4,000 years of **ADE**

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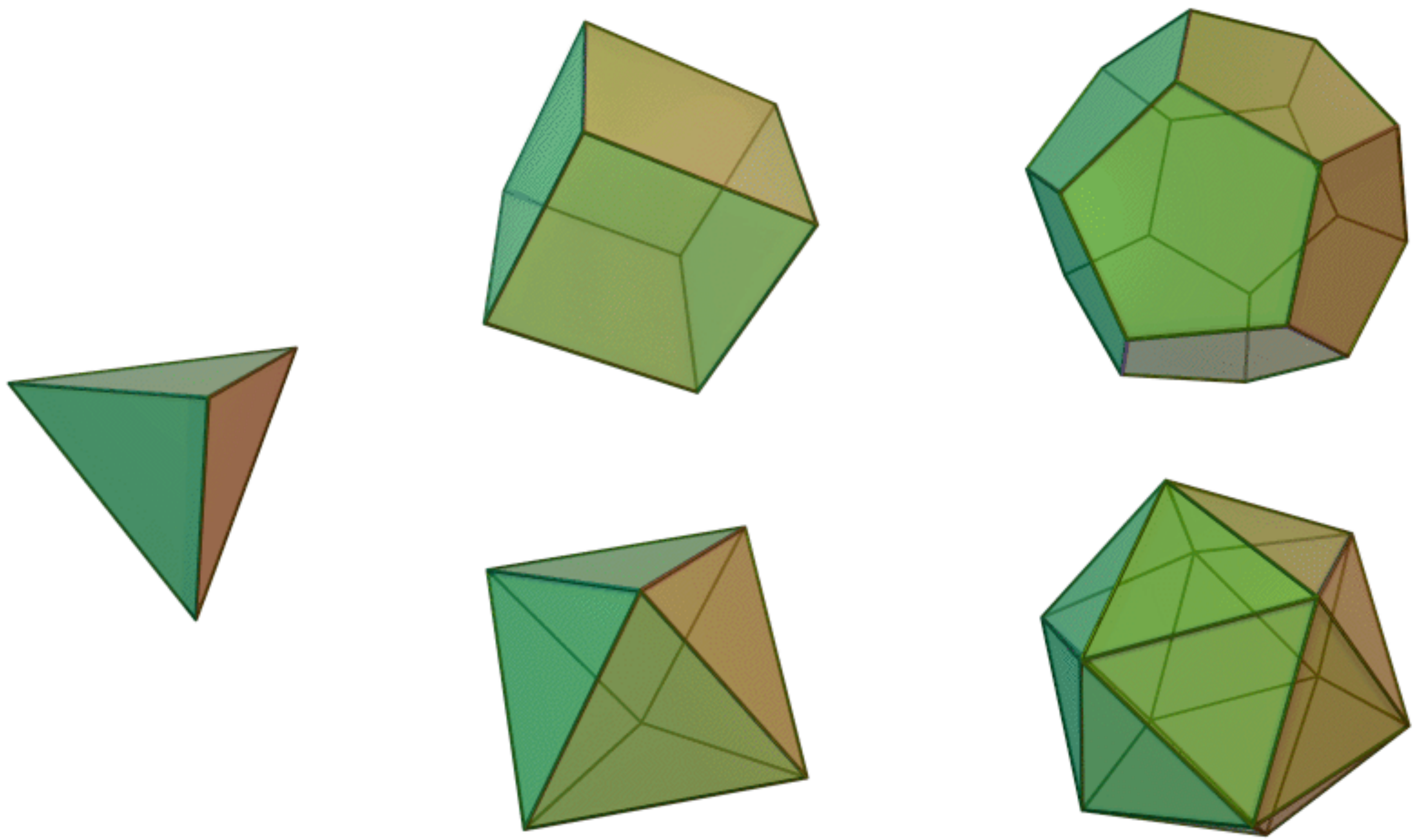
4,000 years ago in Scotland...

Platonic solids 1,500 years before Plato?

... or mathematical hoax?

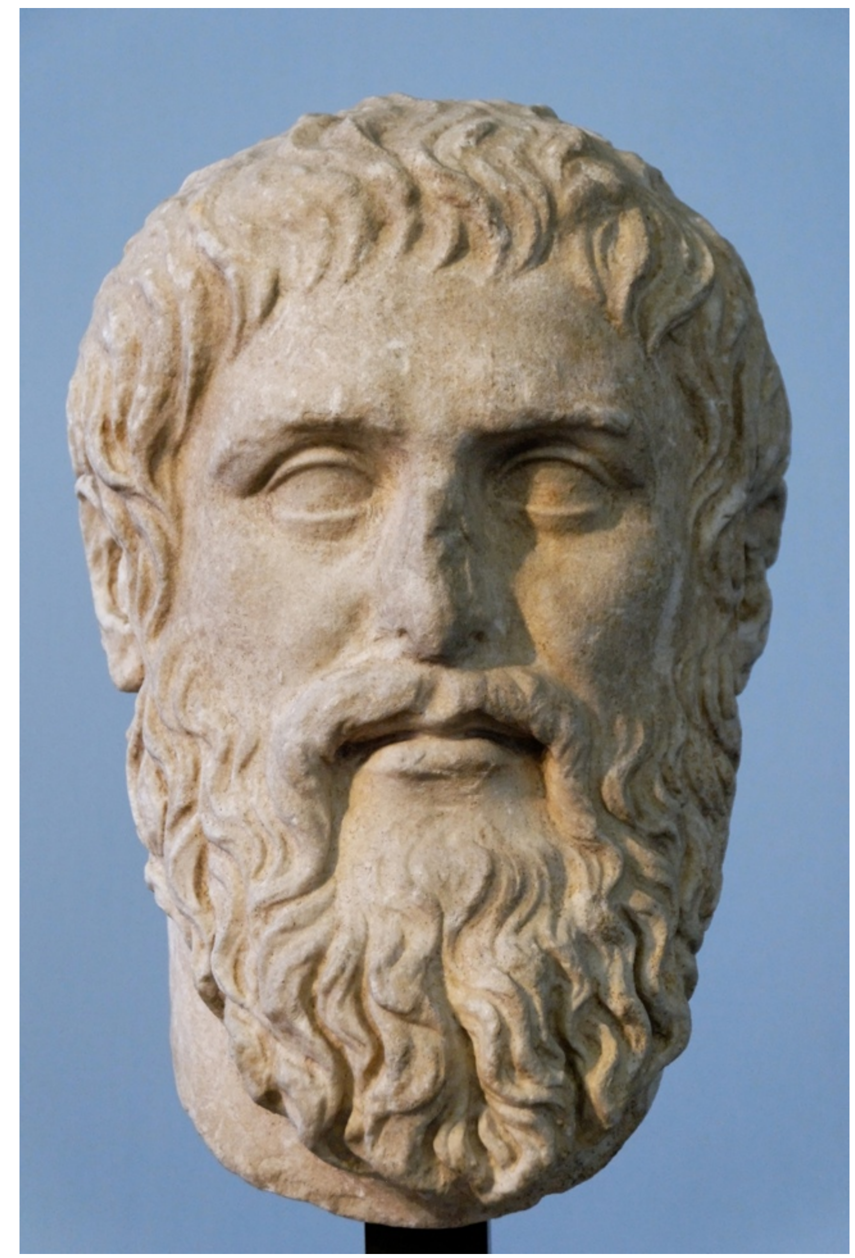
Platonic solids



Classified by Theaetetus
(≅ 417 BC - 369 BC)

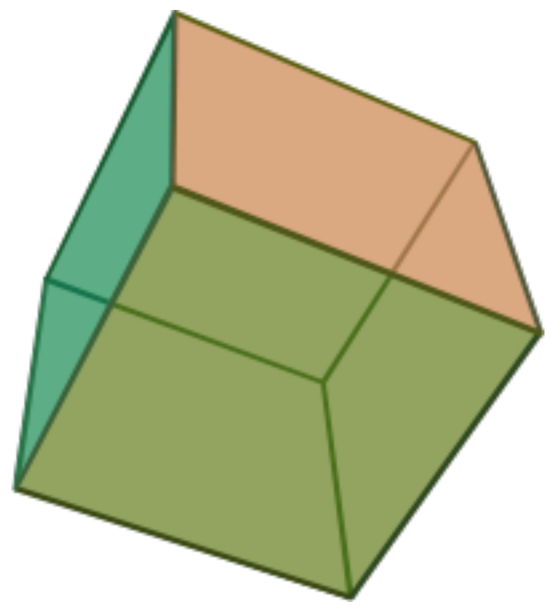
Mentioned by Plato
in the *Timaeus* (≅360 BC).

Hence the name.

