M341 Ordinary Differential Equations
(Autumn 2000/Spring 2001)

Answers to selected problems

Here are answers and some solutions to selected problems.

1.4 These systems decouple:

(a) $x(t) = (k_1 e^t, k_2 e^t, k_3 e^t)$,
(b) $x(t) = (k_1 e^t, k_2 e^{-2t}, k_3)$, and
(c) $x(t) = (k_1 e^t, k_2 e^{-2t}, k_3 e^{2t})$.

1.5 $(a, b, c) = (1, 1, 3)$.

1.7 This is very similar to the case $n = 1$ discussed in the lecture. Since the matrix is diagonal, the equation decouples into n equations: $x_i' = a_i x_i$ for $i = 1, \ldots, n$ where the a_i are the diagonal entries of A. We now apply the result for $n = 1$ derived in the lecture. The unique solution is

$$x_i(t) = x_i(0) e^{a_i t} \ .$$

1.8 From the previous problem, the most general solution of this equation is $x_i(t) = x_i(0) e^{a_i t}$, where the a_i are the diagonal entries of A. A necessary and sufficient condition that all solutions satisfy $\lim_{t\to\infty} x(t) = 0$ is that $\lim_{t\to\infty} e^{a_i t} = 0$ for all i, which in turn force the a_i to be negative (or, if complex, to have negative real parts).

1.10 (b) Any solutions u, v such that $u(0)$ and $v(0)$ are linearly independent.

2.1 A force field F given by $F(x, y) = (F_x, F_y)$ is conservative if and only $F_x = \partial V/\partial x$ and $F_y = \partial V/\partial y$, for some function $V : \mathbb{R}^2 \to \mathbb{R}$. A necessary (and in this case sufficient) condition for this to be the case is the commutativity of the partial derivatives: $\partial F_x/\partial y = \partial F_y/\partial x$. Clearly (a) and (c) satisfy this, but (b) does not.

For (a), we have that $(\partial V/\partial x, \partial V/\partial y) = (x^2, 2y^2)$. Therefore (up to a constant of integration) $V(x, y) = \frac{1}{3}x^3 + \frac{2}{3}y^3$.

Similarly for (c), $(\partial V/\partial x, \partial V/\partial y) = (-x, 0)$, so that (again up to a constant) $V(x, y) = -\frac{1}{2}x^2$.

2.4 (c) is gradient with function $U(x, y) = (x^2 - y^2)/2$. It is also hamiltonian with function $H(x, y) = -xy$.

2.6 The vector field $F(x, y) = (F_x, F_y)$ is gradient if $F_x = \partial U/\partial x$ and $F_y = \partial U/\partial y$ for some function U. A necessary (and sufficient in this case) condition for the existence of U is that $\partial F_x/\partial y = \partial F_y/\partial x$. Applying this to the vector field in the problem, this condition implies $b = c$. The function U is then given (up to a constant) by $U(x, y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}dy^2$.

On the other hand, $F(x, y) = (F_x, F_y)$ is hamiltonian if $F_x = \partial H/\partial y$ and $F_y = -\partial H/\partial x$ for some function H. A necessary (and sufficient) condition for the existence of H is that $\partial F_x/\partial x + \partial F_y/\partial y = 0$. Applying this to the vector field in the problem, this condition implies $a = -d$. The hamiltonian function is (up to a constant) $H(x, y) = \frac{1}{2}by^2 + axy - \frac{1}{2}cx^2$.

Finally, it is both gradient and hamiltonian if both $a = -d$ and $b = c$: $F(x, y) = (ax + by, bx - ay)$.

2.7 Although it was not asked, we first show that any two points on the sphere S_R of radius R in $\mathbb{R}^n$ can be joined by a curve on the sphere. Indeed, let x and y be two points on S_R; i.e., $\|x\| = \|y\| = R$. Let $c : [0, 1] \to \mathbb{R}^n$ be the curve in $\mathbb{R}^n$ defined by $c(t) = ty + (1 - t)x$. This is a straight line joining x and y. Suppose first that $c(t)$ does not pass by the origin. Now, consider the new curve $\gamma : [0, 1] \to \mathbb{R}^n$ given by

$$\gamma(t) = \frac{Rc(t)}{\|c(t)\|} \ .$$

Clearly $\|\gamma(t)\| = R$ for all t, hence it is a curve on the sphere, and moreover $\gamma(0) = c(0) = x$ and $\gamma(1) = c(1) = y$. If $c(t)$ passes by the origin, just deform it a little so that it does not (it will stop being a straight line, but that's OK) and define γ in the same way. Alternatively, pick a third point z on the sphere and join x to z and z to y by the above method. (The curve $\gamma(t)$ can be interpreted geometrically as the projection of the curve $c(t)$ onto the sphere. Indeed, imagine a source of light at the origin and a spherical screen at radius R. Then $\gamma(t)$ is the projection onto that screen of the straight line $c(t)$. This makes sense provided that $c(t)$ does not pass through the origin.)

To show that V is constant on the sphere S_R we will show that it takes the same value on any two points. Let x, y be any two points on the sphere and let $\gamma : [0, 1] \to \mathbb{R}^n$ be any curve on the sphere S_R joining them. Since γ lies on the sphere, $\|\gamma(t)\| = R$ for all t. We will show that the potential V is constant along γ, whence it takes the same value at x and at y. Since x and y are arbitrary, we are done.

Let $V \circ \gamma : [0, 1] \to \mathbb{R}$ be the composition $t \mapsto V(\gamma(t))$. Differentiating with respect to t we find

$$\begin{aligned}
\frac{dV(\gamma(t))}{dt} &= \langle \operatorname{grad} V(\gamma(t)), \gamma'(t) \rangle && \text{(by the chain rule)} \\
&= -\langle F(\gamma(t)), \gamma'(t) \rangle && \text{(since } F = -\operatorname{grad} V\text{)} \\
&= -h(\gamma(t)) \langle \gamma(t), \gamma'(t) \rangle && \text{(since } F \text{ is central)} \\
&= -\tfrac{1}{2} h(\gamma(t)) \frac{d}{dt} \|\gamma(t)\|^2 && \text{(by the Leibniz rule)} \\
&= 0 \ , && \text{(since } \|\gamma(t)\|^2 = R^2 \text{ for all } t.\text{)}
\end{aligned}$$

2.8 The hamiltonian functions are as follows:

(a) $H(x, y) = \frac{1}{2}y^2 + (x - 1)^2$,
(b) $H(x, y) = x^2 - xy + y^2$,
(c) $H(x, y) = x^2 - 3xy + y^2$, and
(d) $H(x, y) = \frac{1}{2}(x - y)^2$.

2.9 The gradient functions are the negative of the hamiltonian functions in the previous exercise.

3.1 (a) $(x(t), y(t)) = (0, 3e^{2t})$.

(b) First of all, because the initial conditions are given at $t = 1$, it is convenient to define $x(t) = x_1(t+1)$ and $y(t) = x_2(t+1)$. In this way the initial value problem becomes

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

The characteristic polynomial of the matrix A defining the vector field is $p_A(\lambda) = \lambda^2 - 3\lambda + 1$, whose roots are $\lambda_\pm = \frac{1}{2}(3 \pm \sqrt{5})$. The eigenvectors corresponding to these eigenvalues are found to be, respectively, $v_\pm = (\frac{1}{2}(1 \pm \sqrt{5}), 1)$. Let S^{-1} denote the matrix whose columns are v_- and v_+ respectively. One has

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & \frac{1}{2}(\sqrt{5}+1) \\ 1 & \frac{1}{2}(\sqrt{5}-1) \end{pmatrix} \quad \text{and} \quad S^{-1} = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) & \frac{1}{2}(1+\sqrt{5}) \\ 1 & 1 \end{pmatrix} .$$

As discussed in Lecture, $(x(t), y(t))$ is found by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = S^{-1} \begin{pmatrix} e^{\lambda_- t} & 0 \\ 0 & e^{\lambda_+ t} \end{pmatrix} S \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

which after a little bit of simplification becomes

$$\begin{aligned} x(t) &= e^{3t/2} \left(\cosh \tfrac{\sqrt{5}}{2} t + \tfrac{3}{\sqrt{5}} \sinh \tfrac{\sqrt{5}}{2} t \right) \\ y(t) &= e^{3t/2} \left(\cosh \tfrac{\sqrt{5}}{2} t + \tfrac{1}{\sqrt{5}} \sinh \tfrac{\sqrt{5}}{2} t \right) . \end{aligned}$$

Finally we solve for x_1 and x_2 by shifting t to $t-1$.

3.2 $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$.

3.4 The eigenvalues should be positive.

3.6 (b) $b > 0$ and therefore $c < b^2/4$.

3.7 (a,c) The general solution of this system will be a linear combination

$$x(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\mu t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

If $c_1 = c_2 = 0$, the solution curve is constant: $x(t) = 0$ for any choice of λ and μ. From now on assume that this is not the case.

If $0 < \lambda < \mu$, $\lim_{t\to\infty} \|x(t)\|$ is unbounded. If $c_2 \neq 0$, the solution curves asymptotically become straight lines with direction $\pm(1,1)$, whereas if $c_2 = 0$ the solutions are straight lines with direction $\pm(1,0)$ for all t.

If $\lambda < \mu < 0$, then $\lim_{t\to\infty} \|x(t)\| = 0$. As $t \to \infty$, the solution curves approach the origin along the $\pm(1,1)$ directions provided that $c_2 \neq 0$. If $c_2 = 0$, then the solution curves are straight lines coming from the $\pm(1,0)$ directions.

The phase portraits can be gleaned from the plots of the vector fields which are appended to these solutions.

3.8 By complex conjugating the characteristic polynomial, prove that if λ is a root, then so is its complex conjugate $\bar{\lambda}$.

3.9 (d) The characteristic polynomial of A is $p_A(\lambda) = \lambda^2 - 2\lambda + 5$, whence the roots are $\lambda_\pm = 1 \pm 2i$, where $i = \sqrt{-1}$, with eigenvectors $(\pm i, 1)$. Forming (the inverse of) the diagonalising matrix S^{-1} and inverting to find S, we obtain

$$S^{-1} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \implies S = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} .$$

The solution of the initial value problem is therefore

$$x(t) = S^{-1} \begin{pmatrix} e^t e^{i2t} & 0 \\ 0 & e^t e^{-i2t} \end{pmatrix} S \begin{pmatrix} 3 \\ -9 \end{pmatrix} = 3e^t \begin{pmatrix} \cos 2t + 3\sin 2t \\ \sin 2t - 3\cos 2t \end{pmatrix} .$$

The phase diagram consists of counterclockwise spirals going away from the origin, as can be gleaned by the sketch of the vector field shown in the appended figure.

3.14 (a) Introduce the new basis $(1, 0, 0)$, $(0, -\sqrt{2}, \sqrt{2})$ and $(1, -2, 1)$, and new coordinates (y_1, y_2, y_3) related to the old coordinates (x_1, x_2, x_3) by

$$x_1 = y_1 + y_3 \qquad x_2 = -\sqrt{2}y_2 - 2y_3 \qquad x_3 = \sqrt{2}y_2 - y_3 .$$

In the new coordinates the differential equation becomes

$$y_1' = y_1 \qquad y_2' = -\sqrt{2}y_3 \qquad y_3' = \sqrt{2}y_2 ,$$

whose general solution is

$$\begin{aligned} y_1 &= c_3 e^t \\ y_2 &= c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) \\ y_3 &= -c_2 \cos(\sqrt{2}t) + c_1 \sin(\sqrt{2}t) , \end{aligned}$$

where (c_1, c_2, c_3) are three arbitrary constants. In terms of the original coordinates,

$$\begin{aligned} x_1 &= c_3 e^t - c_2 \cos(\sqrt{2}t) + c_1 \sin(\sqrt{2}t) \\ x_2 &= (2c_2 - c_1\sqrt{2}) \cos(\sqrt{2}t) - (c_2\sqrt{2} + 2c_1) \sin(\sqrt{2}t) \\ x_3 &= (c_2 + c_1\sqrt{2}) \cos(\sqrt{2}t) + (c_2\sqrt{2} - c_1) \sin(\sqrt{2}t) . \end{aligned}$$

3.15 Let $a_{ij}(t)$ and $b_{ij}(t)$ be the (ij) entries of $A(t)$ and $B(t)$. The (ij) entry of the matrix product is given by

$$\sum_{k=1}^{n} a_{ik}(t) b_{kj}(t) .$$

Since the derivative acts on a matrix entry-wise, the result follows from linearity of the derivative and the product rule on functions:

$$\left(\sum_{k=1}^{n} a_{ik}(t) b_{kj}(t) \right)' = \sum_{k=1}^{n} \left(a_{ik}'(t) b_{kj}(t) + a_{ik}(t) b_{kj}'(t) \right) .$$

The second part just follows by taking the derivative of the identity: $AA^{-1} = I$, so that

$$A'A^{-1} + A\left(A^{-1}\right)' = 0 \ ,$$

which implies the result we wanted to show.

