_n'(x) \, u_m(x) - u_n(x) \, u_m'(x)\right)' \ ,$$

whence integrating both sides of the equation from $x = 0$ to $x = L$, we obtain

$$(\lambda_m^2 - \lambda_n^2) \int_0^L u_m(x) \, u_n(x) \, dx = \left(u_n'(x) \, u_m(x) - u_n(x) \, u_m'(x)\right)\Big|_0^L = 0 \ ,$$

since $u_n(0) = u_n(L) = 0$ and the same for u_m. Therefore we see that unless $\lambda_n^2 = \lambda_m^2$, which is equivalent to $n = m$ (since n, m are positive integers), the integral $\int_0^L u_m(x)\, u_n(x)\, dx$ vanishes. On the other hand, if $m = n$, we have that

$$\int_0^L u_n(x)^2\, dx = \int_0^L \left(\sin \frac{n\pi x}{L}\right)^2 dx = \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi x}{L}\right) dx = \frac{L}{2} .$$

Therefore, in summary, we have the **orthogonality property** of the functions $u_m(x)$:

$$\int_0^L u_m(x)\, u_n(x)\, dx = \begin{cases} \frac{L}{2} , & \text{if } n = m, \text{ and} \\ 0 , & \text{otherwise.} \end{cases} \tag{3.6}$$

Let us now go back to the solution of the remaining initial condition (3.5). This condition can be rewritten as

$$f(x) = \sum_{n=1}^{\infty} b_n\, u_n(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} . \tag{3.7}$$

Let us multiply both sides by $u_m(x)$ and integrate from $x = 0$ to $x = L$:

$$\int_0^L f(x)\, u_m(x)\, dx = \sum_{n=1}^{\infty} b_n \int_0^L u_n(x)\, u_m(x)\, dx ,$$

where we have interchanged the order of integration and summation with impunity.[1] Using the orthogonality relation (3.6) we see that of all the terms in the right-hand side, only the term with $n = m$ contributes to the sum, whence

$$\int_0^L f(x)\, u_m(x)\, dx = b_m \frac{L}{2} ,$$

or in other words,

$$b_m = \frac{2}{L}\int_0^L f(x)\, u_m(x)\, dx , \tag{3.8}$$

a formula due to Euler. Finally, the solution of the wave equation (3.1) with boundary conditions (3.2) and initial conditions (3.3) is

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} , \tag{3.9}$$

[1] This would have to be justified, but in this part of the course we will be much more cavalier about these things. The amount of material that would have to be introduced to be able to justify this procedure is too much for a course at this level and of this length.

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \ .$$

Inserting this expression into the solution (3.9), we find that

$$\begin{aligned}
\psi(x,t) &= \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L f(y) \sin \frac{n\pi y}{L} \, dy \right] \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \\
&= \int_0^L \left[\frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi y}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \right] f(y) \, dy \\
&= \int_0^L K(x,y;t) \, f(y) \, dy \ ,
\end{aligned}$$

where the **propagator** $K(x,y,t)$ is (formally) defined by

$$K(x,y;t) \equiv \sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi y}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \ .$$

To understand why it is called a propagator, notice that

$$\psi(x,t) = \int_0^L K(x,y;t) \, \psi(y,0) \, dy \ ,$$

so that one can obtain $\psi(x,t)$ from its value at $t=0$ simply by multiplying by $K(x,y;t)$ and integrating; hence $K(x,y;t)$ allows us to *propagate* the configuration at $t=0$ to any other time t.

Actually, the attentive reader will have noticed that we never showed that the series $\sum_{n=1}^{\infty} b_n u_n(x)$, with b_n given by the Euler formula (3.8) converges to $f(x)$. In fact, it is possible to show that it does, but the convergence is not necessarily pointwise (and certainly not uniform). We state without proof the following result:

$$\lim_{N\to\infty} \int_0^L \left(f(x) - \sum_{n=1}^{N} b_n \, u_n(x) \right)^2 dx = 0 \ . \tag{3.10}$$

In other words, the function

$$h(x) \equiv f(x) - \sum_{n=1}^{\infty} b_n \, u_n(x)$$

has the property that the integral

$$\int_0^L h(x)^2 \, dx = 0 \ .$$

This however does not mean that $h(x) = 0$, but only that it is zero **almost everywhere**.

To understand this notice consider the (discontinuous) function

$$h(x) = \begin{cases} 1 \ , & \text{for } x = x_0\text{, and} \\ 0 \ , & \text{otherwise.} \end{cases}$$

Then it is clear that the improper integral

$$\int_0^L h(x)^2 \, dx = \lim_{r,s \searrow 0} \int_0^{x_0 - r} + \int_{x_0+s}^L h(x)^2 \, dx = 0 \ .$$

The same would happen if $h(x)$ were zero but at a *finite* number of points.

Of course, if $h(x)$ were continuous *and* zero almost everywhere, it would have to be identically zero. In this case the convergence of the series (3.7) would be pointwise. This is the case if $f(x)$ is itself continuous.

Expanding (3.10), we find that

$$\int_0^L f(x)^2 \, dx - 2 \sum_{n=1}^{\infty} b_n \int_0^L f(x) \, u_n(x) \, dx + \sum_{n,m=1}^{\infty} b_n \, b_m \int_0^L u_n(x) \, u_m(x) \, dx = 0 \ .$$

Using (3.8) and (3.6) we can simplify this a little

$$\int_0^L f(x)^2 \, dx - 2 \sum_{n=1}^{\infty} \frac{L}{2} \, b_n^2 + \sum_{n=1}^{\infty} \frac{L}{2} \, b_n^2 = 0 \ ,$$

whence

$$\sum_{n=1}^{\infty} b_n^2 = \frac{2}{L} \int_0^L f(x)^2 \, dx \ .$$

Since $f(x)^2$ is continuous, it is integrable, and hence the right-hand side is finite, whence the series $\sum_{n=1}^{\infty} b_n^2$ also converges. In particular, it means that $\lim_{n \to \infty} b_n = 0$.

3.1.2 The Fourier series of a periodic function

We have seen above that a continuous function $f(x)$ defined on the interval $[0, L]$ and vanishing at the boundary, $f(0) = f(L) = 0$, can be expanded in terms of the functions $u_n(x) = \sin(n\pi x/L)$. In this section we will generalise this and consider similar expansions for periodic functions.

To be precise let $f(x)$ be a complex-valued function of a real variable which is periodic with period L: $f(x+L) = f(x)$ for all x. Periodicity means that $f(x)$ is uniquely determined by its behaviour within a period. In other words, if we know f in the interval $[0, L]$ then we know $f(x)$ everywhere. Said differently, any function defined on $[0, L]$, obeying $f(0) = f(L)$ can be extended to the whole real line as a periodic function. More generally, the interval $[0, L]$ can be substituted by any one period $[x_0, x_0 + L]$, for some x_0 with the property that $f(x_0) = f(x_0+L)$. This is not a useless generalisation: it will be important when we discuss the case of $f(x)$ being a discontinuous function. The strength of the Fourier expansion is that it treats discontinuous functions (at least those with a finite number of discontinuities in any one period) as easily as it treats continuous functions. The reason is, as we stated briefly above, that the convergence of the series is not pointwise but rather in the sense (3.10), which simply means that it converges pointwise almost everywhere.

The functions $e_n(x) \equiv \exp(i2\pi nx/L)$ are periodic with period L, since $e_n(x + L) = e_n(x) \exp(i2\pi n) = e_n(x)$. Therefore we could try to expand

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n\, e_n(x)\ ,} \tag{3.11}$$

for some complex coefficients $\{c_n\}$. This series is known as a **trigonometric** of **Fourier series** of the periodic function f, and the $\{c_n\}$ are called the **Fourier coefficients**. Under complex conjugation, the exponentials $e_n(x)$ satisfy $e_n(x)^* = e_{-n}(x)$, and also the following orthogonality property:

$$\int_0^L e_m(x)^*\, e_n(x)\, dx = \int_0^L e^{i2\pi(n-m)x/L}\, dx = \begin{cases} L\ , & \text{if } n = m, \text{ and} \\ 0\ , & \text{otherwise.} \end{cases}$$

Therefore if we multiply both sides of (3.11) by $e_m(x)^*$ and integrate, we find the following formula for the Fourier coefficients:

$$c_m = \frac{1}{L} \int_0^L e_m(x)^*\, f(x)\, dx\ .$$

It is important to realise that the exponential functions $e_n(x)$ satisfy the orthogonality relation for any one period, not necessarily $[0, L]$:

$$\int\limits_{\substack{\text{one}\\ \text{period}}} e_m(x)^*\, e_n(x)\, dx = \begin{cases} L\ , & \text{if } n = m, \text{ and} \\ 0\ , & \text{otherwise;} \end{cases} \tag{3.12}$$

whence the Fourier coefficients can be obtained by integrating over any one period:

$$\boxed{c_m = \frac{1}{L} \int\limits_{\substack{\text{one}\\ \text{period}}} e_m(x)^* \, f(x) \, dx \ .} \tag{3.13}$$

Again we can state without proof that the series converges pointwise almost everywhere within any one period, in the sense that

$$\lim_{N\to\infty} \int\limits_{\substack{\text{one}\\ \text{period}}} \left| f(x) - \sum_{n=-N}^{N} c_n \, e_n(x) \right|^2 dx = 0 \ ,$$

whenever the $\{c_n\}$ are given by (3.13).

There is one special case where the series converges pointwise and uniformly. Let $g(z)$ be a function which is analytic in an open annulus containing the unit circle $|z| = 1$. We saw in Section 2.3.4 that such a function is approximated uniformly by a Laurent series of the form

$$g(z) = \sum_{n=-\infty}^{\infty} b_n \, z^n \ ,$$

where the $\{b_n\}$ are given by equations (2.53) and (2.54). Evaluating this on the unit circle $z = e^{i\theta}$, we have that

$$g(e^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n \, e^{in\theta} \ , \tag{3.14}$$

and the coefficients $\{b_n\}$ are given by

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) \, e^{-in\theta} \, d\theta \ , \tag{3.15}$$

which agrees precisely with the Fourier series of the function $g(e^{i\theta})$ which is periodic with period 2π. We can rescale this by defining $\theta = 2\pi x/L$ where x is periodic with period L. Let $f(x) \equiv g(\exp(i2\pi x/L))$, which is now periodic with period L. Then the Laurent series (3.14) becomes the Fourier series (3.11) where the Laurent coefficients (3.15) are now give by the Fourier coefficients (3.13).

Some examples

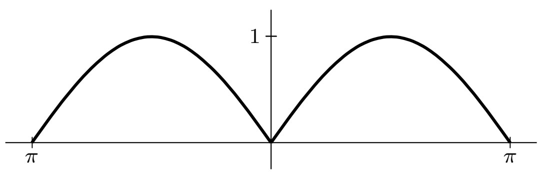


Figure 3.1: Plot of $|\sin x|$ for $x \in [-\pi, \pi]$.

Let us now compute some examples of Fourier series. The first example is the function $f(x) = |\sin x|$. A graph of this function shows that it is periodic with period π, as seen in Figure 3.1. We therefore try an expansion of the form

$$|\sin x| = \sum_{n=-\infty}^{\infty} c_n \, e^{i2nx} \ ,$$

where the coefficients $\{c_n\}$ are given by

$$c_n = \frac{1}{\pi} \int_0^{\pi} |\sin x| \, e^{-i2nx} \, dx = \frac{1}{\pi} \int_0^{\pi} \sin x \, e^{-i2nx} \, dx \ .$$

We can expand $\sin x$ into exponentials to obtain

$$\begin{aligned}
c_n &= \frac{1}{2\pi i} \int_0^{\pi} \left(e^{ix} - e^{-ix}\right) \, e^{-i2nx} \, dx \\
&= \frac{1}{2\pi i} \left[\int_0^{\pi} e^{-i(2n-1)x} \, dx - \int_0^{\pi} e^{-i(2n+1)x} \, dx \right] \\
&= \frac{1}{2\pi i} \left[\frac{i}{2n-1} \left(e^{-i(2n-1)\pi} - 1\right) - \frac{i}{2n+1} \left(e^{-i(2n+1)\pi} - 1\right) \right] \\
&= \frac{1}{2\pi i} (-2i) \left[\frac{1}{2n-1} - \frac{1}{2n+1} \right] \\
&= -\frac{2}{\pi} \, \frac{1}{4n^2-1} \ .
\end{aligned}$$

Therefore,

$$|\sin x| = \sum_{n=-\infty}^{\infty} -\frac{2}{\pi} \, \frac{1}{4n^2-1} \, e^{i2nx} = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4}{\pi} \, \frac{1}{4n^2-1} \cos 2nx \ .$$

Notice that this can be used in order to compute infinite sums. Evaluating this at $x = 0$, we have that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \tfrac{1}{2} \ ,$$

whereas evaluating this at $x = \pi/2$, we have that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{2-\pi}{4} \ .$$

Of course, we could have summed these series using the residue theorem, as explained in Section 2.4.5.

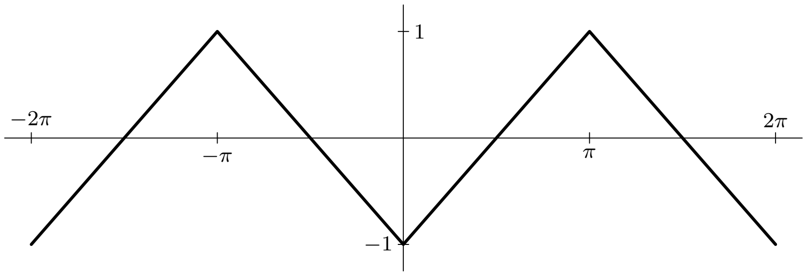


Figure 3.2: Plot of $f(x)$ for $x \in [-2\pi, 2\pi]$.