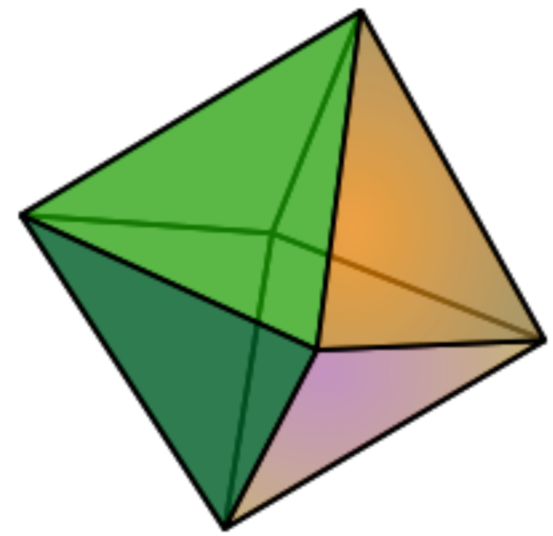


The Classical Elements

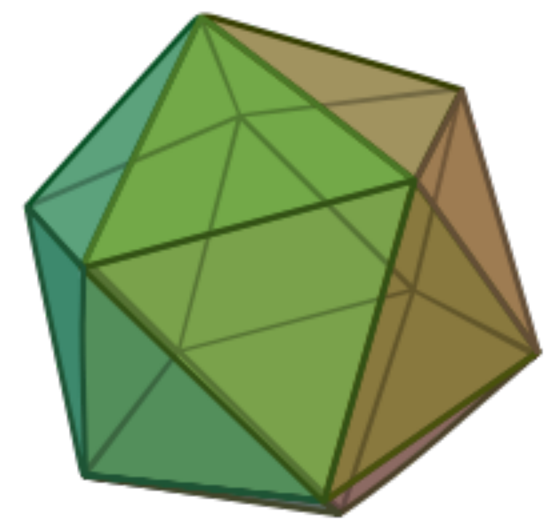
Fire



Earth



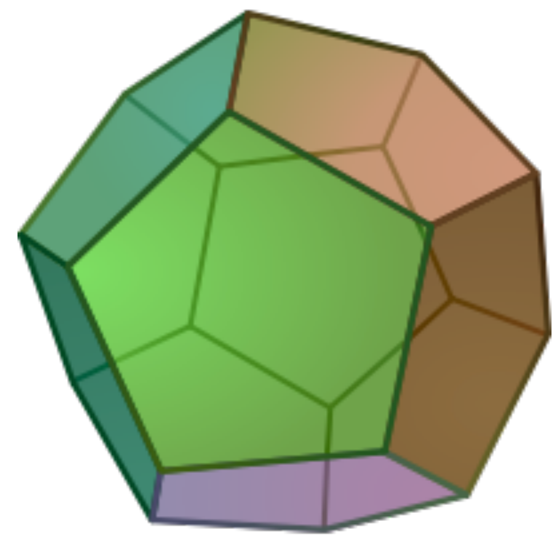
Air



Water

What about the dodecahedron???

Prediction: a new element!



Ether

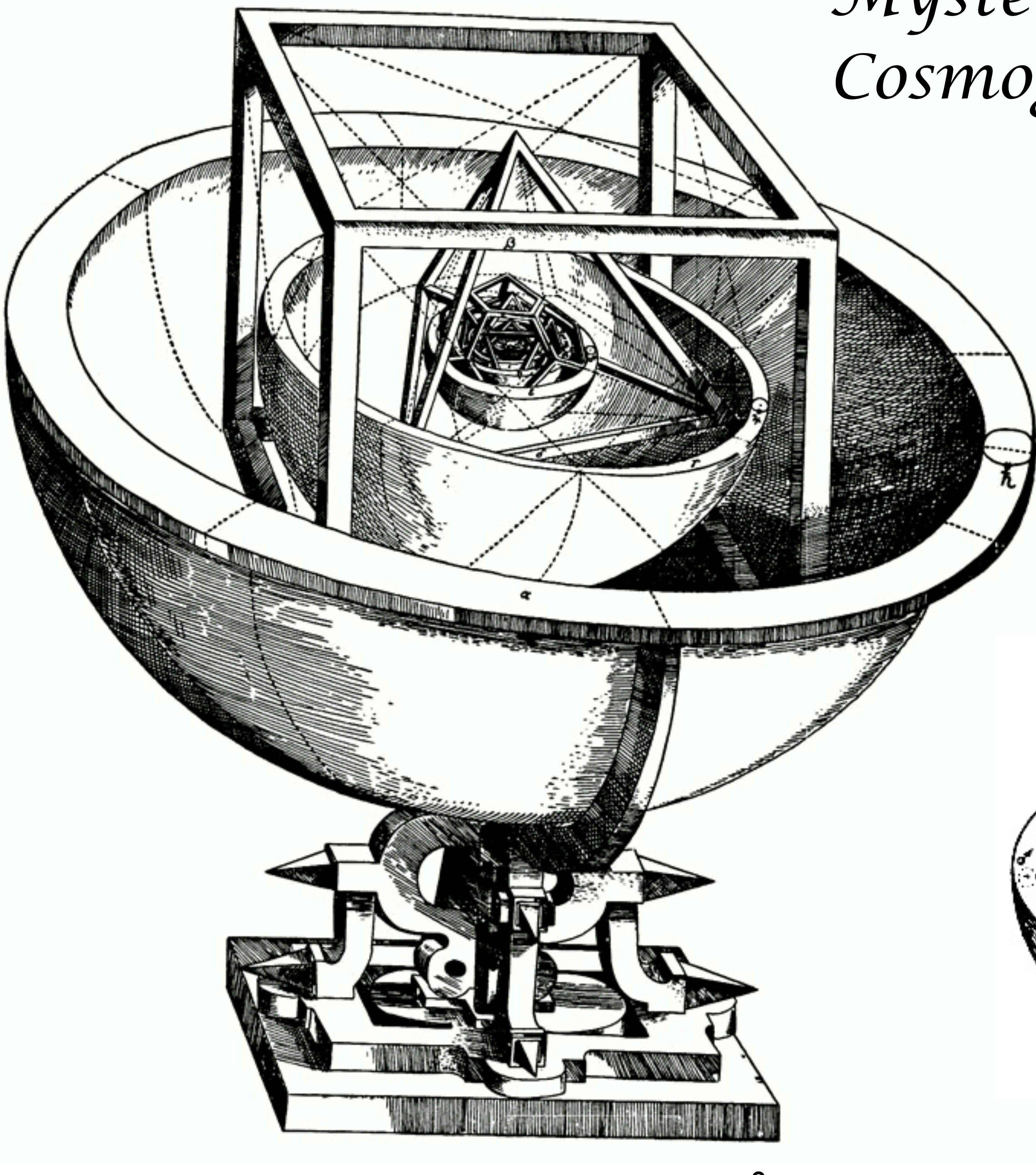
(no relation to the 19th century ether)

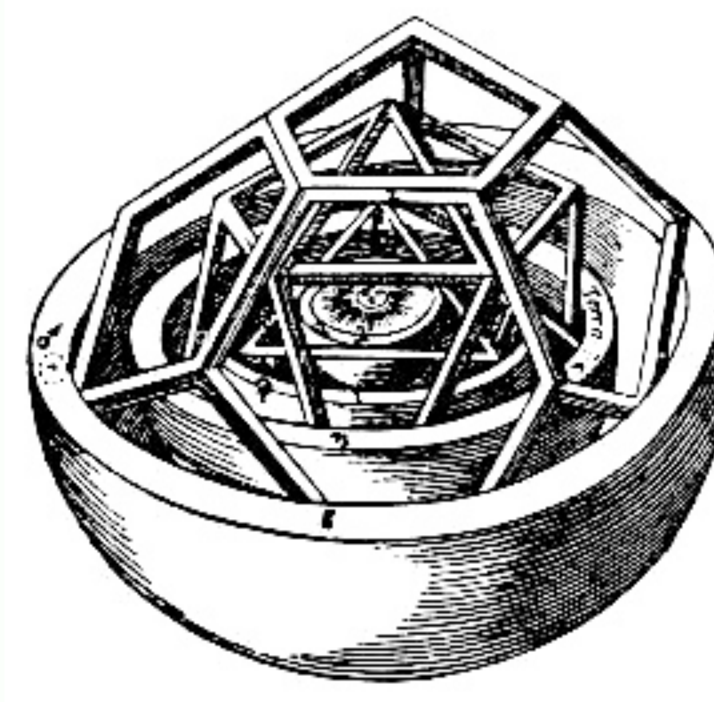
Planetary model

Johannes Kepler
(1571-1630)

