A NEW EXPLICIT CONSTRUCTION OF W_3 FROM THE AFFINE ALGEBRA $A_2^{(1)}$

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ABSTRACT

We provide an explicit construction of the W_3 extended conformal algebra starting from the affine algebra $A_2^{(1)}$. Starting with the current algebra of $A_2^{(1)}$ we prove, by explicit construction, that the cohomology of a BRST operator associated to a set of first class constraints admits a representation of W_3. We prove this at the level of operator algebras with the help of a free field representation for $A_2^{(1)}$.

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§1 Introduction

Extended conformal algebras are associative operator product algebras which contain the Virasoro algebra as a subalgebra and are finitely generated by, in addition to the Virasoro generator, holomorphic Virasoro primaries in such a way that the following closure property is satisfied: that in the singular part of the operator product expansion (OPE) of these fields there appear only Virasoro descendents of the identity and of these primary fields themselves as well as normal ordered products thereof. Clearly if we dropped the requirement of these algebras being finitely generated we could always satisfy the closure property by augmenting the number of generators by whichever new primaries appear in the right hand side of the OPE, but then we would be dealing with objects far too general to offer any hopes of classification. The study of extended conformal algebras is interesting because they are the only hope of classifying rational conformal field theories with $c \geq 1$, this being the central poblem in conformal field theory.

The systematic study of extended conformal algebras was initiated by Zamolodchikov in **[1]** , where he analysed the possible associative operator algebras generated by a stress tensor (generating a Virasoro subalgebra) and one or more holomorphic primary fields of half-integral conformal weight $s \leq 3$ with the above closure property. He found a lot of already existing conformal field theories: free fermions ($s = \frac{1}{2}$), affine Lie algebras ($s = 1$), superVirasoro algebras ($s = \frac{3}{2}$), direct product of Virasoro algebras ($s = 2$); as well as two new algebras ($s = \frac{5}{2}$ and $s = 3$) which, unlike the others, are not Lie (super)algebras. The case $s = \frac{5}{2}$ satisfies the associativity condition for a specific value of the central charge ($c = -\frac{13}{14}$); whereas the case $s = 3$ yields an associative operator algebra—called W_3—for all values of the central charge. This extended algebra has been the focus of a lot of recent work and, in particular, it has been shown to be the symmetry algebra of the 3-state Potts model at criticality[2].

In **[3]** Fateev and Zamolodchikov investigated the degenerate representations and the minimal models associated to W_3 after developing the Feigin-Fuchs (or

Coulomb gas) representation of W_3 in terms of two free bosons with background charge. In this paper they also discovered some suggestive relations between the representation theory of W_3 and of the affine algebra $A_2^{(1)}$. In a later work Fateev and Lykyanov[4] generalised this work to the so-called W_n algebras obtained by extending the Virasoro algebra by holomorphic primary fields of weights $3, 4, \ldots, n$. They obtained a Feigin-Fuchs representation for these algebras in terms of $n-1$ free bosons with background charge by what amounts to a quantum analog of the Miura transformations in the theory of Korteweg-de Vries equations. They also found similar suggestive relations between W_n and $A_{n-1}^{(1)}$ as had been found in **[3]** for the case $n = 3$.

Other systematic approaches to extended conformal algebras are the so-called Casimir algebras. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of rank ℓ. Then the center of the universal enveloping algebra of $\mathfrak{g}$ is ℓ-dimensional and is spanned by the casimirs of $\mathfrak{g}$. Associated to each casimir we can define an operator (also referred to as the casimir with a little abuse of notation) in the universal enveloping algebra of the affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$. This is a generalisation of the Sugawara construction which from the quadratic casimir obtains the Sugawara stress tensor. It turns out that these operators (except for the Sugawara tensor itself) are primary of weight equal to the order of the casimir with respect to the Virasoro algebra generated by the Sugawara tensor. However, and except for specific values of the central charge, the operator algebra generated by the casimirs does not close in the sense described above. For example[5], in the case of the affine algebra $A_2^{(1)}$, in the OPE of the cubic casimir with itself there appears a new primary field which only decouples for $c = 2$. The way to make a closed operator algebra from the casimirs relies in a coset construction analogous to that of Goddard-Kent-Olive[6] for $A_1^{(1)}$. For example in the case of $A_2^{(1)}$, the authors of **[7]** obtained the W_3 operator algebra by constructing a weight three primary operator in the universal enveloping algebra of $A_2^{(1)} \times A_2^{(1)}$ which commutes with the diagonal $A_2^{(1)}$ subalgebra: hence a weight three primary field in the coset theory $(A_2^{(1)} \times A_2^{(1)})/A_2^{(1)}$. This operator turns out to form a closed operator algebra with the coset Virasoro generator if and only if one of the $A_2^{(1)}$ factors is at level 1. The weight three primary is constructed out of the cubic casimirs of the two factors but also contains mixed terms which are rather mysterious. In fact, the explicit construction of the operators generating the casimir algebras associated to other affine algebras is still lacking and the very existence of the higher casimir algebras has not been proven except for $A_2^{(1)}$, although there is certainly some evidence of it coming from character formulas for affine algebras[8].

In **[8]** , Bouwknegt continued the approach of Zamolodchikov by investigating the possible extensions of the Virasoro algebra by a holomorphic primary field of weight s. He argued that, if one demands associativity of the resulting operator algebra for generic values of the Virasoro central charge, the only solutions with integer s can occur for $s = 1, 2, 3, 4, 6$, which he suggests are related to the casimir algebras resulting from $A_1 \times D_1$, $A_1 \times A_1$, A_2, B_2, and G_2 respectively. The coset construction (or any other construction, for that matter) of these casimir algebras is known explicitly only for the first three cases. The explicit form of the algebra (as an abstract associative algebra) is known in all cases: **[9]** for the case of spin 4 and **[10]** for the case of spin 6.