3.22 (a)

$$\begin{aligned} x(t) &= (c_2 - tc_1)e^{2t} \\ y(t) &= c_1 e^{2t} \end{aligned}$$

(b)

$$\begin{aligned} x(t) &= e^{2t}(c_1 \cos t - c_2 \sin t) \\ y(t) &= e^{2t}(c_2 \cos t + c_1 \sin t) \end{aligned}$$

3.23 (a)

$$\begin{aligned} x(t) &= (2t+1)e^{2t} \\ y(t) &= -2e^{2t} \end{aligned}$$

(b)

$$\begin{aligned} x(t) &= 2e^{2t} \sin t \\ y(t) &= -2e^{2t} \cos t \end{aligned}$$

3.25 Consider the restriction of A to the eigenspace with eigenvalue λ and use Problem 3.24.

3.29 (a) sink; (b) source; (c) source; (d) none of these; (e) none of these.

3.30 (a) Only if $a < -2$ are there any values of such k and in this case for $k > \sqrt{-2a}$.

(b) No values of k.

3.33 Such a T has a real eigenvalue. Now study T on this eigenspace.

3.35 (a) $x(t) = \frac{1}{17}(-4\cos t + \sin t) - \frac{4}{17}e^{4t} + e^{4t}c$.

(b) $x(t) = -\frac{1}{16}(4t+1) + \frac{1}{16}e^{4t} + e^{4t}c$.

(c) $x(t) = c_1 \cos t + c_2 \sin t$ and $y(t) = -c_1 \sin t + c_2 \cos t + 2t$.

3.40 The above second-order homogeneous equation is equivalent to the first-order system $x' = Ax$ where

$$x = (s, s') \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \ .$$

The characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 + a\lambda + b \ ,$$

whose roots are given by

$$\frac{-a \pm \sqrt{a^2 - 4b}}{2} \ .$$

Introduce the discriminant $\Delta = a^2 - 4b$. We must distinguish several cases:

$\Delta > 0$. There are two distinct real roots $\lambda < \mu$, say. The general solution takes the form

$$s(t) = c_1 e^{\lambda t} + c_2 e^{\mu t} = e^{\lambda t}\left(c_1 + c_2 e^{(\mu-\lambda)t}\right) \ .$$

Clearly the equation $s(t) = 0$ has at most one solution.

$\Delta = 0$. In this case there is one real root, $\lambda = -a/2$ which is repeated. Because A is never diagonal, the system describes an improper node, whence the general solution takes the form

$$s(t) = e^{\lambda t}\left(c_1 + c_2 t\right) \ .$$

The equation $s(t) = 0$ has again at most one solution.

$\Delta < 0$. In this case there are two distinct complex conjugate eigenvalues: $\alpha \pm i\beta$, say, with $\beta \neq 0$. The general solution is given by

$$s(t) = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right) \ .$$

In this case the equation $s(t) = 0$ has always an infinite number of solutions, due to the periodicity of the trigonometric functions.

Therefore the answers are:

(a) $a^2 \geq 4b$

(b) $a^2 \geq 4b$

(c) $a^2 < 4b$

3.42 $a = 0$ and $b > 0$ and the period is $\sqrt{b}/2\pi$.

4.1 (a) $f(x) = x + 2$ and $x_0(t) = 2$. Then

$$x_1(t) = 2 + \int_0^t f(x_0(s))ds = 2 + \int_0^t 4ds = 2 + 4t \ .$$

Iterating,

$$x_2(t) = 2 + \int_0^t (4 + 4s)ds = 2 + 4t + 2t^2 \ .$$

Once again,

$$x_3(t) = 2 + \int_0^t (4 + 4s + 2s^2)ds = 2 + 4t + 2t^2 + \tfrac{2}{3}t^3 \ ;$$

whence by induction

$$x_n(t) = 4\left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}\right) - 2 \ .$$

Hence

$$x(t) = \lim_{n\to\infty} x_n(t) = 4e^{2t} - 2 \ .$$

(b) Here $x_0(t) = 0$, $x_1(t) = \int_0^t 0 ds = 0$ and, in fact, $x_n(t) = 0$ for all n, whence in the limit $x(t) = 0$.

(c) $x(t) = t^{-3}$.

(e) Integrate the equation

$$x'(s) = 1/(2x(s))$$

from $s = 1$ to $s = t$ to obtain

$$x(t) = 1 + \int_1^t \frac{ds}{2x(s)} \ .$$

To set up the Picard iteration scheme, let $x_0(t) = 1$ be the constant function. Then

$$x_1(t) = 1 + \int_1^t \frac{ds}{2} = \frac{1+t}{2} \ .$$

Similarly,

$$x_2(t) = 1 + \int_1^t \frac{ds}{1+s} = 1 + \log\left((1+t)/2\right) \ .$$

The next term is already not an elementary function.

This equation is separable and the solution is $x(t) = \sqrt{t}$, valid for $t \geq 0$.

4.2 If $\det A = 0$, then there there is a nonzero vector v such that $Av = 0$. Every point in the line containing v is a critical point. Clearly they are not isolated.

4.8 Each of these systems is linear and has an isolated critical point, so the classification that we arrived at in lecture applies. Recall that if we write the system in matrix form:

$$x' = A\,x \ ,$$

then we can identify the type of critical point from the values of the trace τ and the determinant Δ of the matrix A.

(a) In this case $\tau = 5$, $\Delta = 6$. In this case $\tau^2 > 4\Delta$, whence we are below the critical parabola. This means we have an *unstable node*.

(b) We have $\tau = 4$, $\Delta = 3$, so that again $\tau^2 > 4\Delta$. Again we have an *unstable node*.

(c) Now $\tau = 0$, $\Delta = -1$. Again $\tau^2 > 4\Delta$, so we are below the critical parabola. This means we have an *unstable saddle point*.

(d) In this case $\tau = 0$, $\Delta = 9$. In this case $\tau^2 < 4\Delta$, whence we are above the critical parabola. This means we have a *stable centre*.

(e) In this case $\tau = -6$, $\Delta = 9$. We are on the critical parabola, since $\tau^2 = 4\Delta$. Because the matrix is not proportional to the identity matrix, we have a *stable improper node*.

(f) Finally, we find $\tau = 6$, $\Delta = 18$, and $\tau^2 < 4\Delta$, whence we are above the critical parabola. We have an *unstable spiral*.

4.9 We write the equation in matrix form

$$x' = A\,x + B\ , \quad \text{with} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e \\ f \end{pmatrix}\ .$$

(a) The condition $ad \neq bc$ says that A is invertible; hence there is a unique critical point x_0 given by

$$x_0 = -A^{-1}\,B\ .$$

(b) Introducing $\bar{x} := x - x_0$, we can write the above system as

$$\bar{x} = A\bar{x}\ .$$

Notice that the same matrix A appears as in the original system; hence $\bar{a} = a$, $\bar{b} = b$, etc. Since A is invertible, the new system has an isolated critical point at the origin.

(c) This system has

$$A = \begin{pmatrix} 2 & -2 \\ 11 & -8 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 10 \\ 49 \end{pmatrix}\ .$$

We check that $\det A = 6$, whence there is an isolated critical point at

$$x_0 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}\ .$$

The trace τ and determinant Δ of A are given by $\tau = -6$ and $\Delta = 6$. This means that $\tau^2 > 4\Delta$, whence it lies below the critical parabola. This means that the critical point is a *stable node*.

4.10 **(a)** If A is symmetric and positive-definite, it is diagonalisable and both eigenvalues are positive. This means that all equilibria are sources, hence unstable. We can have a *focus* or a *node*.

(b) As before but now both eigenvalues are negative. The equilibria are sinks, hence asymptotically stable: we can have a *focus* or a *node*.

(c) If A is skew-symmetric, and assuming that it is not identically zero, there is only one possibility: a neutrally stable *centre*.

(d) If $A = B+C$ where B is symmetric negative-definite, and C is skew-symmetric (and assumed nonzero), we see that $\operatorname{tr} A = \operatorname{tr} B < 0$, since C is traceless. At the same time, $\det A = \det B + \det C > 0$ since both B and C have positive determinant. As a result all critical points are *stable*: one can have a *spiral* or a *node*.

4.12 The two systems can be written as follows:

$$\begin{cases} x' = y + \varepsilon x(x^2 + y^2) \\ y' = -x + \varepsilon y(x^2 + y^2) \end{cases},$$

where $\varepsilon = 1$ for the first system and -1 for the second system. The critical points are given by the solutions to the system of equations:

$$\begin{cases} y + \varepsilon x(x^2 + y^2) = 0 \\ -x + \varepsilon y(x^2 + y^2) = 0 \end{cases}.$$

Multiplying the first equation by y and the second by x and subtracting one equation from the other, we obtain $x^2 + y^2 = 0$, whence the only possible critical point is the origin. We notice that the origin is indeed a critical point. Both systems linearise to the same system:

$$\begin{cases} x' = y \\ y' = -x \end{cases} ,$$

which corresponds to a neutrally stable centre. We now solve the nonlinear system by introducing polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$. The resulting equations decouple:

$$r' = \varepsilon r^3 \quad \text{and} \quad \theta' = -1 ,$$

and are solved by

$$\theta(t) = \theta(0) - t \quad \text{and} \quad r(t) = \frac{r(0)}{\sqrt{1 - 2\varepsilon r(0)^2 t}} .$$

Clearly, if $\varepsilon = 1$, then $r(t)$ increases, whence the critical point is unstable. On the other hand, if $\varepsilon = -1$, $r(t)$ decreases and the critical point is stable. Notice that in the unstable case we cannot define the trajectory for arbitrarily large t; indeed, $0 \leq t < \frac{1}{2}r(0)^{-2}$. In other words, the vector field is incomplete, since it reaches the limit point in finite time.

4.14 **(a)** The equation is clearly equivalent to

$$\begin{cases} x' = y \\ y' = -\omega^2 x - 2\mu y \end{cases} .$$

Write it in matrix form

$$x' = A\,x \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\mu \end{pmatrix} .$$

The trace τ of the matrix is -2μ, whereas the determinant Δ is $\omega^2 > 0$. Hence A is invertible and therefore there is a unique critical point at the origin.

(b) The nature of the critical point depends on μ and ω. There are four cases to consider:

(i) Here $\mu = 0$. Hence $\tau = 0$ and $\Delta = \omega^2$. Here $\tau^2 < 4\Delta$, whence we are above the critical parabola and we have a *stable centre*.

(i) Here $0 < \mu < \omega$. Therefore $\tau - -2\mu < 0$ and $\Delta = \omega^2 > 0$. Also $\tau^2 = 4\mu^2 < 4\Delta$, so we are above the critical parabola and we have a *stable spiral*.

(iii) Here $\tau^2 = 4\Delta$, so we are on the critical parabola, while again having $\tau < 0$ and $\delta > 0$. The matrix is clearly not proportional to the identity, so that what we have is a *stable improper node*.

(iv) Now $\mu > \omega$, so that we are below the critical parabola but still within the zone of stability: so what we have is a *stable node*.