As a second example, let us consider the function $f(x)$ defined in the interval $[-\pi, \pi]$ by

$$f(x) = \begin{cases} -1 - \frac{2}{\pi}\,x \ , & \text{if } -\pi \le x \le 0, \text{ and} \\ -1 + \frac{2}{\pi}\,x \ , & \text{if } 0 \le x \le \pi. \end{cases} \tag{3.16}$$

and extended periodically to the whole real line. A plot of this function for $x \in [-2\pi, 2\pi]$ is shown in Figure 3.2. It is clear from the picture that $f(x)$ has periodicity 2π, whence we expect a Fourier series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx} \ ,$$

where the coefficients are given by

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \\ &= \frac{1}{2\pi} \left[\int_{-\pi}^{0} (-1 - \frac{2}{\pi}\, x)\, e^{-inx}\, dx + \int_{0}^{\pi} (-1 + \frac{2}{\pi}\, x)\, e^{-inx}\, dx \right] \\ &= \frac{1}{2\pi} \int_{0}^{\pi} (-1 + \frac{2}{\pi}\, x)\, \left[e^{inx} + e^{-inx} \right]\, dx \\ &= -\frac{1}{\pi} \int_{0}^{\pi} \cos(nx)\, dx + \frac{2}{\pi^2} \int_{0}^{\pi} x\, \cos(nx)\, dx \ . \end{aligned}$$

We must distinguish between $n = 0$ and $n \neq 0$. Performing the elementary integrals for both of these cases, we arrive at

$$c_n = \begin{cases} \frac{2}{\pi^2\, n^2} \left[(-1)^n - 1 \right] \ , & \text{for } n \neq 0, \text{ and} \\ 0 \ , & \text{for } n = 0. \end{cases} \tag{3.17}$$

Therefore we have that

$$\begin{aligned}
f(x) &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{\pi^2\, n^2}\, [(-1)^n - 1]\, e^{inx} \\
&= \sum_{n=1}^{\infty} \frac{4}{\pi^2\, n^2}\, [(-1)^n - 1]\, \cos nx \\
&= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} -\frac{8}{\pi^2\, n^2}\, \cos nx \\
&= \sum_{\ell=0}^{\infty} \frac{8}{\pi^2}\, \frac{-1}{(2\ell+1)^2}\, \cos(2\ell+1)\, x \ .
\end{aligned}$$

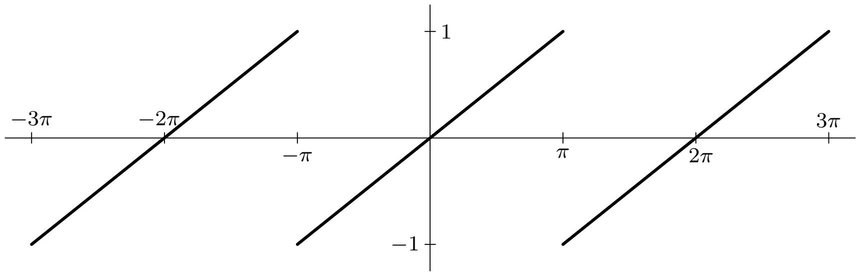


Figure 3.3: Plot of $g(x)$ for $x \in [-3\pi, 3\pi]$.

Finally we consider the case of a discontinuous function:

$$g(x) = \frac{x}{\pi}\ , \qquad \text{where } x \in [-\pi, \pi],$$

and extended periodically to the whole real line. The function has period 2π, and so we expect a series expansion of the form

$$g(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx}\ ,$$

where the Fourier coefficients are given by

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\, e^{-inx}\, dx \\
&= \frac{1}{2\pi^2} \int_{-\pi}^{\pi} x\, e^{-inx}\, dx \ .
\end{aligned}$$

We must distinguish the cases $n = 0$ and $n \neq 0$. In either case we can perform the elementary integrals to arrive at

$$c_n = \begin{cases} 0 \ , & \text{if } n = 0, \\ \frac{i}{\pi n} (-1)^n \ , & \text{otherwise.} \end{cases}$$

Therefore,

$$g(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{i}{\pi n} (-1)^n e^{inx} = \sum_{n=1}^{\infty} \frac{-2}{\pi n} (-1)^n \sin n x \ .$$

Now notice something curious: the function $g(x)$ is discontinuous at $x = (2\ell + 1)\pi$. Evaluating the series at such values of x we see that because $\sin n(2\ell + 1)\pi = 0$, the series sums to zero for these values. In other words, $g(x)$ is only equal to the Fourier series at those values x where $g(x)$ is continuous. At the values where $g(x)$ is discontinuous, the Fourier series can be shown to converge to the mean of the left and right limits of the function: in this case, $\lim_{x \searrow \pi} g(x) = -1$ and $\lim_{x \nearrow \pi} g(x) = 1$, and the average is 0, in agreement with what we just saw.

3.1.3 Some properties of the Fourier series

In this section we explore some general properties of the Fourier series of a complex periodic function $f(x)$ with period L.

Let us start with the following observation. If $f(x)$ is real, then the Fourier coefficients obey $c_n^* = c_{-n}$. This follows from the following. Taking the complex conjugate of the Fourier series for $f(x)$, we have

$$f(x)^* = \left(\sum_{n=-\infty}^{\infty} c_n \, e_n(x) \right)^* = \sum_{n=-\infty}^{\infty} c_n^* \, e_{-n}(x) \ ,$$

where we have used that $e_n(x)^* = e_{-n}(x)$. Since $f(x)$ is real, $f(x) = f(x)^*$ for all x, whence

$$\sum_{n=-\infty}^{\infty} c_n \, e_n(x) = \sum_{n=-\infty}^{\infty} c_n^* \, e_{-n}(x) = \sum_{n=-\infty}^{\infty} c_{-n}^* \, e_n(x) \ .$$

Multiplying both sides of the equation by $e_m^*(x)$, integrating over one period and using the orthogonality relation (3.12), we find that $c_m = c_{-m}^*$.

Fourier sine and cosine series

Suppose that $f(x)$ is periodic and also even, so that $f(-x) = f(x)$. Then this means that

$$f(x) = \tfrac{1}{2}\left[f(x) + f(-x)\right] \ .$$

If we substitute its Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e_n(x) \ ,$$

we see that

$$f(x) = \sum_{n=-\infty}^{\infty} \tfrac{1}{2}\, c_n \left[e_n(x) + e_n(-x)\right] = \sum_{n=-\infty}^{\infty} c_n \, \cos \lambda_n\, x \ ,$$

where $\lambda_n = 2\pi n/L$. Now we use the fact that $\cos\lambda_{-n}x = \cos\lambda_n x$ to rewrite the series as

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left[c_n + c_{-n}\right] \cos\lambda_n\, x = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \, \cos\lambda_n\, x \ ,$$

where $a_n \equiv [c_n + c_{-n}]$. Using (3.13) we find the following expression for the $\{a_n\}$:

$$a_n = \frac{2}{L} \int_{\substack{\text{one}\\ \text{period}}} \cos\lambda_n x\, f(x)\, dx \ .$$

The above expression for $f(x)$ as a sum of cosines is known as a **Fourier cosine series** and the $\{a_n\}$ are the **Fourier cosine coefficients**.

Similarly, one can consider the Fourier series of an odd periodic function $f(-x) = -f(x)$. Now we have that

$$f(x) = \tfrac{1}{2}\left[f(x) - f(-x)\right] \ ,$$

which, when we substitute its Fourier series, becomes

$$f(x) = \sum_{n=-\infty}^{\infty} \tfrac{1}{2}\, c_n \left[e_n(x) - e_n(-x)\right] = \sum_{n=-\infty}^{\infty} i\, c_n \, \sin\lambda_n x \ .$$

Now we use the fact that $\sin\lambda_{-n}x = -\sin\lambda_n x$, and that $\lambda_0 = 0$, to rewrite the series as

$$f(x) = \sum_{n=1}^{\infty} i\ \left[c_n - c_{-n}\right] \sin\lambda_n x = \sum_{n=1}^{\infty} b_n \, \sin\lambda_n x \ ,$$

where $b_n \equiv i\,[c_n - c_{-n}]$. Using (3.13) we find the following expression for the $\{b_n\}$:

$$b_n = \frac{2}{L} \int\limits_{\substack{\text{one}\\\text{period}}} \sin\lambda_n x\, f(x)\, dx \ .$$

The above expression for $f(x)$ as a sum of sines is known as a **Fourier sine series** and the $\{b_n\}$ are the **Fourier sine coefficients**.

Any function can be decomposed into the sum of an odd and an even function and this is reflected in the fact that the complex exponential $e_n(x)$ can be decomposed into a sum of a cosine and a sine: $e_n(x) = \cos\lambda_n x + i\,\sin\lambda_n x$. Therefore for $f(x)$ periodic, we have

$$\begin{aligned} f(x) = \sum_{n=-\infty}^{\infty} c_n\, e_n(x) &= \sum_{n=-\infty}^{\infty} c_n\, [\cos\lambda_n x + i\,\sin\lambda_n x] \\ &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\, \cos\lambda_n x + \sum_{n=1}^{\infty} b_n\, \sin\lambda_n x \ , \end{aligned}$$

where the first two terms comprise a Fourier cosine series and the last term is a Fourier sine series.

Parseval's identity

Let $f(x)$ be a complex periodic function and let us compute the following integral

$$\|f\|^2 \equiv \frac{1}{L} \int\limits_{\substack{\text{one}\\\text{period}}} |f(x)|^2\, dx \ ,$$

using the Fourier series.

$$\|f\|^2 = \frac{1}{L} \int\limits_{\substack{\text{one}\\\text{period}}} \left| \sum_{n=-\infty}^{\infty} c_n\, e_n(x) \right|^2 dx \ .$$

Expanding the right-hand side and interchanging the order of integration and summation, we have

$$\|f\|^2 = \frac{1}{L} \sum_{n,m=-\infty}^{\infty} c_n^*\, c_m \int\limits_{\substack{\text{one}\\\text{period}}} e_n(x)^*\, e_m(x)\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 \ ,$$

where we have used the orthogonality relation (3.12). In other words, we have derived **Parseval's identity**:

$$\boxed{\sum_{n=-\infty}^{\infty} |c_n|^2 = \|f\|^2 \ .} \tag{3.18}$$

Explain the Fourier series as setting up an isometry between L^2 and ℓ^2.

The Dirac delta "function"

Let us insert the expression (3.13) for the Fourier coefficients back into the Fourier series (3.11) for a periodic function $f(x)$:

$$f(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{L} \int\limits_{\substack{\text{one}\\\text{period}}} e_n(y)^* \, f(y) \, dy \right] e_n(x) \ .$$

Interchanging the order of summation and integration, we find

$$f(x) = \int\limits_{\substack{\text{one}\\\text{period}}} \left[\sum_{n=-\infty}^{\infty} \frac{1}{L} \, e_n(y)^* \, e_n(x) \right] f(y) \, dy = \int\limits_{\substack{\text{one}\\\text{period}}} \delta(x-y) \, f(y) \, dy \ ,$$

where we have introduced the **Dirac delta "function"**

$$\boxed{\delta(x-y) \equiv \sum_{n=-\infty}^{\infty} \frac{1}{L} \, e_n(y)^* \, e_n(x) = \sum_{n=-\infty}^{\infty} \frac{1}{L} \, e^{i(x-y)2\pi n/L} \ .} \tag{3.19}$$

Despite its name, the delta function is *not* a function, even though it is a limit of functions. Instead it is a **distribution**. Distributions are only well-defined when integrated against sufficiently well-behaved functions known as **test functions**. The delta function is the distribution defined by the condition:

$$\int\limits_{\substack{\text{one}\\\text{period}}} \delta(x-y) \, f(y) \, dy = f(x) \ .$$

In particular,

$$\int\limits_{\substack{\text{one}\\\text{period}}} \delta(y) \, dy = 1 \ ,$$

hence it depends on the region of integration. This is clear from the above expression which has an explicit dependence on the period L. In the following section, we will see another delta functions adapted to a different region of integration: the whole real line.

3.1.4 Application: steady-state response

We now come to one of the main applications of the Fourier series: finding steady-state solutions to differential equations.

Consider a system governed by a differential equation

$$\frac{d^2\phi(t)}{dt^2} + a_1 \frac{d\phi(t)}{dt} + a_0\,\phi(t) = e^{i\omega t} \; .$$

The function $\phi(t)$ can be understood as the response of the system which is being driven by a sinusoidal force $e^{i\omega t}$. After sufficient time has elapsed, or assuming that we have been driving the system in this fashion for an infinitely long time, say, for all $t < 0$, a realistic system will be in a so-called **steady state**: in which $\phi(t) = A(\omega)e^{i\omega t}$. The reason is that energy dissipates in a realistic system due to damping or friction, so that in the absence of the driving term, the system will tend to lose all its energy: so that $\phi(t) \to 0$ in the limit as $t \to \infty$. To find the steady-state response of the above system one then substitutes $\phi(t) = A(\omega)e^{i\omega t}$ in the equation and solves for $A(\omega)$:

$$\frac{d^2\phi(t)}{dt^2} + a_1 \frac{d\phi(t)}{dt} + a_0\,\phi(t) = A(\omega)\left(-\omega^2 + i\,a_1\,\omega + a_0\right)\, e^{i\omega t} = e^{i\omega t} \; ,$$

whence

$$A(\omega) = \frac{1}{-\omega^2 + i\,a_1\,\omega + a_0} \; .$$

In practice, one would like however to analyse the steady-state response of a system which is being driven not by a simple sinusoidal function but by a general periodic function $f(t)$, with period T. This suggests that we expand the driving force in terms of a Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i2\pi nt/T} \; ,$$

where the coefficients are given by

$$c_n = \frac{1}{T} \int\limits_{\substack{\text{one}\\ \text{period}}} f(t)\, e^{-i2\pi nt/T}\, dt \; .$$

Above we found the steady-state response of the system for the sinusoidal forces $\exp(i2\pi nt/T)$, namely

$$\phi_n(t) = \frac{1}{-4\pi^2 n^2/T^2 + i\,a_1\,2\pi\,n/T + a_0}\,e^{i2\pi nt/T}\ .$$

Because the equation is linear, we see that the response to a force which is a linear combination of simple sinusoids will be the same linear combination of the responses to the simple sinusoidal forces. Assuming that this can be extended to infinite linear combinations[2], we see that since $\phi_n(t)$ solves the differential equation for the driving force $\exp(i2\pi nt/T)$, then the series

$$\phi(t) = \sum_{n=-\infty}^{\infty} c_n\,\phi_n(t)$$

solves the differential equation for the driving force

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T}\ .$$

As an example, let us consider the differential equation

$$\frac{d^2\phi(t)}{dt^2} + 2\,\frac{d\phi(t)}{dt} + 2\,\phi(t) = f(t)\ ,$$

where $f(t)$ is the periodic function defined in (3.16). This function has period $T = 2\pi$ and according to what was said above above, the solution of this equation is

$$\phi(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{-n^2 + 2i\,n + 2}\,e^{int}\ ,$$

where the coefficients c_n are given in (3.17). Explicitly, we have

$$\phi(t) = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{2\,((-1)^n - 1)}{\pi^2\,n^2}\,\frac{e^{int}}{-n^2 + 2i\,n + 2}\ .$$

[2]Fourier series, since they contain an infinite number of terms, are limiting cases of linear combinations and strictly speaking we would have to justify that, for example, the derivative of the series is the series of termwise derivatives. This would follow if the series were uniformly convergent, for example. In the absence of general theorems, which will be the case in this course, one has to justify this *a posteriori*.