In this paper we offer an explicit construction of W_3 from $A_2^{(1)}$ different from the coset construction of **[7]** . Our method is founded on its classical analogue for which it is possible to prove that a classical version of W_3 (as fundamental Poisson brackets of a Poisson manifold) arises from the symplectic reduction of the infinite dimensional Poisson manifold defined by the dual of the affine algebra $A_2^{(1)}$ relative to the action of a unipotent subgroup. This observation was made precise by Drinfeld and Sokolov[11] after a previous observation by Reiman and Semenov-Tyan-Shanskii. It is interesting to notice that also in our construction the cubic casimir of A_2 plays a fundamental role. In fact, the operators which generate the W_3 algebra are induced from (the BRST completions of) deformations (in a sense made more precise later on) of the Sugawara stress tensor and of the cubic casimir.

Let us very briefly describe this method for the case of $A_n^{(1)}$. The dual space

M of $A_n^{(1)}$ has a canonical Poisson structure which is invariant under the action of the corresponding loop group. Let $\mathfrak{n}_+$ denote the subalgebra of A_n generated by the elements associated to the positive roots and $\widehat{\mathfrak{n}}_+$ the corresponding affine algebra. The action of the corresponding loop group is Poisson and gives rise to a moment mapping $J : M \to \widehat{\mathfrak{n}}_+^*$. Drinfeld and Sokolov consider the level set M_o of J coresponding to the following constraints:

$$E_{-\alpha}(z) = \begin{cases} 1, & \text{if } \alpha \in \Delta, \\ 0, & \text{otherwise;} \end{cases} \tag{1.1}$$

where α is a positive root, $E_{-\alpha}$ the generator of A_n corresponding to the negative root $-\alpha$, and Δ is a choice of simple roots. Equivariance of the moment mapping implies that M_o is stabilized by the loop group corresponding to $\widehat{\mathfrak{n}}_+$ which allows us to quotient by its action and obtain a manifold $\widetilde{M}$ which inherits a natural Poisson structure from that on M. In order to write the fundamental Poisson brackets of $\widetilde{M}$ explicitly we need to coordinatize $\widetilde{M}$. Since $\widetilde{M}$ is defined as a quotient there are no preferred coordinates. In order for $\widetilde{M}$ to inherit coordinates from M_o it is necessary to exhibit it as a submanifold of M_o, *i.e.* , to fix a gauge. Different choices of a gauge slice will give different coordinates and, hence, different fundamental Poisson brackets. Drinfeld and Sokolov show there exists a choice of gauge slice in which the fundamental Poisson brackets are those of n free bosons (for A_n); whereas there exists a different gauge slice in which the fundamental Poisson brackets represent the Gelfand-Dickey algebras which, for A_n is a classical version of W_{n+1}, the case $n = 1$ coresponding to the Virasoro algebra. Moreover the change of coordinates from one gauge slice to another (*i.e.* , the corresponding gauge transformation) is nothing but the Miura transformation of the KdV theory.

The quantum version of this Poisson reduction was carried out explicitly for the case of A_1 (*i.e.* , Virasoro) by Bershadsky and Ooguri[**12**]. In this paper we do the analogous thing for A_2 yielding the W_3 algebra of Zamolodchikov. The quantum analogue of the Poisson reduction is treated, as in **[12]** , within the framework of BRST cohomology. Bershadsky and Ooguri also proved, using the recent work of Felder[13] and of Bernard and Felder[14], that the correspondence took irreducible representations into irreducible representations, at least in the case of completely degenerate representations. In our case there are no results available of this isnd for W_3 nor for A_2. If such results existed, however, the same method of **[12]** can be extended to our case to prove that at least for degenerate representations, the correspondence takes irreducible representations of $A_2^{(1)}$ into irreducible representations of W_3.

This paper is organised as follows. Section 2 contains the explicit construction of the W_3 algebra. The idea behind the construction is the following. In the BRST framework physical operators are equivalence classes of BRST invariant operators: two such operators being called equivalent when their difference is a BRST (anti)commutator. In this spirit we construct two BRST invariant operators which, in cohomology (*i.e.* , up to BRST (anti)commutators), obey the operator algebra of W_3. In fact, we show this by first computing the BRST cohomology with the use of a free field representation for $A_2^{(1)}$ and then showing that in cohomology the operators agree with those in the Feigin-Fuchs representation of W_3 obtained by Fateev and Zamolodchikov. The BRST invariant operators we construct are such that their ghost independent parts are deformations (via the addition of terms of lower order in currents) of the casimirs of $A_2^{(1)}$. Section 3 contains the computation of the BRST cohomology in the free field representation. It turns out that the results of that section generalise considerably (in fact, to arbitrary simple Lie algebras), but we leave the precise statements of the theorems as wells as their proofs to a separate publication where we also hope to discuss some applications. Finally in section 4 we offer some conclusions and discuss some obvious open problems.