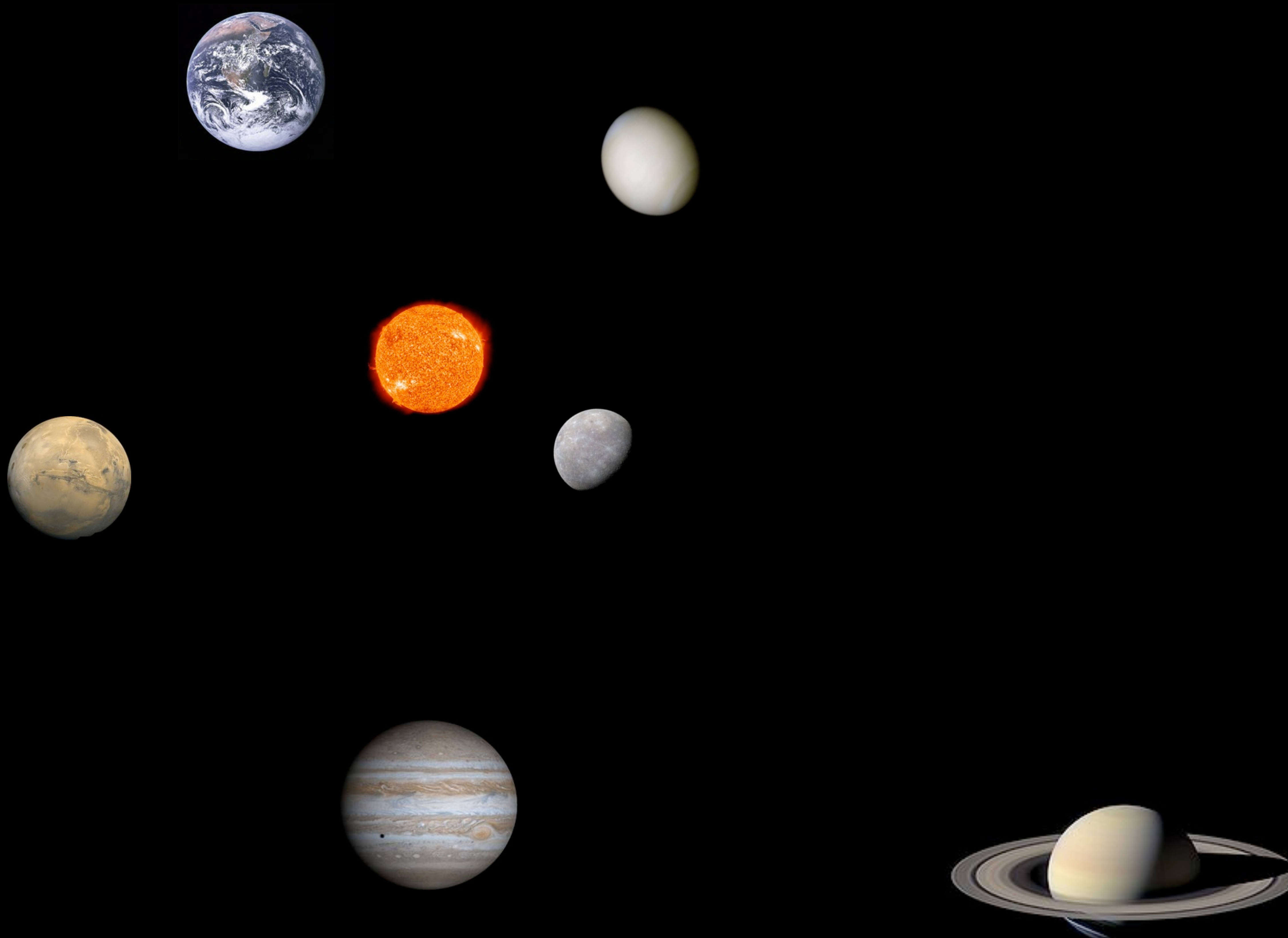
KEPLER based a model of the solar system on the platonic solids.

Mysterium Cosmographicum (1600)





Circumradius/Inradius ~ ratio of sizes of orbits



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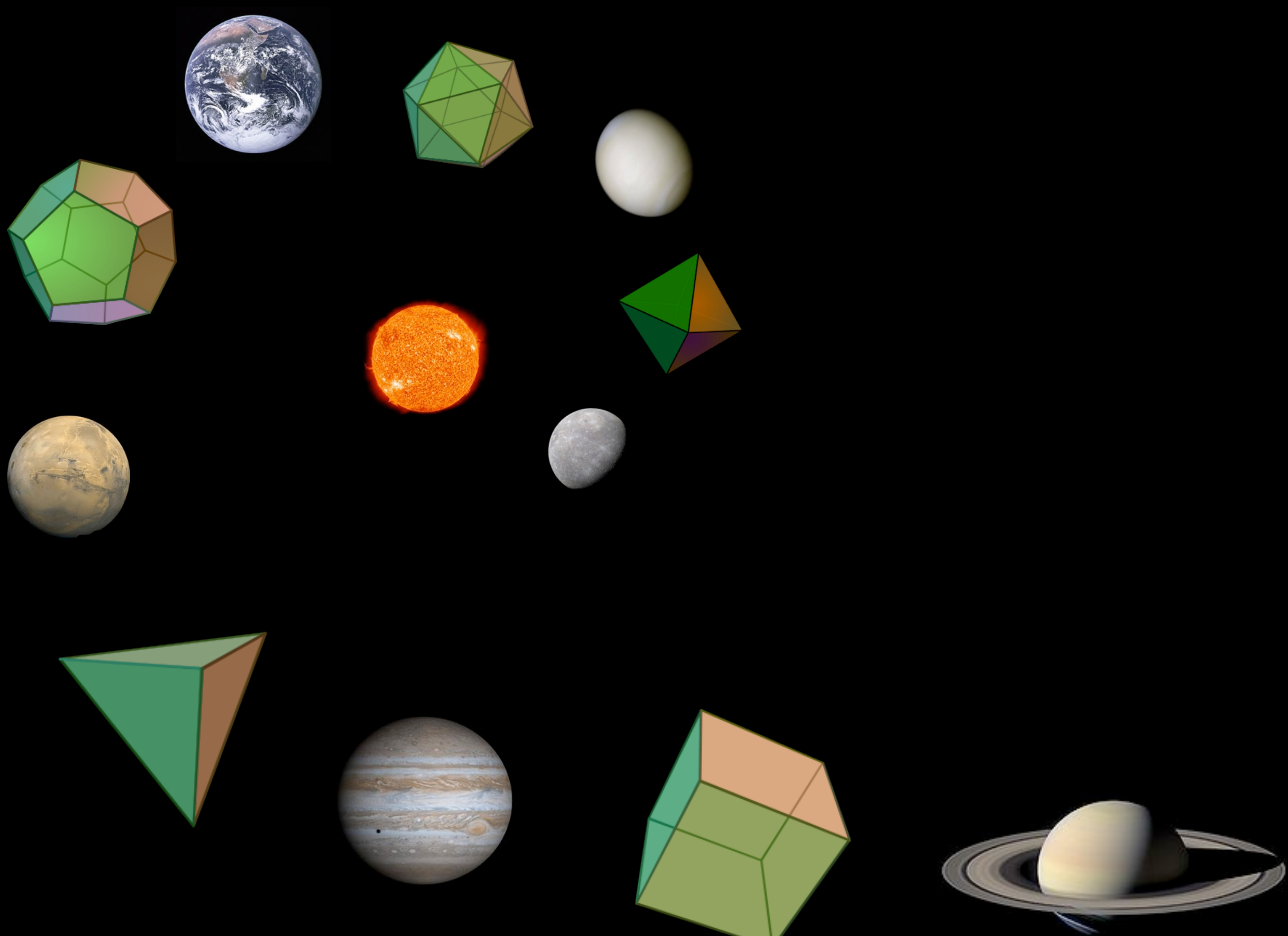