We would now have to check that $\phi(t)$ is twice differentiable. It could not be differentiable three times because, by the defining equation, the second derivative is given by

$$\frac{d^2\phi(t)}{dt^2} = f(t) - 2\,\frac{d\phi(t)}{dt} - 2\,\phi(t)\ ,$$

and $f(t)$ is not differentiable. The twice-differentiability of $\phi(t)$ follows from the uniform convergence of the above series for $\phi(t)$. To see this we apply the Weierstrass M-test:

$$\left|\frac{2\ ((-1)^n - 1)}{\pi^2\, n^2}\ \frac{e^{int}}{-n^2 + 2i\,n + 2}\right| \leq \frac{8}{\pi^2\, n^4}\ ,$$

and the series

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{8}{\pi^2\, n^4}$$

is absolutely convergent. Every time we take a derivative with respect to t, we bring down a factor of $i\,n$, hence we see that the series for $\phi(t)$ can be legitimately differentiated termwise only twice, since the series

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{8}{\pi^2\, n^3} \qquad \text{and} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{8}{\pi^2\, n^2}$$

are still absolutely convergent, but the series

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{8}{\pi^2\, n}$$

is not.

Green's functions

Let us return for a moment to the general second order differential equation treated above:

$$\frac{d^2\phi(t)}{dt^2} + a_1\,\frac{d\phi(t)}{dt} + a_0\,\phi(t) = f(t)\ , \tag{3.20}$$

where $f(t)$ is periodic with period T and can be expanded in a Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i2\pi nt/T}\ .$$

Then as we have just seen, the solution is given by

$$\phi(t) = \sum_{n=-\infty}^{\infty} \frac{c_n\, e^{i2\pi nt/T}}{-4\pi^2 n^2/T^2 + i\, a_1\, 2\pi\, n/T + a_0} \ ,$$

where the coefficients c_n are given by

$$c_n = \frac{1}{T} \int\limits_{\substack{\text{one}\\ \text{period}}} f(t)\, e^{-i2\pi nt/T}\, dt \ .$$

Inserting this back into the solution, we find

$$\phi(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int\limits_{\substack{\text{one}\\ \text{period}}} f(\tau)\, e^{-i2\pi n\tau/T}\, d\tau \right] \frac{e^{i2\pi nt/T}}{-4\pi^2 n^2/T^2 + i\, a_1\, 2\pi\, n/T + a_0}$$

Interchanging the order of summation and integration,

$$\phi(t) = \int\limits_{\substack{\text{one}\\ \text{period}}} \left[\frac{1}{T} \sum_{n=-\infty}^{\infty} \frac{e^{i2\pi n(t-\tau)/T}}{-4\pi^2 n^2/T^2 + i\, a_1\, 2\pi\, n/T + a_0} \right] f(\tau)\, d\tau \ ,$$

which we can write as

$$\phi(t) = \int\limits_{\substack{\text{one}\\ \text{period}}} G(t-\tau)\, f(\tau)\, d\tau \ , \tag{3.21}$$

where

$$G(t) \equiv \sum_{n=-\infty}^{\infty} \frac{T\, e^{i2\pi nt/T}}{-4\pi^2 n^2 + i\, a_1\, 2\pi\, nT + a_0\, T^2}$$

is the **Green's function** for the above equation. It is defined (formally) as the solution of the differential equation

$$\frac{d^2 G(t)}{dt^2} + a_1 \frac{dG(t)}{dt} + a_0\, G(t) = \delta(t) \ , \tag{3.22}$$

where

$$\delta(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nt/T}$$

is the Dirac delta "function." In other words, the Green's function is the response of the system to a delta function. It should be clear that if $G(t)$ satisfies (3.22) then $\phi(t)$ given by (3.21) satisfies the original equation (3.20):

$$\begin{aligned}
&\frac{d^2\phi(t)}{dt^2} + a_1\,\frac{d\phi(t)}{dt} + a_0\,\phi(t) \\
&\qquad = \int\limits_{\substack{\text{one}\\ \text{period}}} \left(\frac{d^2G(t-\tau)}{dt^2} + a_1\,\frac{dG(t-\tau)}{dt} + a_0\,G(t-\tau)\right) f(\tau)\,d\tau \\
&\qquad = \int\limits_{\substack{\text{one}\\ \text{period}}} \delta(t-\tau)\,f(\tau)\,d\tau = f(t)\ .
\end{aligned}$$

3.2 The Fourier transform

In the previous section we have seen how to expand a periodic function as a trigonometric series. This can be thought of as a decomposition of a periodic function in terms of elementary modes, each of which has a definite frequency allowed by the periodicity. If the function has period L, then the frequencies must be integer multiples of the fundamental frequency $k = 2\pi/L$. In this section we would like to establish a similar decomposition for functions which are not periodic. A non-periodic function can be thought of as a periodic function in the limit $L \to \infty$. Clearly, the larger L is, the less frequently the function repeats, until in the limit $L \to \infty$ the function does not repeat at all. In the limit $L \to \infty$ the allowed frequencies become a continuum and the Fourier sum goes over to a Fourier integral. In this section we will discuss this integral as well as some of its basic properties, and apply it to a variety of situations: solution of the wave equation and steady-state solutions to differential equations. As in the previous section we will omit most of the analytic details which are necessary to justify the cavalier operations we will be performing.

3.2.1 The Fourier integral

Consider a function $f(x)$ defined on the real line. If $f(x)$ were periodic with period L, say, we could try to expand $f(x)$ in a Fourier series converging to it almost everywhere within each period

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{i2\pi nx/L}\ ,$$

where the coefficients $\{c_n\}$ are given by

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-i2\pi nx/L}\, dx \ , \tag{3.23}$$

where we have chosen the period to be $[-L/2, L/2]$ for convenience in what follows. Even if $f(x)$ is *not* periodic, we can still define a function

$$f_L(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{i2\pi nx/L} \ , \tag{3.24}$$

with the same $\{c_n\}$ as above. By construction, this function $f_L(x)$ is periodic with period L and moreover agrees with $f(x)$ for almost all $x \in [-L/2, L/2]$. Then it is clear that as we make L larger and larger, then $f_L(x)$ and $f(x)$ agree (almost everywhere) on a larger and larger subset of the real line. One should expect that in the limit $L \to \infty$, $f_L(x)$ should converge to $f(x)$ in some sense. The task ahead is to find reasonable expressions for the limit $L \to \infty$ of the expression (3.24) of $f_L(x)$ and of the coefficients (3.23).

The continuum limit in detail.

This prompts us to define the **Fourier (integral) transform** of the function $f(x)$ as

$$\boxed{\mathcal{F}\{f\}(k) \equiv \hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \ ,} \tag{3.25}$$

provided that the integral exists. Not every function $f(x)$ has a Fourier transform. A sufficient condition is that it be **square-integrable**; that is, so that the following integral converges:

$$\|f\|^2 \equiv \int_{-\infty}^{\infty} |f(x)|^2\, dx \ .$$

If in addition of being square-integrable, the function is continuous, then one also has the **inversion formula**

$$\boxed{f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk \ .} \tag{3.26}$$

More generally, one has the **Fourier inversion theorem**, which states that if $f(x)$ is square-integrable, then the Fourier transform $\hat{f}(k)$ exists and moreover

$$\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \begin{cases} f(x), & \text{if } f \text{ is continuous at } x, \text{ and} \\ \frac{1}{2}\left[\lim_{y\nearrow x} + \lim_{y\searrow x}\right] f(y), & \text{otherwise.} \end{cases}$$

In other words, at a point of discontinuity, the inverse transform produces the average of the left and right limiting values of the function f. This was also the case with the Fourier series. In any case, assuming that the function $f(x)$ is such that its points of discontinuity are isolated, then the inverse transform will agree with $f(x)$ everywhere but at the discontinuities.

Some examples

Before discussing any general properties of the Fourier transform, let us compute some examples.

Let $f(x) = 1/(4+x^2)$. This function is clearly square-integrable. Indeed, the integral

$$\|f\|^2 = \int_{-\infty}^{\infty} \frac{1}{(4+x^2)^2}\, dx$$

can be computed using the residue theorem as we did in Section 2.4.3. We will not do the calculation in detail, but simply remark that $\|f\|^2 = \pi/16$. Therefore its Fourier transform exists:

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{4+x^2}\, dx \ .$$

We can compute this integral using the residue theorem. According to equation (2.64), we have that for $k < 0$, we pick up the residues of the poles in the upper half-plane, whereas for $k > 0$ we pick up the poles in the lower half-plane. The function $\exp(ikz)/(4+z^2)$ has simple poles at $z = \pm 2i$. Therefore we have

$$\begin{aligned} \hat{f}(k) &= \begin{cases} \frac{1}{2\pi}\, 2\pi i \ \mathrm{Res}(2i)\ , & \text{if } k \leq 0, \text{ and} \\ \frac{1}{2\pi}\, (-2\pi i)\ \mathrm{Res}(-2i)\ , & \text{if } k \geq 0; \end{cases} \\ &= \begin{cases} \frac{1}{4}\, e^{2k}\ , & \text{if } k \leq 0, \text{ and} \\ \frac{1}{4}\, e^{-2k}\ , & \text{if } k \geq 0; \end{cases} \\ &= \frac{1}{4}\, e^{-2|k|}\ . \end{aligned}$$

We can also verify the inversion formula. Indeed,

$$\begin{aligned}
f(x) &\stackrel{?}{=} \int_{-\infty}^{\infty} \frac{1}{4}\, e^{-2|k|}\, e^{ikx}\, dk \\
&= \int_{-\infty}^{0} \frac{1}{4} e^{2k}\, e^{ikx}\, dk + \int_{0}^{\infty} \frac{1}{4} e^{-2k}\, e^{ikx}\, dk \\
&= \int_{0}^{\infty} \frac{1}{4} e^{-2k-ikx}\, dk + \int_{0}^{\infty} \frac{1}{4} e^{-2k+ikx}\, dk \\
&= \frac{1}{4}\left[\frac{1}{2+ix} + \frac{1}{2-ix}\right] \\
&= \frac{1}{4+x^2}\ .
\end{aligned}$$

As our second example consider the Fourier transform of a **pulse**:

$$f(x) = \begin{cases} 1\ , & \text{for } |x| < \pi, \text{ and} \\ 0\ , & \text{otherwise.} \end{cases} \tag{3.27}$$

It is clearly square-integrable, with $\|f\|^2 = 2\pi$. Its Fourier transform is given by

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx}\, dx = \frac{\sin \pi k}{\pi k}\ .$$

We will not verify the inversion formula in this example. If we were to do this we would be able to evaluate the integral in the inversion formula for $x \neq \pm\pi$ and we would obtain $f(x)$ for those values. The residue methods fail at the discontinuities $x = \pm\pi$, and one has to appeal to more advanced methods we will not discuss in this course.

Finally consider the Fourier transform of a **finite wave train**:

$$f(x) = \begin{cases} \sin x\ , & \text{for } |x| \leq 6\pi; \text{ and} \\ 0\ , & \text{otherwise.} \end{cases}$$

This function is clearly square-integrable, since

$$\|f\|^2 = \int_{-6\pi}^{6\pi} (\sin x)^2\, dx = 6\pi\ .$$

Its Fourier transform is given by

$$
\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \\
&= \frac{1}{2\pi} \int_{-6\pi}^{6\pi} \sin x\, e^{-ikx}\, dx \\
&= \frac{1}{4\pi i} \int_{-6\pi}^{6\pi} \left(e^{ix} - e^{-ix}\right)\, e^{-ikx}\, dx \\
&= \frac{1}{4\pi i} \int_{-6\pi}^{6\pi} \left(e^{i(1-k)x} - e^{-i(1+k)x}\right)\, dx \\
&= \frac{1}{4\pi i} \left[\frac{i}{k-1}\left(e^{-ik6\pi} - e^{ik6\pi}\right) - \frac{i}{1+k}\left(e^{-ik6\pi} - e^{ik6\pi}\right)\right] \\
&= \frac{i \sin 6\pi k}{2\pi} \left[\frac{1}{1-k} + \frac{1}{1+k}\right] \\
&= \frac{i \sin 6\pi k}{\pi(1-k^2)} \; .
\end{aligned}
$$

We will not verify the inversion formula for this transform; although in this case the formula holds for all x since the original function is continuous.

3.2.2 Some properties of the Fourier transform

In this section we will discuss some basic properties of the Fourier transform. All the basic properties of Fourier series extend in some way to the Fourier integral. Although we will not discuss all of them, it would be an instructive exercise nevertheless to try and guess and prove the extensions by yourself.

The first basic property is that if $\hat{f}(k)$ is the Fourier transform of $f(x)$, then $\hat{f}(-k)^*$ is the Fourier transform of $f(x)^*$. This follows simply by taking the complex conjugate of the Fourier integral (3.25):

$$
\hat{f}(k)^* = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)^*\, e^{ikx}\, dx = \mathfrak{F}\left\{f(x)^*\right\}(-k) \; ,
$$

whence $\hat{f}(-k)^* = \mathfrak{F}\left\{f(x)^*\right\}(k)$. Therefore we conclude that if $f(x)$ is real, then $\hat{f}(k)^* = \hat{f}(-k)$.

Suppose that $f'(x) = \frac{df(x)}{dx}$ is also square-integrable. Its Fourier transform is given by

$$
\mathfrak{F}\left\{f'(x)\right\}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f'(x)\, e^{-ikx}\, dx \; .
$$

Let us integrate by parts:

$$\mathcal{F}\{f'(x)\}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (ik)\, f(x)\, e^{-ikx}\, dx \ ,$$

where we have dropped the boundary terms since $f(x)$ is square-integrable and hence vanishes in the limit $|x| \to \infty$. In other words,

$$\mathcal{F}\{f'(x)\}(k) = ik\, \mathcal{F}\{f(x)\}(k) \ . \tag{3.28}$$

More generally, if the n-th derivative $f^{(n)}(x)$ is square-integrable, then

$$\boxed{\mathcal{F}\left\{f^{(n)}(x)\right\}(k) = (ik)^n\, \mathcal{F}\{f(x)\}(k) \ .} \tag{3.29}$$

This is one of the most useful properties of the Fourier transform, since it will allow us to turn differential equations into algebraic equations.