§2 The Basic Construction

We start by setting down our conventions for $A_2 \cong \mathfrak{sl}_3$. The algebra is generated by $\{h^\pm, J_i^\pm\}_{i=1}^3$ with the following brackets:

$$\begin{aligned}
&[h^+\,,\,J_1^\pm] = \pm J_1^\pm && [h^+\,,\,J_2^\pm] = \pm J_2^\pm \\
&[h^+\,,\,J_3^\pm] = \pm 2J_3^\pm && [h^-\,,\,J_1^\pm] = \pm\sqrt{3}J_1^\pm \\
&[h^-\,,\,J_2^\pm] = \mp\sqrt{3}J_2^\pm && [h^-\,,\,J_3^\pm] = 0 \\
&[J_1^+\,,\,J_1^-] = \frac{1}{2}(h^+ + \sqrt{3}h^-) && [J_2^+\,,\,J_2^-] = \frac{1}{2}(h^+ - \sqrt{3}h^-) \\
&[J_3^+\,,\,J_3^-] = h^+ && [J_1^\pm\,,\,J_2^\pm] = \pm J_3^\pm \\
&[J_1^\pm\,,\,J_3^\mp] = \mp J_2^\mp && [J_2^\pm\,,\,J_3^\mp] = \pm J_1^\mp
\end{aligned} \tag{2.1}$$

all others being zero. We fix an invariant symmetric bilinear form $(,)$ on the algebra given by

$$(h^\pm, h^\pm) = 2 \text{ and } (J_i^+, J_i^-) = 1\ \forall i\ . \tag{2.2}$$

Let $A_2^{(1)}$ denote the corresponding (untwisted) affine algebra. Its Lie bracket is encoded in the operator product expansion of the currents $X(z)$ for $X \in \mathfrak{sl}_3$:

$$X(z)\,Y(w) = \frac{k(X,Y)}{(z-w)^2} + \frac{[X\,,\,Y](w)}{z-w} + \text{reg}\ . \tag{2.3}$$

The Sugawara stress tensor is given by

$$T_S(z) = \frac{1}{2(k+3)} g^{ab}\left(X_a X_b\right)(z)\ , \tag{2.4}$$

where $\{X_a\}$ is any basis for $\mathfrak{sl}_3$, g^{ab} is the inverse of $g_{ab} \equiv (X_a, X_b)$, and where $()$ indicates normal ordering. Our conventions for normal ordering are those described in the appendix of **[5]** . The currents $X(z)$ are primary fields of weight one with respect to the Sugawara stress tensor which obeys the Virasoro algebra with central charge $c_S = 8k/(k+3)$.

In the case of $\mathfrak{sl}_3$, the constraints (1.1) are given by:

$$J_1^-(z) = J_2^-(z) = 1 \qquad J_3^-(z) = 0\ . \tag{2.5}$$

Since we are interested in inducing the structure of a Virasoro module in the resulting quantum theory, we must impose these constraints in a conformally covariant fashion: *i.e.* , we need to deform the Sugawara stress tensor so that the conformal weights of $J_1^-(z)$ and $J_2^-(z)$ are zero. If we demand that the constraints remain primary and that the deformation consists of adding pieces of lower order in currents, it is easy to see that the unique such deformation is given by

$$T_{\rm def}(z) \equiv T_S(z) - \partial h^+(z) \ , \tag{2.6}$$

which still satisfies the Virasoro algebra with central charge equal to $8k/(k+3) - 24k$. Relative to $T_{\rm def}$ all the currents remain primary except for h^+. In fact,

$$T_{\rm def}(z)\, X(w) = \frac{2k(X,h^+)}{(z-w)^3} + \frac{\Delta_X X(w)}{(z-w)^2} + \frac{\partial X(w)}{z-w} + \text{reg} \ , \tag{2.7}$$

where the conformal weigths Δ are given by

$$\Delta_{h^\pm} = 1 \qquad \Delta_{J_1^\pm} = 1 \pm 1 \qquad \Delta_{J_2^\pm} = 1 \pm 1 \qquad \Delta_{J_3^\pm} = 1 \pm 2 \ . \tag{2.8}$$

In order to impose the constraints (2.5) à la BRST we introduce three fermionic (b,c) systems of weights $(0,1)$, $(0,1)$, and $(-1,2)$ with operator product expansions

$$b_i(z)\, c_j(w) = \frac{\delta_{ij}}{z-w} + \text{reg} \ ; \tag{2.9}$$

and we define the BRST operator

$$d \equiv \oint_{C_0} \frac{dz}{2\pi i}\, j(z) \ , \tag{2.10}$$

where

$$j(z) = -c_1(z) - c_2(z) + \sum_{i=1}^{3} \left(c_i J_i^-\right)(z) + (c_1 c_2 b_3)(z) \ . \tag{2.11}$$

Since the constraint algebra closes, the BRST operator is square-zero: $d^2 = 0$. Its cohomology is therefore defined and this will, in fact, be the space of quanta of the constrained theory. The operators which survive upon reduction are those which (anti)commute with the BRST operator modulo those which can be written as $[d\,,\,\text{something}]$.

Adding the stress tensor $T^{\rm gh}$ of the ghosts

$$T^{\rm gh}(z) = (\partial b_1\, c_1) + (\partial b_2\, c_2) + 2\,(\partial b_3\, c_3) + (\partial b_3 \partial c_3) \tag{2.12}$$

to $T_{\rm def}$ we obtain a stress tensor $T_{\rm tot}$ which obeys a Virasoro algebra with central charge $8k/(k+3) - 24k - 30$ and relative to which the BRST current $j(z)$ is a primary field of weight one. As a consequence of this fact the total stress tensor $T_{\rm tot}$ is BRST invariant and hence makes the BRST cohomology into a Virasoro module. We can write the central charge in a more suggestive fashion:

$$c = 50 - 24(k+3) - \frac{24}{k+3}\ , \tag{2.13}$$

which, assuming that $k+3 = p/q$ can be written as

$$c = 2\left(1 - \frac{12(p-q)^2}{pq}\right)\ , \tag{2.14}$$

which is precisely the central charge for the minimal series of the W_3 algebra if p, q are relatively prime natural numbers. Notice however that in this case, the level cannot be an integer.