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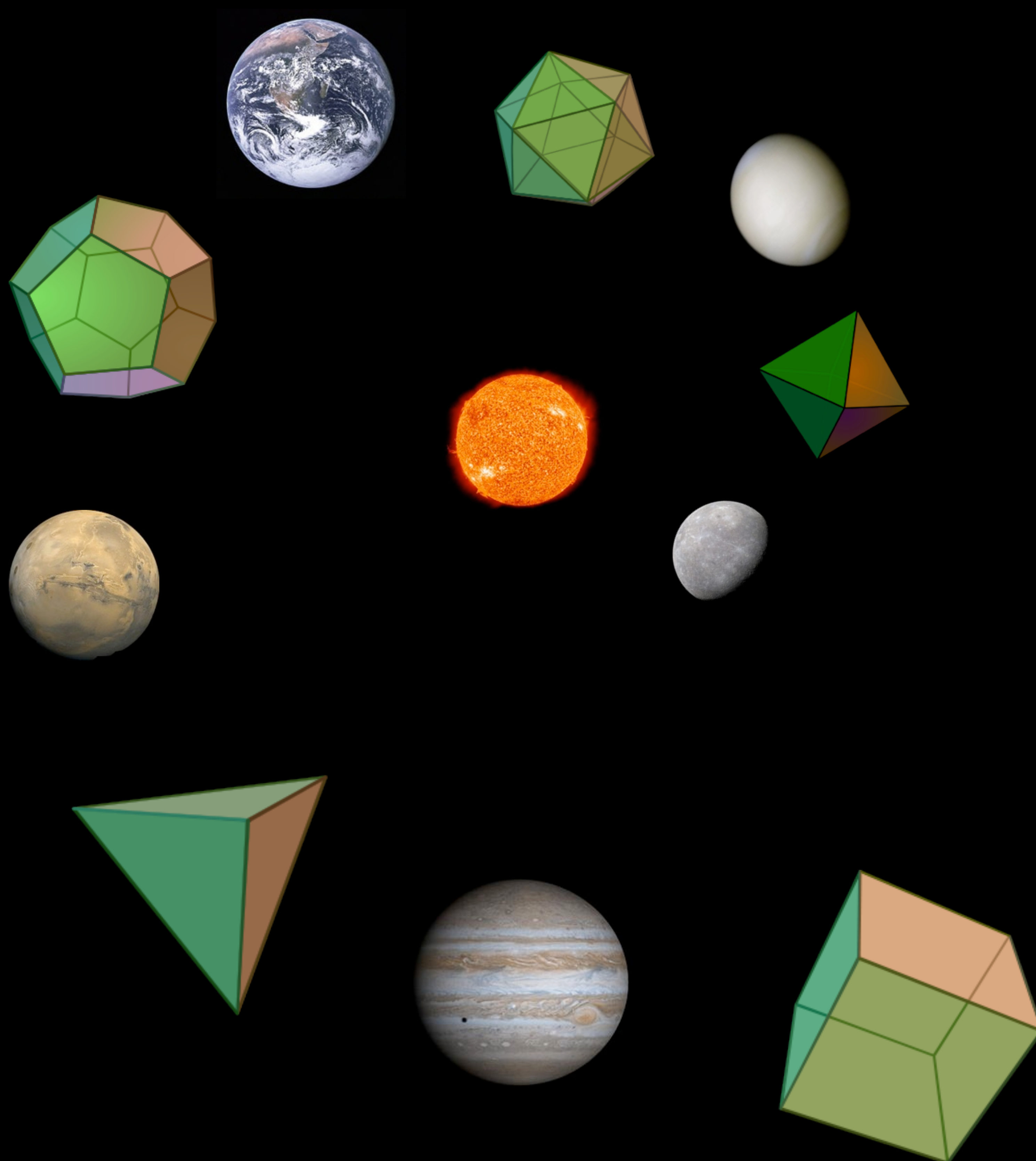