4.17 The system

$$\begin{cases} x' = x + y - x(x^2 + y^2) \\ y' = -x + y - y(x^2 + y^2) \end{cases}$$

has a unique critical point at the origin. Linearising the system there we see that in matrix form it is given by

$$x' = A\,x \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

The trace $\tau = 2$ and the determinant $\Delta = 2$, whence the critical point is an *unstable node*. Nevertheless the nonlinear system has a limit cycle. To see this notice that multiplying the first equation by x and the second by y, we obtain that

$$\dot{E} = 2E(1 - E) \ , \quad \text{where} \quad E = x^2 + y^2 \ .$$

This shows that $\dot{E} = 0$ when $E = 0$ and $E = 1$. The former case corresponds to the critical point, whereas the latter corresponds to the limit cycle. Notice that when $0 < E < 1$, $\dot{E} > 0$ so that solutions which start within the unit disk evolve outwards towards the boundary, whereas when $E > 1$, $\dot{E} < 0$ so that solutions which start outside the unit disk evolve inwards towards the same boundary. We can solve the system in polar coordinates. Noticing that $E = r^2$, we have that $\dot{r} = r(1 - r^2)$ and similarly one obtains that $\dot{\theta} = -1$. These equations can be integrated to yield:

$$\theta(t) = \theta(0) - t \quad \text{and} \quad r(t) = \frac{r(0)}{\sqrt{r(0)^2 - (r(0)^2 - 1)e^{-2t}}} \ .$$

Hence we see that if $r(0) < 1$ the solutions spiral out towards $r = 1$, whereas if $r(0) > 1$ they spiral in towards $r = 1$.

4.18 Multiply the first equation by x and the second by y and add the two equations, to obtain

$$r' = r\left(3 - e^{r^2}\right) \ , \quad \text{where} \quad r^2 = x^2 + y^2 \ .$$

This says that $r' = 0$ when $r = \sqrt{\log 3}$, so that this is where expect the periodic solution. (There is a trivial periodic solution at the origin, which is the only critical point of the system; but this is not what the problem asks!) Multiplying the first equation by y and the second by x and subtracting the two equations results in

$$\theta' = 1 \implies \theta(t) = \theta(0) + t \ .$$

Therefore the unique solution with initial conditions $(r(0) = \sqrt{\log 3}, \theta(0))$ will have constant $r(t) \equiv \sqrt{\log 3}$ and will have period 2π.

4.19 This is a conservative system. Multiplying the equation by x' we notice that the result is simply $\dot{E}$ where the energy E is given by

$$E := \tfrac{1}{2}(x')^2 + \tfrac{1}{2}x^2 - \tfrac{1}{3}\lambda x^3 - x \ ,$$

from where we read off the potential function:

$$U := \tfrac{1}{2}x^2 - \tfrac{1}{3}\lambda x^3 - x \ .$$

Critical points occur on the x-axis, at the stationary points of the potential:

$$\left.\frac{dU}{dx}\right|_{x_0} = 0 \implies x_0 = \frac{1 \pm \sqrt{1-4\lambda}}{2\lambda} \ .$$

There are two regimes of interest: $0 < \lambda < \frac{1}{4}$ and $\lambda > \frac{1}{4}$. In the latter regime there are no (real) critical points, whereas in the former there are two critical points. There are no closed orbits in the case of no critical points. For $0 < \lambda < \frac{1}{4}$, the potential function is a minimum at $x_- := \frac{1-\sqrt{1-4\lambda}}{2\lambda}$ and a maximum at $x_+ := \frac{1+\sqrt{1-4\lambda}}{2\lambda}$. This means that x_- is a stable centre and x_+ is an unstable saddle point. There will be closed trajectories around the centre.

4.20 **(a)** If x is a solution to the system $x' = A\,x$ then taking the transpose of this equation we see that x^T is a solution of $(x^T)' = x^T\,A^T$. Therefore,

$$\begin{aligned}\dot{E} = (x^T x)' = (x^T)'x + x^T x' = x^T A^T x + x^T A x \\ = x^T(A^T + A)x \ ,\end{aligned}$$

which vanishes if A is skew-symmetric.

(b) Similarly, if now $E = x^T M x$, then

$$\begin{aligned}\dot{E} = (x^T M x)' = (x^T)' M x + x^T M x' \\ = x^T A^T M x + x^T M A x = x^T(A^T M + M A)x \ ,\end{aligned}$$

which, if $A^T M + MA$ is negative-definite, implies that $\dot{E} < 0$. If in addition M is positive-definite, then $E > 0$ everywhere except at the origin $x = 0$. This means that E is a Liapunov function and (at least when $n = 2$) we have seen that the origin is then an asymptotically stable critical point.

(c) This part of the problem is a simple application of the fact that if a matrix M is positive or negative-definite (or semi-definite), then so is $S^T M S$ for any invertible matrix S. This is very easy to prove. To be concrete, let us do the positive-definite case. By definition, M is positive-definite if and only $x^T M x > 0$ for all nonzero vectors x, hence in particular for vectors x of the form $x = Sy$, where S in invertible. (Every x is of this form: simply take $y = S^{-1}x$.) Therefore, $y^T S^T M S y > 0$ for all y, whence $S^T M S$ is positive-definite.

Let us now apply this fact to solve this part of the problem. By hypothesis there exists a positive-definite matrix M such that $MA + A^T M$ is negative (semi-)definite. Substituting $A = S^{-1}NS$ into the previous expression we deduce that

$$MS^{-1}NS + S^T N^T (S^{-1})^T M$$

is negative (semi-)definite. Since S is invertible, we can rewrite the above expression as follows

$$S^T \left((S^{-1})^T M S^{-1} N + N^T (S^{-1})^T M S^{-1} \right) S \ .$$

Now, by the remarks above negative (semi-)definiteness of the above matrix is equivalent to the matrix

$$(S^{-1})^T M S^{-1} N + N^T (S^{-1})^T M S^{-1}$$

being negative (semi-)definite, and by the same token that the matrix $P := (S^{-1})^T M S^{-1}$ is positive-definite, since so is M.

(d) The above result simply states that the existence of a Liapunov function is independent of the coordinates used to write it down. In other words, to analyse the Liapunov stability of the critical points of a linear system, we are free to make a linear change of variables to take the matrix A defining the system to one of the normal forms we classified in class. We now discuss each normal form N in turn and find a positive-definite matrix P such that $PN + N^T P$ is negative (semi-)definite. The corresponding Liapunov function is then $E = x^T P x$.

(i) Consider first the case of a diagonalisable matrix with real eigenvalues:

$$N = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{with } \lambda_2 \leq \lambda_1 < 0.$$

This matrix is already symmetric and negative-definite. So that we can take $P = I$, the identity matrix.

(ii) Consider now the case of a non-diagonalisable matrix with complex eigenvalues:

$$N = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad \text{with } \alpha \leq 0.$$

Notice that $N + N^T$ is negative semi-definite, whence we can again take $P = I$.

(iii) Finally we consider the case of a non-diagonalisable matrix with one real eigenvalue with multiplicity 2:

$$N = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \qquad \text{with } \lambda < 0.$$

In this case, we can try the following symmetric matrix

$$P = \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix} \implies PN + N^T P = \begin{pmatrix} 2\mu^2\lambda & 1 \\ 1 & 2\lambda \end{pmatrix} \ .$$

This matrix is negative-definite provided that its determinant is positive, which requires $2\mu\lambda > 1$.

4.21 Consider the second order equation

$$x'' + \kappa x' + x = 0 \ .$$

(a) This equation is equivalent to the following system:

$$\begin{cases} x' = y \\ y' = -x - \kappa y \end{cases} ,$$

which is a linear system with a unique critical point at the origin.

(b) Consider $E(x,y) = x^2 + y^2$. Its derivative along trajectories is given by

$$\dot{E}(x,y) = \frac{\partial E}{\partial x} y + \frac{\partial E}{\partial y}(-x - \kappa y) = 2xy - 2xy - 2\kappa y^2 \leq 0 ,$$

hence the origin is stable.

(c) We can analyse the stability properties of the origin directly since the system is linear and has an isolated critical point at the origin. The matrix defining the system is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\kappa \end{pmatrix} ,$$

which has trace $\tau = -\kappa$ and determinant $\Delta = 1$. We can therefore recognise four distinct ranges of values of $\kappa \geq 0$:

$\kappa = 0$ We have $(\tau, \Delta) = (0,1)$, so the origin is a centre.

$0 < \kappa < 2$ We now have $(\tau, \Delta) = (-\kappa, 1)$ and $\kappa^2 < 4$, whence the origin is a stable spiral.

$\kappa = 2$ Here $\tau^2 = 4\Delta$, and we have a stable improper node on the critical parabola.

$\kappa > 2$ Finally we have $\tau^2 > 4\Delta$, whence we are below the critical parabola but still in the stable region. Therefore the origin is a stable node.

4.22 **(a)** Consider the system

$$\begin{cases} x' = -3x^3 - y \\ y' = x^5 - 2y^3 \end{cases}$$

Some experimentation leads us to consider the function $E(x,y) = \frac{1}{6}x^6 + \frac{1}{2}y^2$, which vanishes at the origin, is positive-definite everywhere else and whose derivative along a trajectory is given by

$$\dot{E}(x,y) = -3x^8 - 2y^4 ,$$

which is negative-definite away from the origin. Hence the origin is an asymptotically stable critical point.

(b) Some experimentation leads us to the function $E(x,y) = x^2 + y^2$, which vanishes at the origin, is positive-definite everywhere else, and decreases along trajectories:

$$\dot{E}(x,y) = -4x^2 - 2y^4 .$$

Hence by Liapunov stability, the origin is asymptotically stable.

(c) We now have the system

$$\begin{cases} x' = y^2 + xy^2 - x^3 \\ y' = -xy + x^2y - y^3 \end{cases}$$

Again a little experimentation suggests that we try $E(x,y) = x^2+y^2$, which is again decreasing:

$$\dot{E}(x,y) = -2(x^2+y^2)^2 \ .$$

(d) Finally we have the system

$$\begin{cases} x' = x^3y + x^2y^3 - x^5 \\ y' = -2x^4 - 6x^3y^2 - 2y^5 \end{cases}$$

After a little thought we are persuaded to consider the function $E(x,y) = x^2 + \frac{1}{2}y^2$, which is clearly a Liapunov function and, in fact, decreasing:

$$\dot{E}(x,y) = -2(x^3+y^3)^2 \ .$$

Hence the origin is asymptotically stable.

4.23 The van der Pol equation is equivalent to the following system

$$\begin{cases} x' = y \\ y' = -x + \mu y(1-x^2) \end{cases} ,$$

which clearly has an isolated critical point at the origin. The system is almost linear, so we can try to use linearisation. The linear system is given by

$$x' = Ax \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} ,$$

which has determinant 1 and trace μ. Therefore the linear system will be unstable for $\mu > 0$, asymptotically stable for $\mu < 0$ and neutrally stable for $\mu = 0$. Using Poincaré's theorem, we therefore know that the almost linear system will be asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$. For $\mu = 0$, the theorem does not tell us anything, but in this case, the nonlinear system reduces to the linear one, and therefore it is neutrally stable.

Alternatively one can use Liapunov (in)stability. Consider the function $E(x,y) = x^2 + y^2$. This function vanishes at the origin and is positive definite everywhere else on the phase plane. Computing its variation along a solution curve, we find

$$\dot{E}(x,y) = 2xy + 2y\left(-x + \mu y(1-x^2)\right) = 2\mu y^2(1-x^2) \ .$$

Therefore in the punctured unit disk ($0 < x^2 + y^2 < 1$), $\dot{E}$ is positive-definite if $\mu > 0$ and negative-definite if $\mu < 0$. Using the Liapunov stability and instability theorems we deduce that the origin is unstable for $\mu > 0$ and asymptotically stable for $\mu < 0$. For $\mu = 0$, $\dot{E} = 0$ and the system is stable. Since it reduces to a linear system, we see that it is indeed neutrally stable.

4.24 Consider the system

$$\begin{cases} x' = -y + x\,f(x,y) \\ y' = x + y\,f(x,y) \end{cases} ,$$

where f is continuous and continuously differentiable on some disk D centred at the origin.