We also want to show that not only the Virasoro algebra descends to cohomology but that in fact the BRST cohomology affords the structure of a W_3 module. In order to prove this we need to show that in the reduced operator algebra (*i.e.* , the algebra of BRST invariant operators modulo BRST exact operators) we find W_3 as a subalgebra. More precisely we must show that the algebra generated by the $A_2^{(1)}$ currents and the (b,c) fields has a subalgebra which yields W_3 upon reduction. In order to prove this, we find it convenient to embed this operator algebra in a larger one. This larger operator algebra is the algebra generated by the modes of the free fields in the free field realization of $A_2^{(1)}$ given, for example, in **[12]** . We are clearly allowed to do this as long as the operators realising the W_3 algebra are expressed in terms of currents.

The free field realization consists of two free bosons $\{\partial\varphi_i\}_{i=1}^2$ and three bosonic $\{(\beta_i,\gamma_i)\}_{i=1}^3$ systems and the currents are given by:

$$
\begin{aligned}
h^- =&\sqrt{3}(\beta_1\gamma_1-\beta_2\gamma_2)-i\alpha'_+\partial\varphi_2\\
h^+ =&\beta_1\gamma_1+\beta_2\gamma_2+2\beta_3\gamma_3-i\alpha'_+\partial\varphi_1\\
J_1^- =&\beta_1+\gamma_2\beta_3\\
J_2^- =&\beta_2\\
J_3^- =&\beta_3\\
J_1^+ =&-\beta_1\gamma_1^2+\beta_2\gamma_3+(k+1)\partial\gamma_1+i\alpha'_+\gamma_1\vec{e}_1\cdot\partial\vec{\varphi}\\
J_2^+ =&\beta_1(\gamma_1\gamma_2-\gamma_3)-\beta_2\gamma_2^2-\beta_3\gamma_2\gamma_3+k\partial\gamma_2+i\alpha'_+\gamma_2\vec{e}_2\cdot\partial\vec{\varphi}\\
J_3^+ =&\beta_1(\gamma_1^2\gamma_2-\gamma_1\gamma_2)-\beta_2\gamma_2\gamma_3-\beta_3\gamma_3^2+k\partial\gamma_3-(k+1)\partial\gamma_1\gamma_2\\
&+i\alpha'_+\gamma_3\partial\varphi_1-i\alpha'_+\gamma_1\gamma_2\vec{e}_1\cdot\partial\vec{\varphi}
\end{aligned}
\tag{2.15}
$$

where $\alpha'_+=\sqrt{2(k+3)}$, $\vec{e}_1=\frac{1}{2}(1,\sqrt{3})$, $\vec{e}_2=\frac{1}{2}(1,-\sqrt{3})$, $\vec{\varphi}=(\varphi_1,\varphi_2)$; and where the free fields have the following operator product expansions

$$\partial\varphi_i(z)\,\partial\varphi_j(w)=\frac{-\delta_{ij}}{(z-w)^2}+\text{reg} \tag{2.16}$$

$$\beta_i(z)\,\gamma_j(w)=\frac{\delta_{ij}}{z-w}+\text{reg} \tag{2.17}$$

Computing T_{def} in terms of the free fields one finds that it splits into two parts

$$T_{\text{def}}(z)=T_\varphi(z)+T_{\beta,\gamma}(z)\ ; \tag{2.18}$$

where T_φ corresponds to two free bosons—one of them with a background charge:

$$T_\varphi=-\frac{1}{2}((\partial\varphi_1)^2+(\partial\varphi_2)^2)+i2\alpha'_0\partial^2\varphi_1\ , \tag{2.19}$$

where $\alpha'_0=(k+2)/\alpha'_+$; and where $T_{\beta,\gamma}$ is the stress tensor corresponding to three bosonic (β,γ) systems of weights $(0,1)$, $(0,1)$, and $(-1,2)$:

$$T_{\beta,\gamma} = -\partial\beta_1\,\gamma_1 - \partial\beta_2\,\gamma_2 - 2\partial\beta_3\,\gamma_3 - \partial\beta_3\partial\gamma_3\ . \tag{2.20}$$

Notice that the free field expressions for the currents associated to the negative roots do not involve the free bosons. Since these are the only currents entering in the definition of the BRST operator, this operator commutes with the free bosons and, thus, T_φ and $T_{\beta,\gamma} + T^{\rm gh}$ are separately BRST invariant. It is easy to show[**12**] that $T_{\beta,\gamma} + T^{\rm gh}$ can be written as a BRST anticommutator and hence it does not contribute in cohomology. On the other hand the only way that T_φ can be trivial in cohomology is if the BRST cohomology itself vanishes completely. We will prove in the next section that, in fact, the BRST cohomology is isomorphic to the Fock space $\mathcal{H}_\varphi$ of the free bosons. Therefore, under this isomorphism, the operator in cohomology induced by the BRST invariant stress tensor $T_{\rm tot}$ is precisely T_φ. This, is turn, is precisely the stress tensor in the free boson representation for W_3 constructed by Fateev and Zamolodchikov in [**3**] .