Another version of the Dirac delta function

Let $f(x)$ be a continuous square-integrable function. In this case, the Fourier inversion theorem says that the inversion formula is valid, so that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk \ .$$

If we insert the definition of the Fourier transform $\hat{f}(k)$ in this equation, we obtain

$$f(x) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\, e^{-iky}\, dy\right] e^{ikx}\, dk \ .$$

If f is in addition sufficiently well-behaved[3] we can exchange the order of integrations to obtain

$$f(x) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-y)}\, dk\right] f(y)\, dy = \int_{-\infty}^{\infty} \delta(x-y)\, f(y)\, dy \ ,$$

where we have introduced the **Dirac delta function**

$$\delta(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk \ .$$

[3]Technically, it is enough that f belong to the **Schwarz class**, consisting of those infinitely differentiable functions which decay, together with all its derivatives, sufficiently fast at infinity.

Notice that we can also write this as

$$\boxed{\delta(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\, dk \ ,} \tag{3.30}$$

which makes it clear that it is the Fourier transform of the constant function $f(x) = 1$. Of course, this function is not square-integrable, so this statement is purely formal. We should not expect anything better because the Dirac delta function is not a function. This version of the Dirac delta function is adapted to the integral over the whole real line, as opposed to the one defined by equation (3.19), which is adapted to a finite interval.

Parseval's identity revisited

Another result from Fourier series which extends in some fashion to the Fourier integral transform is the one in equation (3.18). We will first attempt to show that the Fourier transform of a square-integrable function is itself square-integrable. Let us compute

$$\begin{aligned}
\|\hat{f}\|^2 &= \int_{-\infty}^{\infty} |\hat{f}(k)|^2\, dk \\
&= \int_{-\infty}^{\infty} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \right|^2 dk \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{4\pi^2} f(x)\, f(y)^*\, e^{-ikx}\, e^{iky}\, dx\, dy\, dk \ .
\end{aligned}$$

Being somewhat cavalier, let us interchange the order of integration so that we do the k-integral first. Recognising the result as $2\pi\delta(x-y)$, with $\delta(x-y)$ the delta function of (3.30), we can simplify this to

$$\|\hat{f}\|^2 = \int_{-\infty}^{\infty} \frac{1}{2\pi} f(x)\, f(x)^*\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \|f\|^2 \ .$$

Therefore since $\|f\|^2$ is finite, so is $\|\hat{f}\|^2$, and moreover their norms are related by **Parseval's identity**:

$$\boxed{\|\hat{f}\|^2 = \frac{1}{2\pi} \|f\|^2 \ ,} \tag{3.31}$$

which is the integral version of equation (3.18).

For many applications this factor of $1/2\pi$ is a nuisance and one redefines the Fourier transform so that

$$\hat{\mathfrak{F}}\{f\}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \ ,$$

and the inversion formula is more symmetrical

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathcal{F}}\{f\}(k)\, e^{ikx}\, dk \ .$$

In this case, Parseval's identity becomes simply

$$\|\hat{\mathcal{F}}\{f\}\|^2 = \|f\|^2 \ .$$

One should mention that the Fourier transform is an isometry from L^2 to L^2.

3.2.3 Application: one-dimensional wave equation

Let us now illustrate the use of the Fourier transform to solve partial differential equations by considering the one-dimensional wave equation (3.1) again. This time, however, we are not imposing the boundary conditions (3.2) for x. Instead we may impose that at each moment in time t, $\psi(x,t)$ is square-integrable, which is roughly equivalent to saying that the wave has a finite amount of energy. As initial conditions we will again impose (3.3), where $f(x)$ is a square-integrable function.

We will analyse this problem by taking the Fourier transform of the wave equation. From equation (3.29) with $n = 2$ we have that

$$\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}\psi(x,t)\right\} = -k^2\, \hat{\psi}(k,t) \ ,$$

where

$$\hat{\psi}(k,t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x,t)\, e^{-ikx}\, dx \ ,$$

is the Fourier transform of $\psi(x,t)$. Similarly, taking the derivative inside the integral,

$$\mathcal{F}\left\{\frac{\partial^2}{\partial t^2}\psi(x,t)\right\} = \frac{\partial^2}{\partial t^2}\hat{\psi}(k,t) \ .$$

Therefore the wave equation becomes

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{\psi}(k,t) = -k^2\hat{\psi}(k,t) \ .$$

The most general solution is given by a linear combination of two sinusoids:

$$\hat{\psi}(k,t) = \hat{a}(k)\, \cos kct + \hat{b}(k)\, \sin kct \ ,$$

where the "constants" $\hat{a}$ and $\hat{b}$ can still depend on k. The first of the initial conditions (3.3) implies that

$$\left.\frac{\partial\hat{\psi}(k,t)}{\partial t}\right|_{t=0} = 0 \ ,$$

whence we have that $\hat{b}(k) = 0$. Using the inversion formula (3.26), we can write

$$\psi(x,t) = \int_{-\infty}^{\infty} \hat{a}(k)\, \cos kct\, e^{ikx}\, dk \ .$$

Evaluating at $t = 0$, we have that

$$\psi(x,t) = f(x) = \int_{-\infty}^{\infty} \hat{a}(k)\, e^{ikx}\, dk \ ,$$

whence comparing with the inversion formula (3.26), we see that $\hat{a}(k) = \hat{f}(k)$, so that

$$\psi(x,t) = \int_{-\infty}^{\infty} \hat{f}(k)\, \cos kct\, e^{ikx}\, dk \ , \tag{3.32}$$

where

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \ .$$

Inserting back this expression into the solution (3.32) and interchanging the order of integration, we have

$$\begin{aligned}
\psi(x,t) &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(y)\, e^{-iky}\, dy \right] \cos kct\, e^{ikx}\, dk \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \cos kct\, e^{ik(x-y)}\, dk \right] f(y)\, dy \\
&= \int_{-\infty}^{\infty} K(x-y,t)\, f(y)\, dy \ ,
\end{aligned}$$

where we have introduced the **propagator** $K(x,t)$ defined by

$$K(x,t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos kct\, e^{ikx}\, dk \ .$$

Notice that $K(x,t)$ clearly satisfies the wave equation (3.1):

$$\frac{\partial^2}{\partial x^2} K(x,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} K(x,t) \ ,$$

with initial conditions

$$\left. \frac{\partial K(x,t)}{\partial t} \right|_{t=0} = 0 \ ,$$

and

$$K(x,0) = \delta(x) \ ,$$

according to (3.30).

3.2.4 Application: steady-state response

Another useful application of the Fourier transform is to solve for the steady-state solutions of linear ordinary differential equations. Suppose that we have a system governed by a differential equation

$$\frac{d^2\phi(t)}{dt^2} + a_1 \frac{d\phi(t)}{dt} + a_0\, \phi(t) = f(t) \ ,$$

where $f(t)$ is some driving term. We saw that when $f(t)$ is periodic we can use the method of Fourier series in order to solve for $\phi(t)$. If $f(t)$ is not periodic, then it makes sense that we try and use the Fourier integral transform. Let us define the Fourier transform $\hat{\phi}(\omega)$ of $\phi(t)$ by

$$\mathcal{F}\left\{\phi(t)\right\}(\omega) \equiv \hat{\phi}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)\, e^{-i\omega t}\, dt \ .$$

Similarly, let $\hat{f}(\omega)$ denote the Fourier transform of $f(t)$. Then we can take the Fourier transform of the differential equation and we obtain an algebraic equation for $\hat{\phi}(\omega)$:

$$-\omega^2\, \hat{\phi}(\omega) + i\, a_1\omega\, \hat{\phi}(\omega) + a_0\, \hat{\phi}(\omega) = \hat{f}(\omega) \ ,$$

which can be readily solved to yield

$$\hat{\phi}(\omega) = \frac{1}{-\omega^2 + i\, a_1\omega + a_0}\, \hat{f}(\omega) \ .$$

Now we can transform back via the inversion formula

$$\phi(t) = \int_{-\infty}^{\infty} \hat{\phi}(\omega)\, e^{i\omega t}\, d\omega = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{-\omega^2 + i\, a_1\omega + a_0}\, \hat{f}(\omega)\, d\omega \ .$$

Using the definition of $\hat{f}(\omega)$, we have

$$\phi(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{-\omega^2 + i\, a_1\omega + a_0} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\, e^{-i\tau\omega}\, d\tau\right] d\omega \ .$$

If, as we have been doing without justification in this part of the course, we interchange the order of integration, we obtain

$$\phi(t) = \int_{-\infty}^{\infty} G(t-\tau)\, f(\tau)\, d\tau \ ,$$

where we have introduced the **Green's function** $G(t)$, defined by

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{-\omega^2 + i\, a_1\omega + a_0}\, d\omega \ .$$

Notice that as in the case of the Fourier series, $G(t)$ satisfies the equation

$$\frac{d^2G(t)}{dt^2} + a_1 \frac{dG(t)}{dt} + a_0 \, G(t) = \delta(t) \ ,$$

so that it is the response of the system to a delta function input.

As a concrete illustration of the method, let us find a steady-state solution to the following differential equation:

$$\frac{d^2\phi(t)}{dt^2} + 2 \frac{d\phi(t)}{dt} + 2 \, \phi(t) = f(t) \ ,$$

where $f(t)$ is the pulse defined in (3.27). Let us first compute the Green's function for this system:

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{-\omega^2 + 2i\,\omega + 2} \, d\omega \ .$$

We can compute this using the residue theorem and, in particular, equation (2.64). The integrand has simple poles at $i \pm 1$, which lie in the upper half-plane. Therefore it follows immediately from equation (2.64), that $G(t) = 0$ for $t \leq 0$. For $t > 0$, we have that

$$G(t) = \frac{1}{2\pi} \, 2\pi \, i \, \left[\mathrm{Res}(i+1) + \mathrm{Res}(i-1) \right] \ .$$

We compute the residues to be

$$\mathrm{Res}(i+1) = -\frac{e^{-t+it}}{2} \qquad \text{and} \qquad \mathrm{Res}(i-1) = \frac{e^{-t-it}}{2} \ ,$$

whence for $t > 0$, we have

$$G(t) = - \, e^{-t} \, \sin t \ .$$

In summary, the Green's function for this system is

$$G(t) = \begin{cases} 0 \ , & \text{for } t < 0, \text{ and} \\ - \, e^{-t} \, \sin t \ , & \text{for } t \geq 0. \end{cases}$$

Notice that although it is continuous at $t = 0$, its first derivative is not continuous there, and hence the second derivative does not exist at $t = 0$. This is to be expected, since the second derivative of $G(t)$ at $t = 0$ is related to the delta function, which is not a function. In any case, we can now integrate this against the pulse $f(t)$ to find the solution:

$$\begin{aligned}\phi(t) &= \int_{-\infty}^{\infty} G(t-\tau)\, f(\tau)\, d\tau \\ &= \int_{-\pi}^{\pi} G(t-\tau)\, d\tau \ .\end{aligned}$$

Taking into account that $G(t) = 0$ for $t < 0$, we are forced to distinguish between three epochs: $t < -\pi$, $-\pi \leq t \leq \pi$, and $t > \pi$, corresponding to the time before the pulse, during the pulse and after the pulse. We can perform the integral in each of these three epochs with the following results:

$$\phi(t) = \begin{cases} 0 \ , & \text{for } t < -\pi, \\ -\frac{1}{2} - \frac{1}{2}e^{-\pi-t}\,(\cos t + \sin t) \ , & \text{for } t \in [-\pi, \pi], \text{ and} \\ e^{-t}\sinh\pi(\cos t + \sin t) \ , & \text{for } t > \pi. \end{cases}$$

Notice that before the pulse the system is at rest, and that after the pulse the response dies off exponentially. This is as we expect for a steady-state response to an input of finite duration.

3.3 The Laplace transform

In the previous section we introduced the Fourier transform as a tool to find steady-state solutions to differential equations. These solutions can be interpreted as the response of a system which has been driven for such a long time that any transient solutions have died out. In many systems, however, one is also interested in the transient solutions, and in any case, mathematically one usually finds the most general solution of the differential equation. The Laplace transform will allow us to do this. In many ways the Laplace transform is reminiscent of the Fourier transform, with the important difference that it incorporates in a natural way the initial conditions.

3.3.1 The Heaviside D-calculus

Let us start by presenting the D-calculus introduced by Heaviside. The justification for this method is the Laplace transform. An example should suffice to illustrate the method, but first we need to introduce a little bit of notation.

Differential operators

The result of taking the derivative of a function is another function: for example, $d/dt\,(t^n) = nt^{n-1}$ or $d/dt \sin t = \cos t$. Therefore we can think of the derivative as some sort of machine to which one feeds a function as input and gets another function in return. Such machines are generally called **operators**. It is convenient to introduce symbols for operators and, in the case of the derivative operator, it is customary to call it D. Therefore, if f is a function, Df is the function one obtains by having D act on f. A function is defined by specifying its values at every point t. In the case of Df we have

$$Df(t) = \frac{df(t)}{dt} \ .$$

Operators can be composed. For example we can consider D^2 to be the operator which acting on a function f gives $D^2 f = D(Df)$, or

$$D^2 f(t) = D(Df)(t) = \frac{dDf(t)}{dt} = \frac{d^2 f(t)}{dt^2} \ .$$

Therefore D^2 is the second derivative. Operators can be multiplied by functions, and in particular, by constants. If a is a constant, the operator aD is defined by

$$(aD)f(t) \equiv a\,Df(t) = a\,\frac{df(t)}{dt} \ .$$

Similarly, if $g(t)$ is a function, then the operator gD is defined by

$$(gD)f(t) \equiv g(t)\,Df(t) = g(t)\,\frac{df(t)}{dt} \ .$$

Operators can also be added: if g and h are functions, then the expression $gD^2 + hD$ is an operator, defined by

$$(gD^2 + hD)f(t) = g(t)\,\frac{d^2 f(t)}{dt^2} + h(t)\,\frac{df(t)}{dt} \ .$$

In other words, linear combinations of operators are again operators. Operators which are formed by linear combinations with function coefficients of D and its powers are known as **differential operators**. A very important property shared by all differential operators is that they are **linear**. Let us consider the derivative operator D, and let $f(t)$ and $g(t)$ be functions. Then,

$$D(f+g)(t) = \frac{d(f(t)+g(t))}{dt} = \frac{df(t)}{dt} + \frac{dg(t)}{dt} = Df(t) + Dg(t) \ .$$

In other words, $D(f+g) = Df + Dg$. Similarly, it is easy to see that this is still true for any power of D and for any linear combination of powers of D. In summary, differential operators are linear.