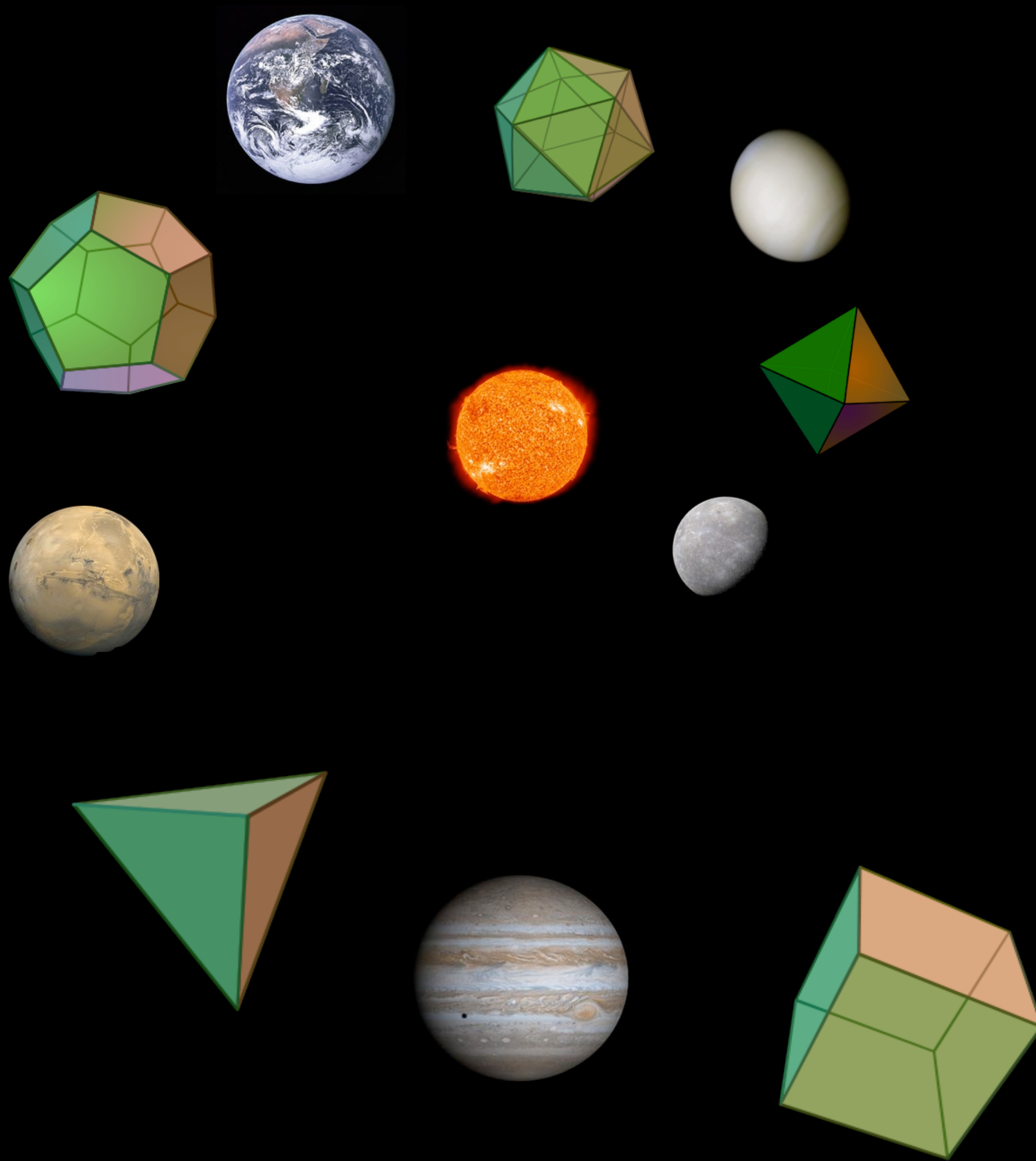
$$R_{\text{circ}}/R_{\text{in}}$$

$$T = 3$$

$$H, O \cong 1.73$$

$$D, I \cong 1.26$$

Circumradius/Inradius ~ ratio of sizes of orbits



$$R_{\rm circ}/R_{\rm in}$$

$$T = 3$$

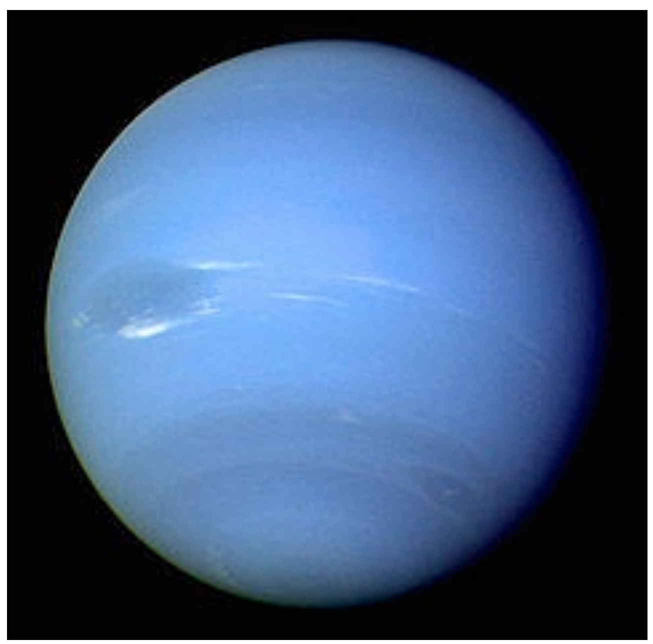
$$H, O \cong 1.73$$

$$D, I \cong 1.26$$

$$\cong \pm 25\%$$

But then Neptune was discovered

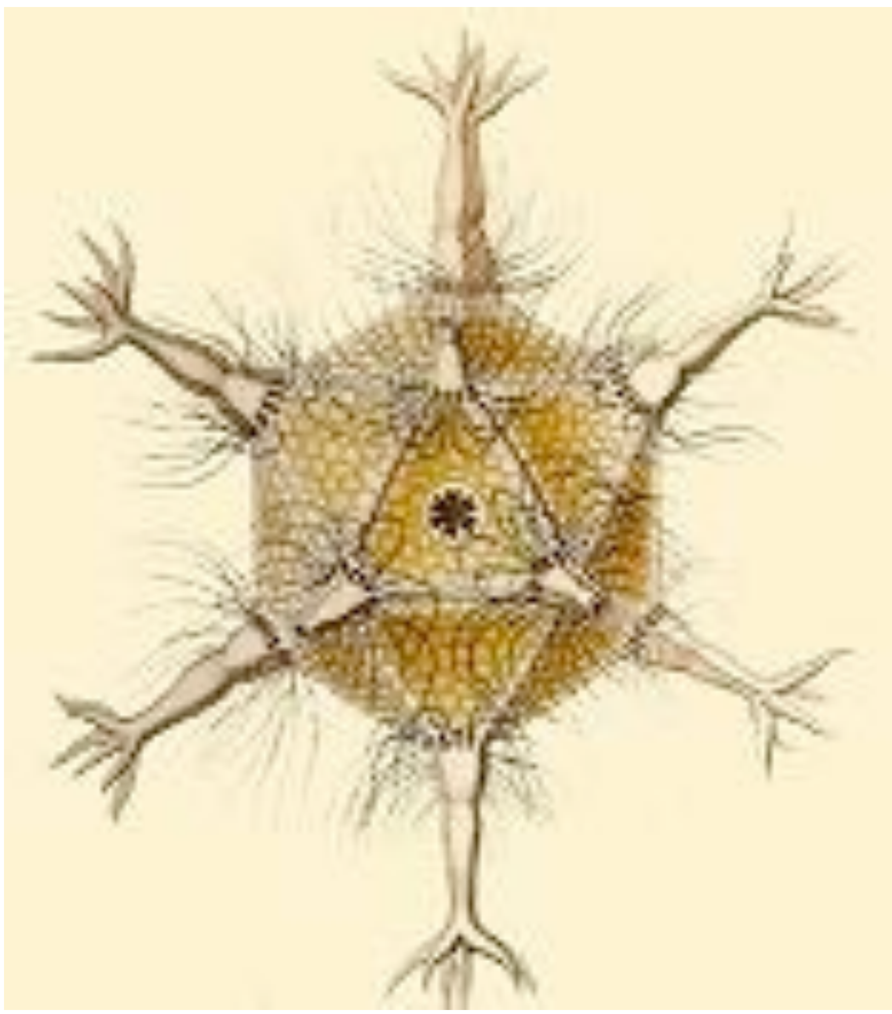
"with the point of a pen"



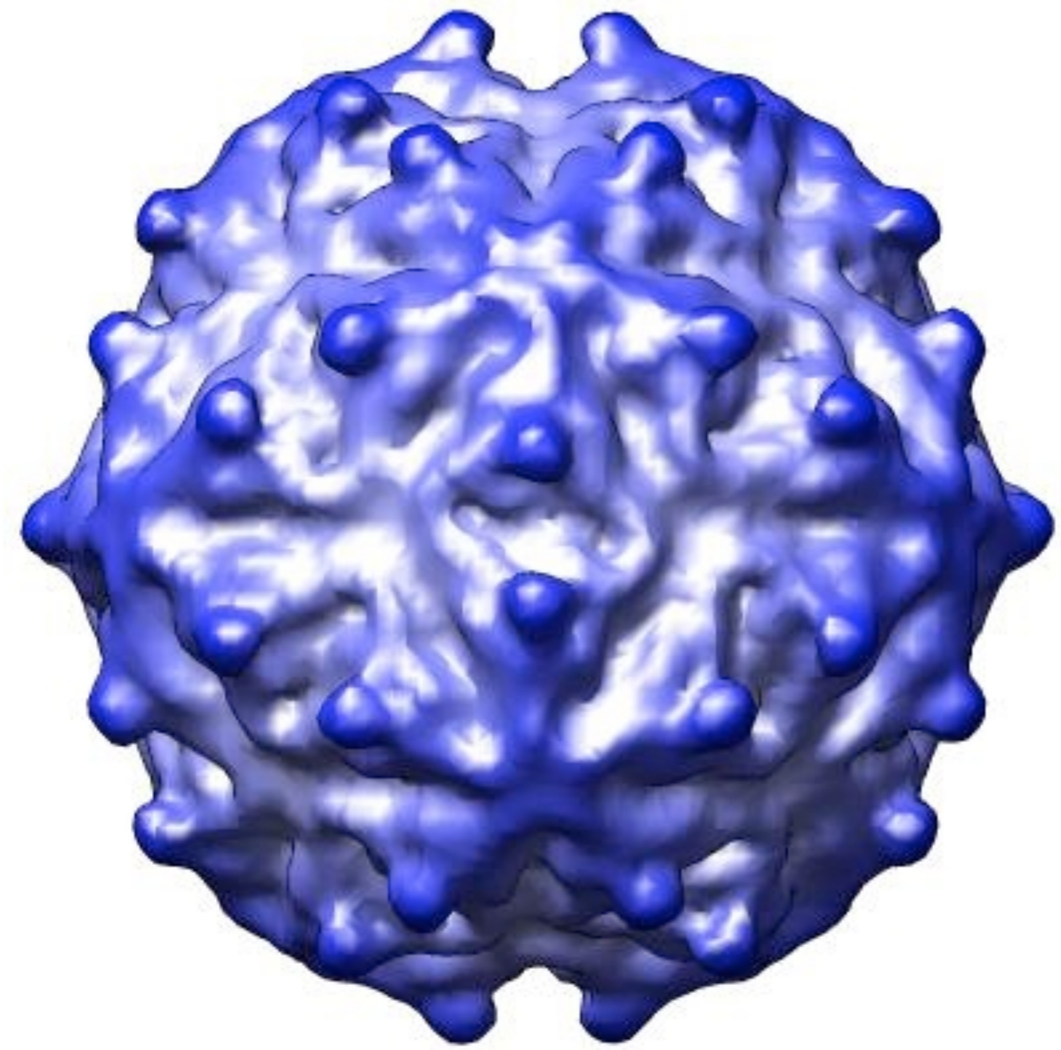
Urbain Le Verrier

(1811-1877)

Even in Biology...



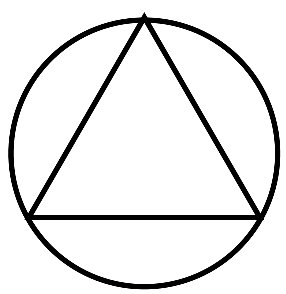
Circogonia icosahedra