In order to complete the construction of the W_3 algebra we need to find a BRST invariant primary field *in terms of currents* which induces in BRST cohomology an operator obeying, together with the operator induced by $T_{\rm tot}$, the operator product algebra defining W_3. By analogy with the work of [**5**] we take as our starting point the cubic Casimir operator[1]

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[1] Our d-symbols are normalised as follows:

$$\begin{aligned}
d^{h^-h^-h^-} &= 1 & d^{h^+J_2^+J_2^-} &= \sqrt{3}\\
d^{h^-h^+h^+} &= -1 & d^{h^-J_2^+J_2^-} &= 1\\
d^{h^+J_1^+J_1^-} &= -\sqrt{3} & d^{h^-J_3^+J_3^-} &= -2\\
d^{h^-J_1^+J_1^-} &= 1 & d^{J_3^\pm J_1^\mp J_2^\mp} &= -2\sqrt{3}
\end{aligned}$$

$$Q^{(3)} = d^{abc} \left(X_a X_b X_c \right) \ , \tag{2.21}$$

which is primary of conformal weight 3 with respect to the Sugawara stress tensor. We then deform it by adding terms of lower order in currents and we complete it to a BRST invariant $T_{\rm tot}$-primary operator by adding ghost dependent terms. Doing this we find, after an extremely tedious albeit straight forward calculation, that the following operator is both BRST invariant and primary of weight 3 with respect to $T_{\rm tot}$:

$$W = Q^{(3)} + (k+3) \left[X^{(0,0)} + X^{(1,1)} + X^{(2,2)} \right] \ , \tag{2.22}$$

where

$$\begin{aligned} X^{(0,0)} =& 3 \left[\left(h^- \partial h^+ \right) + 3 \left(h^+ \partial h^- \right) \right] + 6\sqrt{3} \left[\left(J_1^- \partial J_1^+ \right) + \left(J_2^- \partial J_2^+ \right) \right] \\ & + 4\sqrt{3}(k+1) \left[\left(J_2^+ \partial J_2^- \right) + \left(J_1^+ \partial J_1^- \right) \right] \ , \end{aligned} \tag{2.23}$$

$$\begin{aligned} X^{(1,1)} =& -6\sqrt{3}(k+2) \left[\left(\partial^2 b_2 c_2 \right) - \left(\partial^2 b_2 c_2 \right) \right] + 6 \left[\left(h^- b_3 \partial c_3 \right) - \left(h^- \partial b_3 c_3 \right) \right] \\ & - 18 \left(\partial h^- b_3 c_3 \right) - 4\sqrt{3}(k^2+4k+6) \left[(\partial b_2 \partial c_2) - (\partial b_1 \partial c_1) \right] \\ & - 3(2k+3) \left[\left(h^- \partial b_2 c_2 \right) + \left(h^- \partial b_1 c_1 \right) \right] \\ & + \sqrt{3}(2k+11) \left[\left(h^+ \partial b_2 c_2 \right) - \left(h^+ \partial b_1 c_1 \right) \right] \\ & + 2\sqrt{3}(2k-1) \left[\left(J_2^- \partial b_1 c_3 \right) + \left(J_1^- \partial b_2 c_3 \right) \right] \\ & + 4\sqrt{3}(k+1) \left[\left(J_2^+ \partial b_3 c_1 \right) + \left(J_1^+ \partial b_3 c_2 \right) \right] \\ & - 6\sqrt{3} \left[\left(\partial J_2^+ b_3 c_1 \right) + \left(\partial J_1^+ b_3 c_2 \right) \right] \\ & + 4\sqrt{3}(k+4) \left[\left(J_2^+ b_3 \partial c_1 \right) + \left(J_1^+ b_3 \partial c_2 \right) \right] \ , \end{aligned} \tag{2.24}$$

and

$$X^{(2,2)} = -24 \left[(b_3 \partial b_2 c_2 c_3) - (b_3 \partial b_1 c_1 c_3) \right] \ . \tag{2.25}$$

Moreover, we find that this is the unique (up to BRST coboundaries) such operator. Plugging W into the free field representation we find that it splits into a piece W_φ

which only depends on the free bosons

$$
\begin{aligned}
W_\varphi =& i(\alpha'_+)^3 \left[(\partial\varphi_2\partial\varphi_2\partial\varphi_2) - 3\,(\partial\varphi_2\partial\varphi_1\partial\varphi_1)\right] \\
& - 3(\alpha'_+)^2(k+2)\left[\left(\partial\varphi_2\partial^2\varphi_1\right) + 3\left(\partial\varphi_1\partial^2\varphi_2\right)\right] \\
& + 6i\alpha'_+(k+2)^2\partial^3\varphi_2
\end{aligned} \tag{2.26}
$$

and a messy second part which is BRST exact.

Under the isomorphism between BRST cohomology and $\mathcal{H}_\varphi$, the operator which W induces in cohomology is precisely W_φ which, up to a multiplicative constant which we are free to chose, coincides with the free boson realization of the weight three primary field of the W_3 algebra as computed by Fateev and Zamolodchikov in **[3]** . Therefore we conclude that the operators induced by T_{tot} and W in BRST cohomology satisfy the operator product algebra of W_3. Therefore we conclude that the reduced operator algebra obtained from $A_2^{(1)}$ contains W_3 as a subalgebra. In this way given any $A_2^{(1)}$-module $\mathfrak{M}$, the BRST cohomology $H_d(\mathfrak{M}\otimes\mathcal{H}_{\text{gh}})$ inherits the structure of a W_3 module.

We shall have more to say about the nature of this correspondence in the concluding section, but now several remarks about the construction of W are in order. First of all, it is worth noticing that the ghost independent part of W is not primary with respect to the deformed stress tensor T_{def}. In fact, if one starts with the most general deformation of $Q^{(3)}$ (obtained by adding local terms of lower order in currents) which is primary with respect to T_{def}, one finds — in our case, the hard way — that it is impossible to promote it to a BRST invariant opearator by only adding ghost dependent pieces.

A second remark is that the expression for W is, of course, not unique since one can always add BRST coboundaries of primary fields (*e.g.* , $\left(J_3^+ b_3\right)$) without altering its cohomology class. We hope that via this method one can find another representative for the cohomology class of W which allows one to recognize a structure which could suggest a calculationally less involved method to generalize this construction to $\mathfrak{sl}_n$ for general n. This should yield, in their free boson representations, the operator product algebras of the W_n algebras of Fateev and Lykyanov[4].