(a) The critical points are the zeros of the vector field characterising the system:

$$y = xf(x,y) \quad \text{and} \quad yf(x,y) = -x \ .$$

Plugging the first equation into the second, we find that $x(1+f(x,y)^2) = 0$, which implies that $x = 0$ and hence that $y = 0$, by the first equation. In other words, the origin is the unique critical point of this system.

(b) Consider the function $E(x,y) = x^2+y^2$. It vanishes at the origin and is positive-definite everywhere else. Its derivative along a trajectory is given by

$$\dot{E}(x,y) = 2(x^2+y^2)f(x,y) \ .$$

Hence if $f(x,y) < 0$ in the punctured disk $D\backslash\{(0,0)\}$, then E' is decreasing and by Liapunov stability, the origin is asymptotically stable.

(Notice that it follows from Liapunov instability that if $f(x,y) > 0$ in the punctured disk, then the origin would be unstable.)

4.25 For each of these systems, we can read off the equation satisfied by the radial coordinate r, from the relation $rr' = xx' + yy'$.

(b) The radial equation in this case is

$$r' = r\sin(1/r) \ .$$

This says that there is a critical point at $r = 0$ and an infinite number of limit cycles at $r = R_n := 1/(n\pi)$. For $R_{2n} > r > R_{2n+1}$ we see that $r' > 0$, whereas for $R_{2n-1} > r > R_{2n}$, $r' < 0$. Hence the critical point is unstable, the limit cycles at $r = R_{2n}$ are stable and the ones at $r = R_{2n+1}$ are unstable.

(c) The radial equation is now

$$r' = r(r^3 - r) = r^2(r^2-1) \ .$$

This says that there is a critical point at $r = 0$ and a limit cycle at $r = 1$. For $r < 1$ we have $r' < 0$, whereas for $r > 1$ we have $r' > 0$. In other words, the critical point is asymptotically stable and the limit cycle at $r = 1$ is unstable.

(d) Finally, the radial equation is $r' = \sin r$, which says that there is a critical point at $r = 0$ and limit cycles at $r = R_n := n\pi$. For $R_{2n-1} < r < R_{2n}$, $r' < 0$; whereas for $R_{2n} < r < R_{2n+1}$, $r' > 0$. In other words, the critical point is unstable, the limit cycles at $r = R_{2n}$ are unstable, whereas the ones at $r = R_{2n-1}$ are stable.

4.26 These equations can be rewritten as a first order equation

$$\begin{cases} x' = u(x, y) \\ y' = v(x, y) \end{cases} .$$

(a) This equation is equivalent to the system

$$\begin{cases} x' = y \\ y' = -y - y^5 + 3x^3 \end{cases} ,$$

which clearly has a unique critical point at the origin. We notice that the quantity

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 + (-1 - 5y^4) < 0 ,$$

whence by the Bendixson negative criterion, there are no limit cycles anywhere in the phase plane.

(b) This equation is equivalent to the equation

$$\begin{cases} x' = y \\ y' = y(x^2 + 1) - x^5 \end{cases} ,$$

which has a unique critical point at the origin. The Bendixson negative criterion says that there are no limit cycles, since the divergence of the vector field

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + (x^2 + 1) > 0$$

is always positive.

(c) The equivalent equation is now given by

$$\begin{cases} x' = y \\ y' = 1 + y^2 + x^2 \end{cases} .$$

This system has no critical points, hence it cannot have limit cycles either.

4.27 **(a)** Consider the function $E(x, y) = 2x^2 + y^2$. It is clearly positive everywhere but at the origin. Its derivative along trajectories is given by

$$\begin{aligned} \dot{E}(x, y) &= 4x(2x - y - 2x^3 - 3xy^2) + 2y(2x + 4y - 4y^3 - 2x^2 y) \\ &= 8(x^2 + y^2)\left(1 - (x^2 + y^2)\right) . \end{aligned}$$

Therefore $\dot{E}(x, y) > 0$ when $x^2 + y^2 < 1$ and $\dot{E}(x, y) < 0$ when $x^2 + y^2 > 1$. Consider therefore the region

$$R = \left\{(x, y) \mid \tfrac{1}{2} \leq E(x, y) \leq 2\right\} .$$

It is clearly closed and bounded and contains the unit circle $x^2+y^2 = 1$. The unit circle separates R into two regions with the circle as a common boundary component to both. If the trajectory starts in the region inside the unit circle, then $\dot{E} > 0$ whereas if it starts in the region outside the unit circle, $\dot{E} < 0$. Hence the trajectory cannot leave the region R. If we prove that the region R has no equilibria, then it would follow from the Poincaré–Bendixson theorem that R contains a limit cycle.

In fact, the above vector field has only one equilibrium point: at the origin. To see this we can argue as follows. From the expression for $\dot{E}(x, y)$ it follows that the only equilibria which are not at the origin must lie in the unit circle $x^2 + y^2 = 1$. Let the vector field be $f(x, y) = (u(x, y), v(x, y))$. Then substituting $x^2 = 1 - y^2$ in the equation $u(x, y) = 0$ we obtain the equation $y(1 + xy) = 0$. This means that either $y = 0$ or $1 + xy = 0$. The equation $v(x, 0) = 0$ forces $x = 0$, hence the only possible equilibria away from the origin lie in the intersection of the hyperbola $xy = -1$ and the unit circle $x^2 + y^2 = 1$, which is empty.

(b) The system is now

$$\begin{cases} x' = 8x - 2y - 4x^3 - 2xy^2 \\ y' = x + 4y - 2y^3 - 3x^2 y \end{cases} .$$

Consider the function $E(x, y) = x^2 + 2y^2$. Its derivative along trajectories is

$$\begin{aligned} E'(x, y) &= 2x(8x - 2y - 4x^3 - 2xy^2) + 4y(x + 4y - 2y^3 - 3x^2 y) \\ &= 8(x^2 + y^2)\left(2 - (x^2 + y^2)\right) . \end{aligned}$$

Therefore E is decreasing for $x^2 + y^2 > 2$ and increasing for $x^2 + y^2 < 2$. This suggests that we consider the region

$$R = \left\{(x, y) \mid \tfrac{3}{2} \leq E(x, y) \leq 3\right\} .$$

It is clearly closed and bounded and contains the circle $x^2 + y^2 = 2$, which separates R into two regions with the circle as common boundary. If the trajectory starts in the region inside the circle, then $E' > 0$ whereas if it starts in the region outside the circle, $E' < 0$. Hence the trajectory cannot leave the region R, and again applying the Poincaré–Bendixson theorem, we conclude that R contains a limit cycle.

5.1 $T_{f_n} \to \delta$ in the (weak) distributional sense means that for all $\varphi \in \mathcal{D}$, $\langle T_{f_n}, \varphi\rangle \to \langle \delta, \varphi\rangle = \varphi(0)$ as real numbers. So we start by comparing them

$$\begin{aligned} \langle T_{f_n}, \varphi\rangle - \langle \delta, \varphi\rangle &= \int_{-\infty}^{\infty} f_n(t)\, \varphi(t)\, dt - \varphi(0) \\ &= \int_{-\infty}^{\infty} n f(nt)\, \varphi(t)\, dt - \int_{-\infty}^{\infty} f(t)\, \varphi(0)\, dt \\ (s = nt) \qquad &= \int_{-\infty}^{\infty} f(s)\, [\varphi(s/n) - \varphi(0)]\; ds . \end{aligned}$$

This implies the estimate

$$|\langle T_{f_n}, \varphi\rangle - \langle \delta, \varphi\rangle| \leq \int_{-\infty}^{\infty} |f(s)|\,|\varphi(s/n) - \varphi(0)|\; ds\ .$$

Now given ε, choose R large enough so that $\int_{|s|>R} |f(s)|dt < \varepsilon$. This is possible because f is absolutely integrable. Similarly, choose δ small enough so that $|\varphi(t)-\varphi(0)| < \varepsilon$ whenever $|t| < \delta$. This is possible because of continuity of φ. Now take N such that $N = \lceil R/\delta \rceil$; that is, the smallest integer greater than or equal to R/δ. Let us rewrite the above estimate as

$$\begin{aligned} |\langle T_{f_n}, \varphi\rangle - \langle \delta, \varphi\rangle| \leq \int_{-R}^{R} |f(s)|\,|\varphi(s/n) - \varphi(0)|\; ds \\ + \int_{|s|>R} |f(s)|\,|\varphi(s/n) - \varphi(0)|\; ds \end{aligned}$$

For the first term in the right-hand side we have

$$\int_{-R}^{R} |f(s)|\,|\varphi(s/n) - \varphi(0)|\; ds \leq \max_{|s|\leq R} |\varphi(s/n) - \varphi(0)| \int_{-\infty}^{\infty} |f(s)|\, ds\ .$$

Now for $n \geq N$ and $|s| \leq R$, $|s/n| \leq R/N \leq \delta$, whence

$$\max_{|s|\leq R} |\varphi(s/n) - \varphi(0)| \leq \varepsilon\ .$$

Similarly, for the second term we have

$$\begin{aligned} &\int_{|s|>R} |f(s)|\,|\varphi(s/n) - \varphi(0)|\; ds \\ &\qquad \leq \max_{s} |\varphi(s/n) - \varphi(0)| \int_{|s|>R} |f(s)|\, ds \\ &\qquad\qquad \leq \varepsilon \max_{s} |\varphi(s/n) - \varphi(0)|\ . \end{aligned}$$

Because f is absolutely integrable, $\int_{-\infty}^{\infty} |f(t)|\, dt = C$ for some number C. Similarly, because φ is a test function, it is bounded $|\varphi(t)| \leq K$ for some K. Now using the triangle inequality $\max |\varphi(s/n) - \varphi(0)| \leq 2K$, whence putting it all together we have

$$|\langle T_{f_n}, \varphi\rangle - \langle \delta, \varphi\rangle| \leq (C + 2K)\varepsilon\ ,$$

which can therefore be made as small as desired.

5.2 We need to show that for all test functions $\varphi \in \mathcal{D}$,

$$\left\langle T_f' - T_{f'}, \varphi\right\rangle = 0\ .$$

By definition, $\langle T_{f'}, \varphi\rangle = \int f'\varphi$, whereas

$$\left\langle T_f', \varphi\right\rangle = -\left\langle T_f, \varphi'\right\rangle = -\int f\varphi'\ .$$

The desired equality follows by integration by parts.

5.3 Let H_a be the shifted Heaviside step function $H_a(t) = H(t-a)$, and let T_{H_a} be the corresponding regular distribution. By definition, for all $\varphi \in \mathcal{D}$,

$$\begin{aligned}\left\langle T'_{H_a}, \varphi \right\rangle = -\left\langle T_{H_a}, \varphi' \right\rangle &= -\int_{-\infty}^{\infty} H_a(t)\varphi'(t)dt \\ &= -\int_a^{\infty} \varphi'(t)dt = -\varphi(t)\Big|_a^{\infty} = \varphi(a) = \langle \delta_a, \varphi \rangle \ ,\end{aligned}$$

where we have used the fact that φ has compact support and the definition of δ_a. Hence $T'_{H_a} = \delta_a$.

5.4 Let f be a smooth function and T a distribution. Then we saw that fT is a distribution. By definition, for all $\varphi \in \mathcal{D}$, the derivative of fT is such that

$$\begin{aligned}\langle (fT)', \varphi \rangle &= -\langle fT, \varphi' \rangle \\ &= -\langle T, f\varphi' \rangle \\ &= -\langle T, (f\varphi)' - f'\varphi \rangle \\ &= \langle T', f\varphi \rangle + \langle T, f'\varphi \rangle \\ &= \langle fT' + f'T, \varphi \rangle \ .\end{aligned}$$

5.5 Following the definitions, for all $\varphi \in \mathcal{D}$,

$$\begin{aligned}\left\langle f\delta^{(n)}, \varphi \right\rangle &= \left\langle \delta^{(n)}, f\varphi \right\rangle \\ &= (-1)^n \left\langle \delta, (f\varphi)^{(n)} \right\rangle \\ &= (-1)^n (f\varphi)^{(n)}(0) \\ &= (-1)^n \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(0)\varphi^{(n-i)}(0) \ ,\end{aligned}$$

where we have used the Leibniz (product) rule. Now notice that the right-hand side is simply

$$(-1)^n \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(0) \left\langle \delta, \varphi^{(n-i)} \right\rangle = \left\langle \sum_{i=0}^{n} (-1)^i \binom{n}{i} f^{(i)}(0)\delta^{(n-i)}, \varphi \right\rangle \ ,$$

which yields the result. Now we specialise to obtain the two corollaries. First, if $f(t) = t^m$, then $f^{(i)}(0) = 0$ unless $i = m$, and $f^{(m)}(0) = m!$. This immediately yields the first corollary. The second follows similarly: if now $f(t) = \exp(-\lambda t)$, then $f^{(m)}(0) = (-1)^m \lambda^m$, yielding the second.