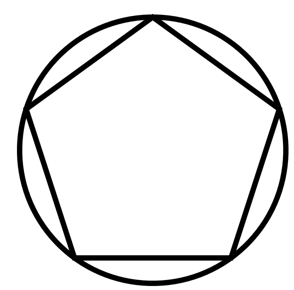
Pariacoto virus

Finite rotation groups

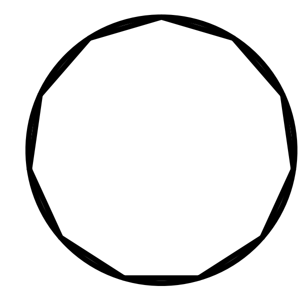
In 2 dimensions, finite rotation groups are cyclic.



C_3



C_5



C_{11}

In terms of complex numbers,

$$C_N = \left\{ e^{i2\pi k/N} \middle| k = 0, 1, \ldots, N-1 \right\}$$

And now in 3D

EULER: every (nontrivial) rotation about the origin fixes a line in $\mathbb{R}^3$.

That line intersects the unit sphere at two points: the **poles** of the rotation.

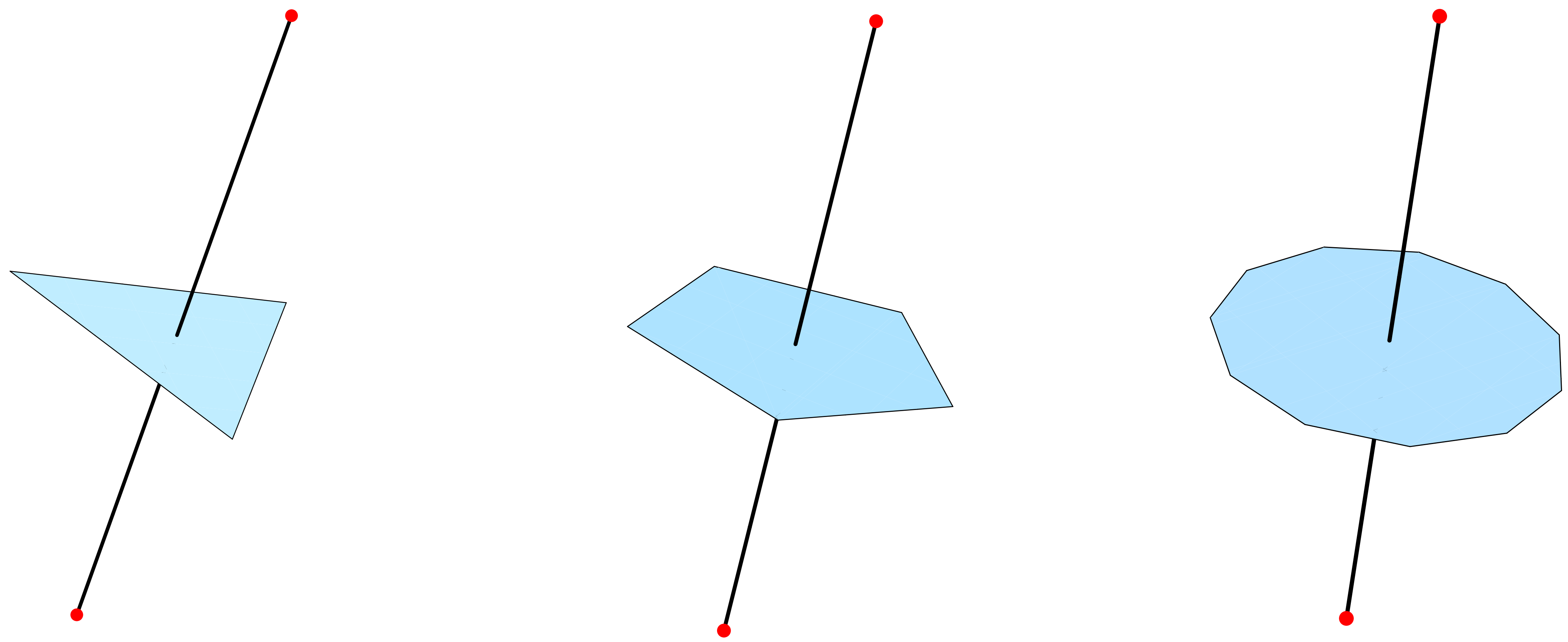
A finite subgroup **G** of rotations has a finite set **P** of poles. Moreover **G** acts on **P**:

$$x\ell = \ell \qquad \Longrightarrow \qquad yxy^{-1}y\ell = y\ell$$

The action of **G** partitions **P** into orbits.

There are two cases.

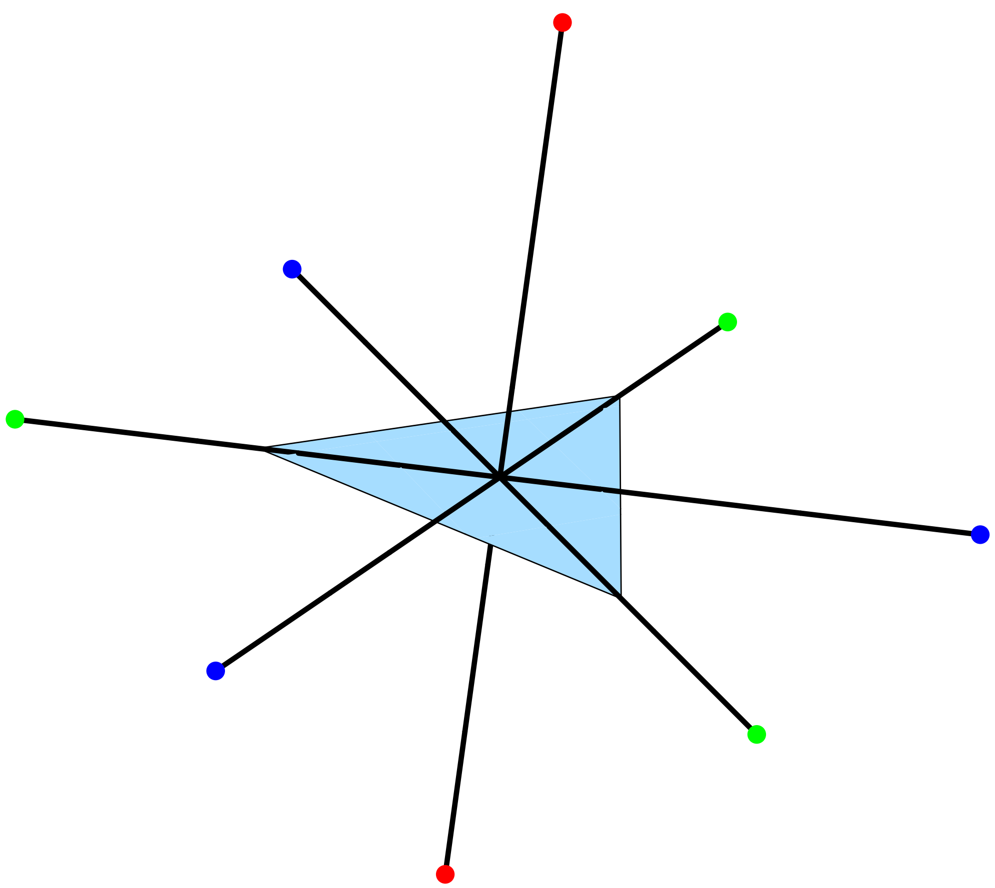
2 orbits **Cyclic groups**



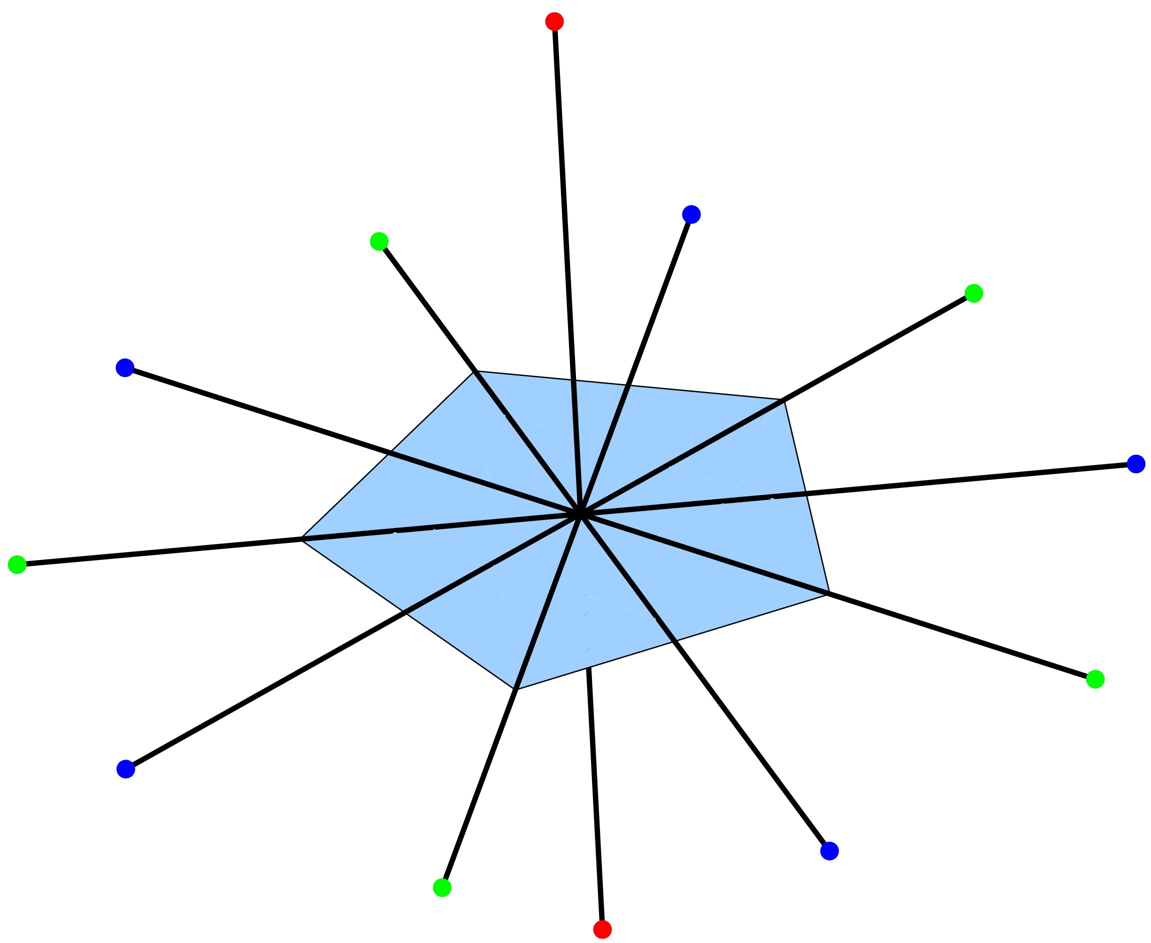
C_3 C_5 C_{11}

3 orbits

Dihedral groups

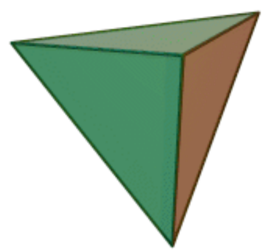


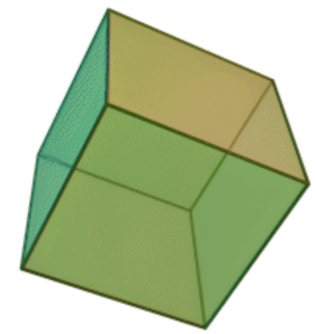
D_6



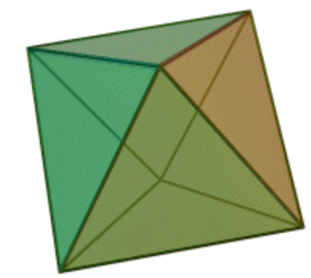
D_{10}

Polyhedral groups



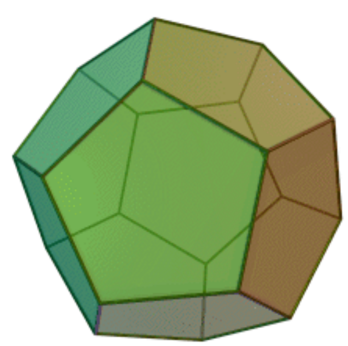


$\cong$

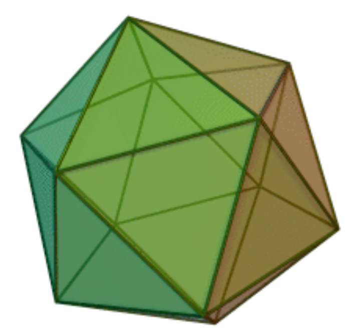


T_{12}

O_{24}



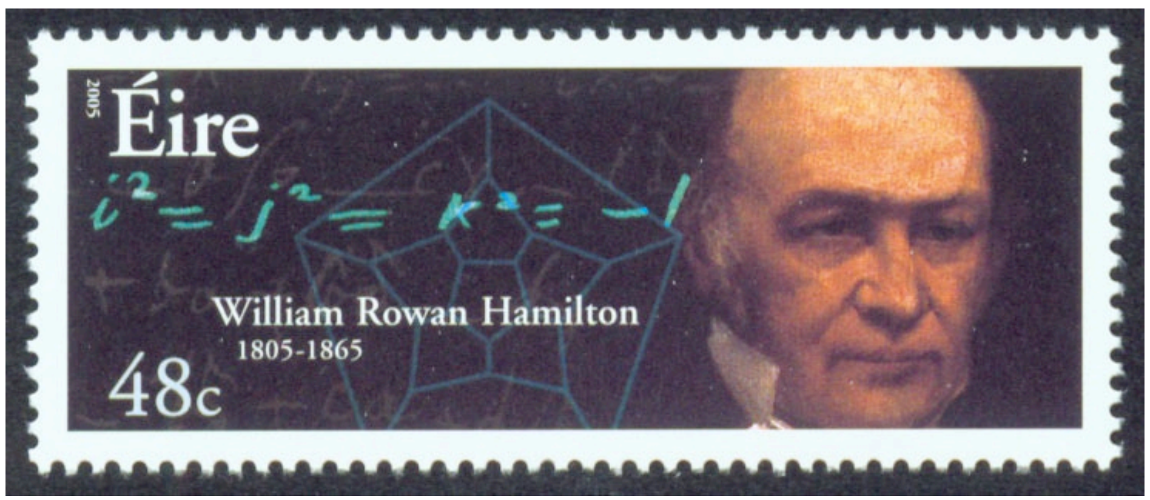
$\cong$

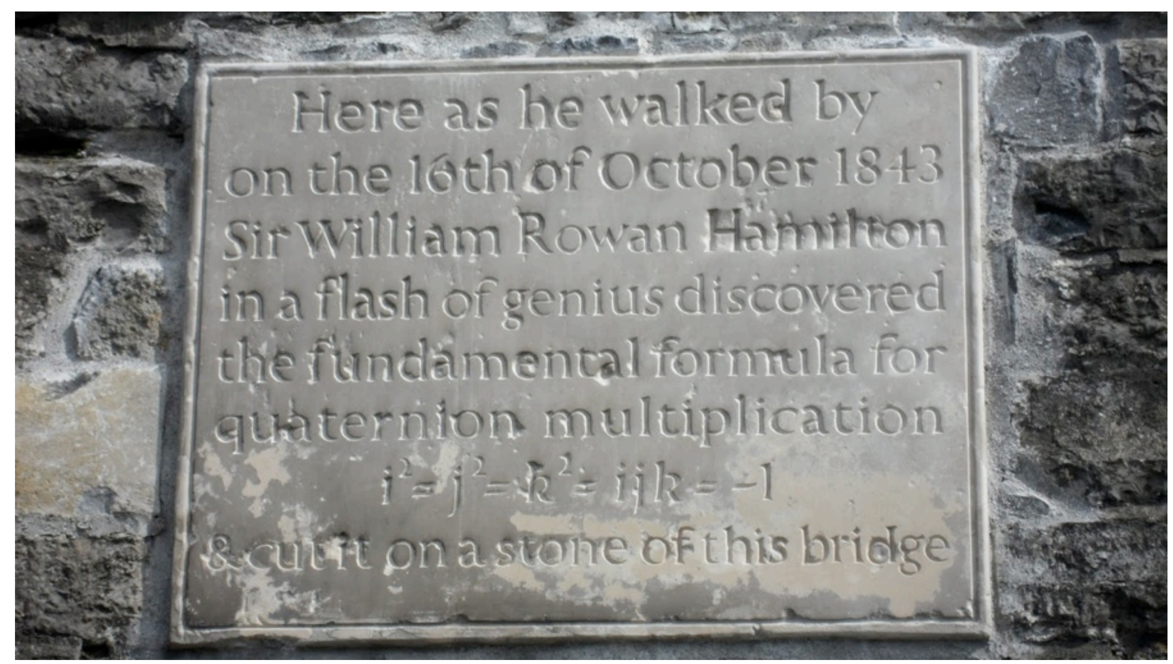


I_{60}

Quaternions

3d rotations are easily written using quaternions.





Add another imaginary unit j to $\mathbb{C}$ declaring that

$$ij = -ji$$

and let $\mathbb{H}$ denote the vector space consisting of

$$\boldsymbol{x} = x_0 + x_1 i + x_2 j + x_3 ij \qquad x_i \in \mathbb{R}$$

with the obvious associative multiplication.

This defines

the real division algebra of **quaternions**.

There is a notion of **conjugation**

$$\overline{\boldsymbol{x}} = x_0 - x_1 i - x_2 j - x_3 ij \qquad \text{and} \qquad \overline{\boldsymbol{x}\boldsymbol{y}} = \overline{\boldsymbol{y}}\,\overline{\boldsymbol{x}}$$

and of **norm**

$$|\boldsymbol{x}|^2 := \overline{\boldsymbol{x}}\boldsymbol{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

The quaternion algebra is **normed**:

$$|\boldsymbol{x}\boldsymbol{y}| = |\boldsymbol{x}||\boldsymbol{y}|$$

Unit-norm quaternions form a group: $\mathrm{Sp}(1)$

Topologically it is the 3-sphere.

A quaternion is **real** if

$$\overline{\boldsymbol{x}} = \boldsymbol{x} \qquad \Longrightarrow \qquad \boldsymbol{x} = x_0$$

and it is **imaginary** if

$$\overline{\boldsymbol{x}} = -\boldsymbol{x} \qquad \Longrightarrow \qquad \boldsymbol{x} = x_1 i + x_2 j + x_3 ij$$

$\mathrm{Sp}(1)$ acts on the imaginary quaternions:

$$\boldsymbol{x} \mapsto \boldsymbol{u}\boldsymbol{x}\boldsymbol{u}^{-1} = \boldsymbol{u}\boldsymbol{x}\overline{\boldsymbol{u}}$$

linearly and isometrically; in fact, by **rotations**.

Indeed all rotations can be obtained this way.

The map from quaternions to rotations is 2-to-1; the quaternions cover the rotations twice.

There are finite subgroups of unit quaternions covering the finite subgroups of rotations.

These are the **ADE** subgroups of quaternions.

$\mathsf{A}_{2n} \mapsto C_{2n+1}$ $\mathsf{E}_6 \mapsto T_{12}$

$\mathsf{A}_{2n+1} \mapsto C_{n+1}$ $\mathsf{E}_7 \mapsto O_{24}$

$\mathsf{D}_n \mapsto D_{2(n-2)}$ $\mathsf{E}_8 \mapsto I_{60}$

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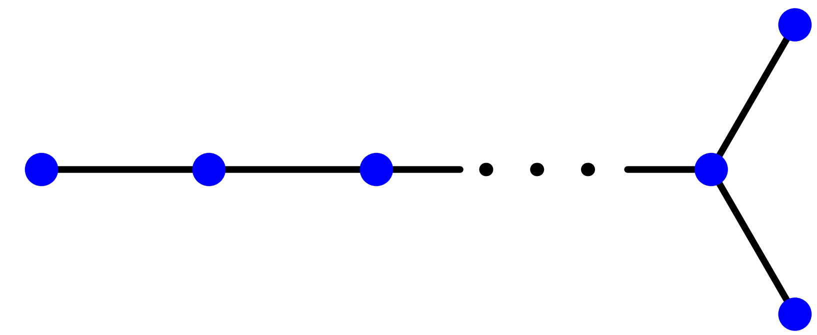
All but the first are double covers.