A third remark is that the appearance of the cubic Casimir is unavoidable. If one works backwards from the expression for W_φ and one tries to construct a BRST invariant operator in terms of currents which reproduces it in cohomology, one is led immediately to the cubic terms in $Q^{(3)}$ which consist purely of Cartan generators. In order to then promote it to a BRST invariant operator one is forced to introduce futher terms cubic in currents which, at the end of the day, recover precisely the cubic terms appearing in the cubic Casimir (and no other cubic terms can appear).

The fourth and final remark is that we have not determined whether the operator algebra of $T_{\rm tot}$ and W is already that of W_3 before descending to BRST cohomology; or, in fact, whether this could be achieved by the addition of BRST coboundaries. This is an interesting question that might be worth pursuing if only to try and obtain some insight into the form of W.

§3 The Cohomology of the BRST Operator

In this section we conclude the construction by providing a proof of the isomorphism between the BRST cohomology and the Fock space of the free bosons in the free field representation of $A_2^{(1)}$. In other words, if $\mathcal{H} \equiv \mathcal{H}_\varphi \otimes \mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{bc}$ is the full Fock space of the fields in the free field representation of $A_2^{(1)}$, then we will prove that

$$H_d(\mathcal{H}) \cong \mathcal{H}_\varphi \; . \tag{3.1}$$

In terms of the free fields, the BRST operator d corresponding to the constraints (2.5) is the closed contour integral around the origin of the following current:

$$j(z) = -c_1(z) - c_2(z) + \sum_{i=1}^{3} \left(\beta_i c_i\right)(z) + \left(c_1 c_2 b_3\right)(z) + \left(c_1 \gamma_2 \beta_3\right)(z) \; . \tag{3.2}$$

Since it is independent of $\partial\varphi_i$ the BRST operator is the identity in $\mathcal{H}_\varphi$ and therefore

$$H_d(\mathcal{H}) = \mathcal{H}_\varphi \otimes H_d(\mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{bc}) \ , \tag{3.3}$$

whence (3.1) is equivalent to proving that

$$H_d(\mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{bc}) \cong \mathbb{C} \ , \tag{3.4}$$

to which the rest of this section is devoted. The idea of the proof is very simple: we will filter $C \equiv \mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{bc}$ in such a way that the computation of $H_d(C)$ is reduced to the computation of the cohomology of a simpler operator.

To this end let us define a grading on C. We define the following *filtration degree*:

$$\operatorname{fdeg} b_i = \operatorname{fdeg} \beta_i = -1 \qquad \operatorname{fdeg} c_i = \operatorname{fdeg} \gamma_i = 1 \ . \tag{3.5}$$

Assigning zero filtration degree to the $\mathfrak{sl}_2$ invariant vacuum Ω_0 defines fdeg on all of C, since this is generated by the modes of $\{b_i, c_i, \beta_i, \gamma_i\}$ acting on Ω_0. This vacuum is annihilated by the following modes: $(\beta_i)_n$ and $(b_i)_n$ for $i = 1, 2$ and for all $n > 0$; by $(\gamma_i)_n$ and $(c_i)_n$ for $i = 1, 2$ and for all $n \geq 0$; by $(\beta_3)_n$ and $(b_3)_n$ for all $n > 1$; and by $(\gamma_3)_n$ and $(c_3)_n$ for all $n \geq -1$.

Let us define $C_n \equiv \{\omega \in C \mid \operatorname{fdeg} \omega = n\}$. Then $C = \bigoplus_{n\in\mathbb{Z}} C_n$. According to this grading we can split d into two terms of different filtration degrees: $d = d_0 + d_1$ where d_i is the closed contour integral around the origin of $j_i(z)$, where

$$j_0(z) = \sum_{i=1}^{3} (\beta_i c_i)(z) \tag{3.6}$$

and

$$j_1(z) = -\, c_1(z) - c_2(z) + (c_1 c_2 b_3)(z) + (c_1 \gamma_2 \beta_3)(z) \ . \tag{3.7}$$

Because the filtration degree is compatible with the operator algebra, breaking up the equation $d^2 = 0$ into its homogeneous terms we find that $d_0^2 = \{d_0\, , \, d_1\} = d_1^2 =$

0 separately. Therefore the cohomology of d_0 is defined. We will compute it next and later show that it is isomorphic to the cohomology of d.

As a technical aside for readers familiar with spectral sequences let us mention that if we define $F^pC \equiv \bigoplus_{n\geq p} C_n$, then (FC, d) becomes a filtered complex. This filtration gives rise to a spectral sequence whose E_1 term is isomorphic to the cohomology of d_0. Since d_1 is d_0-exact — namely $j_1 = -[d_0\,, \gamma_1 + \gamma_2 + (c_1\gamma_2 b_3)]$ — the differential induced by d on E_1 is identically zero and, hence, the spectral sequence degenerates at the E_1 term. However, the filtration is not bounded and, therefore, one cannot conclude[2] that the limit term is isomorphic to the cohomology of d. However the case at hand is not too pathological, it seems, and by bounding the filtration from above we will see that the cohomology of d_0 is indeed isomorphic to that of d; although we will not use the existence of a convergent spectral sequence to prove it.

We now proceed with the proof of the following preliminary result.

Lemma. $H_{d_0}(C) \cong \mathbb{C}$.