The highest power of D which occurs in a differential operator is called the **order** of the differential operator. This agrees with the nomenclature used for differential equations. In fact, a second order ordinary differential equation, like this one

$$a(t)\,\frac{d^2 f(t)}{dt^2} + b(t)\,\frac{df(t)}{dt} + c(t)\,f(t) = h(t) \;,$$

can be rewritten as an operator equation $K f(t) = h(t)$, where we have introduced the second order differential operator $K = a\,D^2 + b\,D + c$.

An example

Suppose we want to solve the following differential equation

$$\frac{d^2 f(t)}{dt^2} + 3\,\frac{df(t)}{dt} + 2\,f(t) = e^{it} \;. \tag{3.33}$$

We first write it down as an operator equation:

$$(D^2 + 3D + 2)\,f(t) = e^{it} \;.$$

Next we will manipulate the operator formally as if D were a variable and not an operator:

$$D^2 + 3D + 2 = (D+2)(D+1) \;;$$

whence formally

$$f(t) = \frac{1}{(D+2)(D+1)}\,e^{it} = \left[\frac{1}{D+1} - \frac{1}{D+2}\right]\,e^{it} \;, \tag{3.34}$$

where we have used a partial fraction expansion: remember we are treating D as if it were a variable z, say. Now we do something even more suspect and expand each of the simple fractions using a geometric series:

$$\frac{1}{D+1} = \sum_{j=0}^{\infty} (-1)^j D^j \qquad \text{and} \qquad \frac{1}{D+2} = \sum_{j=0}^{\infty} (-1)^j \frac{1}{2^{j+1}}\,D^j \;.$$

Now notice that $D\,e^{it} = i\,e^{it}$; hence

$$\frac{1}{D+1}\,e^{it} = \sum_{j=0}^{\infty} (-1)^j D^j\,e^{it} = \sum_{j=0}^{\infty} (-1)^j i^j\,e^{it} = \frac{1}{i+1}\,e^{it} \;,$$

$$\frac{1}{D+2}\,e^{it} = \sum_{j=0}^{\infty} (-1)^j \frac{1}{2^{j+1}}\,D^j\,e^{it} = \sum_{j=0}^{\infty} (-1)^j \frac{i^j}{2^{j+1}}\,e^{it} = \frac{1}{i+2}\,e^{it} \;.$$

Therefore into equation (3.34), we obtain

$$f(t) = \left[\frac{1}{i+1} - \frac{1}{i+2}\right] e^{it} = \frac{1-3i}{10}\, e^{it} ,$$

which can be checked to obey equation (3.33) by direct substitution.

Of course this is only a **particular solution** to the differential equation (3.33). In order to obtain the most general solution we have to add to it the **complementary solution**, which is the most general solution of the **associated homogeneous equation**:

$$K f(t) = (D+1)(D+2) f(t) = \frac{d^2 f(t)}{dt^2} + 3\,\frac{df(t)}{dt} + 2\, f(t) = 0 .$$

The reason for this is that if $g(t)$ solves the equation $Kg(t) = 0$, and $Kf(t) = e^{it}$, then, by linearity, $K(f+g)(t) = Kf(t) + Kg(t) = e^{it} + 0 = e^{it}$. To find the complementary solution, notice that

$$(D+1)(D+2)f(t) = 0$$

has two kinds of solutions:

$$(D+1)f_1(t) = 0 \qquad \text{and} \qquad (D+2)f_2(t) = 0 .$$

These first order equations can be read off immediately:

$$f_1(t) = a\, e^{-t} \qquad \text{and} \qquad f_2(t) = b\, e^{-2t} ,$$

where the constants a and b are to be determined from the initial conditions: $f(0)$ and $f'(0)$, say. In summary, we have the following general solution to the differential equation (3.33):

$$f(t) = \frac{1-3i}{10}\, e^{it} + a\, e^{-t} + b\, e^{-2t} , \tag{3.35}$$

which can be checked explicitly to solve the differential equation (3.33). Notice that the first term corresponds to the steady-state response and the last two terms are transient.

3.3.2 The Laplace transform

The D-calculus might seem a little suspect, but it can be justified by the use of the **Laplace transform**, which we define as follows

$$\boxed{\mathcal{L}\,\{f\}\,(s) \equiv F(s) = \int_0^\infty f(t)\, e^{-st}\, dt ,} \tag{3.36}$$

provided that the integral exists. This might restrict the values of s for which the transform exists.

A function $f(t)$ is said to be of **exponential order** if there exist real constants M and α for which

$$|f(t)| \leq M\, e^{\alpha t} \ .$$

It is not hard to see that if $f(t)$ is of exponential order, then the Laplace transform $F(s)$ of $f(t)$ exists provided that $\mathrm{Re}(s) > \alpha$.

To see this let us estimate the integral

$$|F(s)| = \int_0^\infty f(t)\, e^{-st}\, dt \ \leq \int_0^\infty |f(t)|\, |e^{-st}|\, dt \leq \int_0^\infty M\, e^{\alpha t}\, e^{-\mathrm{Re}(s)t}\, dt \ .$$

Provided that $\mathrm{Re}(s) > \alpha$, this integral exists and

$$|F(s)| \leq \frac{M}{\mathrm{Re}(s) - \alpha} \ .$$

Notice that in particular, in the limit $\mathrm{Re}(s) \to \infty$, $F(s) \to 0$. This can be proven in more generality: so that if a function $F(s)$ does not approach 0 in the limit $\mathrm{Re}(s) \to \infty$, it cannot be the Laplace transform of any function $f(t)$.

We postpone a more complete discussion of the properties of the Laplace transform until later, but for now let us note the few properties we will need to justify the D-calculus solution of the differential equation (3.33) above.

The first important property is that the Laplace transform is **linear**. Clearly, if $f(t)$ and $g(t)$ are functions whose Laplace transforms $F(s)$ and $G(s)$ exist, then for those values of s for which *both* $F(s)$ and $G(s)$ exist, we have that

$$\mathcal{L}\left\{f + g\right\}(s) = \mathcal{L}\left\{f\right\}(s) + \mathcal{L}\left\{g\right\}(s) = F(s) + G(s) \ .$$

Next let us consider the function $f(t) = \exp(at)$ where a is some complex number. This function is of exponential order, so that its Laplace transform exists provided that $\mathrm{Re}(s) > \mathrm{Re}(a)$. This being the case, we have that

$$\mathcal{L}\left\{e^{at}\right\}(s) = \int_0^\infty e^{at}\, e^{-st}\, dt = \frac{1}{s-a} \ . \tag{3.37}$$

Suppose now that $f(t)$ is a differentiable function. Let us try to compute the Laplace transform of its derivative $f'(t)$. By definition,

$$\mathcal{L}\left\{f'\right\}(s) = \int_0^\infty f'(t)\, e^{-st}\, dt \ ,$$

which can be integrated by parts to obtain

$$\begin{aligned}\mathcal{L}\{f'\}(s) = \int_0^\infty s\, f(t)\, e^{-st}\, dt + f(t)\, e^{-st}\Big|_0^\infty \\ = s\,\mathcal{L}\{f\}(s) - f(0) + \lim_{t\to\infty} f(t)\, e^{-st}\ .\end{aligned}$$

Provided the last term is zero, which might imply conditions on f and/or s, we have that

$$\boxed{\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0)\ .} \tag{3.38}$$

We can iterate this expression in order to find the Laplace transform of higher derivatives of $f(t)$. For example, the Laplace transform of the second derivative is easy to find by understanding $f''(t)$ as the first derivative of $f'(t)$ and iterating the above formula:

$$\begin{aligned}\mathcal{L}\{f''\}(s) &= \mathcal{L}\{(f')'\}(s) \\ &= s\,\mathcal{L}\{f'\}(s) - f'(0) \\ &= s\,\left(s\,\mathcal{L}\{f\}(s) - f(0)\right) - f'(0) \\ &= s^2\,\mathcal{L}\{f\}(s) - s\, f(0) - f'(0)\ ,\end{aligned}$$

provided that $f(t)\exp(-st)$ and $f'(t)\exp(-st)$ both go to zero in the limit $t\to\infty$.

The D-calculus justified

We are now ready to justify the D-calculus solution of the previous section. This serves also to illustrate how to solve initial value problems using the Laplace transform.

Consider again the differential equation (3.33):

$$\frac{d^2 f(t)}{dt^2} + 3\,\frac{df(t)}{dt} + 2\, f(t) = e^{it}\ ,$$

and let us take the Laplace transform of both sides of the equation. Since the Laplace transform is linear, we can write this as

$$\mathcal{L}\{f''\}(s) + 3\,\mathcal{L}\{f'\}(s) + 2\,\mathcal{L}\{f\}(s) = \mathcal{L}\left\{e^{it}\right\}(s)\ .$$

Letting $F(s)$ denote the Laplace transform of f, we can use equations (3.37) and (3.38) to rewrite this as

$$s^2F(s) - sf(0) - f'(0) + 3\,(sF(s) - f(0)) + 2F(s) = \frac{1}{s-i}\ ,$$

which can be solved for $F(s)$:

$$F(s) = \frac{1}{s^2+3s+2}\left[\frac{1}{s-i} + (s+3)f(0) + f'(0)\right] .$$

Expanding this out, and factorising $s^2 + 3s + 2 = (s + 1)(s + 2)$, we have

$$F(s) = \frac{1}{(s-i)(s+1)(s+2)} + \frac{(s+3)f(0)+f'(0)}{(s+1)(s+2)} .$$

We now decompose this into partial fractions:

$$F(s) = \frac{\frac{1}{10}(1-3i)}{s-i} + \frac{2f(0)+f'(0)-\frac{1}{2}(1-i)}{s+1} + \frac{\frac{1}{5}(2-i)-f(0)-f'(0)}{s+2} .$$

Using linearity again and (3.37) we can recognise this as the Laplace transform of the function

$$\begin{aligned} f(t) = \frac{1-3i}{10}\,e^{it} + \left(2f(0)+f'(0)-\frac{1-i}{2}\right)\,e^{-t} \\ + \left(\frac{2-i}{5}-f(0)-f'(0)\right)\,e^{-2t} , \end{aligned}$$

which agrees with (3.35) and moreover displays manifestly the dependence of the coefficients a and b in that expression in terms of the initial conditions.

The inverse Laplace transform

The Laplace transform is applicable to a wide range of initial value problems. The main difficulty stems from inverting the transform, which might be difficult. In practice one resorts to tables of Laplace transforms, like Table 3.1 below; but if this does not work, there is an inversion formula, as for the Fourier transform, which we will state without proof. It says that if $F(s)$ is the Laplace transform of a function $f(t)$, then one can recover the function (except maybe at points of discontinuity) by

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(s)\,e^{st}\,ds ,$$

where the integral is meant to be a contour integral along the imaginary axis. In other words, parametrising $s = iy$, we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(iy)\,e^{iyt}\,dy .$$

It may happen, however, that Laplace transform $F(s)$ does not make sense for $\mathrm{Re}(s)=0$, because the integral (3.36) does not converge. Suppose instead that there is some positive real number a such that the Laplace transform of $f(t)\,e^{-at}$ does exist for $\mathrm{Re}(s)=0$. In this case, we can use the inversion formula to obtain

$$f(t)\,e^{-at} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathcal{L}\left\{f(t)\,e^{-at}\right\}(s)\,e^{st}\,ds\ .$$

Using the shift formula (3.43), $\mathcal{L}\left\{f(t)\,e^{-at}\right\}(s) = F(s+a)$, whence, multiplying by e^{at} on both sides of the inversion formula:

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(s+a)\,e^{(s+a)t}\,ds\ .$$

Changing variables of integration to $u=s+a$, we have

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(u)\,e^{ut}\,du\ , \tag{3.39}$$

which can now be interpreted as a contour integral along the line $u=a$. In other words, we can for free shift the original contour of integration to the right until $F(s)$ makes sense on it.

3.3.3 Basic properties of the Laplace transform

We shall now discuss the basic properties of the Laplace transform. We have already seen that it is linear and we computed the transform of a simple exponential function $\exp(at)$ in equation (3.37). From this simple result, we can compute the Laplace transforms of a few simple functions related to the exponential.

Let ω be a real number. From the fact that $\exp(i\omega t) = \cos\omega t + i\,\sin\omega t$, linearity of the Laplace transform implies that

$$\begin{aligned}\mathcal{L}\left\{e^{i\omega t}\right\}(s) &= \mathcal{L}\left\{\cos\omega t\right\}(s) + i\,\mathcal{L}\left\{\sin\omega t\right\}(s)\\ &= \frac{1}{s-i\omega} = \frac{s}{s^2+\omega^2} + i\,\frac{\omega}{s^2+\omega^2}\ ,\end{aligned}$$

from where we can read off the Laplace transforms of $\cos\omega t$ and $\sin\omega t$. Notice that these expressions are valid for $\mathrm{Re}(s)>0$, since this is the condition for the existence of the Laplace transform of the exponential.

Similarly, let β be a real number and recall the trigonometric identities (2.16), from where we can deduce that

$$\cosh\beta t = \cos i\beta t \qquad \text{and} \qquad \sinh\beta t = -i\sin i\beta t\ .$$

As a result, we immediately see that the Laplace transforms of the hyperbolic functions are given by

$$\mathfrak{L}\left\{\cosh\beta t\right\}(s) = \frac{s}{s^2-\beta^2} \qquad \text{and} \qquad \mathfrak{L}\left\{\sinh\beta t\right\}(s) = \frac{\beta}{s^2-\beta^2} \ ,$$

where the condition is now $\mathrm{Re}(s) > |\beta|$.

Putting $a = 0$ in (3.37), we see that the Laplace transform of the constant function $f(t) = 1$, is given by

$$\mathfrak{L}\left\{1\right\}(s) = \frac{1}{s} \ ,$$

which is valid for $\mathrm{Re}(s) > 0$.