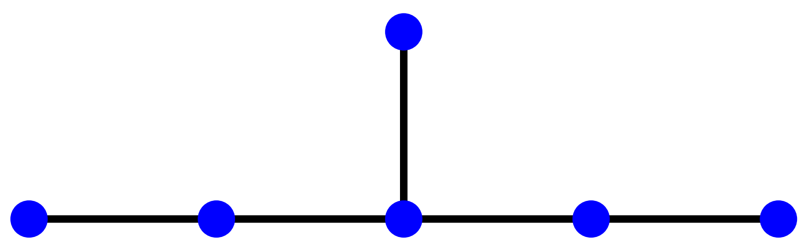
They are named after some famous graphs:

A_n

D_n

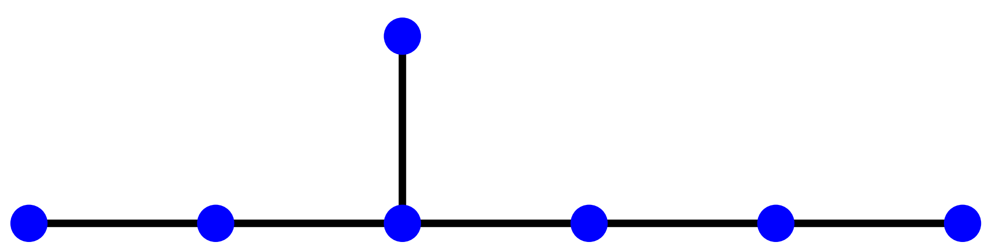


E_6

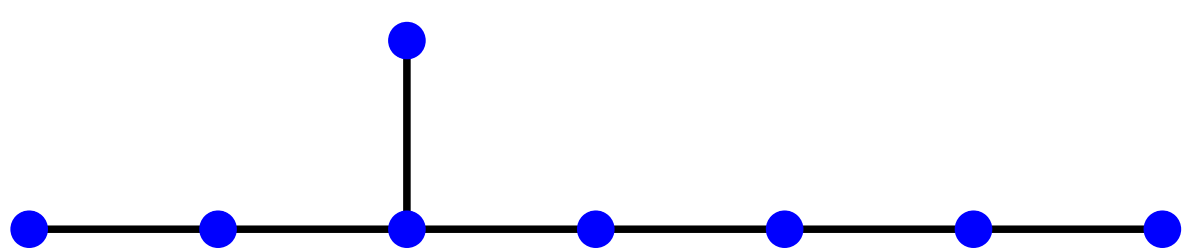


These are the (simply-laced) **Dynkin** diagrams.

E_7



E_8

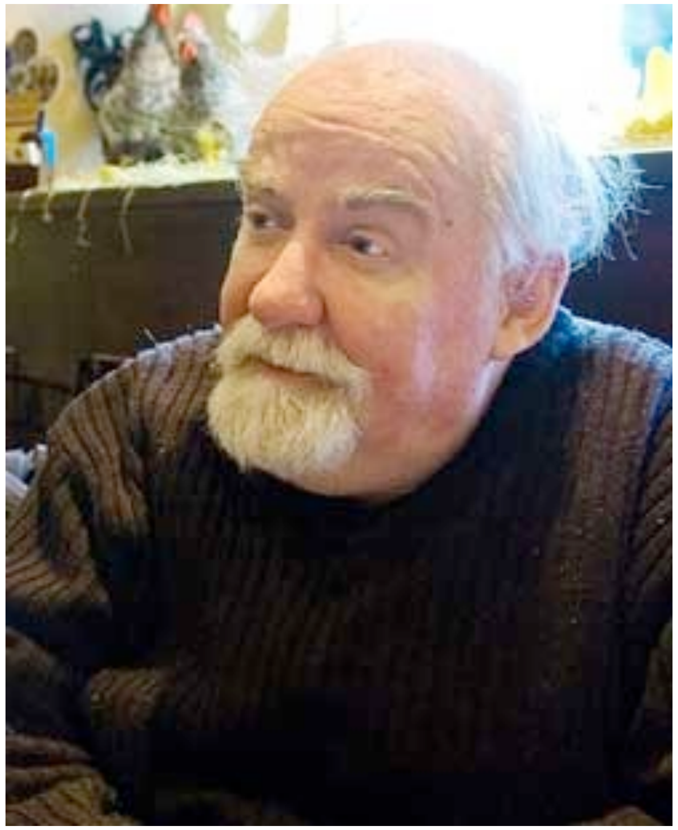


The McKay Correspondence

How to assign a graph to a finite subgroup of quaternions?

Look at its representations!

$$\rho : G \to \mathrm{GL}(V)$$



John McKay

A representation is **irreducible** if there is no proper subspace **W** $\subset$ **V** which is stable under **G**.

A finite group has a finite number of irreducible complex representations:

$$R_0, R_1, \ldots, R_N$$

trivial one-dimensional irrep

Finite subgroups of quaternions come with a two-dimensional complex representation **R**, coming from left quaternion multiplication:

$$x_0 + x_1 i + x_2 j + x_3 ij \quad \mapsto \quad \begin{pmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{pmatrix}$$

G a finite subgroup of quaternions

Define a graph **Γ** associated to **G**

vertices: complex irreps of **G**

$R_j \subset R \otimes R_i \quad \Longrightarrow \quad$ ∃ an **edge** *(ij)*

drop the vertex of the trivial representation

ADE Dynkin diagram!

e.g., $C_N = \left\langle e^{i2\pi/N} \right\rangle$

All complex irreps of an abelian group are one-dimensional, distinguished by a charge **k**

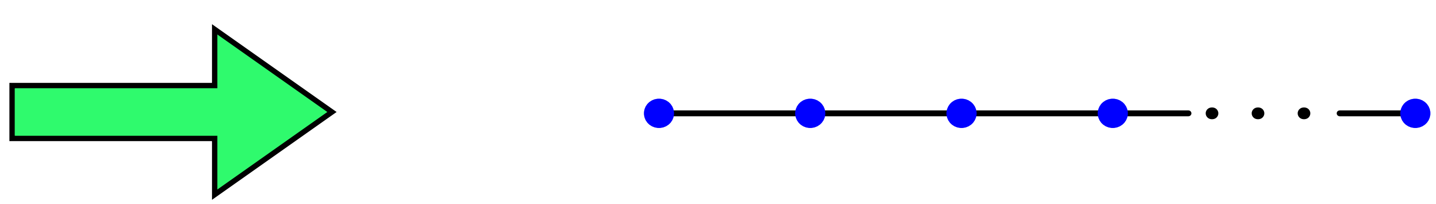
$$R_k : \quad e^{i2\pi/N} \mapsto e^{i2\pi k/N} \qquad k \in \mathbb{Z}/N\mathbb{Z}$$

The two-dimensional representation **R** is

$$R \cong R_1 \oplus R_{-1}$$

Since charge is additive:

$$R_k \otimes R \cong R_{k+1} \oplus R_{k-1}$$



A_{N-1}

Other **ADE** classifications

Simply-laced complex simple Lie algebras:

$\mathsf{A}_n \leftrightarrow \mathfrak{su}(n+1)$

$\mathsf{D}_n \leftrightarrow \mathfrak{so}(2n)$

$\mathsf{E}_6 \leftrightarrow \mathfrak{e}_6$

$\mathsf{E}_7 \leftrightarrow \mathfrak{e}_7$

$\mathsf{E}_8 \leftrightarrow \mathfrak{e}_8$

Élie Cartan
(1869-1951)

Wilhelm Killing
(1847-1923)

- Finite-type quivers (Gabriel's theorem)
- Kleinian surface singularities (Brieskorn)
- and many others...

Arnold in 1976 asked whether there is a connection between these objects which could explain why they are classified by the same combinatorial data.

Vladimir Arnold
(1937-2010)

ADE in Physics

In our* recent work on the AdS_4/CFT_3 correspondence, we came across a new **ADE** classification.

AdS_4/CFT_3 posits a correspondence between certain 7-dimensional manifolds and certain superconformal quantum field theories in 3 dimensions. We were interested in classifying a subclass of such geometries corresponding to theories with "$N \geq 4$ supersymmetry".

* Paul de Medeiros, JMF, Sunil Gadhia, Elena Méndez-Escobar, **arXiv:0909.0163**

A little twist

Given an automorphism of **G**

$$\tau : G \to G \qquad \tau(\boldsymbol{u}_1\boldsymbol{u}_2) = \tau(\boldsymbol{u}_1)\tau(\boldsymbol{u}_2)$$

we can have **G** act on $\mathbb{H}^2$

$$\boldsymbol{u} \cdot (\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{u}\boldsymbol{x}, \tau(\boldsymbol{u})\boldsymbol{y})$$

preserving and acting freely on the unit sphere in $\mathbb{H}^2$

The resulting quotients

$$X = S^7/G$$

are **all** the **smooth** manifolds for which the eleven-dimensional manifold

$$\mathrm{AdS}_4 \times X$$

is an **M-theory universe** preserving at least **half of the supersymmetry**.

(Non-smooth quotients are classified by fibred products of **ADE** groups. See Paul de Medeiros, JMF **arXiv:1007.4761**.)

Perhaps you will see a time when this will seem as scientifically naive as Plato or Kepler seem to us today!

Thank you!