Proof: Let us introduce the following notation. For every kind of field let us define the following creation ($a^\dagger$) and annihilation (a) operators:

$$\left.\begin{aligned} a^\dagger(\beta_i)_p &\equiv (\beta_i)_{-p} \\ a(\beta_i)_p &\equiv -(\gamma_i)_p \end{aligned}\right\} \forall p \geq 0,\ i = 1,2$$
$$\left.\begin{aligned} a^\dagger(\beta_3)_p &\equiv (\beta_3)_{-p} \\ a(\beta_3)_p &\equiv -(\gamma_3)_p \end{aligned}\right\} \forall p \geq -1$$
$$\left.\begin{aligned} a^\dagger(\gamma_i)_p &\equiv (\gamma_i)_{-p} \\ a(\gamma_i)_p &\equiv (\beta_i)_p \end{aligned}\right\} \forall p \geq 1,\ i = 1,2 \qquad (3.8)$$
$$\left.\begin{aligned} a^\dagger(\gamma_3)_p &\equiv (\gamma_3)_{-p} \\ a(\gamma_3)_p &\equiv (\beta_3)_p \end{aligned}\right\} \forall p \geq 2$$

[2] We thank Jim Stasheff for confirming this observation.

and the same for b and c except for the minus signs in the definitions of $a(b_k)$. With this notation and if X is a free field

$$[a(X)_p \, , \, a^\dagger(X)_q]_\pm = \delta_{pq} \ , \tag{3.9}$$

for allowed p, q. A short calculation further shows that

$$d_0 = \sum_{\substack{i,p \\ \text{allowed}}} \Big(a^\dagger(\beta_i)_p a(b_i)_p + a^\dagger(c_i)_p a(\gamma_i)_p \Big) \ . \tag{3.10}$$

Let us define the operator K as follows

$$K \equiv \sum_{\substack{i,p \\ \text{allowed}}} \Big(a^\dagger(b_i)_p a(\beta_i)_p + a^\dagger(\gamma_i)_p a(c_i)_p \Big) \ . \tag{3.11}$$

This operator obeys $\{d_0 \, , \, K\} = N$, where N is the full number operator diagonalised by the basis states

$$\prod_{\substack{i,p \\ \text{allowed}}} \Big[a^\dagger(\beta_i)_p \Big]^{k_{i,p}} \Big[a^\dagger(b_i)_p \Big]^{k'_{i,p}} \Big[a^\dagger(\gamma_i)_p \Big]^{l_{i,p}} \Big[a^\dagger(c_i)_p \Big]^{l'_{i,p}} \cdot \Omega_0 \tag{3.12}$$

with eigenvalue

$$\sum_{\substack{i,p \\ \text{allowed}}} \big(k_{i,p} + k'_{i,p} + l_{i,p} + l'_{i,p} \big) < \infty \ , \tag{3.13}$$

where $k_{i,p}$ and $l_{i,p}$ take integer non-negative values and $k'_{i,p}$ and $l'_{i,p}$ are either 0 or 1. Since N commutes with d_0, the cohomology of d_0 splits into a direct sum

$$H_{d_0}(C) = \bigoplus_{n \geq 0} H_{d_0}(C^{(n)}) \ , \tag{3.14}$$

where $C^{(n)}$ is the eigenspace of N with eigenvalue n. But since N is d_0-exact, the cohomology resides in $C^{(0)}$; for any d_0-cocycle $\omega \in C^{(n \neq 0)}$ is also a coboundary: $\omega = d_0 \frac{1}{n} K \, \omega$. But $C^{(0)}$ is spanned by the vacuum which is thus a non-trivial d_0-cocycle. ■

A remark is in order: notice that d_0 is the BRST charge corresponding to abelian constraints. As such we could have invoked the Kugo-Ojima quartet mechanism to deduce the lemma. However, we feel that the above proof is conceptually more transparent.

Before proving the isomorphism between the cohomologies of d and d_0 we need a technical remark. Since the stress tensor $T = T_{\beta,\gamma} + T^{\mathrm{gh}}$ is BRST exact, a similar argument to the one used in the proof of the lemma shows that all the BRST cohomology resides in the zero eigenspace of L_0, since L_0 is also BRST exact and diagonalisable. States with zero conformal weight are finite linear combinations of basis states of the form (3.12) which themselves have zero conformal weight. This imposes a linear relation among the occupation numbers which, since for a conformal field $\phi(z)$ one has that $[L_0\, ,\, \phi_n] = -n\phi_n$, is given by

$$\begin{aligned} 0 &= \sum_{\substack{i,p \\ \text{allowed}}} p\left(k_{i,p} + k'_{i,p} + l_{i,p} + l'_{i,p}\right) \\ &= \sum_{i,p\geq 2} p\left(k_{i,p} + k'_{i,p} + l_{i,p} + l'_{i,p}\right) + \sum_{i=1}^{2}\left(k_{i,1} + k'_{i,1} + l_{i,1} + l'_{i,1}\right) \\ &\quad + k_{3,1} + k'_{3,1} - k_{3,-1} - k'_{3,-1}\ , \end{aligned} \tag{3.15}$$

where we have also used that the vacuum has zero conformal weight due to $\mathfrak{sl}_2$ invariance. This relation implies that the filtration degree of a basis state of zero conformal weight is given by

$$\sum_{i,p\geq 0} (-1-p)\left(k_{i,p} + k'_{i,p}\right) + \sum_{i,p\geq 1} (1-p)\left(l_{i,p} + l'_{i,p}\right) \leq 0\ . \tag{3.16}$$

Therefore we can conclude that, since L_0 has zero filtration degree, no state $\omega \in C$ of zero conformal weight can contain homogeneous terms of positive filtration degree. In other words, the filtration degree of zero conformal weight states is bounded above. This remark will be crucial in the proof of the following theorem.