5.6 Let $f(t) = t^{k-1}/(k-1)!$. Then we have to show that for all $\varphi \in \mathcal{D}$,

$$\left\langle (f(t)T_H)^{(k)}, \varphi \right\rangle = \langle \delta, \varphi \rangle = \varphi(0) \ .$$

By definition,

$$\left\langle (f(t)T_H)^{(k)}, \varphi \right\rangle = (-1)^k \left\langle f(t)T_H, \varphi^{(k)} \right\rangle$$
$$= (-1)^k \left\langle T_H, f(t)\varphi^{(k)} \right\rangle$$
$$= (-1)^k \int_0^\infty f(t)\,\varphi^{(k)}(t)\,dt \ .$$

We now integrate by parts $k-1$ times and notice that we do not pick up any boundary terms because $f^{(i)}(0) = 0$ for $i \neq k-1$. In the end we obtain

$$(-1)^k(-1)^{k-1} \int_0^\infty f^{(k-1)}(t)\,\varphi'(t)\,dt = -\int_0^\infty \varphi'(t)\,dt$$
$$= -\varphi(t)\Big|_0^\infty$$
$$= \varphi(0) \ .$$

5.7 By definition, if $T \in \mathcal{D}'$ and for all $\varphi \in \mathcal{D}$, we have

$$\langle T'', \varphi \rangle = \langle T, \varphi'' \rangle \ .$$

In particular for $T = T_f$, with f given in the problem, we have

$$\langle T_f, \varphi'' \rangle = \int_{-\infty}^\infty f(t)\,\varphi''(t)\,dt$$
$$= -\int_{-\infty}^{-1} \varphi''(t)\,dt + \int_{-1}^1 t\,\varphi''(t)\,dt + \int_1^\infty \varphi''(y)\,dt$$
$$= \varphi(-1) - \varphi(1)$$
$$= \langle \delta_{-1} - \delta_1, \varphi \rangle \ ,$$

in the notation of Problem 1. Therefore, we have that

$$T_f'' = \delta_{-1} - \delta_1 \ .$$

5.8 These equations are to be interpreted as distributions, of course. The first equation says that for $f(t) = |t|$,

$$\left\langle T_f', \varphi \right\rangle = -\langle T_f, \varphi' \rangle$$
$$= -\int_{-\infty}^\infty |t|\,\varphi'(t)\,dt$$
$$= \int_{-\infty}^0 t\,\varphi'(t)\,dt - \int_0^\infty t\,\varphi'(t)\,dt$$
$$= -\int_{-\infty}^0 \varphi(t)\,dt + \int_0^\infty \varphi(t)\,dt \ ,$$

where we have integrated by parts and dropped the boundary terms in order to go from the third to the fourth lines. But notice that

$$\int_{-\infty}^\infty (H(t) - H(-t))\varphi(t)\,dt = \int_{-\infty}^\infty H(t)\varphi(t)\,dt - \int_{-\infty}^\infty H(-t)\varphi(t)\,dt$$
$$= \int_0^\infty \varphi(t)\,dt - \int_{-\infty}^0 \varphi(t)\,dt \ .$$

For the second equation we have that for $f(t) = \exp(-k|t|)$,

$$\begin{aligned}
\langle T_f'', \varphi \rangle &= \langle T_f, \varphi'' \rangle \\
&= \int_{-\infty}^{0} e^{kt}\, \varphi''(t)\, dt + \int_{0}^{\infty} e^{-kt}\, \varphi''(t)\, dt \\
&= e^{kt}\varphi'(t)\Big|_{-\infty}^{0} - ke^{kt}\varphi\Big|_{-\infty}^{0} + \int_{-\infty}^{0} k^2\, e^{kt}\, \varphi(t)\, dt \\
&\quad + e^{-kt}\varphi'(t)\Big|_{0}^{\infty} + ke^{-kt}\varphi\Big|_{0}^{\infty} + \int_{0}^{\infty} k^2\, e^{-kt}\, \varphi(t)\, dt \\
&= -2k\varphi(0) + \int_{-\infty}^{\infty} k^2\, e^{-k|t|}\, \varphi(t)\, dt \\
&= \langle -2k\delta + k^2 T_f, \varphi \rangle \ .
\end{aligned}$$

This says that $F(t) = -1/2k \exp -k|t|$ is a fundamental solution for the operator $L = D^2 - k^2$. To find the Green's function we add to it any function in the kernel of L in such a way that the resulting function $G(t)$ vanishes for all $t < 0$. The kernel of L is two-dimensional and has basis $\exp \pm kt$. Therefore the most general fundamental solution for L is given by

$$-\tfrac{1}{2k} e^{-k|t|} + c_1 e^{-kt} + c_2 e^{kt} \ .$$

Demanding that this function vanish for $t < 0$ fixes $c_1 = 0$ and $c_2 = 1/2k$, whence the Green's function is given by

$$G(t) = \begin{cases} 0 & \text{for } t < 0, \\ \frac{1}{k}\sinh kt & \text{for } t > 0. \end{cases}$$

A solution for the inhomogeneous equation $Lx = f$ is then given by the convolution $G \star f$. For the function in question, since $f(s)$ is only different from zero for $0 < s < 1$, we have to distinguish three regions: $t < 0$, $0 < t < 1$ and $t > 1$. Clearly for $t < 0$ the fact that $f(s) = 0$ for $s < 0$ means that $x(t) = 0$. For $0 < t < 1$, we have

$$(G \star f)(t) = \int_0^t G(t-s) s ds = \int_0^t \frac{s}{k} \sinh k(t-s)\, ds \ .$$

Finally for $t > 1$ we have

$$(G \star f)(t) = \int_0^1 G(t-s) s ds = \int_0^1 \frac{s}{k} \sinh k(t-s)\, ds \ .$$

Performing the integrals (you are allowed to use Maple if you want to, although they are easy), a solution to the inhomogeneous equation is given by

$$x(t) = \begin{cases} 0 & t \leq 0 \\ \frac{\sinh kt}{k^3} - \frac{t}{k^2} & 0 < t \leq 1 \\ \frac{\sinh kt}{k^3} + \frac{\sinh k(1-t)}{k^3} - \frac{\cosh k(1-t)}{k^2} & t > 1 \end{cases}$$

This is plotted below.

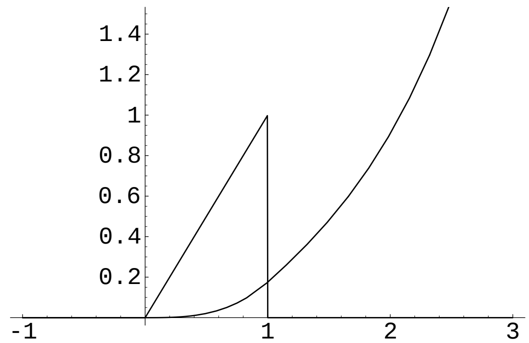

Figure 1: $x(t)$ and $f(t)$ for $-1 \leq t \leq 3$.

5.9 As discussed in lecture, the Green's function takes the form $G(t) = g(t)H(t)$, where $g(t)$ is the unique solution to the linear homogeneous ODE $Lg = 0$ subject to the initial conditions $g(0) = 0$ and $g'(0) = 1$. It is trivial to find $g(t)$. We have to consider three cases

$$
\begin{aligned}
g(t) &= 2e^{-at/2}\,\frac{\sinh \frac{1}{2}\sqrt{a^2-4b}t}{\sqrt{a^2-4b}} && \text{for } a^2 > 4b \\
g(t) &= 2e^{-at/2}\,\frac{\sin \frac{1}{2}\sqrt{4b-a^2}t}{\sqrt{4b-a^2}} && \text{for } a^2 < 4b \\
g(t) &= te^{-at/2} && \text{for } a^2 = 4b\ .
\end{aligned}
$$

In our case, $a = b = 1$, whence $a^2 < 4b$. The solution of the inhomogeneous equation is given by the convolution of G and f:

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} G(t-s)\, f(s)\, ds \\
&= \int_{-\infty}^{\infty} g(t-s)H(t-s)\, f(s)\, ds \\
&= \int_0^1 g(t-s)H(t-s)\, ds \\
&= \int_0^1 e^{-(t-s)/2}\,\frac{\sin(t-s)\sqrt{3}/2}{\sqrt{3}/2}\, H(t-s)\, ds\ .
\end{aligned}
$$

We must distinguish three regimes $t < 0$, $0 \leq t \leq 1$ and $t > 1$. Clearly for $t < 0$ the integrand vanishes, whence so does the integral and $x(t) = 0$ here. For $0 \leq t \leq 1$, we have

$$
\begin{aligned}
x(t) &= \int_0^t e^{-(t-s)/2}\,\frac{\sin(t-s)\sqrt{3}/2}{\sqrt{3}/2}\, ds \\
&= 1 - e^{-t/2}\left(\cos t\sqrt{3}/2 + \frac{1}{\sqrt{3}}\sin t\sqrt{3}/2\right)\ .
\end{aligned}
$$

For $t > 1$, we have

$$\begin{aligned} x(t) &= \int_0^1 e^{-(t-s)/2} \frac{\sin(t-s)\sqrt{3}/2}{\sqrt{3}/2}\, ds \\ &= e^{-(t-1)/2}\cos(t-1)\sqrt{3}/2 - e^{-t/2}\cos t\sqrt{3}/2 \\ &\quad + e^{-(t-1)/2}\frac{1}{\sqrt{3}}\sin(t-1)\sqrt{3}/2 - e^{-t/2}\frac{1}{\sqrt{3}}\sin t\sqrt{3}/2\ . \end{aligned}$$

A plot of $x(t)$ and $f(t)$ is given below.

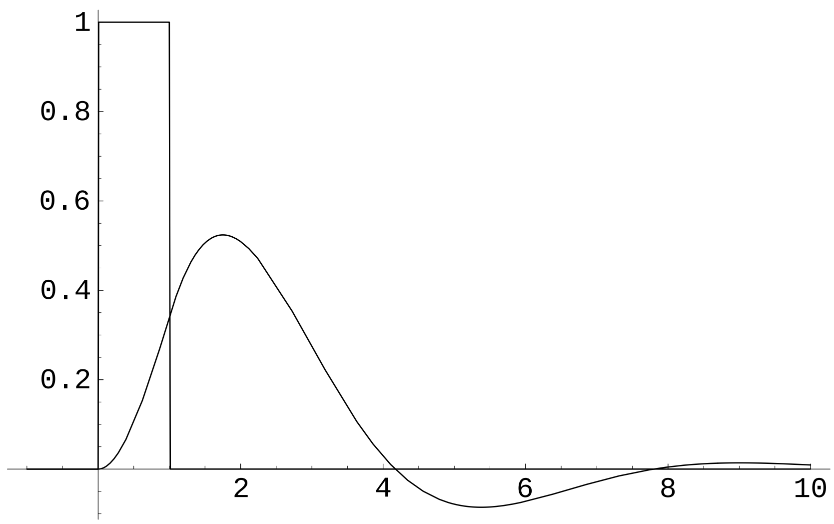

Figure 2: $x(t)$ and $f(t)$ for $-1 \le t \le 10$.

Although the graph of $x(t)$ seems "pretty smooth", it is not even twice differentiable. It is easy to show that $x(t)$ and $x'(t)$ are continuous, but $x''(t)$ is not – this follows from the differential equation, which says that $x''(t) = f(t) - x'(t) - x(t)$, and the fact that $f(t)$ is not continuous.