Suppose that $f(t)$ has Laplace transform $F(s)$. Then by taking derivatives with respect to s of the expression (3.36) for $F(s)$, we arrive at

$$\mathfrak{L}\left\{t^n f(t)\right\}(s) = (-1)^n F^{(n)}(s) \ , \tag{3.40}$$

which is valid for those values of s for which the Laplace transform $F(s)$ of $f(t)$ exists. In particular, if we take $f(t) = 1$, we arrive at

$$\mathfrak{L}\left\{t^n\right\}(s) = (-1)^n \frac{d^n}{ds^n}\frac{1}{s} = \frac{n!}{s^{n+1}} \ , \tag{3.41}$$

valid for $\mathrm{Re}(s) > 0$.

How about if n is a negative integer? Let us consider the Laplace transform of $g(t) \equiv f(t)/t$, and let us call it $G(s)$. From equation (3.40) for $n = 1$, we have that

$$\mathfrak{L}\left\{f(t)\right\}(s) = \mathfrak{L}\left\{tg(t)\right\}(s) = -G'(s) \ ,$$

so that $G(s)$ in an antiderivative for $-F(s)$; that is,

$$G(s) = -\int_a^s F(\sigma)\, d\sigma \ .$$

If we demand that $G(s)$ vanishes in the limit $s \to \infty$, then we must choose $a = \infty$, and hence

$$\mathfrak{L}\left\{f(t)/t\right\}(s) = \int_s^\infty F(\sigma)\, d\sigma \ . \tag{3.42}$$

Another important property of the Laplace transform is the **shifting formula**:

$$\mathfrak{L}\left\{e^{at} f(t)\right\}(s) = \mathfrak{L}\left\{f(t)\right\}(s-a) = F(s-a) \ , \tag{3.43}$$

which is evident from the definition (3.36) of the Laplace transform. Related to this property is the following. Given a function $f(t)$, let τ be a positive real constant, and introduce the notion of the **delayed function** $f_\tau(t)$, defined by

$$f_\tau(t) = \begin{cases} f(t-\tau) \ , & \text{for } t \geq \tau, \text{ and} \\ 0 \ , & \text{otherwise.} \end{cases} \tag{3.44}$$

In other words, the delayed function is the same as the original function, but it has been translated in time by τ, hence the name. The Laplace transform of the delayed function is given by

$$\begin{aligned} \mathfrak{L}\left\{f_\tau\right\}(s) &= \int_0^\infty f_\tau(t)\, e^{-st}\, dt \\ &= \int_\tau^\infty f(t-\tau)\, e^{-st}\, dt \\ &= e^{-s\tau} \int_0^\infty f(u)\, e^{-su}\, du \\ &= e^{-s\tau} \mathfrak{L}\left\{f\right\}(s) \ , \end{aligned}$$

where we have changed the variable of integration from t to $u = t - \tau$. In other words,

$$\mathfrak{L}\left\{f_\tau\right\}(s) = e^{-s\tau} F(s) \ . \tag{3.45}$$

Although the delta function $\delta(t-\tau)$ is not a function, we can nevertheless attempt to compute its Laplace transform:

$$\mathfrak{L}\left\{\delta(t-\tau)\right\}(s) = \int_0^\infty \delta(t-\tau)\, e^{-st}\, dt = \begin{cases} e^{-s\tau} \ , & \text{if } \tau \geq 0, \text{ and} \\ 0 \ , & \text{otherwise.} \end{cases}$$

Introducing the **Heaviside step function** $\theta(t)$, defined as

$$\theta(t) = \begin{cases} 1 \ , & \text{for } t \geq 0, \text{ and} \\ 0 \ , & \text{for } t < 0; \end{cases}$$

we see that

$$\mathfrak{L}\left\{\delta(t-\tau)\right\}(s) = \theta(\tau)\, e^{-s\tau} \ .$$

Finally let us consider the Laplace transforms of integrals and derivatives of functions. In the previous section we derived equation (3.38) for the Laplace transform of the derivative $f'(t)$ of a function $f(t)$. Iterating this expression we can find a formula for the Laplace transform of the n-th derivative of a function:

$$\mathfrak{L}\left\{f^{(n)}\right\}(s) = s^n\, F(s) - \sum_{k=0}^{n-1} s^{n-1-k}\, f^{(k)}(0) \ , \tag{3.46}$$

where by $f^{(0)}$ we mean the original function f. This formula is valid whenever $\lim_{t\to\infty} f^{(n)}(t)\exp(-st) = 0$. How about integration? Consider the function

$$g(t) = \int_0^t f(\tau)\, d\tau \ .$$

What is its Laplace transform? We know that since $g(t)$ is an antiderivative for f, $g'(t) = f(t)$ and moreover, from the definition, that $g(0) = 0$. Therefore we can compute the Laplace transform of $f(t) = g'(t)$, in two ways. On the one hand it is simply $F(s)$, but using (3.38) we can write

$$\mathcal{L}\left\{f\right\}(s) = \mathcal{L}\left\{g'\right\}(s) = s\mathcal{L}\left\{g\right\}(s) - g(0) = s\mathcal{L}\left\{g\right\}(s) \ ,$$

whence

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}(s) = \frac{F(s)}{s} \ .$$

These properties are summarised in Table 3.1.

3.3.4 Application: stability and the damped oscillator

In this section we will use the Laplace transform to characterise the notion of stability of a dynamical system which is governed by a linear ordinary differential equation.

Many systems are governed by differential equations of the form

$$K\, f(t) = u(t) \ , \tag{3.47}$$

where K is an n-th order differential operator which we will take, for simplicity, to have constant coefficients and such that the coefficient of the term of highest degree is 1 ; that is,

$$K = D^n + a_{n-1}\, D^{n-1} + \cdots + a_1\, D + a_0 \ .$$

The differential equation (3.47) describes the **output response** $f(t)$ of the system to an **input** $u(t)$. For the purposes of this section we will say that a system is **stable** if in the absence of any input all solutions are transient; that is,

$$\lim_{t\to\infty} f(t) = 0 \ ,$$

regardless the initial conditions.

Often one extends the notion of stability to systems for which $f(t)$ remains bounded as $t \to \infty$: for example, if the solutions oscillate; but we will not do this here. In any case, the method we will employ extends trivially to this weaker notion of stability.

Function	Transform	Conditions
$f(t)$	$F(s)$	convergence
e^{at}	$\dfrac{1}{s-a}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$	$\omega \in \mathbb{R}$ and $\mathrm{Re}(s) > 0$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$	$\omega \in \mathbb{R}$ and $\mathrm{Re}(s) > 0$
$\cosh\beta t$	$\dfrac{s}{s^2-\beta^2}$	$\beta \in \mathbb{R}$ and $\mathrm{Re}(s) > \|\beta\|$
$\sinh\beta t$	$\dfrac{\beta}{s^2-\beta^2}$	$\beta \in \mathbb{R}$ and $\mathrm{Re}(s) > \|\beta\|$
t^n	$\dfrac{n!}{s^{n+1}}$	$n = 0, 1, \ldots$ and $\mathrm{Re}(s) > 0$
$e^{at}\, f(t)$	$F(s-a)$	convergence
$t^n\, f(t)$	$(-1)^n\, F^{(n)}(s)$	same as for $F(s)$
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(\sigma)\, d\sigma$	same as for $F(s)$
$f_\tau(t)$	$e^{-s\tau}\, F(s)$	$\tau > 0$ and same as for $F(s)$
$\delta(t-\tau)$	$\theta(t-\tau)\, e^{-s\tau}$	none
$f^{(n)}(t)$	$\displaystyle s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k}\, f^{(k)}(0)$	$\displaystyle\lim_{t\to\infty} f^{(k)}(t)e^{-st} = 0$
$\displaystyle\int_0^t f(\tau)\, d\tau$	$\dfrac{F(s)}{s}$	same as for $F(s)$

Table 3.1: Some Laplace transforms

Stability can be analysed using the Laplace transform. In order to see this let us take the Laplace transform of the equation (3.47). Letting $F(s)$ and $U(s)$ denote the Laplace transforms of $f(t)$ and $u(t)$ respectively, we have

$$(s^n + a_{n-1}\, s^{n-1} + \cdots + a_1\, s + a_0) F(s) = U(s) + P(s) \ , \tag{3.48}$$

where $P(s)$ is a polynomial in s of order at most $n-1$ depending on the initial conditions: $f^{(k)}(0)$ for $k = 0, 1, \ldots, n-1$. In fact, a little bit of algebra using equation (3.46) shows that

$$P(s) = \sum_{i=0}^{n-1} p_i\, s^i \ , \qquad \text{with } p_i = \sum_{j=0}^{n-i-1} a_{j+i+1} f^{(j)}(0) \ ,$$

with the conventions that $a_n = 1$. We will not need its explicit expression, however. We can solve for $F(s)$ in the transformed equation (3.48):

$$F(s) = \frac{U(s)}{s^n + \cdots + a_0} + \frac{P(s)}{s^n + \cdots + a_0} \ .$$

Notice that the first term in the right-hand side of the equation depends on the input, whereas the second term depends on the initial conditions. Moreover the common denominator depends only on the differential operator K; that is, it is intrinsic to the system. It is convenient to define the function

$$H(s) = \frac{1}{s^n + \cdots + a_0} \ .$$

It is called the **transfer function** of the system and it encodes a great deal of information about the qualitative dynamics of the system. In particular we can will be able to characterise the stability of the system by studying the poles of the transfer function in the complex s-plane.

Let us start with the case of a first order equation:

$$(D + a_0) f(t) = u(t) \ .$$

Taking the Laplace transform and solving for the Laplace transform $F(s)$ of $f(t)$ we have

$$F(s) = \frac{U(s)}{s + a_0} + \frac{f(0)}{s + a_0} \ .$$

In the absence of any input ($u = 0$), the solution of this equation is given by

$$f(t) = f(0)\, e^{-a_0\, t} \ .$$

This solution is transient provided that $\mathrm{Re}(a_0) > 0$. This is equivalent to saying that the pole $-a_0$ of the transfer function $1/(s + a_0)$ lies in the left half of the plane.

Let us now consider a second order equation:

$$(D^2 + a_1 D + a_0) f(t) = u(t) \ .$$

Taking the Laplace transform and solving for $F(s)$, we find

$$F(s) = H(s)\, U(s) + H(s)\ [(s + a_1)\, f(0) + f'(0)] \ , \tag{3.49}$$

where the transfer function is given by

$$H(s) = \frac{1}{s^2 + a_1\, s + a_0} \ .$$

The poles of $H(s)$ occur at the zeros of $s^2 + a_1\, s + a_0$. Two possibilities can occur: the zeros are simple and distinct: $s_\pm$ say, or there is one double zero at s_0. In either case we will decompose the right-hand side of the transformed equation (3.49) with $u(t)$ and hence $U(s)$ set to zero, into partial fractions. In the case of distinct zeros, we have

$$F(s) = \frac{A_1}{s - s_+} + \frac{A_2}{s - s_-} \ ,$$

where A_1 and A_2 are constants depending on $f(0)$ and $f'(0)$. The transform is trivial to invert:

$$f(t) = A_1\, e^{s_+\, t} + A_2\, e^{s_-\, t} \ ,$$

which is transient for all A_1 and A_2 if and only if $\mathrm{Re}(s_\pm) < 0$; in other words, if and only if the poles of the transfer function lie in the left side of the plane. On the other hand, if the zero is double, then we have

$$F(s) = \frac{B_1}{s - s_0} + \frac{B_2}{(s - s_0)^2} \ ,$$

where again B_1 and B_2 are constants depending on $f(0)$ and $f'(0)$. We can invert the transform and find that

$$f(t) = B_1\, e^{s_0\, t} + B_2\, t\, e^{s_0\, t} \ ,$$

which is transient for all B_1 and B_2 if and only if $\mathrm{Re}(s_0) < 0$; so that s_0 lies in the left side of the plane.

In fact this is a general result: a system is stable if all the poles of the transfer function lie in the left side of the plane. A formal proof of this statement is not hard, but takes some bookkeeping, so we will leave it as an exercise for the industrious reader.

Notice that if one relaxes the condition that the solutions should be transient for *all* initial conditions, then it may happen that for certain types of initial conditions non-transient solutions have a zero coefficient. The system may therefore seem stable, but only because of the special choice of initial conditions.

The damped harmonic oscillator

Stability is not the only property of a system that can be detected by studying the poles of the transfer function. With some experience one can detect change in the qualitative behaviour of a system by studying the poles. A simple example is provided by the damped harmonic oscillator.

This system is defined by two parameters μ and ω, both positive real numbers. The differential equation which governs this system is

$$(D^2 + 2\mu\, D + \omega^2)\, f(t) = u(t)\ .$$

The transfer function is

$$H(s) = \frac{1}{s^2 + 2\mu\, s + \omega^2}\ ,$$

which has poles at

$$s_\pm = -\mu \pm \sqrt{\mu^2 - \omega^2}\ .$$

We must distinguish three separate cases:

(a) (**overdamped**) $\mu > \omega$
In this case the poles are real and negative:

$$s_\pm = -\mu \left(1 \mp \sqrt{1 - \frac{\omega^2}{\mu^2}}\right)\ .$$

(b) (**critically damped**) $\mu = \omega$
In this case there is a double pole, real and negative: $s_+ = s_- = -\mu$.

(c) (**underdamped**) $\mu < \omega$
In this case the poles are complex:

$$s_\pm = -\mu \pm i\omega\sqrt{1 - \frac{\mu^2}{\omega^2}}\ .$$

Hence provided that μ is positive, the system is stable.

Suppose that we start with the system being overdamped so that the ratio $\varrho \equiv \omega/\mu$ is less than 1: $\varrho < 1$. As we increase ϱ either by increasing ω or decreasing μ, the poles of the transfer function, which start in the negative real axis, start moving towards each other, coinciding when $\varrho = 1$. If we continue increasing ϱ so that it becomes greater than 1, the poles move vertically away from each other keeping their real parts constant. It is the transition from real to complex poles which offers the most drastic qualitative change in the behaviour of the system.

3.3.5 Application: convolution and the tautochrone

In this section we discuss a beautiful application of the Laplace transform. We also take the opportunity to discuss the convolution of two functions.