Theorem. *Let C^{L_0} denote the subspace of C of zero conformal weight. Then*

$$H_d(C) \cong H_d(C^{L_0}) \cong H_{d_0}(C) \ .$$

Proof: The first isomorphism follows from the above remarks. To prove the second isomorphism it is clearly enough to show that every L_0-invariant d-cocycle is d-cohomologous to a multiple of the vacuum. Let $\psi \in C^{L_0}$ be a d-cocycle. We say that ψ has *order* p if ψ contains no homogeneous term of filtration degree less than p. Clearly, by the above remarks, p is a non-positive integer. If $p = 0$ then $d\psi = 0$ implies that $d_0\psi = 0$. By the lemma, $\psi = \alpha\Omega_0 + d_0\xi$ where ξ is some vector of zero conformal weight and zero filtration degree. Since $d_1\xi$ has also zero conformal weight (since $[d_1 \, , \, L_0] = 0$) but positive filtration degree it must be zero, whence $d_0\xi = d\xi$. Therefore ψ is cohomologous to a multiple of the vacuum. Therefore the theorem is true for cocycles of order zero. Suppose that the theorem has been proven for cocycles of order up to $p + 1$. Then let ψ be a d-cocycle of order $p < 0$. Under the grading of C^{L_0} induced by the filtration degree, ψ splits as $\psi = \psi_p \oplus \psi_{p+1} \oplus \cdots \oplus \psi_0$, where $\psi_n \in C_n^{L_0}$. Since $d\psi = 0$, $d_0\psi_p = 0$ which, since $p < 0$, implies (by the lemma) that $\psi_p = d_0\xi_p$ for some $\xi_p \in C_p^{L_0}$. Let $\psi' = \psi - d\xi_p$. Then ψ' is a d-cocycle of order $p + 1$ which, by the induction hypothesis, is cohomologous to a multiple of the vacuum. Therefore the theorem holds for oder p and by backwards induction we are done. ■

It turns out that the analogous statement to (3.1) for a general simple finite dimensional Lie algebra is also true and that the proof follows essentially the same lines as in the case of $\mathfrak{sl}_3$. The precise statement of the theorem, as well as its proof and some applications will be discussed elsewhere[15].

§4 Conclusions

In this paper we have provided a (quantum) homological construction of the W_3 extended conformal algebra starting from the affine algebra $A_2^{(1)}$, different from the coset construction of **[7]** . It is instructive to compare the two constructions since they both have their merits and their shortcomings.

The coset construction seems to be more natural when it comes to building the minimal unitary representations of W_3 since the relevant values of the central charge are obtained starting from $A_2^{(1)} \times A_2^{(1)}$ at level $(k, 1)$, where $k \in \mathbb{N}$, which corresponds to unitary representations of $A_2^{(1)} \times A_2^{(1)}$. Similarly the fusion rules for the minimal models of W_3 behave qualitatively like the $\otimes$-decomposition of irreducible representations of $A_2 \times A_2$, which suggests a closer connection to the coset construction than to the homological one.

On the other hand, the homological construction is more economical in the sense that the space of states of the resulting quantum theory (the zero ghost number BRST cohomology, in the homological construction) is smaller than the resulting space in the coset construction. In fact, a naive count of the degrees of freedom shows that the coset construction yields $\dim A_2 = 8$ degrees of freedom whereas the homological construction yields only $\operatorname{rank} A_2 = 2$ degrees of freedom corresponding precisely to the two Virasoro primary fields generating W_3.

Also the homological construction seems to be more natural when discussing free field representations since it intertwines directly between the Feigin-Fuchs representations for $A_2^{(1)}$ and W_3. Thus constructions which rely on Feigin-Fuchs representations (*e.g.* , computation of Kač determinant, Fock space resolutions) are bound to correspond via the homological reduction. This is exemplified for the case of $A_1^{(1)} \leftrightarrow$ Virasoro in **[12]** . In particular there is good evidence to support the fact that at least for completely degenerate representations the homological reduction of an irreducible $A_2^{(1)}$ module yields an irreducible module of W_3. This is to be contrasted with the coset construction where one needs to project out the irreducible modules after a detailed analysis of branching relations.

Unfortunately both constructions are computationally involved and neither invites an immediate generalization to more complicated algebras. The homological construction, however, may offer some hope in this context provided that a Coulomb gas representation exists for the extended algebra. It can be proven[15] that the analogous result to the theorem in section 3 is still true for an arbitrary finite dimensional simple Lie algebra $\mathfrak{g}$. Thus this construction provides one with a resolution of the Fock space of the rank $\mathfrak{g}$ free bosons in terms of the free field Fock space of $\widehat{\mathfrak{g}}$ and the ghosts (*i.e.* , $\mathcal{F} \equiv \mathcal{H}_\varphi \otimes \mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{\rm gh}$). Then general results in homological algebra guarantee the existence of a lift of any operator in $\mathcal{H}_\varphi$ to a BRST invariant operator in $\mathcal{F}$. However there is nothing that guarantees that the operator can be written in terms of currents (and ghosts, of course) in a local way. Work is in progress towards this with the hope that one may at least give an existence proof for general $\mathfrak{g}$. Nevertheless, even if this existence proof were possible there is no ingredient in the method which would suggest that the lifted operators are in any way related to the casimirs of $\mathfrak{g}$. This is clearly something to be understood from a different perspective.

Finally we should mention that there are two other cases that are still tractable computationally. These are the ones corresponding to the other rank 2 algebras: $B_2 = C_2$ and G_2 which should yield generalizations of W_3 with one extra primary field each of spins 4 and 6 respectively. Work towards this is in progress[16].

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