5.10 Throughout this problem, $\varphi, \psi, \chi \in \mathcal{D}$ are test functions: smooth with compact support. The convolution $\varphi \star \psi$ is the function defined by

$$(\varphi \star \psi)(t) := \int_{-\infty}^{\infty} \varphi(t-s)\, \psi(s)\, ds\ . \tag{1}$$

(a) Let $\operatorname{supp}\varphi = [a, b]$ and $\operatorname{supp}\psi = [c, d]$. Then in the expression for the convolution, the integral will vanish unless $c \le s \le d$ and $a \le t - s \le b$. This means that $t \le b + s \le b + d$ and $t \ge a + s \ge a + c$, whence $\operatorname{supp}\varphi \star \psi = [a + c, b + d]$.

(b) Since φ and ψ are smooth, we can take the derivative inside the integral:

$$\begin{aligned}(\varphi\star\psi)'(t) &= \int_{-\infty}^{\infty} \frac{d}{dt}\varphi(t-s)\,\psi(s)\,ds\\ &= \int_{-\infty}^{\infty} \varphi'(t-s)\,\psi(s)\,ds\\ &= (\varphi'\star\psi)(t)\ .\end{aligned}$$

On the other hand notice that $\frac{d}{dt}\varphi(t-s) = -\frac{d}{ds}\varphi(t-s)$, whence

$$\begin{aligned}(\varphi\star\psi)'(t) &= -\int_{-\infty}^{\infty} \frac{d}{ds}\varphi(t-s)\,\psi(s)\,ds\\ &= \int_{-\infty}^{\infty} \varphi(t-s)\,\psi'(s)\,ds\\ &= (\varphi\star\psi')(t)\ ,\end{aligned}$$

where we have integrated by parts and dropped the boundary terms.

(c) From part (a) we see that $\varphi\star\psi$ has compact support, whereas from part (b) we see that it is infinitely differentiable. Notice that part (b) says something stronger: $\varphi\star\psi$ will be infinitely differentiable provided at least one of φ or ψ is. This is why convolution with a smooth function is known as a *smoothing operator*.

(d) Change variables in the integral (1): $u = t - s$, to get:

$$(\varphi\star\psi)(t) = \int_{-\infty}^{\infty} \varphi(u)\,\psi(t-u)\,du = (\psi\star\varphi)(t)\ ,$$

by the Shakespeare theorem. Associativity follows similarly:

$$\begin{aligned}((\varphi\star\psi)\star\chi)\,(t) &= \int_{-\infty}^{\infty} (\varphi\star\psi)(t-s)\,\chi(s)\,ds\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(t-s-u)\,\psi(u)\,\chi(s)\,du\,ds\\ (v=u+s) \qquad &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(t-v)\,\psi(v-s)\,\chi(s)\,ds\,dv\\ &= \int_{-\infty}^{\infty} \varphi(t-v)\ (\psi\star\chi)\,(v)\,dv\\ &= (\varphi\star(\psi\star\chi))\,(t)\ .\end{aligned}$$

Now define the inner product of two test functions by:

$$\langle\varphi,\psi\rangle := \int_{-\infty}^{\infty} \varphi(t)\psi(t)dt\ .$$

(e) With $\varphi^\vee(t) := \varphi(-t)$, using equation (1),

$$\begin{aligned}(\varphi^\vee\star\psi)(0) &= \int_{-\infty}^{\infty} \varphi^\vee(-s)\,\psi(s)\,ds\\ &= \int_{-\infty}^{\infty} \varphi(s)\,\psi(s)\,ds\\ &= \langle\varphi,\psi\rangle\ \ .\end{aligned}$$

(f) We first prove that $(\varphi \star \psi)^\vee = \varphi^\vee \star \psi^\vee$:

$$\begin{aligned}
(\varphi \star \psi)^\vee(t) &= \varphi \star \psi(-t) \\
&= \int_{-\infty}^{\infty} \varphi(-t-s)\,\psi(s)\,ds \\
(u=-s) \qquad &= \int_{-\infty}^{\infty} \varphi(-t+u)\,\psi(-u)\,du \\
&= \int_{-\infty}^{\infty} \varphi^\vee(t-u)\,\psi^\vee(u)\,du \\
&= (\varphi^\vee \star \psi^\vee)(t)\ .
\end{aligned}$$

Now from part (e) and using associativity of the convolution,

$$\begin{aligned}
\langle \varphi \star \psi, \chi \rangle &= ((\varphi \star \psi)^\vee \star \chi)(0) \\
&= ((\varphi^\vee \star \psi^\vee) \star \chi)(0) \\
&= (\varphi^\vee \star (\psi^\vee \star \chi))(0) \\
&= \langle \varphi, \psi^\vee \star \chi \rangle\ .
\end{aligned}$$

5.11 **(a)** We have to show that for all $T \in \mathcal{D}'$, $\Phi^* T$ is both linear and continuous, so that it is again in $\mathcal{D}'$. Linearity is obvious because Φ is linear:

$$\begin{aligned}
\langle \Phi^* T, c_1\varphi_1 + c_2\varphi_2 \rangle &= \langle T, \Phi(c_1\varphi_1 + c_2\varphi_2) \rangle \\
&= \langle T, c_1\Phi(\varphi_1) + c_2\Phi(\varphi_2) \rangle \\
&= c_1 \langle T, \Phi(\varphi_1) \rangle + c_2 \langle T, \Phi(\varphi_2) \rangle \\
&= c_1 \langle \Phi^* T, \varphi_1 \rangle + c_2 \langle \Phi^* T, \varphi_2 \rangle\ .
\end{aligned}$$

As for continuity, notice that if $\varphi_n \to 0$ then

$$\langle \Phi^* T, \varphi_n \rangle = \langle T, \Phi(\varphi_n) \rangle\ .$$

Now, Φ continuous implies that $\Phi(\varphi_n) \to 0$; and T continuous implies that $\langle T, \Phi(\varphi_n) \rangle \to 0$. Hence $\langle \Phi^* T, \varphi_n \rangle \to 0$ and $\Phi^* T$ is continuous.

(b) To show that Δ_a and Θ_b map test functions to test functions, we have to show that they take smooth functions to smooth functions and that they take compactly supported functions to compactly supported functions. That they preserve smoothness if obvious: let $(\Delta_a\varphi)' = a\Delta_a\varphi'$ and $(\Theta_b\varphi)' = \Theta_b\varphi'$. Also, if φ has support $[c,d]$, then $\Delta_a\varphi$ has support $[c/a, d/a]$ (if $a > 0$) or $[d/a, c/a]$ (if $a < 0$), and $\Theta_b\varphi$ has support $[c+b, d+b]$, and in either case it remains compact.

It remains to show that they are linear and continuous. Linearity is clear: $\Delta_a(c_1\varphi_1 + c_2\varphi_2)(t) = (c_1\varphi_1 + c_2\varphi_2)(at) = c_1\varphi_1(at) + c_2\varphi_2(at) = c_1\Delta_a\varphi_1(t) + c_2\Delta_a\varphi_2(t) = (c_1\Delta_a\varphi_1 + c_2\Delta_a\varphi_2)(t)$. The same goes for Θ_b.

Continuity is also clear. Suppose that $\varphi_m \to 0$. This means that there is some R such that $\varphi_m(t) = 0$ for $|t| > R$, and that for each k, the k-th derivatives $\varphi_m^{(k)} \to 0$ uniformly. The functions $\Delta_a\varphi_m$ vanish for $|at| > R$ or equivalently $|t| > R/|a|$, so the first condition for convergence is satisfied. Notice that $(\Delta_a \varphi_m)^{(k)}(t) = a^k \varphi_m^{(m)}(at)$, whence $(\Delta_a \varphi_m)^{(k)} \to 0$ uniformly as well. Therefore $\Delta_a \varphi_m \to 0$. The same goes for Θ_b.

(c) Let T_f be a regular distribution. By definition,

$$\begin{aligned} \langle \Delta_a^* T_f, \varphi \rangle &= \langle T_f, \Delta_a \varphi \rangle \\ &= \int_{-\infty}^{\infty} f(t)\, \varphi(at)\, dt \\ &= \int_{-\infty}^{\infty} f(u/a)\, \varphi(u)\, du/|a| \\ &= \left\langle T_{\Delta_a^* f}, \varphi \right\rangle \ , \end{aligned}$$

where we have let $u = at$ and where

$$\Delta_a^* f(t) = \frac{1}{|a|} f(t/a) \ .$$

Similarly, by definition

$$\begin{aligned} \langle \Theta_b^* T_f, \varphi \rangle &= \langle T_f, \Theta_b \varphi \rangle \\ &= \int_{-\infty}^{\infty} f(t)\, \varphi(t-b)\, dt \\ &= \int_{-\infty}^{\infty} f(u+b)\, \varphi(u)\, du \\ &= \left\langle T_{\Theta_b^* f}, \varphi \right\rangle \ , \end{aligned}$$

where we have let $u = t - b$ and where

$$\Theta_b^* f(t) = f(t+b) \ .$$

5.12 Notice that

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h} \langle \Theta_h^* T - T, \varphi \rangle &= \lim_{h\to 0} \frac{1}{h} \langle T, \Theta_b \varphi - \varphi \rangle \\ &= \left\langle T, \lim_{h\to 0} \frac{1}{h} (\Theta_b \varphi - \varphi) \right\rangle \end{aligned}$$

where we have used continuity of T to pass the limit inside. But now,

$$\lim_{h\to 0} \frac{1}{h} (\Theta_b \varphi - \varphi)(t) = \lim_{h\to 0} \frac{1}{h} (\varphi(t-h) - \varphi(t)) = -\varphi'(t) \ ,$$

whence

$$\lim_{h\to 0} \frac{1}{h} \langle \Theta_h^* T - T, \varphi \rangle = -\langle T, \varphi' \rangle \ ,$$

which is precisely $\langle T', \varphi \rangle$.

5.13 Let f be a test function which is never negative and still obeys $\int_{-\infty}^{\infty} f(t) dt = 1$. Let $\varphi_n := f_n \star \varphi$. We want to show that $\varphi_n - \varphi \to 0$ in $\mathcal{D}$.

This first convergence condition says that $\varphi_n - \varphi$ have a common support for all n. Indeed, let $\operatorname{supp} \varphi = [a, b]$ and $\operatorname{supp} f = [c, d]$. Then $\operatorname{supp} f_n =$

$[c/n, d/n]$, and hence $\operatorname{supp} f_n \star \varphi = [a + c/n, b + d/n] \subseteq [a + c, b + d]$. So it is satisfied.

The second convergence condition says that for fixed k, $\varphi_n^{(k)} \to \varphi^{(k)}$ uniformly as $n \to \infty$. Since $\varphi_n^{(k)} = f_n \star \varphi^{(k)}$ (see Problem 9 (b)), this second condition will be satisfied if for *every* test function ψ, $f_n \star \psi \to \psi$ uniformly as $n \to \infty$.

Let R be such that $f(s) = 0$ for $|s| > R$. Then $f_n(s) = 0$ for $|s| > R/n$. Therefore,

$$|f_n \star \psi(t) - \psi(t)| = \left| \int_{-R/n}^{R/n} f_n(s)\, \psi(t-s)\, ds - \int_{-R/n}^{R/n} f_n(s)\, \psi(t)\, ds \right|$$

where we have used that $\int_{-R/n}^{R/n} f_n(s)\, ds = 1$. Estimating the integral, we have

$$\begin{aligned} |f_n \star \psi(t) - \psi(t)| &\leq \int_{-R/n}^{R/n} f_n(s) |\psi(t-s) - \psi(t)|\, ds \\ &\leq \max_{|s| \leq R/n} |\psi(t-s) - \psi(s)| \ . \end{aligned}$$

Because test functions have compact support they are uniformly continuous, hence by taking n large enough, we can make $|\psi(t-s) - \psi(t)|$ as small as desired for $|s| \leq R/n$ *uniformly* in t, hence $f_n \star \psi(t) \to \psi(t)$ uniformly in t as well.