The convolution

Suppose that $f(t)$ and $g(t)$ are two functions with Laplace transforms $F(s)$ and $G(s)$. Consider the product $F(s)\,G(s)$. Is this the Laplace transform of any function? It turns out it is! To see this let us write the product $F(s)\,G(s)$ explicitly:

$$F(s)\,G(s) = \left(\int_0^\infty f(u)\,e^{-su}\,du\right)\left(\int_0^\infty g(v)\,e^{-sv}\,dv\right) .$$

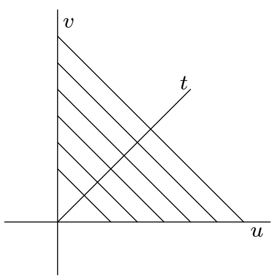
We can think of this as a double integral in the positive quadrant of the (u,v)-plane:

$$F(s)\,G(s) = \iint e^{-s(u+v)}\,f(u)\,g(v)\,du\,dv \ . \tag{3.50}$$

If this were the Laplace transform of anything, it would have to be of the form

$$F(s)\,G(s) \stackrel{?}{=} \int_0^\infty h(t)\,e^{-st}\,dt \ . \tag{3.51}$$

Comparing the two equations we are prompted to define $t = u + v$. In the positive quadrant in the (u,v)-axis, t runs from 0 to ∞: lines of constant t having slope -1. Therefore we see that integrating (u,v) in the positive quadrant is the same as integrating (t,v) where t runs from 0 to ∞ and for every t, v runs from 0 to t:



$$\iint k(u,v)\,du\,dv = \int_0^\infty dt \int_0^t k(t-v,v)\,dv$$

In other words, we can rewrite equation (3.50) as

$$F(s)\,G(s) = \int_0^\infty e^{-st} \int_0^t f(t-v)\,g(v)\,dv \ .$$

Comparing with equation (3.51), we see that this equation is true provided that

$$h(t) = \int_0^t f(t-v)\, g(v)\, dv \ .$$

This means that $h(t)$ is the **convolution** of f and g. The convolution is often denoted $f \star g$:

$$\boxed{(f \star g)(t) \equiv \int_0^t f(t-\tau)\, g(\tau)\, d\tau \ ,} \tag{3.52}$$

and it is characterised by the **convolution theorem**:

$$\boxed{\mathcal{L}\left\{f \star g\right\}(s) = F(s)\, G(s) \ .} \tag{3.53}$$

Notice that $f \star g = g \star f$. This is clear from the fact that $F(s)\, G(s) = G(s)\, F(s)$, but can also be checked directly by making a change of variables $\tau = t - \sigma$ in the integral in (3.52).

Abel's mechanical problem and the tautochrone

As an amusing application of the convolution theorem for the Laplace transform, let us consider Abel's mechanical problem. In short, the problem can be described as follows. Consider a bead of mass m which can slide down a wire frame under the influence of gravity but without any friction. Suppose that the bead is dropped from rest from a height h. Let $\tau(h)$ denote the time it takes to slide down to the ground. If one knows the shape of the wire it is a simple matter to determine the function $\tau(h)$, and we will do so below. Abel's mechanical problem is the inverse: given the function $\tau(h)$ determine the shape of the wire. As we will see below, this leads to an integral equation which has to be solved. In general integral equations are difficult to solve, but in this particular case, the integral is in the form of a convolution, whence its Laplace transform factorises. It is precisely this feature which makes the problem solvable.

To see what I mean, consider the following integral equation for the unknown function $f(t)$:

$$f(t) = 1 + \int_0^t f(t-\tau)\, \sin\tau\, d\tau \ . \tag{3.54}$$

We can recognise the integral as the convolution of the functions $f(t)$ and $\sin t$, whence taking the Laplace transform of both sides of the equation, we have

$$F(s) = \frac{1}{s} + F(s)\frac{1}{s^2+1} \ ,$$

which we can immediately solve for $F(s)$:

$$F(s) = \frac{1}{s} + \frac{1}{s^3} ,$$

which is the Laplace transform of the function

$$f(t) = 1 + \tfrac{1}{2}\, t^2 .$$

One can verify directly that this function obeys the original integral equation (3.54).

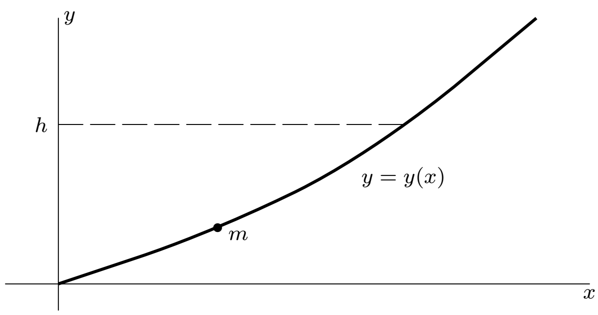


Figure 3.4: Abel's mechanical problem

In order to set up Abel's mechanical problem, it will prove convenient to keep Figure 3.4 in mind. We will assume that the wire has no torsion, so that the motion of the bead happens in one plane: the (x, y) plane with y the vertical displacement and x the horizontal displacement. We choose our axes in such a way that wire touches the ground at the origin of the plane: $(0, 0)$. The shape of the wire is given by a function $y = y(x)$, with $y(0) = 0$. Let ℓ denote the length *along* the wire from the origin to the point $(x, y = y(x))$ on the wire. We drop the bead from rest from a height h. Because there is no friction, energy is conserved. The kinetic energy of the bead at any time t after being dropped is given by

$$T = \tfrac{1}{2}\, m \left(\frac{d\ell}{dt}\right)^2 ,$$

whereas the potential energy is given by

$$V = -mg\,(h - y) .$$

Conservation of energy says that $T + V$ is a constant. To compute this constant, let us evaluate this at the moment the bead is dropped, $t = 0$. Because it is dropped from rest, $d\ell/dt = 0$ at $t = 0$, and hence $T = 0$.

Since at $t = 0$, $y = h$ the potential energy also vanishes and we have that $T + V = 0$. This identity can be rewritten as

$$\tfrac{1}{2}\, m \left(\frac{d\ell}{dt}\right)^2 = mg\,(h - y) \ ,$$

from which we can find a formula for $d\ell/dt$:

$$\frac{d\ell}{dt} = -\sqrt{2g\,(h-y)} \ , \tag{3.55}$$

where we have chosen the negative sign for the square root, because as the bead falls, ℓ decreases. Now, the length element along the wire is given by

$$d\ell = \sqrt{dx^2 + dy^2} \ ,$$

where dx and dy are not independent since we have a relation $y = y(x)$. Inverting this relation gives x as a function of y and we can use this to write

$$d\ell = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \equiv f(y)\, dy \ , \tag{3.56}$$

which defines the function $f(y)$. Clearly, $f(y)$ encodes the information about the shape of the wire: knowing $f(y)$ for all y allows us to solve for the dependence of x on y and viceversa. Indeed, suppose that $f(y)$ is known, then solving for dx/dy, we have that

$$\frac{dx}{dy} = \sqrt{f(y)^2 - 1} \ ,$$

from where we have

$$dx = \sqrt{f(y)^2 - 1}\, dy \ , \tag{3.57}$$

which can then be integrated to find x as a function of y, and by inverting this, y as a function of x.

Let us rewrite equation (3.55) as

$$dt = -\frac{1}{\sqrt{2g\,(h-y)}}\, d\ell \ .$$

and insert equation (3.56) in this equation, to obtain

$$dt = -\frac{f(y)}{\sqrt{2g\,(h-y)}}\, dy \ .$$

Finally we integrate this along the trajectory of the bead, as it falls from $y = h$ at $t = 0$ until $y = 0$ at $t = \tau(h)$:

$$\int_0^{\tau(h)} dt = -\int_h^0 \frac{f(y)}{\sqrt{2g\,(h-y)}}\, dy \ ,$$

whence

$$\boxed{\tau(h) = \frac{1}{\sqrt{2g}} \int_0^h f(y)\, \frac{1}{\sqrt{h-y}}\, dy \ .} \tag{3.58}$$

This formula gives us how long it takes for the bead to fall along the wire from a height h: so if we know the shape of the wire, and hence $f(y)$, we can compute $\tau(h)$ just by integrating. On the other hand, suppose that we are given $\tau(h)$ and we want to solve for the shape of the wire. This means solving equation (3.58) for $f(y)$ and then finding $y = y(x)$ from $f(y)$. The latter half of the problem is a first order differential equation, but the former half is an integral equation. In general this problem would be quite difficult, but because we notice that the integral in the right hand side is in the form of a convolution, we can try to solve this by using the Laplace transform.

Before doing so, however, let us check that we have not made a mistake, by testing the integral expression for $\tau(h)$ in a some cases where we know the answer. Suppose, for instance, that the wire is completely vertical. This means that $dx/dy = 0$, whence $f(y) = 1$. In this case, equation (3.58) simplifies enormously, and we get

$$\tau(h) = \frac{1}{\sqrt{2g}} \int_0^h \frac{dy}{\sqrt{h-y}} = \sqrt{\frac{2h}{g}} \ ,$$

as expected from elementary newtonian mechanics. Similarly, if the wire is inclined θ degrees from the horizontal, so that $y(x) = \tan\theta\, x$. Then $dx/dy = \cot\theta$, and hence $f(y)$ is given by

$$f(y) = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (\cot\theta)^2} = \csc\theta \ .$$

Therefore, the time taken to fall is simply $\csc\theta$ times the vertical time of fall:

$$\tau(h) = \csc\theta \sqrt{\frac{2h}{g}} \ ,$$

which, since $\csc(\pi/2) = 1$, agrees with the previous result.

Let us now take the Laplace transform of both sides of equation (3.58), thinking of them both as functions of h; in other words, the Laplace transform $F(s)$ of a function $g(h)$ is given by

$$G(s) = \int_0^\infty e^{-sh}\, g(h)\, dh \ .$$

(This is Shakespeare's theorem yet again!) Applying this to equation (3.58), we find

$$T(s) = \frac{1}{\sqrt{2g}} F(s)\, \mathfrak{L}\left\{\frac{1}{\sqrt{h}}\right\}(s) \ ,$$

where $T(s)$ is the Laplace transform of the function τ, and $F(s)$ is the Laplace transform of the function f. The Laplace transform of the function $1/\sqrt{h}$ was worked out in the problems and the result is:

$$\mathfrak{L}\left\{\frac{1}{\sqrt{h}}\right\}(s) = \sqrt{\frac{\pi}{s}} \ . \tag{3.59}$$

We can then solve for $F(s)$ in terms of $T(s)$ as follows:

$$F(s) = \sqrt{\frac{2g}{\pi}}\, \sqrt{s}\, T(s) \ , \tag{3.60}$$

which can in principle be inverted to solve for f, either from Table 3.1 or, if all else fails, from the inversion formula (3.39).

Let us apply this to solving for the shape that the wire must have for it to have the curious property that no matter what height we drop the bead from, it will take the same amount of time to fall to the ground. Such a shape is known as the *tautochrone*. Clearly, the tautochrone is such that $\tau(h) = \tau$ is constant, whence its Laplace transform is $T(s) = \tau/s$. Into equation (3.60), we get

$$F(s) = \sqrt{\frac{2g}{\pi}}\, \sqrt{s}\, \frac{\tau}{s} = \sqrt{2g}\, \frac{\tau}{\pi} \sqrt{\frac{\pi}{s}} \ ,$$

where we have rewritten it in a way that makes it easy to invert. From equation (3.59) we immediately see that

$$f(y) = \sqrt{2g}\, \frac{\tau}{\pi} \frac{1}{\sqrt{y}} \ .$$

To reconstruct the formula for the shape of the wire, we apply equation (3.57) to obtain

$$dx = \sqrt{\frac{2g\tau^2}{\pi^2} \frac{1}{y} - 1}\, dy \ ,$$

which can be integrated to

$$x = \int_0^y \sqrt{\frac{2g\tau^2}{\pi^2}\frac{1}{y} - 1}\, dy = \int_0^y \frac{\sqrt{\frac{2g\tau^2}{\pi^2} - y}}{\sqrt{y}}\, dy \ . \tag{3.61}$$

Notice that the constant of integration is fixed to 0 since the wire is such that when $x = 0$, $y = 0$. This integral can be performed by a trigonometric substitution. First of all let us define

$$b \equiv \frac{2g\tau^2}{\pi^2} \ ,$$

and let $y = b\,(\sin\phi)^2$, so that

$$dy = 2b\,\sin\phi\,\cos\phi\, d\phi \ .$$

Into the integral in (3.61), we find

$$x = \int_0^{\phi(y)} 2b\,(\cos\phi)^2\, d\phi = \frac{b}{2}\,\left[2\phi(y) + \sin 2\phi(y)\right] \ ,$$

where

$$y = b\,(\sin\phi(y))^2 = \frac{b}{2}\,(1 - \cos 2\phi(y)) \ .$$

If we define $a = b/2$ and $\theta = 2\phi(y)$, we have the following parametric representation for the curve in the (x, y) plane defining the wire:

$$x = a(\theta + \sin\theta) \qquad \text{and} \qquad y = a(1 - \cos\theta) \ .$$

This curve is called a **cycloid**. It is the curve traced by a point in the rim of a circle of radius a rolling upside down without sliding along the line $y = a$, as shown in the Figure 3.5.

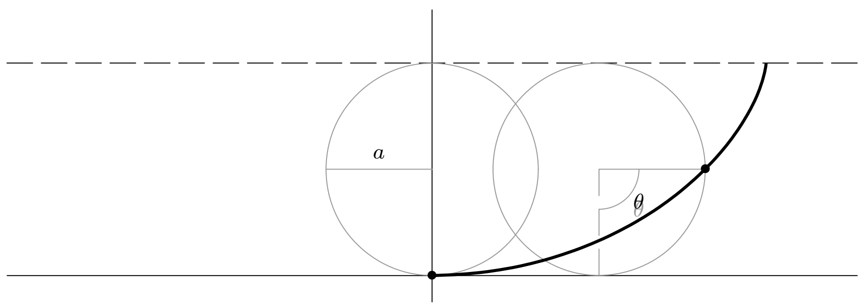


Figure 3.5: The cycloid

The cycloid also has another interesting property: it is the *brachistochrone*, namely the shape of the wire for which the time $\tau(h)$ is minimised. Although the proof is not hard, we will not do it here.

3.3.6 The Gamma and Zeta functions

This section falls outside the main scope of these notes, but since it allows a glimpse at some of the deepest and most beautiful aspects of mathematics, I could not resist the temptation to include it.