6.2 The Laplace transform of $f(t)$ is given by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \ .$$

Let us break the integral into periods:

$$\begin{aligned} F(s) &= \left(\int_0^T + \int_T^{2T} + \int_{2T}^{3T} + \cdots \right) f(t)\, e^{-st}\, dt \\ &= \sum_{n=0}^\infty \int_{nT}^{nT+T} f(t)\, e^{-st}\, dt \ . \end{aligned}$$

Now, in the n-th integral, let us make the change of variables $t = \tau n T$, so that τ always goes from 0 to T:

$$F(s) = \sum_{n=0}^\infty \int_0^T f(nT + \tau)\, e^{-s(\tau + nT)}\, d\tau \ .$$

Using the periodicity of f, so that $f(\tau + nT) = f(\tau)$, we have

$$F(s) = \sum_{n=0}^\infty \int_0^T f(\tau)\, e^{-s(nT+\tau)}\, d\tau = \sum_{n=0}^\infty e^{-snT} \int_0^T f(\tau)\, e^{-s\tau}\, d\tau \ .$$

We can now sum the geometric series, which converges provided that $\mathrm{Re}(s) > 0$, and obtain the desired answer:

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T f(\tau)\, e^{-s\tau}\, d\tau \ .$$

As an application consider the function $f(t)$ which is 1 in the intervals $[2n, 2n+1]$ and zero everywhere else. Clearly this is a periodic function with period 2. Therefore its Laplace transform is given by

$$F(s) = \frac{1}{1-e^{-2s}} \int_0^2 f(\tau)\, e^{-s\tau}\, d\tau = \frac{1}{1-e^{-2s}} \int_0^1 e^{-s\tau}\, d\tau = \frac{1}{s} \frac{1-e^{-s}}{1-e^{-2s}} \ .$$

6.3 Following the hint, let us take the Laplace transform of both sides of the equation. To compute the Laplace transform of the RHS we use linearity and the convolution theorem, once recognising the second term as the convolution $h \star f$ where $h(t) = t$. Letting $F(s)$ denote the Laplace transform of $f(t)$, we have

$$F(s) = \frac{1}{s} - \frac{1}{s^2} F(s) \ ,$$

where we have used Table 1 (see Notes) to write down the Laplace transforms of 1 and t, respectively. The above formula exists for all s with $\mathrm{Re}(s) > 0$ such that $F(s)$ exists. We can now solve the resulting algebraic equation for $F(s)$ to obtain

$$F(s) = \frac{s}{1+s^2} \ .$$

Looking up in the table, we see that this is the Laplace transform of $f(t) = \cos t$. (One can check that this indeed solves the integral equation.)

6.4 Consider the second order ordinary differential equation

$$f''(t) + \omega^2 f(t) = u(t),$$

subject to the initial conditions $f(0) = f'(0) = 0$. Let us take the Laplace transform. If let $F(s)$ and $U(s)$ denote the Laplace transforms of $f(t)$ and $u(t)$, then with the above initial conditions, we find

$$s^2\, F(s) + \omega^2\, F(s) = U(s) \ ,$$

whence

$$F(s) = \frac{U(s)}{s^2+\omega^2} = \frac{1}{\omega}\, U(s)\, \frac{\omega}{s^2+\omega^2} \ .$$

From the Table we see that this is the Laplace transform of the function

$$f(t) = \frac{1}{\omega} \int_0^t u(\tau)\, \sin\omega(t-\tau)\, d\tau \ .$$

6.6 (a) Consider the function

$$f(t) = 3\cos 2t - 8e^{-2t} \ .$$

By linearity, the Laplace transform $F(s)$ of $f(t)$ is the sum of the Laplace transforms of each of the terms:

$$F(s) \equiv \mathcal{L}\left\{f(t)\right\} = 3\mathcal{L}\left\{\cos 2t\right\} - 8\mathcal{L}\left\{e^{-2t}\right\} \ ,$$

which can be read from the Table:

$$\mathcal{L}\{\cos 2t\}(s) = \frac{s}{s^2+4} \qquad \text{and} \qquad \mathcal{L}\left\{e^{-2t}\right\}(s) = \frac{1}{s+2} ;$$

whence

$$F(s) = \frac{3s}{s^2+4} - \frac{8}{s+2} .$$

The transform of the first function is valid for $\mathrm{Re}(s) > 0$ and that of the second for $\mathrm{Re}(s) > -2$. Therefore they are *both* valid for $\mathrm{Re}(s) > 0$.

(c) Consider now the function

$$f(t) = \begin{cases} 1 , & \text{for } t < 1, \text{ and} \\ 0 , & \text{for } t \geq 1. \end{cases} ,$$

whose Laplace transform is given by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^1 e^{-st}\, dt = \frac{1}{s}\left(1 - e^{-s}\right) .$$

The transform is valid for *all* s: the singularity at $s = 0$ is removable, since $\lim_{s\to 0} F(s) = 1$.

(d) Now we have

$$f(t) = (\sin t)^2 .$$

Using a trigonometric identity, $(\sin t)^2 = \frac{1}{2}(1 - \cos 2t)$, and linearity of the Laplace transform, we have that

$$F(s) \equiv \mathcal{L}\left\{(\sin t)^2\right\}(s) = \tfrac{1}{2}\mathcal{L}\{1\}(s) - \tfrac{1}{2}\mathcal{L}\{\cos 2t\}(s) ,$$

which we have already worked out in class:

$$F(s) = \tfrac{1}{2}\,\frac{1}{s} - \tfrac{1}{2}\,\frac{s}{s^2+4} .$$

Again we have condition $\mathrm{Re}(s) > 0$ from the first transform and $\mathrm{Re}(s) > -2$ from the second. Hence both are valid whenever $\mathrm{Re}(s) > 0$.

(e) Finally, we have

$$f(t) = \begin{cases} 0 , & \text{for } t < 1, \\ 1 , & \text{for } 1 \leq t \leq 2, \text{ and} \\ 0 , & \text{for } t > 2. \end{cases} ,$$

whose Laplace transform is given by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_1^2 e^{-st}\, dt = \frac{e^{-s}}{s}\left(1 - e^{-s}\right) .$$

Again the transform is valid for all s. Notice that the singularity at $s = 0$ is removable, since $\lim_{s\to 0} F(s) = 1$.

6.7 (a) Consider

$$F(s) = \frac{1}{s^2+4} \ .$$

Comparing with the list of transforms we saw in class we notice that

$$F(s) = \tfrac{1}{2}\,\frac{2}{s^2+4} = \tfrac{1}{2}\,\mathrm{Im}\,\frac{1}{s-2i} \ ,$$

whence

$$F(s) = \mathcal{L}\left\{\tfrac{1}{2}\,\mathrm{Im}\,e^{2it}\right\} = \mathcal{L}\left\{\tfrac{1}{2}\sin 2t\right\} \ .$$

(b) Now consider

$$F(s) = \frac{4}{(s-1)^2} \ .$$

Comparing with the results in class,

$$F(s) = 4\frac{1}{(s-1)^2} = \mathcal{L}\left\{4t\,e^t\right\} \ .$$

(c) Now we have

$$F(s) = \frac{s}{s^2+4s+4} = \frac{s}{(s+2)^2} \ .$$

Into partial fractions,

$$F(s) = \frac{s+2-2}{(s+2)^2} = \frac{1}{s+2} - \frac{2}{(s+2)^2} \ .$$

We can now read off the inverse transforms:

$$F(s) = \mathcal{L}\left\{e^{-2t} - 2t\,e^{-2t}\right\} \ .$$

(d) Consider now

$$F(s) = \frac{1}{s^3+3s^2+2s} = \frac{1}{s(s+1)(s+2)} \ .$$

Again we expand into partial fractions:

$$F(s) = \tfrac{1}{2}\,\frac{1}{s} - \frac{1}{s+1} + \tfrac{1}{2}\,\frac{1}{s+2} \ ,$$

whence we can read off the inverse transforms:

$$F(s) = \mathcal{L}\left\{\tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}\right\} \ .$$

(e) Finally we have

$$F(s) = \frac{s+3}{s^2+4s+7} \ .$$

The denominator factorises as $(s+2+i\sqrt{3})(s+2-i\sqrt{3})$, whence

$$\begin{aligned} F(s) &= \frac{s+3}{(s+2+i\sqrt{3})(s+2-i\sqrt{3})} \\ &= \frac{s+2}{(s+2+i\sqrt{3})(s+2-i\sqrt{3})} + \frac{1}{(s+2+i\sqrt{3})(s+2-i\sqrt{3})} \\ &= \frac{s+2}{(s+2)^2+(\sqrt{3})^2} + \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s+2)^2+(\sqrt{3})^2} \ , \end{aligned}$$

from where we can read off the inverse transforms:

$$F(s) = \mathfrak{L}\left\{e^{-2t}\cos\sqrt{3}t + \frac{1}{\sqrt{3}}e^{-2t}\sin\sqrt{3}t\right\}$$
$$= \mathfrak{L}\left\{e^{-2t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right)\right\} \ .$$

6.8 The method is the same in all cases: we take the Laplace transform of the equation taking into account the initial conditions, solve the corresponding algebraic equation, and invert back. Consider the following differential equation

$$\frac{d^2 f(t)}{dt^2} + a_1 \frac{df(t)}{dt} + a_0 f(t) = u(t) \ , \tag{2}$$

where a_i are constants and $u(t)$ is some function. Taking the Laplace transform of this differential equation we have

$$\left(s^2 F(s) - s f(0) - f'(0)\right) + a_1 \left(sF(s) - f(0)\right) + a_0 F(s) = U(s) \ ,$$

where $F(s) = \mathfrak{L}\{f(t)\}$ and $U(s) = \mathfrak{L}\{u(t)\}$. We can solve this algebraic equation for $F(s)$ in terms of $U(s)$ and the initial conditions

$$F(s) = \frac{U(s) + (s + a_1)f(0) + f'(0)}{s^2 + a_1 s + a_0} \ , \tag{3}$$

which we can then try to invert back. Let us apply this to the following differential equations.

(a) Consider first the equation

$$\frac{d^2 f(t)}{dt^2} - 5\frac{df(t)}{dt} + 6f(t) = 0 \ ,$$

subject to the initial conditions $f(0) = 1$ and $f'(0) = -1$. This equation is of the form (2) with $a_1 = -5$ and $a_0 = 6$ and $u(t) = 0$. Therefore into (3) we see that

$$F(s) = \frac{(s-5) - 1}{s^2 - 5s + 6} = \frac{s - 6}{(s-2)(s-3)} \ .$$

Decomposition into partial fractions, we have

$$F(s) = \frac{4}{s-2} - \frac{3}{s-3} \ ,$$

which we recognise as the Laplace transform of the function

$$f(t) = 4\,e^{2t} - 3\,e^{3t} \ .$$

(b) Consider the following differential equation

$$\frac{d^2 f(t)}{dt^2} - \frac{df(t)}{dt} - 2f(t) = e^{-t}\sin 2t \ , \tag{4}$$

Taking the Laplace transform of this differential equation we have

$$(s^2F(s) - sf(0) - f'(0)) - (sF(s) - f(0)) - 2F(s) = U(s) \ ,$$

where $F(s)$ is the Laplace transform of $f(t)$ and $U(s)$ the Laplace transform of $u(t)$, which is given by

$$U(s) = \frac{2}{(s+1)^2+4} \ ,$$

whence we find

$$F(s) = \frac{2}{((s+1)^2+4)\,(s^2-s-2)} = \frac{2}{((s+1)^2+4)\,(s+1)(s-2)} \ .$$