It is possible to consider the Laplace transform of complex powers t^z, with z some complex number. We see that

$$\mathcal{L}\left\{t^z\right\}(s) = \int_0^\infty t^z\, e^{-st}\, dt = \frac{1}{s^{z+1}} \int_0^\infty u^z\, e^{-u}\, du \ ,$$

where we have changed the variable of integration from t to $u = st$. Let us introduce the **Euler Gamma function**

$$\boxed{\Gamma(z) \equiv \int_0^\infty t^{z-1}\, e^{-t}\, dt \ ,} \tag{3.62}$$

which converges for $\mathrm{Re}(z) > 0$. Then we have that

$$\mathcal{L}\left\{t^z\right\}(s) = \frac{\Gamma(z+1)}{s^{z+1}} \ .$$

Comparing with equation (3.41), we see that $\Gamma(n+1) = n!$, whence we can think of the Gamma function as a way to define the factorial of a complex number. Although the integral representation (3.62) is only defined for $\mathrm{Re}(z) > 0$ it is possible to extend $\Gamma(z)$ to a holomorphic function with only isolated singularities in the whole complex plane: simple poles at the nonpositive integers.

To see this notice that for $\mathrm{Re}(z) > 0$, we can derive a recursion relation for $\Gamma(z)$ extending the well-known $n! = n\,(n-1)!$ for positive integers n. Consider

$$\Gamma(z+1) = \int_0^\infty t^z\, e^{-t}\, dt \ .$$

Integrating by parts,

$$\Gamma(z+1) = \int_0^\infty z\, t^{z-1}\, e^{-t}\, dt - t^z\, e^{-t}\Big|_0^\infty = z\,\Gamma(z) + \lim_{t\to 0} t^z\, e^{-st} \ .$$

Provided that $\mathrm{Re}(z) > 0$, the boundary term vanishes and we have

$$\Gamma(z+1) = z\,\Gamma(z) \ . \tag{3.63}$$

Turning this equation around, we have that

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \ .$$

Since $\Gamma(1) = 1$, which incidentally justifies the usual claim that $0! = 1$, we see that $\Gamma(z)$ has a simple pole at $z = 0$ with residue 1. Using this recursion relation repeatedly, we see that $\Gamma(z)$ has simple poles at all the nonpositive integers, with residue

$$\operatorname{Res}(\Gamma; -k) = \frac{(-1)^k}{k!} ,$$

and these are all the singularities.

The Gamma function is an extremely important function in mathematics, not least of all because it is intimately related to another illustrious function: the **Riemann Zeta function** $\zeta(z)$, defined for $\operatorname{Re}(z) > 1$ by the converging series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} .$$

To see the relation notice that

$$\int_0^\infty t^{z-1}\, e^{-nt}\, dt = \frac{1}{n^z} \int_0^\infty u^{z-1}\, e^{-u}\, du = \frac{\Gamma(z)}{n^z} ,$$

where we have changed variables of integration from t to $u = nt$. Summing both sides of this identity over all positive integers n, we have, on the one hand

$$\sum_{n=1}^{\infty} \frac{\Gamma(z)}{n^z} = \Gamma(z)\,\zeta(z) ,$$

and on the other

$$\sum_{n=1}^{\infty} \int_0^\infty t^{z-1}\, e^{-nt}\, dt = \int_0^\infty t^{z-1} \sum_{n=1}^{\infty} e^{-nt}\, dt = \int_0^\infty t^{z-1}\, \frac{1}{e^t - 1}\, dt .$$

where we have interchanged the summation inside the integral, and summed the geometric series. (This can be justified, although we will not do so here.) As a result we have the following integral representation for the Zeta function

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt .$$

The only source of singularities in the integral is the zero of $e^t - 1$ at the origin, so we can split the integral into two as follows:

$$\zeta(z) = \frac{1}{\Gamma(z)} \left[\int_0^1 \frac{t^{z-1}}{e^t - 1}\, dt + \int_1^\infty \frac{t^{z-1}}{e^t - 1}\, dt \right] .$$

It is possible to show that $\Gamma(z)$ has no zeros, whence $1/\Gamma(z)$ is entire. Similarly, the second integral $\int_1^\infty$ is also entire since the integrand is continuous there. Hence the singularity structure of the Zeta function is contained in the first integral. We can do a Laurent expansion of the integrand around $t = 0$:

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + O(t^3) \ ,$$

where only odd powers of t appear after the first. Therefore integrating termwise, which we can do because Laurent series converge uniformly, we have that

$$\int_0^1 \frac{t^{z-1}}{e^t - 1} \, dt = \frac{1}{z-1} - \frac{1}{2}\frac{1}{z} + \frac{1}{12}\frac{1}{z+1} + \cdots \ , \tag{3.64}$$

where the terms which have been omitted are all of the form $a_k/(z+k)$ where k is a positive odd integer. This shows that the integral $\int_0^1$ has simple poles at $z = 1$, $z = 0$, and $z = -k$ with k a positive odd integer. Because the integral is multiplied by $1/\Gamma(z)$, and the Gamma function has simple poles at the nonnegative integers we see immediately that

- $\zeta(z)$ has a simple pole at $z = 1$ with residue $\Gamma(1) = 1$, and is analytic everywhere else; and
- $\zeta(-2n) = 0$ where n is any positive integer: these are the zeros of $1/\Gamma(z)$ which are not cancelled by the poles in (3.64).

The celebrated **Riemann hypothesis** states that all other zeros of $\zeta(z)$ occur in the line $\mathrm{Re}(z) = \frac{1}{2}$. Now that Fermat's Last Theorem has been proven, the Riemann hypothesis remains the most important open problem in mathematics today.

The importance of settling this hypothesis stems from the intimate relationship between the Zeta function and the theory of numbers. The key to this relationship is the following infinite product expansion for $\zeta(z)$, valid for $\mathrm{Re}(z) > 1$:

$$\frac{1}{\zeta(z)} = \prod_{\substack{\text{primes} \\ p}} \left(1 - \frac{1}{p^z}\right) \ ,$$

which follows from the unique factorisation of every positive integer into a product of primes. To see this notice that since, for $\mathrm{Re}(z) > 1$, one has

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} + \cdots \ ,$$

then it follows that

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \frac{1}{8^z} + \frac{1}{10^z} + \cdots \ ;$$

whence

$$\left(1-\frac{1}{2^z}\right)\zeta(z) = 1+\frac{1}{3^z}+\frac{1}{5^z}+\frac{1}{7^z}+\frac{1}{9^z}+\cdots\ .$$

In other words we have in the right-hand side only those terms $1/n^z$ where n is odd. Similarly,

$$\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{2^z}\right)\zeta(z) = 1+\frac{1}{5^z}+\frac{1}{7^z}+\frac{1}{11^z}+\frac{1}{13^z}+\cdots\ ,$$

where now we have in the right-hand side only those terms $1/n^z$ where n is not divisible by 2 or by 3. Continuing in this fashion, we have that

$$\prod_{\substack{\text{primes}\\ p}}\left(1-\frac{1}{p^z}\right)\zeta(z) = 1\ .$$

By the way, this shows that $\zeta(z)$ has no zeros for $\mathrm{Re}(z) > 1$.

The Zeta function and its generalisations also play a useful role in physics: particularly in quantum field theory, statistical mechanics, and, of course, in string theory. In fact, together with the heat kernel, introduced in the problems, the (generalised) Zeta function proves invaluable in computing determinants and traces of infinite-dimensional matrices

Areas of spheres

As a minor application of the Gamma function, let us compute the area of a unit sphere in n dimensions, for $n \geq 2$.

What do we mean by a unit sphere in n dimensions? The unit sphere in n dimensions is the set of points in n-dimensional euclidean space which are a unit distance away from the origin. If we let $(x_1, x_2, \ldots, x_n)$ be the coordinates for euclidean space, the unit sphere is the set of points which satisfy the equation

$$\sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \ldots x_n^2 = 1\ .$$

In $n = 2$ dimensions, the unit "sphere" is a circle, whereas in $n = 3$ dimensions it is the usual sphere of everyday experience. For $n > 3$, the sphere is harder to visualise, but one can still work with it via the algebraic description above.

What do we mean by its area? We mean the $n-1$-dimensional area: so if $n = 2$, we mean the circumference of the circle, and if $n = 3$ we mean the usual area of everyday experience. Again it gets harder to visualise for $n > 3$, but one can again tackle the problem algebraically as above.

Clearly every point in n-dimensional space lies on some sphere: if it is a distance r away from the origin then, by definition, it lies on the sphere of radius r. There are an uncountable number of spheres in euclidean space, one for every positive real number. All these spheres taken together with the origin (a "sphere" of zero radius) make up all of euclidean space. A simple scaling argument shows that if we double the radius, we multiply the area of the sphere by 2^{n-1}. More generally, the area of the sphere at radius r will be r^{n-1} times the area of the unit sphere. Therefore the volume element in n-dimensions is

$$d^n x = r^{n-1}\, dr\, d\Omega \ ,$$

where $d\Omega$ is the area element of the unit sphere. We will now integrate the function $\exp(-r^2)$ over all of the euclidean space. We can compute this integral in either of two ways. On the one hand,

$$\begin{aligned}
I &\equiv \int \cdots \int e^{-r^2}\, d^n x \\
&= \int \cdots \int e^{-x_1^2 - x_2^2 - \cdots - x_n^2}\, dx_1 dx_2 \cdots dx_n \\
&= \left(\int_{-\infty}^{\infty} e^{-x_1^2}\, dx_1\right)\left(\int_{-\infty}^{\infty} e^{-x_2^2}\, dx_2\right) \cdots \left(\int_{-\infty}^{\infty} e^{-x_n^2}\, dx_n\right) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^n \ ,
\end{aligned}$$

which is computed to give $(\sqrt{\pi})^n$ after using the elementary gaussian result: $\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$. On the other hand,

$$I = \int \cdots \int e^{-r^2}\, r^{n-1}\, dr\, d\Omega = \left(\int_0^{\infty} e^{-r^2}\, r^{n-1}\, dr\right)\left(\int \cdots \int d\Omega\right) \ .$$

The integral of $d\Omega$ is simply the area A of the unit sphere, which is what we want to calculate. The radial integral can be calculated in terms of the Gamma function after changing the variable of integration from r to $t = r^2$:

$$\int_0^{\infty} e^{-r^2}\, r^{n-1}\, dr = \int_0^{\infty} e^{-t}\, \tfrac{1}{2}\, t^{(n-2)/2}\, dt = \frac{\Gamma(n/2)}{2} \ .$$

Equating both ways to compute the integral, we arrive at the following formula for the area $A(n)$ of the unit sphere in n dimensions:

$$A(n) = \frac{2\, \pi^{n/2}}{\Gamma(n/2)} \ . \tag{3.65}$$

To see that this beautiful formula is not obviously wrong, let us see that it reproduces what we know. For $n = 2$, by the area of the unit sphere we mean the circumference of the unit circle, that is 2π. In $n = 3$, we expect the area of the standard unit sphere and that is 4π. Let us see if our expectations are born out. According to the formula,

$$A(2) = \frac{2\pi}{\Gamma(1)} = 2\pi \ ,$$

as expected. For $n = 3$ the formula says

$$A(3) = \frac{2\,\pi^{3/2}}{\Gamma(3/2)} \ .$$

We can compute the half-integral values of the Gamma function as follows. First we have that

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-1/2}\, e^{-t}\, dt \ .$$

Changing variables to $t = u^2$, we have

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\, du = \int_{-\infty}^\infty e^{-u^2}\, du = \sqrt{\pi} \ .$$

Now using the recursion relation (3.63), we have that

$$\Gamma(k + \tfrac{1}{2}) = \frac{2k-1}{2}\,\frac{2k-3}{2}\cdots\frac{1}{2}\,\Gamma(\tfrac{1}{2}) = \frac{(2k-1)!!}{2^k}\sqrt{\pi} \ .$$

In particular, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, whence

$$A(3) = \frac{2\pi^{3/2}}{\Gamma(3/2)} = 4\pi \ ,$$

as expected.

How about for $n = 1$? This case is a little special: in one dimension the unit sphere consists of the points ± 1. So that it is a zero-dimensional set. Is there an intrinsic notion of area for a zero-dimensional set? If we evaluate the above formula for $A(n)$ at $n = 1$, we get an answer: $A(1) = 2$, which is *counting* the number of points: in other words, zero-dimensional area is simply the **cardinality** of the set: the number of elements. This is something that perhaps we would not have expected. As someone said once, some formulae are more clever than the people who come up with them.

Now that we trust the formula, we can compute a few more values to learn something new. First let us simplify the formula by evaluating the

Gamma function at the appropriate values. Distinguishing between odd and even dimensions, we find

$$A(n) = \begin{cases} \dfrac{2\,\pi^\ell}{(\ell-1)!} \;, & \text{for } n = 2\ell, \text{ and} \\ \dfrac{2^{\ell+1}\,\pi^\ell}{(2\ell-1)!!} \;, & \text{for } n = 2\ell+1. \end{cases} \tag{3.66}$$

The next few values are

$$A(4) = 2\,\pi^2 \;, \quad A(5) = \frac{8\pi^3}{3} \;, \quad A(6) = \pi^3 \;, \quad A(7) = \frac{16\pi^3}{15} \;, \quad A(8) = \frac{\pi^4}{3} \;.$$

In case you are wondering whether this is at all useful, it actually comes in handy when normalising electromagnetic fields in higher-dimensional field theories so that they have integral fluxes around charged objects (e.g., branes and black holes).

Let us end with another nice formula. How about the n-dimensional volume $V(n)$ of the unit ball, i.e., the interior of the unit sphere? We can compute this by integrating the areas of the spheres from radius 0 to radius 1. The area of the sphere of radius r will be r^{n-1} times the area of the sphere of unit radius, so that the volume is then

$$V(n) = \int_0^1 A(n) r^{n-1}\, dr = \frac{A(n)}{n} \;.$$

Using the formula (3.65), we see that

$$V(n) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma((n+2)/2)} \;,$$

where we have used the recursion formula (3.63). Because the the unit sphere is inscribed inside the cube of length 2, the ratio of the volume of the unit ball to that of the cube circumscribing it is given by

$$\varrho(n) = \frac{V(n)}{2^n} = \frac{\pi^{n/2}}{2^n\,\Gamma(\frac{n}{2}+1)} \;.$$

If we were to plot this as a function of n we notice that it starts at 1 for $n = 1$ and then decreases quite fast, so that the ball takes up less and less of the volume of he cube which circumscribes it.