After a little algebra, we can rewrite this into partial fractions,

$$\begin{aligned} F(s) &= \frac{2}{39}\frac{1}{s-2} - \frac{1}{6}\frac{1}{s+1} + \frac{1}{26}\frac{3s-1}{(s+1)^2+4} \\ &= \frac{2}{39}\frac{1}{s-2} - \frac{1}{6}\frac{1}{s+1} + \frac{1}{26}\frac{3(s+1)-4}{(s+1)^2+4} \ , \end{aligned}$$

whose last term we can rewrite as

$$\frac{1}{26}\frac{3(s+1)-4}{(s+1)^2+4} = \frac{3}{26}\frac{s+1}{(s+1)^2+4} - \frac{1}{13}\frac{2}{(s+1)^2+4} \ ,$$

which allows us to perform the inverse transform. In fact we can easily see that

$$F(s) = \frac{2}{39}\frac{1}{s-2} - \frac{1}{6}\frac{1}{s+1} + \frac{3}{26}\frac{s+1}{(s+1)^2+4} - \frac{1}{13}\frac{2}{(s+1)^2+4}$$

is the Laplace transform of the function

$$f(t) = \tfrac{2}{39}\,e^{2t} - \tfrac{1}{6}\,e^{-t} + \tfrac{3}{26}e^{-t}\,\cos 2t - \tfrac{1}{13}\,e^{-t}\,\sin 2t \ .$$

(c) Finally we have

$$\frac{d^2f(t)}{dt^2} - 3\frac{df(t)}{dt} + 2f(t) = \begin{cases} 0 \ , & \text{for } 0 \le t < 3, \\ 1 \ , & \text{for } 3 \le t \le 6, \text{ and} \\ 0 \ , & \text{for } t > 6, \end{cases}$$

subject to the initial conditions $f(0) = f'(0) = 0$. Comparing with (2) we have $a_1 = -3$ and $a_0 = 2$. The Laplace transform $U(s)$ of the function $u(t)$ given above, is given by (cf. Problem 1 (e))

$$U(s) = \frac{1}{s}\left(e^{-3s} - e^{-6s}\right) \ .$$

Into (3), we have that

$$F(s) = \frac{e^{-3s} - e^{-6s}}{s(s^2-3s+2)} = \frac{e^{-3s} - e^{-6s}}{s(s-1)(s-2)} \ .$$

Into partial fractions we have

$$F(s) = \left(e^{-3s} - e^{-6s}\right)\left(\frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2}\right) .$$

From the lecture we have that

$$e^{-as}G(s) = \mathcal{L}\left\{\theta(t-a)\,g(t-a)\right\}(s) \qquad \text{where } G(s) = \mathcal{L}\left\{g(t)\right\}(s),$$

where θ is the Heaviside step function. Therefore we can read off the function $f(t)$ whose Laplace transform is $F(s)$:

$$f(t) = \theta(t-3)\left(\tfrac{1}{2} - e^{t-3} + \tfrac{1}{2}\,e^{2(t-3)}\right) \\ \qquad\qquad - \theta(t-6)\left(\tfrac{1}{2} - e^{t-6} + \tfrac{1}{2}\,e^{2(t-6)}\right) ,$$

or equivalently

$$f(t) = \begin{cases} 0 \,, & \text{for } t < 3; \\ \tfrac{1}{2} - e^{t-3} + \tfrac{1}{2}\,e^{2(t-3)} \,, & \text{for } 3 \leq t \leq 6; \text{ and} \\ e^{t-6} + \tfrac{1}{2}e^{2(t-6)} - e^{t-3} - \tfrac{1}{2}e^{2(t-3)} \,, & \text{for } t > 6. \end{cases}$$

7.1 By definition $h(t) = f(t)/g(t)$, or equivalently $g(t)h(t) = f(t)$, or in terms of their series expansions:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n t^n &= \left(\sum_{n=0}^{\infty} b_n t^n\right)\left(\sum_{n=0}^{\infty} c_n t^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n} b_\ell c_{n-\ell}\right) t^n \,, \end{aligned}$$

where we have used the Cauchy product of the power series. Comparing the two series termwise we get the following sequence of relations:

$$\sum_{\ell=0}^{n} b_\ell c_{n-\ell} = a_n \qquad \text{for } n = 0, 1, \ldots$$

which since $b_0 \neq 0$, can be turned into a recurrence relation for the $\{c_n\}$:

$$c_n = \frac{1}{b_0}\left(a_n - \sum_{\ell=1}^{n} b_\ell c_{n-\ell}\right) .$$

Notice that only $\{c_0, c_1, \ldots, c_{n-1}\}$ appear in the right-hand side of the equation. The first few terms can be easily written down as follows:

$$\begin{aligned} c_0 &= \frac{a_0}{b_0} \\ c_1 &= \frac{a_1 b_0 - a_0 b_1}{b_0^2} \\ c_2 &= \frac{a_2 b_0^2 - a_1 b_0 b_1 + a_0 b_1^2 - a_0 b_0 b_2}{b_0^3} \\ c_3 &= \frac{a_3 b_0^3 - a_2 b_0^2 b_1 + a_1 b_0 b_1^2 - a_0 b_1^3 - a_1 b_0^2 b_2 + 2 a_0 b_0 b_1 b_2 - a_0 b_0^2 b_3}{b_0^4} \,. \end{aligned}$$

7.2 By definition of f^{-1}, $t = f^{-1}(f(t))$. In terms of power series, we have

$$t = \sum_{n=0}^{\infty} b_n \left(\sum_{m=0}^{\infty} a_m t^m - a_0 \right)^n = \sum_{n=0}^{\infty} b_n \left(\sum_{m=1}^{\infty} a_m t^m \right)^n$$
$$= b_0 + b_1 \sum_{n=1}^{\infty} a_n t^n + b_2 \sum_{\ell,m=1}^{\infty} a_\ell a_m t^{\ell+m} + b_3 \sum_{\ell,m,p=1}^{\infty} a_\ell a_m a_p t^{\ell+m+p} + \cdots$$

Assuming that $a_1 \neq 0$, this allows us to solve for the $\{b_n\}$ in terms of the $\{a_n\}$:

$$b_0 = 0 \qquad b_1 = \frac{1}{a_1} \qquad b_2 = -\frac{a_2}{a_1^3} \qquad b_3 = \frac{2a_2^2 - a_1 a_3}{a_1^5} \ .$$

7.3 For each of these problems we will try a power series solution of the form:

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \ ,$$

for which

$$x'(t) = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n \quad \text{and} \quad x''(t) = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n \ .$$

Inserting these expressions into each of the differential equations, we can obtain recurrence relations for the coefficients, which have unique solutions once a_0 and a_1 are specified.

(a) For $x'' + x' - tx = 0$, one finds the following recurrence relations:

$$a_{n+2} = \frac{a_{n-1} - (n+1)a_{n+1}}{(n+1)(n+2)} \quad \text{for } n \geq 0,$$

with the proviso that $a_{-1} = 0$. The power series solutions are obtained by solving these recurrence relations:

$$x(t) = a_0 \left(1 + \tfrac{1}{6}t^3 - \tfrac{1}{24}t^4 + \tfrac{1}{120}t^5 + \tfrac{1}{240}t^6 - \tfrac{1}{630}t^7 + \cdots\right)$$
$$+ a_1 \left(t - \tfrac{1}{2}t^2 + \tfrac{1}{6}t^3 + \tfrac{1}{24}t^4 - \tfrac{1}{30}t^5 + \tfrac{1}{90}t^6 - \tfrac{1}{1680}t^7 + \cdots\right) \ .$$

(b) For $(1+t^2)x'' + 2tx' - 2x = 0$, one finds the following recurrence relations:

$$a_{n+2} = -\frac{n-1}{n+1}\, a_n \quad \text{for } n \geq 0.$$

We notice that the "a_1" series truncates immediately, since $a_3 = 0$, and hence so are all a_{2k+1}, for $k > 1$. (In fact, it is obvious that $x(t) = t$ is a solution!) The general solution is then

$$x(t) = a_1 t + a_0 \left(1 + t^2 - \tfrac{1}{3}t^4 + \tfrac{1}{5}t^6 - \tfrac{1}{7}t^8 + \cdots\right)$$
$$= a_1 t + a_0 \left(1 + t \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1}\right) \ ,$$

which we should recognise as

$$x(t) = a_1 t + a_0 \left(1 + t \tan^{-1} t\right) \ .$$

(c) Finally, for $x'' + tx' + x = 0$, we obtain the following recurrence relations:

$$a_{n+2} = -\frac{a_n}{n+2} \quad \text{for } n \geq 0.$$

These relations can be easily solved to yield the following solution:

$$x(t) = a_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{2 \cdot 4 \cdots 2k} \right) + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{1 \cdot 3 \cdots 2k+1} \ .$$

We can recognise the "a_0" series as

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{2 \cdot 4 \cdots 2k} = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{2^k k!} = e^{-t^2/2} \ .$$

7.5 Consider Hermite's equation:

$$x'' - 2tx' + 2px = 0 \ ,$$

where p is a constant.

(a) The coefficients are polynomial in t, and hence they are analytic everywhere. We therefore conclude that $t = 0$ is an ordinary point and that the radius of convergence of analytic solutions is infinite.

(b) The recurrence relation for the coefficients is found as in lecture:

$$a_{n+2} = \frac{2(n-p)}{(n+1)(n+2)} a_n \ .$$

(c) This relation does not mix the odd and even coefficients, and says that when p is a non-negative integer $a_{p+2} = 0$. Hence if p is even, then we can obtain a polynomial solution setting $a_1 = 0$ and $a_0 \neq 0$ and finding that all $a_{2n+1} = 0$ and that $a_{>p} = 0$. As a result the series truncates to an even polynomial of order p. Similarly, when p is odd, we set $a_0 = 0$ and $a_1 \neq 0$. We see that all $a_{2n} = 0$ and that $a_{>p} = 0$. Hence the series truncates to an odd polynomial of order p.

(d) Let p be a non-negative integer. If p is even we put $a_0 = 1$ and $a_1 = 0$, and if p is odd we put $a_0 = 0$ and $a_1 = 1$. In either case we call the resulting polynomial solution H_p. The first few can be solved by the recurrence relation:

$$\begin{aligned} H_0(t) &= 1 & H_1(t) &= t \\ H_2(t) &= 1 - 2t^2 & H_3(t) &= t - \tfrac{2}{3}t^3 \\ H_4(t) &= 1 - 4t^2 + \tfrac{4}{3}t^4 & H_5(t) &= t - \tfrac{4}{3}t^3 + \tfrac{4}{15}t^5 \ . \end{aligned}$$

(e) Consider the inner product on the space of polynomials:

$$\langle f, g \rangle := \int_{-\infty}^{\infty} e^{-t^2} f(t) g(t) \, dt \ .$$

(i) Let $\mathcal{H}$ be the operator $\mathcal{H}f(t) = 2tf'(t) - f''(t)$, and let f, g be two polynomials.

$$\begin{aligned}\langle \mathcal{H}f, g\rangle &= \int_{-\infty}^{\infty} e^{-t^2}\left(2tf'(t) - f''(t)\right) g(t)\, dt \\ &= -\int_{-\infty}^{\infty} \left(e^{-t^2} f'(t)\right)' g(t)\, dt \\ &= \int_{-\infty}^{\infty} e^{-t^2} f'(t) g'(t)\, dt \ ,\end{aligned}$$

where in the last line we have integrated by parts and dropped the boundary terms, since they vanish at $\pm\infty$. Now notice that the above RHS is symmetric in f and g, whence

$$\langle \mathcal{H}f, g\rangle - \langle \mathcal{H}g, f\rangle = 0 \implies \langle \mathcal{H}f, g\rangle = \langle f, \mathcal{H}g\rangle \ ,$$

using the symmetry of the inner product. This shows that $\mathcal{H}$ is self-adjoint.

(ii) Now notice that H_p solves Hermite's equation for p a nonnegative integer, and that this can be rewritten as

$$H_p''(t) - 2tH_p'(t) + 2pH_p(t) = 0 \iff \mathcal{H}H_p(t) = 2pH_p(t) \ ,$$

whence H_p is an eigenfunction of $\mathcal{H}$ with eigenvalue $2p$.

(iii) Now consider

$$\begin{aligned}\langle \mathcal{H}H_p, H_q\rangle &= 2p\,\langle H_p, H_q\rangle \\ \langle H_p, \mathcal{H}H_q\rangle &= 2q\,\langle H_p, H_q\rangle \ .\end{aligned}$$

Since $\mathcal{H}$ is self-adjoint, both expressions are the same, so that

$$2(p-q)\,\langle H_p, H_q\rangle = 0 \ ,$$

whence if $p \neq q$, then H_p and H_q are orthogonal.